Stability of Local Minima and Stable Nonconstant Equilibria *

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1 Introduction

In 1979, H. Matano [19] gave an important result on the existence of stable nonconstant equilibrium solutions of reaction-diffusion equations with homogeneous Neumann boundary conditions

$$\begin{cases} u_t = \Delta u + k\, f(u) & \text{in } D\,, \\ u_\nu = 0 & \text{on } \partial D\,, \end{cases} \tag{1}$$

for some nonconvex domains D. For the same question, more results were obtained later, with different methods, by other authors like J.K. Hale and J.M. Vegas, [11], S. Jimbo, [14] and E. Yanagida, [28].

In 1995 we wrote a short note [5] pointing out that the proof of H. Matano's results could also be extended, with the suitable changes, to prove existence of stable nonconstant equilibria of diffusion equations with nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u & \text{in } D\,, \\ u_\nu = k\, f(u) & \text{on } \partial D\,. \end{cases} \tag{2}$$

In that note we mentioned that problem (2) is also a model of reaction and diffusion like (1), but when the reaction happens only at the boundaries of the container, for example because of the presence of a solid catalyzer. A detailed justification of the apparence of (2) in a combustion problem appears at the end of the paper [16] and earlier motivations in [3]. Other types of stable nonconstant equilibrium solutions for problem (2), in the case that ∂D has several connected components, were obtained also in [3] for one space dimension and in [6] for the multidimensional case (see also [17]).

The proofs in our note were presented, as we said, following the paths of [19], with the suitable changes. But there was a point where we used a really different argument, reliying on the one-dimensionality of central manifolds for local minima instead of using the monotonicity of the flow. And we have recently realized that our argument can also be applied to existence of stable nonconstant equilibria also for non-monotonic problems like the strongly damped wave equation

$$\begin{cases} u_{tt} - a\,\Delta u_t + b\,u_t = \Delta u + k\, f(u) & \text{in } D\,, \\ u_\nu = 0 & \text{on } \partial D\,, \end{cases} \tag{3}$$

for $a, b > 0$, that has the same equilibrium solutions as (1) but which stability, at least in the critical cases, it is no clear if it is the same for the two cases.

Our results on stability will use the fact that the equilibria that we find are local minima of suitable Lyapunov functionals J, that in problems (1), (2) and (3) are the well known energy functionals. But since (1) and (2) are the strict gradient flows of these functionals, we have been asking ourselves if local minima are automatically stable in gradient flows. The general answer is that they need not to be so, except if the phase-space is one-dimensional, and we have constructed a counter-example in two dimensions, that we include in the present paper. This example makes more clear the use in our stability arguments of the condition for the central manifold to be one-dimensional. It is also worth noting that our example is smooth (of class $\mathcal{C}^\infty$) but that it also proved that it can not be real-analytic.

So the main purposes of the present paper are to present a complete proof of the results of [5] for the problem (2), with special emphasis on our argument on the central manifold for the local minima, to show that this argument can be used for equation (3), and to present an example of instability of local minima for gradient flows in finite dimensions. Section 2 below is completely devoted to the abstract reasoning on stability of local minima using the central manifold, in section 3 this argument is applied to (2), so obtaining the results of [5], in section 4 problems (1) and (3) are discussed, and section 5 presents the example of instability of local minima for gradient flows in two dimensions.

2 A result on stability of local minima

The initial value problem for semilinear equations of type

$$u_t = Au + F(u) , \tag{4}$$

where A is linear and F is nonlinear, gives a dynamical system in a Banach space X in several functional settings. Must of them fit into the following framework: there are four Banach spaces $X_0 \subseteq X \subseteq Y \subseteq Y_0$, X_0 is the domain of A as a closed operator of the space Y_0 and F is a regular map from X to Y. In [24], in a formulation which is suitable for wave-like equations, A is the generator of a $\mathcal{C}^0$ semigroup in Y and $X = Y = Y_0$. For parabolic-like equations one possible formulation is that of [23] or [1], where A generates an analytic semigroup in $X_0 = X$ and $Y \not\subseteq Y_0$, and another possibility is that of [12] where A also generates an analytic semigroup on X_0, $X = D(A^\alpha)$ for some $0 < \alpha < 1$ and $Y = Y_0$.

In this section we suppose that we are in one of these cases and that the following hypotheses are satisfied.

(H1) The equation (4) defines a local semidynamical system $T(t)$ in the space X (defined either for strict or mild solutions of (4)).

(H2) There is a Lyapunov functional $J : X \to \mathbb{R}$ that is a continuous function that decreases strictly except at equilibria.

(H3) The equation (4) is such that if $e_0 \in X_0$ is an equilibrium point ($Ae_0 + F(e_0) = 0$) then the following property holds for the spectrum of the linear part $L = A + DF(e_0)$: if $\sigma(L) \subset \{\mathrm{Re}\ \lambda \leq 0\}$ but $\sigma(L) \cap \{\mathrm{Re}\ \lambda = 0\} \neq \emptyset$ then $\sigma(L) \cap \{\mathrm{Re}\ \lambda = 0\} = \{0\}$ and 0 is an algebraically simple eigenvalue.

(H4) In addition of (H3) we will suppose, as it often happens, that

(i) If $\sigma(L) \subset \{\text{Re } \lambda < 0\}$ then e_0 is asymptotically stable.

(ii) If $\sigma(L) \cap \{\text{Re } \lambda > 0\} \neq \emptyset$ then there exists a nonconstant solution $u(t)$ of (4) such that $u(t) \to e_0$ as $t \to -\infty$.

(iii) If $\sigma(L) \subset \{\text{Re } \lambda \leq 0\}$ but $\sigma(L) \cap \{\text{Re } \lambda = 0\} = \{0\}$ (as in (H3)) then there exists a local central manifold $\mathcal{M}$, which is invariant, one-dimensional and tangent in e_0 to the eigenvector associated to the eigenvalue $\lambda = 0$ with the property that e_0 is Lyapunov stable in X if and only if it is stable in $\mathcal{M}$.

(See [4] for a recent and more general approach to local properties near equlibria like (H4)).

Under these assumptions we are going to prove the following general theorem, that will be applied to several specific problems in the next sections.

Theorem 2.1 *Let the hypotheses (H1)-(H4) hold and let e_0 be a local minimum of the functional J. Then e_0 is a stable equilibrium of (4).*

Proof. As e_0 is a local minimum of the functional J it cannot decrease in time. Then, by the hypothesis (H2), e_0 must be an equilibrium point of (4).

To see that e_0 is stable we have to consider three cases depending on the location of the spectrum of the linear operator L.

The case $\sigma(L) \cap \{\text{Re } \lambda > 0\} \neq \emptyset$ is not possible by the hypothesis (H4(ii)), because it implies that e_0 can not be a local minimum of the functional J.

