# ON THE SLOPE OF FIBRED SURFACES

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## 0 Introduction

Let $f : S \longrightarrow B$ be a projective, surjective morphism from a complex smooth surface $S$ onto a complex smooth curve $B$. We set $F$ for the general fibre of $f$ and assume it is connected. Let $g = g(F)$ and $b = g(B)$. We will assume that $f$ is relatively minimal, i.e., that there is no $(-1)-$rational curve on fibres. We usually call $f$ a *fibration* or a *minimal genus $b$ pencil of curves of genus $g$*. We say that $f$ is *smooth* if all its fibres are smooth, that it is *isotrivial* it all its smooth fibres are reciprocally isomorphic, and that it is *locally trivial* if it is smooth and isotrivial.

Our results enable to study the geographical problem of $f$ (that is, to relate numerical invariants of $F$, $S$ and $B$) through the control of some geometrical properties of the general fibre $F$ or the influence of some *global* properties of $S$ such as the relative irregularity $q_f = q - b$. Now we recall the basic relative invariants for $f$. Let $\omega_S$, $\omega_B$ be the canonical sheaves and $K_S$, $K_B$ canonical divisors of $S$ and $B$ respectively. As usual we consider $p_g = h^0(S, \omega_S)$, $q = h^1(S, \omega_S)$, $\chi\mathcal{O}_S = p_g - q + 1$ and denote as $e(X)$ the topological Euler characteristic of X. Then we set:

$$K^2_{S/B} = (K_S - f^*K_B)^2 = K^2_S - 8(b-1)(g-1)$$

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$$\chi_f = \deg f_*\omega_{S/B} = \chi\mathcal{O}_S - (b-1)(g-1)$$
$$e_f = e(S) - e(B)e(F) = e(S) - 4(b-1)(g-1).$$

We have the following classical results:

**Theorem 0.1** *Let $f : S \longrightarrow B$ be a minimal genus $b$ pencil of curves of genus $g \geq 2$. Then*

*(i) (Noether) $12\chi_f = e_f + K^2_{S/B}$.*

*(ii) (Zeuthen-Segre) $e_f \geq 0$. Moreover, $e_f = 0$ if and only if $f$ is smooth.*

*(iii) (Arakelov) $K^2_{S/B} \geq 0$. Moreover, $K^2_{S/B} = 0$ if and only if $f$ is isotrivial.*

*(iv) $\chi_f \geq 0$. Moreover, $\chi_f = 0$ if and only if $f$ is locally trivial.*

When $f$ is not locally trivial, Xiao (cf. [15]) defines the slope of $f$ as

$$\lambda(f) = \frac{K^2_{S/B}}{\chi_f}.$$

It follows immediately from Noether's equality that $0 \leq \lambda(f) \leq 12$. We are mostly concerned with a lower bound of the slope. The main known result is:

**Theorem 0.2** *(Cornalba-Harris, Xiao). If $g \geq 2$ and $f$ is not locally trivial, then $\lambda(f) \geq 4 - \frac{4}{g}$.*

After that, the first problem was to investigate the influence of some properties of the fibration on the behaviour of the slope. The first direction is to study the influence of the relative irregularity $q_f = q(S) - b$. The main known result is:

**Theorem 0.3** *(Xiao ) If $q > b$ then $\lambda(f) \geq 4$. If $\lambda(f) = 4$ and $q > b$ then $q = b + 1$ and $f_*\omega_{S/B}$ is semistable.*

The other most considered problem is the study of how properties of the general fibre $F$ influence. Mostly due to the work of Konno (cf. [9],[11]; see also [6] and [14] for other references) we know the Clifford index (or the gonality) of the general fibre has some meaning in the lower bound of the slope. There are evidences for this. For example, it is known that equality $\lambda(f) = 4 - \frac{4}{g}$ only holds when $F$ is hyperelliptic. When $F$ is trigonal, a better bound is known (see Remark 3.5). On the other hand the following theorem shows that if $g >> 0$ and $F$ has general Clifford Index then $\lambda(f) \simeq 6$ and so, together with 0.3, it forces to understand the case $q_f > 0$, $g >> 0$ and $4 < \lambda(f) < 6$.

