A note on a conjecture of Xiao

Miguel A. Barja[1]. Departament de Matemàtica Aplicada I. Universitat Politècnica de Catalunya. Barcelona. Spain
Francesco Zucconi[2]. Dipartamento di Matematica e Informatica. Università degli Studi di Udine. Udine. Italy

When $f : S \to B$ is a surjective morphism of a complex, smooth surface S onto a complex, smooth, genus b curve B, such that the fibre F of f has genus g, it is well known that $f_*\omega_{S/B} = \mathcal{E}$ is a locally free sheaf of rank g and degree $d = \mathcal{X}\mathcal{O}_S - (b-1)(g-1)$ and that f is not an holomorphic fibre bundle if and only if $d > 0$. In this case the *slope*, $\lambda(f) = \frac{K_S^2 - 8(b-1)(g-1)}{d}$, is the natural invariant associated by Xiao to f (cf. [6]). In [6][Conjecture 2] he conjectured that $\mathcal{E}$ has no locally free quotient of degree zero if $\lambda(f) < 4$. We give a partial affirmative answer to this conjecture:

Theorem 1. *Let $f : S \longrightarrow B$ be a relatively minimal fibration with general fibre F. Let $b = g(B)$ and assume that $g = g(F) \geq 2$ and that f is not locally trivial.*

Then $\mathcal{E} = f_*\omega_{S/B}$ is ample provided one of the following conditions hold

(i) F is non hyperelliptic.

(ii) $b \leq 1$.

(iii) $g(F) \leq 3$.

Proof. (i) If $q(S) > b$ the result follows from [6] Theorem 3.3. Now assume $q(S) = b$. By Fujita's decomposition theorem (see [3][Proposition 1.2], see also [4])

$$\mathcal{E} = \mathcal{A} \oplus \mathcal{F}_1 \oplus ... \oplus \mathcal{F}_r$$

where $h^0(B, (\mathcal{A} \oplus \mathcal{F}_1 \oplus ... \oplus \mathcal{F}_r)^*) = 0$, $\mathcal{A}$ is an ample sheaf and $\mathcal{F}_i$ are non trivial stable degree zero sheaves. Then we only must prove that $\mathcal{F}_i = 0$. If F is not hyperelliptic and rank $(\mathcal{F}_i) \geq 2$ the claim is the content of [6] Proposition 3.1. If rank$(\mathcal{F}_i) = 1$ we can use

[1]Partially supported by CICYT PS93-0790 and HCM project n.ERBCHRXCT-940557.
[2]Partially supported by HCM project n.ERBCHRXCT-940557.

[2] §4.2 or [1] Theorem 3.4 to conclude that $\mathcal{F}_i$ is torsion in $\mathrm{Pic}^0(B)$. Hence it induces an étale base change

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{B} & \xrightarrow{\sigma} & B \end{array}$$

By flatness $\widetilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \sigma^*(f_*\omega_{S/B})$. Since σ is étale $\lambda(f) = \lambda(\widetilde{f})$ and $\sigma^*(\mathcal{F}_i) = \mathcal{O}_{\widetilde{B}}$ is a direct summand of $\widetilde{f}_*\omega_{\widetilde{S}/\widetilde{B}}$. In particular by [3] $q(\widetilde{S}) > \widetilde{b} = g(\widetilde{B})$ hence $\lambda(\widetilde{f}) \geq 4$ by [6] Theorem 3.3: a contradiction.

(ii) If $b = 0$ the claim is trivial. If $b = 1$, any stable degree zero sheaf has rank one, then as in (i) we conclude.

(iii) If $g = 2$ and $\mathcal{E} \neq \mathcal{A}$, then $\mathcal{E} = \mathcal{A} \oplus \mathcal{L}$ where $\mathcal{L}$ torsion and we are done. The only non trivial case if $g = 3$ is $\mathcal{E} = \mathcal{A} \oplus \mathcal{F}$ where $\mathcal{A}$ an ample line bundle and $\mathcal{F}$ a stable, degree zero, rank two vector bundle. Then $K^2_{S/B} \geq (2g-2)\deg \mathcal{A} = 4d$ and we are done by [6][Theorem 2] □

Theorem 3.3 of [6] Xiao says that if $q(S) > b$ and $\lambda(f) = 4$ then $\mathcal{E} = \mathcal{F} \oplus \mathcal{O}_B$, where $\mathcal{F}$ is a semistable sheaf. We have the following improvement:

Theorem 2. *Let $f : S \longrightarrow B$ be a relatively minimal non locally trivial fibration. If $\lambda(f) = 4$ then $\mathcal{E} = f_*\omega_{S/B}$ has at most one degree zero, rank one quotient $\mathcal{L}$*

Moreover, in this case $\mathcal{E} = \mathcal{A} \oplus \mathcal{L}$ with $\mathcal{A}$ semistable and L torsion.

Proof. As in the previous theorem the torsion subsheaf $\mathcal{L}$ becomes the trivial one after an étale base change; thus

$$\widetilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \widetilde{\mathcal{A}} \oplus \mathcal{O}_{\widetilde{B}}, \quad \widetilde{\mathcal{A}} = \sigma^*\mathcal{A}.$$

By [6][Theorem 3.3], $\widetilde{\mathcal{A}}$ is semistable. Then $\mathcal{A}$ is also semistable by [5][Proposition 3.2]. □

References

[1] Barja, M.A. *On the slope and geography of fibred surfaces and threefolds.* Ph. D. Thesis. 1998.

[2] P. Deligne. *Théorie de Hodge II.* Publ. Math. I.H.E.S., **40** (1971), 5-58.

[3] Fujita, T. *On Kahler fibre spaces over curves.* J. Math. Soc. Japan, **30** (1978), 779–794.

[4] Fujita, T.*The sheaf of relative canonical forms of a Kahler fiber space over a curve.* Proc. Japan. Acad., **54**, Ser.A (1978), 183–184.

[5] Y. Miyaoka. *The Chern classes and Kodaira dimension of a minimal variety.* Algebraic Geometry, Sendai (1985); Adv. Study in Pure Math. **10** (1987), 449-476.

[6] Xiao, G. *Fibered algebraic surfaces with low slope.* Math. Ann. **276** (1987), 449–466.