In the case that $\sigma(L) \subset \{\text{Re } \lambda < 0\}$, by the hypothesis (H4(i)) e_0 is asymptotically stable.

For the case $\sigma(L) \subset \{\text{Re } \lambda \leq 0\}$ but $\sigma(L) \cap \{\text{Re } \lambda = 0\} = \{0\}$, (H4(iii)) says that if the equilibrium point e_0 is stable inside $\mathcal{M}$ (which is one-dimensional) then e_0 is also stable in X.

But let us see that in dimension one a local minimum of a Lyapunov function is always stable. We can consider, without loss of generality, that $\mathcal{M}$ is the interval $-r < x < r$, the equilibrium point $e_0 = 0$ and $J(0) = 0$. We are going to see that 0 is stable from the right and the same arguments prove the stability from the left. We consider two cases depending on whether 0 is or not a strict minimum of the functional J in $[0, r)$.

First we consider that 0 is a strict minimum of J in $[0, r)$. As $J(0) = 0$ there exists $r_1 < r$ such that $J(x) > 0$ in $(0, r_1]$. Given $\varepsilon > 0$, let J_ε be the minimum of J on $[\varepsilon, r_1]$. As J is a continuous function there exists $\delta > 0$ such that for $x \leq \delta$ we have $J(x) < J_\varepsilon$. So, if $x \in [0, \delta]$, as J decreases in time, $J(T(t)x) < J_\varepsilon$ for all $t \geq 0$. Then, $T(t)x \notin [\varepsilon, r_1]$ for all $t \geq 0$ and 0 is stable because $T(t)x \in [0, \varepsilon]$.

To finish off let us consider the case where $x = 0$ is not a strict minimum in $[0, r)$. In this case there exists a sequence $x_n \to 0$, as $n \to \infty$, such that $J(x_n) = 0$. These x_n are equilibria. For any $\varepsilon > 0$ there is an equilibrium x_ε such that $0 < x_\varepsilon < \varepsilon$. Then the end points of the interval $[0, x_\varepsilon]$ are equilibria. So the interval $[0, x_\varepsilon]$ is a positively invariant set. Now, taking $\delta = x_\varepsilon$ and $x \in [0, \delta]$ we have $T(t)x \in [0, \varepsilon]$, for all $t > 0$. That is, 0 is stable.

Therefore the equilibrium point e_0 always is stable inside $\mathcal{M}$ and so it is stable in X. □

Remark 2.1 Without the hypothesis (H3) the central manifold $\mathcal{M}$ could have dimension bigger than one, and then inside $\mathcal{M}$ the local minima would not need to be stable. This could be true even for the gradient flow of J, as the example of section 5 shows.

3 The Diffusion Equation with Nonlinear Boundary Conditions

We are going to present the result of existence of nonconstant stable equilibria for (5). It was already announced by the authors, together with a sketch of the proof, in [5]. We consider the problem of a diffusion equation with nonlinear boundary conditions

$$\begin{cases} u_t = \Delta u & \text{in } D\,, \\ u_\nu = k\,f(u) & \text{on } \partial D\,, \end{cases} \tag{5}$$

with $D \subset \mathbb{R}^n$ a bounded domain with regular boundary ∂D and the function $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$. Following the approach of H. Amann [1], the problem (5) admits a semilinear formulation, that is, it can be written in the form

$$u_t = Au + F(u)\,. \tag{6}$$

Here A is the linear part and it maps W^1_p on $(W^1_{p'})'$ (the dual space of W^1_p) with $p > n$, $p \geq 2$ and $1/p + 1/p' = 1$ and it is defined as follows: $Au(v) = -\int_\Omega (\nabla u\,\nabla v + uv)\,dx$. The nonlinear function F maps W^1_p not only on $(W^1_{p'})'$ but on a smaller space E and it is defined by $F(u) = u + \gamma'_{p'} f(\gamma_p u)$, where γ_p and $\gamma'_{p'}$ denote the trace on the boundary in L^p and the dual of the trace operator on the boundary in $L^{p'}$.

Using interpolation results (see [2]) one can see that A is the infinitessimal generator of an analytic semigroup $\{e^{-At}\,,\ t \geq 0\}$ in W^1_p. Also we have a dynamical system

$$T(t)u_0 = e^{At}u_0 + \int_0^t e^{A(t-\tau)} F(T(\tau)u_0)\,d\tau\ ,$$

for a given $u_0 \in W^1_p$, in the space W^1_p. (See [6], [7] for the details). So the equation (6) satisfies the hypothesis $(H1)$. (See [20], [21] for a different functional setting).

Let us consider the functional $J : W^1_p \to \mathbb{R}$ defined by

$$J(u) = \int_D \frac{1}{2}(\nabla u)^2\,dx - \int_{\partial D} k\,\varphi(u)\,d\ell\,, \tag{7}$$

where $\varphi(u) = \int_0^u f(s)\,ds$. It can be proved that J is continuous and twice differentiable with continuity. As $u_t \in W^1_p$ we can derive $J(u)$ with respect to t and aplying Green's formula we obtain

$$\frac{d}{dt}J(u) = -\int_D (u_t)^2\,dx \leq 0\ .$$

So J is decreasing in time except at equilibria and the hypothesis (H2) holds. (See [7] for all the details).

For the spectrum of the linear operator $L = A + DF(e_0)$ (e_0 an equilibrium point of (6)) we know (theorem 2.3 in [6]) that the first eigenvalue of $\sigma(L)$ is

$$\lambda_0 = \sup_{\substack{u \in W_2^1 \\ u \neq 0}} \frac{\int_D -(\nabla u)^2 \, dx + \int_{\partial D} f'(e_0) u^2 \, d\ell}{\int_D u^2 \, dx} .$$

If $\sigma(L) \subset \{\mathrm{Re}\,\lambda \leq 0\}$ but $\sigma(L) \cap \{\mathrm{Re}\,\lambda = 0\} \neq \emptyset$ necessarily $\sigma(L) \cap \{\mathrm{Re}\,\lambda = 0\} = \{0\}$ because only real eigenvalues are possible. Then Proposition 3.2 in [6] proves that if $\lambda_0 = 0$ then λ_0 is a simple eigenvalue. So hypothesis (H3) holds for (3.2).

For the hypothesis (H4), Theorem 2.2 in [6] gives a principle of stability and instability that ensures the hypotheses (H4(i)) and (H4(ii)). Also hypothesis (H4(iii)) holds when the nonlinearity f is a $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$ function (see [26]).

So all hypotheses of section 2 hold and we are going to use Theorem 2.1 to prove that there exists a nonconstant stable equilibrium solution for (5) under some additional assumptions on f and D.

The main result in this section is the next theorem.

Theorem 3.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying*

(i) $f(a) = f(0) = f(b) = 0$ *for some* $a < 0 < b$.