**Theorem 0.4** *(Harris-Eisenbud, Konno). Assume that $g$ is odd and that the general fibre $F$ is of general Cifford index. If $f$ is not a semistable fibration, assume that Green's conjecture on syzygies of the canonical curves holds. Then*

$$\lambda(f) \geq 6\frac{g-1}{g+1}.$$

An explicit sharp lower bound for $\lambda(f)$ depending on the Clifford index of $F$ should not be easy and should depend on other parameters. Indeed, when the fibre is trigonal, the behaviour of $\lambda(f)$ depends on the fact that $F$ is *general* or not in the set of trigonal curves (see [14]). A similar behaviour should hold for tetragonal fibrations: there are tetragonal fibrations with $\lambda(f) = 4$ for any genus $g$ (see [1]), although the general fibre is always bielliptic.

In §1 we give an idea of Xiao's method, which is our main tool. In §2 we study the behaviour of $\lambda(f)$ when the general fibre $F$ is a double cover in such a way that extends to a double cover of the fibration itself. In §3 we explicit the influence of $q_f$ on $\lambda(f)$ through an increasing function on $q_f$ generalising Theorem 0.3. As natural by-product of this estimates and the previous theorems it seems possible to construct through $\lambda(f)$ a geography for fibrations as in the case of surfaces of general type.

We set aside the double cover case by two reasons: 1) all double covers are special curves of non general Clifford index; as happens in the case of bielliptic curves with respect to tetragonal ones, fibrations which are double covers are candidates to be exceptional in the study of $\lambda(f)$. 2) Xiao's method works very well if the possibility for $f$ to be a double cover is excluded, which also suggest the study of double covers as exceptional. For these fibrations we get:

**Theorem 0.5** *Let $f : S \to B$ be a genus $g$, relatively minimal, non isotrivial fibration. Assume $f$ is a double cover fibration of a fibration of genus $\gamma$. Then, if $g \geq 4\gamma + 1$ we have*

$$\lambda(f) \geq 4 + 4\frac{(\gamma - 1)(g - 4\gamma - 1)}{(g - 4\gamma - 1)(g - \gamma) + 2(g - 1)\gamma^2}.$$

In our paper there is also a refined version of this theorem but it is more complicate to state: see 2.4. Next theorem is a partial answer to a natural question: how special are fibrations with $\lambda(f) < 4$?

**Theorem 0.6** *Let $f : S \to B$ be a relatively minimal, non isotrivial double cover fibration of $\sigma : V \to B$. Let $F$ and $E$ be the fibres of $f$ and $\sigma$ respectively and let $g = g(F)$, $\gamma = g(E)$. Assume $F$ is not hyperelliptic nor tetragonal, $\gamma \geq 1$ and $g \geq 2\gamma + 11$. Then $\lambda(f) \geq 4$.*

Our next theorem gives an affirmative answer to the expected influence of $q_f$ on the slope as suggested by Theorem 0.3. It gives a bound which is (strictly) increasing and in some cases is assimptotically sharp (see Example 2.8). If $f$ is general, that is, it is not a double cover fibration we have:

**Theorem 0.7** *Let $f : S \longrightarrow B$ be a relatively minimal fibration which is not a double cover fibration. Assume $g = g(F) \geq 5$ and that $f$ is not locally trivial. Let $h = q(S) - b \geq 1$.*

*(i) If $h \geq 2$ and $g \geq \frac{3}{2}h + 2$ then*

$$\lambda(f) \geq \frac{8g(g-1)(4g-3h-10)}{8g(g-1)(g-h-2) + 3(h-2)(2g-1)} \quad \text{if } F \text{ is not trigonal}$$

$$\lambda(f) \geq \frac{4g(g-1)(4g-3h-10)}{4g(g-1)(g-h-2) + (g-4)(2g-1)} \quad \text{if } F \text{ is trigonal}$$

*(ii) if $g < \frac{3}{2}h + 2$ then*

$$\lambda(f) \geq \frac{4g(g-1)(2g-7)}{\frac{4}{3}g(g-1)(g-3) + (g-4)(2g-1)}$$

As an application we obtain a nice relation between $\lambda(f)$ and the existence of other fibration on $S$ onto curves of genus at least 2 (see also Corollaries 3.9, 3.10):

**Theorem 0.8** *Let $f : S \longrightarrow B$ be a relatively minimal, non locally trivial fibration. Let $F$ be a fibre of $f$, $g = g(F)$ and $q = q(S)$. Assume $f$ is not a double cover fibration and that $s = q - b \geq 1$ (i.e., $f$ is not an Albanese fibration). Let $\mathcal{C} = \{\pi_i : S \longrightarrow C_i \text{ fibrations}, c_i = g(C_i) \geq 2, \pi_i \neq f\}_{i \in I}$. Assume $\mathcal{C} \neq \emptyset$ and let $c = max\{c_i | i \in I\}$. Then*