(ii) $0 < uf(u) < u^2$ *for* $a < u < b$ *and* $u \neq 0$.

(iii) Defining $\varphi(u) = \int_0^u f(s)\, ds$, *let us assume that* $\varphi(b) \geq \varphi(a)$.

Let $D \subset \mathbb{R}^n$ with $n \geq 2$ be a smooth bounded domain. Let D_1 and D_2 be two subdomains of D with smooth boundaries and Γ_i be a smooth portion of $\partial D_i \cap \partial D$ with $|\Gamma_i| > 0$ $(i = 1, 2)$ and $\rho_2(D_1)$ and $\rho_2(D_2)$ be the constants given in lemma 3.1 below. Choose $p > n$, so $W_p^1(D) \subset \mathcal{C}(\overline{D})$.

If the set

$$R = \{ v \in W_p^1(D) \ : \ a \leq v \leq b \ on\ \overline{D}, \int_{\Gamma_1} v d\ell < 0 \, , \int_{\Gamma_2} v \, d\ell > 0 \, , \\ J(v) < \varepsilon_0 - k\, \varphi(b)\, |\partial D| \, \}$$

**is nonempty, where J is as above and*

$$\varepsilon_0 = \varphi(b)\ min\ \{ |\Gamma_1| min\ \{k, \rho_2(D_1)\}, \ |\Gamma_2| min\ \{k, \rho_2(D_2)\}\} \, .$$

then problem (5) has at least one stable nonconstant equilibrium solution.

Lemma 3.1 *Let Ω be a smooth bounded domain. Then there exists a positive constant $\rho_2(\Omega)$, depending only on the domain, such that if $\omega \in W_2^1(\Omega)$ the inequality*

$$\int_{\partial\Omega} w^2\, d\ell \le \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla w)^2\, dx + \frac{1}{|\partial\Omega|}\left(\int_{\partial\Omega} w\, d\ell\right)^2 \tag{8}$$

holds. The optimal constant $\rho_2(\Omega)$ is the second eigenvalue of the Steklov problem

$$\begin{cases} \Delta w^i = 0 & \text{in } \Omega\ , \\ w^i_\nu = \rho_i w^i & \text{on } \partial\Omega\ . \end{cases}$$

Moreover, for a smooth portion Γ of $\partial\Omega$, with $|\Gamma| > 0$, the inequality

$$\int_\Gamma w^2\, d\ell \le \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla w)^2\, dx + \frac{1}{|\Gamma|}(\int_\Gamma w\, d\ell)^2 \tag{9}$$

also holds.

Proof. The second eigenvalue $\rho_2(\Omega)$ of the Stekloff problem can be characterized by

$$\rho_2(\Omega) = \min_{\int_{\partial\Omega} w\, d\ell = 0} \frac{\displaystyle\int_\Omega (\nabla w)^2\, dx}{\displaystyle\int_{\partial\Omega} w^2\, d\ell}\ .$$

(See [13]). Then, for $w \in W_2^1$ satisfying $\int_{\partial\Omega} w\, d\ell = 0$ we have

$$\int_{\partial\Omega} w^2\, d\ell \le \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla w)^2\, dx\ .$$

For any $w \in W_2^1$, let us consider $u = w - \overline{w}$, where $\overline{w} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} w\, d\ell$. Then $\overline{u} = 0$ and it satisfies the last inequality. Finally, by using the definition of $\overline{u}$ we obtain (8) as we wanted.

We are going to see (9). Let $u \in W_2^1$ be such that $\int_\Gamma u\, d\ell = 0$. Then

$$\int_{\partial\Omega} u^2\, d\ell \le \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla u)^2\, dx + \frac{1}{|\partial\Omega|}(\int_{\partial\Omega\setminus\Gamma} u\, d\ell)^2\ .$$

Now, by the Cauchy-Schwarz inequality

$$\int_{\partial\Omega} u^2\, d\ell \ \le\ \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla u)^2\, dx + \frac{|\partial\Omega \setminus \Gamma|}{|\partial\Omega|} \int_{\partial\Omega\setminus\Gamma} u^2\, d\ell\ .$$

Joining the boundary integrals in this inequality we obtain

$$\int_\Gamma u^2\, d\ell \le \frac{1}{\rho_2(\Omega)} \int_\Omega (\nabla u)^2\, dx\ , \tag{10}$$

for any $u \in W_2^1$, with $\int_\Gamma u\, d\ell = 0$.

Finally, for a given $\omega \in W_2^1$, let us consider $u = w - \overline{w}$, with $\overline{\omega} = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma w\, d\ell$. Clearly $\overline{u} = 0$, (10) holds for $\overline{u}$ and we obtain (9). □

Proof of Theorem 3.1.

Step 1. *The set R is positively invariant under $T(t)$.*

This is a consequence of the maximum principle, the fact that $J(u)$ decreases in time and the fact that D_1 and D_2 are such that (9) is satisfied:

Given an initial condition $u_0 \in R$, as $a \le u_0 \le b$, the maximum principle implies that $a \le T(t)u_0 \le b$, for $t \ge 0$.

Let us assume that there exists $t_i > 0$ such that $\displaystyle\int_{\Gamma_i} T(t_i)u_0\, d\ell = 0$ for $t = t_i$ and $i = 1$ or $i = 2$. Let be $u_i = T(t_i)u_0$ and let us aply the inequality (9) to u_i on D_i. That is,

$$\begin{aligned}\int_{\Gamma_i} u_i^2\, d\ell &\le \frac{1}{\rho_2(D_i)}\int_{D_i}(\nabla u_i)^2\, dx + \frac{1}{|\Gamma_i|}\Big(\int_{\Gamma_i} u_i\, d\ell\Big)^2\\ &= \frac{1}{\rho_2(D_i)}\int_{D_i}(\nabla u_i)^2\, dx\,.\end{aligned}$$

So,

$$\int_{D_i}(\nabla u_i)^2\, dx \ge \rho_2(D_i)\int_{\Gamma_i} u_i^2\, d\ell \ge 2\rho_2(D_i)\int_{\Gamma_i}\varphi(u_i)\, d\ell\,. \tag{11}$$

Let us consider $J(u_i)$:

$$\begin{aligned}J(u_i) &= \frac{1}{2}\int_D(\nabla u_i)^2\, dx - \int_{\partial D} k\,\varphi(u_i)\, d\ell\\ &= \frac{1}{2}\int_{D\setminus D_i}(\nabla u_i)^2\, dx - \int_{\partial D\setminus\Gamma_i} k\,\varphi(u_i)\, d\ell + \frac{1}{2}\int_{D_i}(\nabla u_i)^2\, dx\\ &- \int_{\Gamma_i} k\,\varphi(u_i)\, d\ell\\ &\ge \frac{1}{2}\int_{D_i}(\nabla u_i)^2\, dx - k\,\varphi(b)\,|\partial D\setminus\Gamma_i| - \int_{\Gamma_i} k\,\varphi(u_i)\, d\ell\,.\end{aligned}$$

By the inequality (11) we obtain

$$J(u_i) \ge \rho_2(D_i)\int_{\Gamma_i}\varphi(u_i)\, d\ell - k\,\varphi(b)\,|\partial D\setminus\Gamma_i| - \int_{\Gamma_i} k\,\varphi(u_i)\, d\ell\,. \tag{12}$$

As $u_0 \in R$ and J is decreasing in time,

$$J(u_i) < \varepsilon_0 - k\,\varphi(b)\,|\partial D|\,. \tag{13}$$

Joining inequalities (12) and (13) we have

$$(\rho_2(D_i) - k)\int_{\Gamma_i}\varphi(T(t_i)u_0)\, d\ell < \varepsilon_0 - k\,\varphi(b)\,|\Gamma_i|\,,$$

or equivalently

$$\begin{aligned}\varepsilon_0 &> (\rho_2(D_i) - k)\int_{\Gamma_i}\varphi(T(t_i)u_0)\, d\ell + k\,\varphi(b)\,|\Gamma_i|\\ &\ge \varphi(b)\,|\Gamma_i|\min\{k, \rho_2(D_i)\}\,,\end{aligned}$$

which is a contradiction with the definition of ε_0.

Finally, as $J(u)$ decreases in time the last condition in R is satisfied and it proves that R is positively invariant under $T(t)$, for $t > 0$.

Step 2. *If R is nonempty, the interior of R is also nonempty and the absolute minimum in R of the functional J is achieved at (at least) one (nonconstant equilibrium) point e_0, which is an interior point of R.*

Let E be the set of equilibrium points of (6). Let us see that $\overline{R} \cap E$ is a compact set in W^1_p. Since $\overline{R} \cap E$ is closed and invariant, and $T(t)$ is compact (see [7]), it will be enough to see that $\overline{R} \cap E$ is bounded. Acording to (6) the equilibrium points satisfy $Ae + F(e) = 0$. It is easy to see that the set $F(\overline{R})$ is bounded in $(W^1_{p'})'$, and since $A : W^1_p \to (W^1_{p'})'$ is invertible one concludes that $\overline{R} \cap E$ is bounded in W^1_p.

If R is nonempty, then so is $\overline{R} \cap E$: Let us take $w_0 \in R$. As R is positively invariant, $\gamma^+(w_0) \in R$. The $w-$limit set of w_0, $w(w_0)$, is contained in $\overline{R} \cap E$. So, $\overline{R} \cap E$ is also nonempty.

Using that J is continuous, we see that there exists e_0, a minimum of J on $\overline{R} \cap E$. Let us see that e_0 is a minimum of J on $\overline{R}$. Let us assume that e_0 is not a minimum of J on $\overline{R}$. Then there exists $u \in \overline{R}$ such that $J(u) < J(e_0)$ and we consider the positive orbit $\gamma^+(u) \subset \overline{R}$ ($\overline{R}$ is also positively invariant). Then the $w-$limit set $w(u)$ would be a subset of $\overline{R} \cap E$ with $J(e) < J(e_0)$ for all $e \in w(u)$, which is a contradiction.

To finish off the proof of step 2 we have to see that $e_0 \in \overset{\circ}{R}$. Let us suppose that $e_0 \in \partial R$ and we are going to arrive to a contradiction. If $e_0 = a$ or $e_0 = b$ for some $x \in \partial D$ then Hopf's maximum principle implies $e_0 = a$ or $e_0 = b$, and this is incompatible with one of the inequalities $\int_{\Gamma_1} v\, d\ell \leq 0$ or $\int_{\Gamma_2} v\, d\ell \geq 0$. If $J(e_0) = \varepsilon_0 - k\varphi(b)|\partial D|$, since $R \neq \emptyset$ we can take $u \in R$, for which $J(e_0) \leq J(u) < \varepsilon_0 - k\varphi(b)|\partial D|$, and this is a contradiction. Finally, if e_0 satisfies $\int_{\Gamma_1} e_0\, d\ell = 0$ or $\int_{\Gamma_2} e_0\, d\ell = 0$, then we can suppose also that $J(e_0) < \varepsilon_0 - k\,\varphi(b)\,|\partial D|$. So we can suppose that $J(e_0) < \varepsilon_0 - k\,\varphi(b)\,|\partial D|$, and if e_0 satisfies $\int_{\Gamma_1} e_0\, d\ell = 0$ or $\int_{\Gamma_2} e_0\, d\ell = 0$, then following the same argument as in step 1 we see that this is impossible.

It is clear that $e_0 \in \overset{\circ}{R}$ is nonconstant.

Step 3.- *The equilibrium e_0 is stable.*

This is a direct consequence of Theorem 2.1 above, since e_0 is a local minimum of J in W^1_p.
□

In the following theorem we give, for any f as above and $k > 0$, the existence of a domain D for which the set R in Theorem 3.1 is nonempty and so we can conclude the existence of nonconstant stable equilibrium solution. Let us note that the domain D will be of a dumbbell type.

Theorem 3.2 *Given any function f and any $k > 0$ satisfying the hypothesis of Theorem 3.1 above, there exists a domain D such that problem (5) has at least one stable nonconstant* *equilibrium solution.*

Proof. We will distinguish between the cases $n = 2$ and $n \geq 3$.

Case $n = 2$.- Let D_1 and D_2 be two domains in $\mathbb{R}^2$ such that $\rho_2(D_i) \geq k$, $i = 1, 2$. Without loss of generality we can assume $|\partial D_1| \leq |\partial D_2|$. Let us assume also that they are so near each other that $\ell = \text{dist}\,(\overline{D}_1, \overline{D}_2) > 0$ is such that

$$k\,\varphi(a)|\partial D_1| - k\,\varphi(b)\,3\ell > 0\ . \tag{14}$$

There exist two points $P_1 \in \partial D_1$ and $P_2 \in \partial D_2$ and a segment S joining P_1 and P_2, of length ℓ such that S does not intersect neither $\overline{D}_1$ nor $\overline{D}_2$ except at the end points.

Let us consider a $\mathcal{C}^1$ function defined in $\mathbb{R}^2$ by

$$w(x,y) = \begin{cases} a & \text{for } (x,y) \in D_1 \\ b & \text{for } (x,y) \in D_2\ , \end{cases}$$

such that $a \leq w(x,y) \leq b$ for any $(x,y) \in \mathbb{R}^2$ and with $|\nabla w(x,y)|$ globally bounded. Let us denote by M this bound.

It is clear that there exists a domain $D \subset \mathbb{R}^2$ such that

(i) $D_1, D_2 \subset D$.

(ii) $S \subset D$.

(iii) ∂D is smooth.