*(i)* $\lambda(f) \geq 4 + \frac{c-1}{g-c}$

*(ii) If, moreover,* $\dim alb(S) = 1$ *(then necessarily $b = 0$) we have*

$$\lambda(f) \geq 4 + \frac{q-1}{g-q}.$$

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# 1 Preliminaries

Here we give a brief run-down of Xiao's method to estimate $\lambda(f)$. Its method uses a result of Harder and Narasimhan and the theorem of Clifford.

Let $\mathcal{E}$ be a locally free sheaf on $B$ and let $\mathcal{GR}(\mathcal{E})$ be the set of the locally free subsheaves of $\mathcal{E}$; it is defined a function $\mu : \mathcal{GR}(\mathcal{E}) \rightarrow \mathbb{Q}$, $\mathcal{F} \mapsto \mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rank}(\mathcal{F})$. We recall that $\mathcal{E}$ is Mumford-stable (respectively Mumford-semistable) if for every proper subbundle $\mathcal{F}$ of $\mathcal{E}$, $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$ we have

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \ \ (\mathrm{resp.}\mu(\mathcal{F}) \leq \mu(\mathcal{E})).$$

The Harder-Narasimhan theorem concerns the maximum value for $\mu$.

**Theorem 1.1** *Let $\mathcal{E}$ be a locally free sheaf on a smooth curve $B$, there exists a unique filtration by subbundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

*such that, for $i = 1, ..., l$, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is the maximal semistable subbundle of $\mathcal{E}/\mathcal{E}_{i-1}$. We put $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$. In particular for every $i = 1, ..., l$, $\mathcal{E}_i/\mathcal{E}_{i-1}$ is the unique subbundle of $\mathcal{E}/\mathcal{E}_{i-1}$ such that for every subbundle $\mathcal{F}$ of $\mathcal{E}/\mathcal{E}_{i-1}$ we have $\mu(\mathcal{F}) \leq \mu_i$ and if $\mu(\mathcal{F}) = \mu_i$ then $\mathcal{F} \subset \mathcal{E}_i/\mathcal{E}_{i-1}$. Moreover $\mu_1 > \mu_2 > ... > \mu_l$.*

**Proof.** See [8]. □

The numbers $\{\mu_i\}_{1\leq i\leq l}$ are called the Harder-Narasimhan slopes of $\mathcal{E}$.

We shall use the following result that relates the Harder-Narasimhan filtration of a direct sum to the filtrations of the summands.

**Proposition 1.2** *Let $\mathcal{E}$, $\mathcal{H}$, $\mathcal{K}$ be locally free sheaves on a smooth curve $B$. Let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_\ell$, $0 = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_{\ell_1}$, $0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \ldots \subset \mathcal{K}_{\ell_2}$ their Harder-Narasimhan filtrations and $\{\mu_i\}_{1\leq i\leq \ell}$, $\{\mu_i^{\mathcal{H}}\}_{1\leq i\leq \ell_1}$, $\{\mu_i^{\mathcal{K}}\}_{1\leq i\leq \ell_2}$ their Harder-Narasimhan slopes. Assume $\mathcal{H} \oplus \mathcal{K} = \mathcal{E}$. Then we can define $\psi : \{0, \cdots, \ell\} \to \{0, 1, \cdots, \ell_1\}$, $\phi : \{0, \cdots, \ell\} \to \{0, 1, \cdots, \ell_2\}$ such that*

*(i) $\psi(0) = \phi(0) = 0$; for $1 \leq i \leq \ell$, $\psi(i) = \psi(i-1)$ if $\mu_t^{\mathcal{H}} \neq \mu_i$ for every $t \in \{1, \cdots, \ell_1\}$, (respectively, $\phi(i) = \phi(i-1)$ if $\mu_s^{\mathcal{K}} \neq \mu_i$ for every $s \in \{1, \cdots, \ell_2\}$) and $\psi(i) = t$ if $\mu_t^{\mathcal{H}} = \mu_i$, (respectively, $\phi(i) = s$ if $\mu_s^{\mathcal{K}} = \mu_i$);*