(iv) Let be $\Gamma_i = \partial D_i \setminus D$, $i = 1, 2$. We suppose that

$$k\,\varphi(a)\,|\Gamma_1| - k\,\varphi(b)\,|\partial D \setminus (\Gamma_1 \cup \Gamma_2)| > \frac{M^2}{2}\,|D \setminus (D_1 \cup D_2)|\ ,$$

which is possible because of (14).

Moreover, $|\Gamma_1| \leq |\Gamma_2|$.

In this case $\varepsilon_0 = k\,\varphi(b)\,|\Gamma_1|$.

To finish off the proof we are going to see that the restriction of $w(x,y)$ on D belongs to R.

By definition, $a \leq w(x,y) \leq b$, $w \in W_p^1$, $\int_{\Gamma_1} w\,d\ell = a\,|\Gamma_1| < 0$ and $\int_{\Gamma_2} w\,d\ell = b\,|\Gamma_2| > 0$. So we only have to prove the energy inequality:

$$\begin{aligned} J(w) &= \frac{1}{2}\int_D (\nabla w)^2\,dx - \int_{\partial D} k\,\varphi(w)\,d\ell \\ &= \frac{1}{2}\int_{D\setminus(D_1\cup D_2)} (\nabla w)^2\,dx - k\,\varphi(a)\,|\Gamma_1| - k\,\varphi(b)\,|\Gamma_2| \\ &- \int_{\partial D\setminus(\Gamma_1\cup\Gamma_2)} k\,\varphi(w)\,d\ell \\ &\leq \frac{M^2}{2}\,|D \setminus (D_1 \cup D_2)| - k\,\varphi(a)\,|\Gamma_1| - k\,\varphi(b)\,|\Gamma_2| \\ &- \int_{\partial D\setminus(\Gamma_1\cup\Gamma_2)} k\,\varphi(w)\,d\ell\ . \end{aligned}$$

Now by the assumption (iv) above and using that $\varphi(w) \geq 0$, we have

$$\begin{aligned} J(w) &< k\,\varphi(a)\,|\Gamma_1| - k\,\varphi(b)\,|\partial D \setminus (\Gamma_1 \cup \Gamma_2)| - k\,\varphi(a)|\Gamma_1| \\ &- k\,\varphi(b)\,|\Gamma_2| \\ &= -k\,\varphi(b)\,|\partial D \setminus \Gamma_1| = \varepsilon_0 - k\,\varphi(b)\,|\partial D|\,. \end{aligned}$$

So $w \in R$ and R is nonempty. Aplying now Theorem 3.1 we finish the proof for the case $n = 2$.

Case $n > 2$.- In this case we can construct a domain D by similar way as in the previous one. Nevertheless there is an important difference between both cases. While in the case $n = 2$ the "bridge" joining D_1 and D_2 must be short with small area, in this case D_1 and D_2 are not needed to be so near each other because we can make the $(n-1)-$dimensional measure of $\partial D \setminus (\Gamma_1 \cup \Gamma_2)$ small at the same time as the $n-$dimensional measure of $D \setminus (D_1 \cup D_2)$ becames small, admiting values of ℓ not necessarily small. $\square$

4 The reaction-diffusion equation and the strongly damped wave equation

We are going to consider the reaction-diffusion equation

$$\begin{cases} u_t = \Delta u + k\,f(u) & \text{in } D\,, \\ u_\nu = 0 & \text{on } \partial D\,, \end{cases} \tag{15}$$

with $D \subset \mathbb{R}^n$ a bounded domain with regular boundary ∂D and the function $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$.

In [19] H. Matano presented a way of obtaining examples of nonconstant stable equilibria for (15) from which our Theorem 3.1 is a generalisation. But he also used the monotonicity property of the flow and Zorn's lemma. We are going to present the same results but by only using Theorem 2.1. As it will be seen later, this will have the advantadge that the result will admit a straightforward application to a non-monotonic equation, namely the semilinear strongly damped wave equation (3).

The problem (15) admits the semilinear formulation

$$u_t = Au + F(u)\,. \tag{16}$$

Here $A = \Delta$ is the linear operator as a closed operator of $L^p(D)$ (with $p > n$) and with domain $D(A) = W^2_p(D)_N$ (where N stands for the boundary condition). The nonlinear function F maps $W^1_p(D)$ on $L^p(D)$. It is known that A is the infinitessimal generator of an analytic semigroup $\{e^{At}\,,\ t \geq 0\}$ in $D(A)$ and that the equation (16) defines a dynamical system $T(t)$ in the space $D(A^\alpha)$ for $\alpha \in (1/2, 1]$ (such that $D(A^\alpha) \subset W^1_p(D)$). (See [12]). We recall that $D(A^\alpha)$ is dense in $W^1_p(D)$, since $D(A) = W^2_p(D)_N$ is so. Hence, the equation (16) satisfies the first hypothesis (H1).

The energy functional $J : W^1_p \to \mathbb{R}$ defined by

$$J(u) = \int_D \left(\frac{1}{2}(\nabla u)^2 - \varphi(u)\right) dx ,$$

where $\varphi(u) = \int_0^u f(s)\, ds$, is continuous and strictly decreasing except at equilibria (see [19]). So the hypothesis (H2) holds.

With respect to the spectrum of the linear operator $L = A + DF(e_0)$ (e_0 an equilibrium point of (16)), as only real eigenvalues are possible, if $\sigma(L) \subset \{\mathrm{Re}\ \lambda \le 0\}$ with $\sigma(L) \cap \{\mathrm{Re}\ \lambda = 0\} \ne \emptyset$ necessarily $\sigma(L) \cap \{\mathrm{Re}\ \lambda = 0\} = \{0\}$. Now, arguments based on the Krein-Rutman theorem prove the simplicity of $\lambda_1 = 0$ and hypothesis (H3) holds.

Moreover, in [12], one can find the details to ensure the last hypothesis (H4).

So, as (16) satisfies all the hypotheses of section 1 we are going to use Theorem 2.1 in order to prove the existence of a nonconstant stable equilibrium solution. We are going to need some additional assumptions on the function f and the domain D.

The main result in this section for problem (15) is the next theorem:

Theorem 4.1 *(H.Matano, [19]) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying (i), (ii) and (iii) as in Theorem* 3.1.

Let $D \subset \mathbb{R}^n$ with $n \ge 2$ be a smooth bounded domain. Let D_1 and D_2 be two subdomains of D with smooth boundaries and $\lambda_2(D_1)$ and $\lambda_2(D_2)$ be the second eigenvalues of the Neumann problem for $-\Delta$ in D_1 and D_2.

Then the problem (15) has at least one stable nonconstant equilibrium solution e_0 if the set

$$R = \{v \in D(A^\alpha) \ : \ a \le v \le b \ on\ \overline{D}\,,\ \int_{D_1} v dx < 0\,,\ \int_{D_2} v\, dx > 0\,, \\ J(v) < \varepsilon_0 - k\,\varphi(b)\,|D|\,\}$$

is nonempty, where J is as above and

$$\varepsilon_0 = \varphi(b)\ min\ \{|D_1|min\ \{k, \lambda_2(D_1)\},\ |D_2|min\ \{k, \lambda_2(D_2)\}\}\ .$$

Moreover, e_0 is a local minimum of J in $D(A^\alpha)$.

Remark. The domains D_1 and D_2 have to be such that the second Poincare's inequality

$$\frac{1}{\lambda_2(D_i)} \int_{D_i} (\nabla w)^2\, dx + \frac{(\int_{D_i} w\, dx)^2}{\int_{D_i} dx} \ge \int_{D_i} w^2\, dx\ , \quad i = 1, 2\ ,$$

holds for any $w \in W^1_2(D)$.

Proof. The set $R \subset D(A^\alpha)$ is positively invariant under $T(t)$, (see [19]). The same arguments used in step 2 of the proof of Theorem 3.1 prove that if R is nonempty then the interior of R is also nonempty and the absolute minimum in R of the functional J is achieved at one equilibrium point e_0 which is an interior point of R. Finally, as a direct consequence of theorem 2.1 we obtain that the equilibrium e_0 is stable (and clearly nonconstant). □

For any function f as above and any $k > 0$ it can be constructed a domain D for which the set R in Theorem 4.1 is nonempty. So we can conclude the existence of at least one stable nonconstant equilibrium solution for (15) that is also a local minimum of J in $D(A^\alpha)$. A way of constructing these domains can be found in [19]. The domain is of dumbbell type and can be constructed in the following way: Let D_1 and D_2 be two smooth bounded domains with disjoint closures and such that $\lambda_2(D_i) \geq k$, $i = 1, 2$. Then the domain D can be taken as the junction of D_1 and D_2 by means of a "smooth bridge" for which it can only be required to have a "sufficiently small" n-dimensional volume. To see that R is nonempty one can choose a suitable piecewise linear function v. This function belongs to $W_p^1(D)$ but it can be approximated by functions of $D(A^\alpha)$ because of density, and these approximating functions can be chosen in R.

The following theorem gives an example of an extension of Theorem 4.1 to a non-monotonic system, such as a strongly damped wave equation. (See [10] and the references therein for informations concerning the appearence and properties of this equation). We believe that this is a small but significative advantage of the use of Theorem 2.1.

Theorem 4.2 *Under the same hypotheses as in Theorem 4.1, the equilibrium e_0 is also a stable (nonconstant) equilibrium solution of the problem*

$$\begin{cases} u_{tt} - a\,\Delta u_t + b\,u_t = \Delta u + f(u) & \text{in } D\,, \\ u_\nu = 0 & \text{on } \partial D\,, \end{cases} \tag{17}$$

with $a, b > 0$.

Proof. Problem (17) can be written as the first order system $v_t = Bv + G(v)$ with $v = (v_1, v_2) = (u, u_t)$, $B = \begin{pmatrix} 0 & I \\ \Delta & a\Delta - bI \end{pmatrix}$ and $G(v) = (0, f(v_1))$. It is known that B is a closed operator of the space $X = D(\Delta) \times L^p(D) = W_p^2(D)_N \times L^p(D)$ with domain $D(B) = W_p^2(D)_N \times W_p^2(D)_N$ which is the infinitessimal generator of an analytic semigroup (using that $a > 0$, see [8] and [27], and see also [18] for related results). Remember that we take $p > n$ and as in the proof of Theorem 4.1 the Nemitskii operator $u \to f(u)$ is a smooth map from $W_p^2(D)_N$ to $L^p(D)$, so G is a smooth map from X to X.

So we are in the functional framework of the theory of D. Henry [12]. In fact we are in the simplest case, since G maps $D(B^\gamma)$ to X with $\gamma = 0$. So this way we have existence of solutions, uniqueness, a criterium for stability and instability of equilibria by linearization and existence of central manifolds at equilibria with the usual properties.

Our solutions of the initial value problem are continuous functions $v : [0, T] \to X$, smooth for $t > 0$, such that $v(t) \in D(B)$ for $t > 0$ and $v_t = Bv + G(v)$ holds also for $t > 0$. By using this, one can perform the time derivative of the functional along a trajectory

$$\tilde{J}(v) = \int_D \left(\frac{1}{2} u_t^2 + \frac{1}{2} (\nabla u)^2 - \varphi(u) \right) dx$$

and, after performing integration by parts, obtain

$$\frac{d}{dt} \tilde{J}(v(t)) = - \int_D \left(a (\nabla u_t)^2 + b u_t^2 \right) dx .$$

So we see that $\tilde{J}(v(t))$ is monotone decreasing. If $\tilde{J}(v(t))$ is not strictly decreasing, necessarily $u_t = 0$ in some interval $t_1 < t < t_2$ (here we use that $b > 0$), and, because of uniqueness, the whole trajectory $v(t)$ must be an equilibrium.

In order to apply Theorem 2.1 to the equilibrium $v_0 = (e_0, 0)$ we observe that because of Theorem 4.1 e_0 is a local minimum of the parabolic functional $J(u)$, and then v_0 is also a local minimum for our actual modified functional $\tilde{J}(v)$.

It only remains to check the spectral condition (H3) of Theorem 2.1.

The linearized evolution operator around the point $v_0 = (e_0, 0)$ is

$$B_0 = \begin{pmatrix} 0 & I \\ \Delta + f'(e_0) & a\Delta - bI \end{pmatrix} .$$

Because of the lemma 4.1 below if λ is a point of the spectrum of B_0 with Re $\lambda \geq 0$ it must be an eigenvalue.

We will proceed in three steps. In the first step we will show that the points λ of the spectrum of B_0 satisfy Re $\lambda \leq 0$. In the second step, that if one of these points satisfies Re $\lambda = 0$ then it is $\lambda = 0$ and it is geometrically simple. In the third step we will show that $\lambda = 0$ is also algebraically simple.

Going to the first step, if a spectral value λ of B_0 satisfies Re $\lambda \geq 0$, we already said that it has to be an eigenvalue. If we call (v_1, v_2) the corresponding eigenfuction, then $\Delta v_1 + f'(e_0)v_1 + \lambda(a\Delta v_1 - bv_1) = \lambda^2 v_1$. If we multiply this equation by $\overline{v}_1$ and integrate over D we obtain the equation $r_1 + \lambda r_2 = \lambda^2 r_3$, where r_1, r_2, r_3 are real numbers and $r_2 < 0$ and $r_3 > 0$. Since we know that e_0 is a stable equilibrium of (15) all the eigenvalues μ of the operator $\Delta + f'(e_0)$ are $\mu \leq 0$. This means that the quadratic form $\int_D (- \nabla v_1 \nabla \overline{v}_1 + f'(e_0) v_1 \overline{v}_1) \, dx$ is negative semidefinite. So $r_1 \leq 0$, and an elementary analysis of the equation $r_1 + \lambda r_2 = \lambda^2 r_3$ when $r_1 \leq 0$, $r_2 < 0$ and $r_3 > 0$ shows that Re $\lambda \leq 0$.

The second step goes along the same lines. The only possibility of having Re $\lambda = 0$ in the previous equation $r_1 + \lambda r_2 = \lambda^2 r_3$ with $r_1 \leq 0$, $r_2 < 0$ and $r_3 > 0$ is having $r_1 = 0$ and then $\lambda = 0$. Looking then at the equation for v_1 with $\lambda = 0$ we see that v_1 has to be an eigenfunction of $\Delta + f'(e_0)$ with eigenvalue 0. Since we know that all the eigenvalues μ of $\Delta + f'(e_0)$ satisfy $\mu \leq 0$, we conclude that $v_1 = \Phi_0$, the first eigenfunction of $\Delta + f'(e_0)$, that is known to be unique because of the Krein-Rutman arguments. So this eigenvalue $\lambda = 0$ of B_0, if it turns out to exist, it is geometrically simple.

Finally we go to the algebraic multiplicity. We have the eigenfunction $(\Phi_0, 0)$ of B_0 with zero eigenvalue, and we ask ourselves if the equation $B_0(v_1, v_2) = (\Phi_0, 0)$ is solvable. This equation means $v_2 = \Phi_0$, so we ask ourselves about the solvability of $\Delta v_1 + f'(e_0)v_1 + a\,\Delta\Phi_0 - b\,\Phi_0 = 0$. But it is enough to multiply this equation by Φ_0 and integrate over D to see that it is incompatible. And this concludes the proof of Theorem 4.2. $\square$

Lemma 4.1 *The spectrum of B_0 consists of isolated eigenvalues of finite algebraic multiplicities together with the point $\lambda = -1/a$, that is the essential spectrum.*

Proof. Following the appendix of chapter 5 of [12] (in the spirit of [9]) it will be enough to prove the same property for the operator

$$B_1 = \begin{pmatrix} 0 & I \\ \Delta - \dfrac{a}{b}I + \dfrac{1}{a^2}I & a\Delta - bI \end{pmatrix}$$

since B_1 is a relatively compact and bounded perturbation of B_0. Writting $A = (\Delta - (b/a)I)$, then

$$B_1 = \begin{pmatrix} 0 & I \\ A - \dfrac{1}{a^2}I + \dfrac{1}{a^2}I & aA \end{pmatrix} .$$

Defining $w_1 = v_1 + av_2$ and writting B_1 in terms of (v_1, w_1) instead of (v_1, v_2) we get it in the form

$$\begin{pmatrix} -\dfrac{1}{a} & \dfrac{1}{a} \\ 0 & aA \end{pmatrix}$$

from which all the results follow. $\square$

5 Stability and instability of local minima for gradient flows in finite dimensions

For a smooth function $F\colon \mathbb{R}^n \to \mathbb{R}$, its gradient flow is the dynamical system defined by the differential equation

$$x'(t) = -\nabla F(x(t)). \tag{18}$$

The critical points of F are then equilibrium points of (18) and, since F decreases strictly along nonconstant trajectories, local minima of F are equilibrium points that are good candidates to be stable in the sense of Lyapunov. This is really the case for a strict local minimum because F itself is then a Lyapunov function, or also, for a general local minimum if $n = 1$, as it has been seen in section 2 above (see also [7]). The aim of this section is to discuss the general case, when a minimum needs not to be a strict minimum and $n \geq 2$.

In this section we present an example of a $\mathcal{C}^\infty$ function F such that all its local minima are unstable for the gradient flow (18) and a theorem that says that local minima are always stable for real analytic F. Neither the statement of this theorem nor the method of the proof, which is based on Lojasiewicz's inequality, can be considered as completely new in the existing literature. In a more or less hidden form, the statement with an equivalent proof can be found in [25], section 3.

So the situation is somehow the same as for the question of the so called "convergence" or "asymptotic limit" property (each ω-limit set is a singleton): a negative answer for general function F (example of [22]), and a positive answer for real analytic F ([25] Theorem 2). The authors are indebted to Prof. P. Poláčik for this informations concerning the convergence property, which have been the basis of the work that follows.

Our example is in $\mathbb{R}^2$ and we describe its dynamics, in polar coordinates. The point $r = 0$ and the circle $r = 1$ are the equilibria. The point $r = 0$ corresponds to a local strict maximum, and it is a source. The points in $r = 1$ are local (and global) minima, but they are not stable: all the solutions in $0 < r < 1$ approach the point $r = 1, \theta = 0$, except the solution consisting of the segment $0 < r < 1, \theta = \pi$ that approaches the point $r = 1, \theta = \pi$, and all the solutions in $r > 1$ approach the point $r = 1, \theta = -\pi/2$, except the solution consisting of the half-line $r > 1, \theta = \pi/2$, that approaches the point $r = 1, \theta = \pi/2$. So, this is a dissipative system (all the trajectories approach the bounded set $r \leq 1$) with no stable equilibria. We note that this example has the additional property that, although all of the equilibria are unstable, each trajectory approaches a single equilibrium. This shows that stability of local minima is a property that is independent from that of convergence.

For convenience, we use logarithmic-polar coordinates (R, θ) with $R = \log r$ because the system becomes

$$\begin{aligned} R' &= -e^{-2R}\frac{\partial F}{\partial R}\,, \\ \theta' &= -e^{-2R}\frac{\partial F}{\partial \theta}\,. \end{aligned}$$

The geometry of its orbits is the same as that of the simpler system

$$\begin{aligned} R' &= -\tfrac{\partial F}{\partial R}\,, \\ \theta' &= -\tfrac{\partial F}{\partial \theta}\,. \end{aligned} \tag{19}$$

Now we define F, and only for $-1 < R < 1$, because it can be easily extended outside this annulus with the desired properties:

$$\begin{aligned} F &= \phi(R)(2 + \sin\theta) \quad \text{for} \quad R \geq 0\,, \\ F &= \phi(-R)(2 - \cos\theta) \quad \text{for} \quad R \leq 0\,. \end{aligned}$$

with ϕ being the suitable function provided by the following

Lemma 5.1 *There exists a function $\phi(s)$ defined for $s \geq 0$, of class $\mathcal{C}^\infty$ and with a zero of infinite order at $s = 0$, such that $\phi'(s) > 0$ for $s > 0$ and that the quotient $\phi(s)/\phi'(s)$ has a non-integrable singularity at $s = 0$.*

This lemma is proved below and we use it to continue our construction. Observe that even disregarding the zero of infinite order it is impossible for a real analytic function ϕ to have this quotient with the required singularity.

We analyze the system only in the region $R \geq 0$, because in $R \leq 0$ the dynamics is the same, but rotated by an angle of $\pi/2$. System (19) becomes

$$\begin{aligned} R' &= -\phi'(R)(2 + \sin\theta) \\ \theta' &= -\phi(R)\cos\theta \,. \end{aligned}$$

Since $\phi'(R) > 0$ for $R > 0$, it is clear that the trajectories approach $R = 0$. It is also clear that $R > 0, \theta = \pi/2$ and $R > 0, \theta = -\pi/2$ are orbits of the system. So an initial condition (R_0, θ_0) with $R_0 > 0$ and $-\pi/2 < \theta_0 < \pi/2$ evolves inside this region with θ decreasing in time. Let us show that θ approaches $-\pi/2$ as $R \to 0$: supposing the contrary, we would have $\cos\theta > \varepsilon$ for some $\varepsilon > 0$ as long as $R_0 > R > 0$ and then

$$\frac{d\theta}{dR} = \frac{\phi(R)}{\phi'(R)} \frac{\cos\theta}{2 + \sin\theta} > \frac{\phi(R)}{\phi'(R)} \frac{\varepsilon}{3}$$

$$\theta_0 - \theta(R) > \frac{\varepsilon}{3} \int_R^{R_0} \frac{\phi(R)}{\phi'(R)} \, dR \,.$$

So, $\theta(R) \to -\infty$ as $R \to 0$, a contradiction. So $\theta(R) \to -\pi/2$, because the trajectory cannot cross $\theta = -\pi/2$.

The case with $\pi/2 < \theta_0 < 3\pi/2$ can be obtained by symmetry.

Proof of the Lemma. Let $\alpha(t)$ for $0 \leq t \leq 1$ be a function of class $\mathcal{C}^\infty$, $\alpha(t) \geq 0$ with zeroes of infinite order at $t = 0$ and $t = 1$. Suppose also that $\int_0^1 \alpha(t)dt = 1$. We define $\phi'(s) = e^{-2/s} + A(s)$ where

$$A(s) = \begin{cases} n(n+1)e^{-n}\alpha(n(n+1)s - n)\,, & \text{for } \frac{1}{n+1} \leq s \leq 1/n \text{ and } n \text{ odd, and} \\ 0\,, & \text{for } \frac{1}{n+1} \leq s \leq 1/n \text{ and } n \text{ even.} \end{cases}$$

It is clear that $A(s)$ is of class $\mathcal{C}^\infty$ for $s > 0$. Let us see that it has a zero of intinite order at $s = 0$:

$$\left| \frac{d^m}{ds^m} A(s) \right| \leq [n(n+1)]^{m+1} e^{-n} sup \left\{ \left| \frac{d^m}{dt^m} \alpha(t) \right| ; 0 \leq t \leq 1 \right\}$$

if $1/(n+1) \leq s \leq 1/n, n$ odd, and it is clear that this expression tends to zero, for fixed m, as $n \to \infty$.

Observe now that if $1/(n+1) \le s \le 1/n, n$ even, then

$$\begin{aligned}\phi(s) \ge \int_0^s A(t)dt &= \sum_{k=n+1,k\text{ odd}}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} A(t)\,dt = \\ &= \sum_{k=n+1,k\text{ odd}}^{\infty} k(k+1)e^{-k} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \alpha(k(k+1)t-k)\,dt = \sum_{k=n+1,k\text{ odd}}^{\infty} e^{-k} \\ &= \frac{e^{-n+1}}{e^2-1}\,.\end{aligned}$$

So finally we have

$$\begin{aligned}\int_0^1 \frac{\phi(s)}{\phi'(s)}ds &\ge \sum_{n=2,n\text{ even}}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{\phi(s)}{\phi'(s)}\,ds \ge \sum_{n=2,n\text{ even}}^{\infty} \frac{1}{n(n+1)}\frac{e^{-n+1}}{e^2-1}\frac{1}{e^{-2n}} = \\ &= \sum_{n=2,n\text{ even}}^{\infty} \frac{1}{n(n+1)}\frac{e}{e^2-1}e^n = \infty\,.\end{aligned}$$

□

The situation in the analytical case is summarized in the following theorem.

Theorem 5.1 *([25], Section 3) Let $\Omega \subset \mathbb{R}^n$ be an open set and and $F\colon \Omega \to \mathbb{R}$ a real-analytic function. Suppose that $x_m \in \Omega$ is a local minimum of F. Then x_m is a Lyapunov-stable equilibrium point of the gradient flow (18).*

Proof. We suppose that $x_m = 0$ and $F(0) = 0$. There exist $\delta_1 > 0$ such that $\|x\| < \delta_1 \Rightarrow F(x) \ge 0$, and $\delta_2 > 0$ and α with $0 < \alpha < 1$ such that

$$\|x\| < \delta_2 \Rightarrow \|\nabla F(x)\| \ge |F(x)|^\alpha$$

(this is Lojasiewicz's inequality, valid since F is real-analytic, see [15] (Prop. 1 of n. 16) and the comments in [25]).

Let $\varepsilon > 0$ be given, and suppose that $\varepsilon < \delta_1$ and $\varepsilon < \delta_2$. Let $\gamma > 0$ be such that

$$\|x\| < \gamma \;\Rightarrow\; \frac{1}{1-\alpha}F(x)^{1-\alpha} < \frac{\varepsilon}{3}$$

and define $\delta = \min(\varepsilon/3, \gamma)$.

Let x_0 be the initial condition of a solution $x(t)$ of (1) and suppose that $\|x_0\| < \delta$. If x_0 is an equilibrium point of (18), then $\|x(t)\| < \varepsilon$ for all $t > 0$. If it is not so, we can change the variable t by the variable $s = F(x(t))$, and s is a monotone decreasing function of t. Suppose that for some s_1,$\|x(s)\| < \varepsilon$ when $s_1 < s \le F(x_0) = s_0$ and that $\|x(s_1)\| = \varepsilon$ (it is clear that $s_1 \ge 0$). This leads to a contradiction:

$$\|\frac{dx}{ds}\| = \frac{1}{\|\nabla F(x)\|} \le \frac{1}{F(x)^\alpha} = \frac{1}{s^\alpha}$$

$$\|x(s_0) - x(s_1)\| \le \int_{s_1}^{s_0} \frac{d\tau}{\tau^\alpha} \le \int_0^{s_0} \frac{d\tau}{\tau^\alpha} = \frac{F(x_0)^{1-\alpha}}{1-\alpha} < \frac{\varepsilon}{3}\,.$$

So $\|x(s_1)\| \le \|x(s_0)\| + \varepsilon/3$, a contradiction. □

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