*(ii) $\mathcal{E}_i = \mathcal{H}_{\psi(i)} \oplus \mathcal{K}_{\phi(i)}$.*

**Proof.** Call $\pi_{\mathcal{H}} : \mathcal{E} \longrightarrow \mathcal{H}$, $\pi_{\mathcal{K}} : \mathcal{E} \longrightarrow \mathcal{K}$ the natural projections. Let $\mathcal{E}_1^{\mathcal{H}} = \pi_{\mathcal{H}}(\mathcal{E}_1)$, $\mathcal{E}_1^{\mathcal{K}} = \pi_{\mathcal{K}}(\mathcal{E}_1)$; both are locally free since they are torsion free ($\mathcal{E}_1^{\mathcal{H}} \subseteq \mathcal{H}$, $\mathcal{E}_1^{\mathcal{K}} \subseteq \mathcal{K}$). We have $\mathcal{E}_1 \subseteq \mathcal{E}_1^{\mathcal{H}} \oplus \mathcal{E}_1^{\mathcal{K}}$.

Assume $\mathcal{E}_1^{\mathcal{H}} \neq 0$. Since $\mathcal{E}_1$ is semistable and $\mathcal{E}_1^{\mathcal{H}}$ is a quotient, we have that $\mu(\mathcal{E}_1^{\mathcal{H}}) \geq \mu(\mathcal{E}_1) = \mu_1$. From the inclusions $\mathcal{E}_1^{\mathcal{H}} \subseteq \mathcal{H} \subseteq \mathcal{E}$ we get $\mu_1 \leq \mu(\mathcal{E}_1^{\mathcal{H}}) \leq \mu_1^{\mathcal{H}} \leq \mu_1$ since $\mathcal{H}_1, \mathcal{E}_1$ are the maximal unstabilizing sheaves in $\mathcal{H}$ and $\mathcal{E}$ respectively. Hence $\mu_1 = \mu_1^{\mathcal{H}}$ and $\mathcal{E}_1^{\mathcal{H}} \subseteq \mathcal{H}_1 \subseteq \mathcal{E}_1$ by the maximality of $\mathcal{H}_1$ and of $\mathcal{E}_1$ (see 1.1). The same argument works if $\mathcal{E}_1^{\mathcal{K}} \neq 0$.

Assume $\mu_1^{\mathcal{K}} \neq \mu_1$. Then necessarily $\mathcal{E}_1^{\mathcal{K}} = 0$ and $\mathcal{E}_1^{\mathcal{H}} \neq 0$. Hence $\mu_1^{\mathcal{H}} = \mu_1$, $\mathcal{E}_1 \subseteq \mathcal{E}_1^1 \subseteq \mathcal{F}_1$ and then $\mathcal{E}_1 = \mathcal{F}_1 = \mathcal{H}_1 \oplus \mathcal{K}_0$ by maximality. The same argument works if $\mu_1^{\mathcal{H}} \neq \mu_1$.

Assume $\mu_1^{\mathcal{H}} = \mu_1^{\mathcal{K}} = \mu_1$. Then $\mathcal{E}_1 \subseteq \mathcal{E}_1^{\mathcal{H}} \oplus \mathcal{E}_1^{\mathcal{K}} \subseteq \mathcal{H}_1 \oplus \mathcal{K}_1$ with $\mu(\mathcal{H}_1 \oplus \mathcal{K}_1) = \mu_1$. Again by maximality of $\mathcal{E}_1$ we conclude $\mathcal{E}_1 = \mathcal{H}_1 \oplus \mathcal{K}_1$.

The proof concludes by induction dealing with $\mathcal{E}/\mathcal{E}_1 = \mathcal{H}/\mathcal{H}_{\psi(1)} \oplus \mathcal{K}/\mathcal{K}_{\phi(1)}$. □

**Corollary 1.3** *With the above notations we have:*

$$\max\{\mu_1^{\mathcal{H}}, \mu_1^{\mathcal{K}}\} = \mu_1$$
$$\min\{\mu_{\ell_1}^{\mathcal{H}}, \mu_{\ell_2}^{\mathcal{K}}\} = \mu_\ell$$

**Proof.** Obvious. □

We will use the well-known refined version of Clifford's theorem:

**Theorem 1.4 Clifford-plus.** *Let $F$ be a smooth curve of genus $g$. Let $\eta$ be a divisor of degree $d$ such that the linear system $\mid \eta \mid$ has dimension $r-1$ and let $\phi_{|\eta|} : F \to I\!P^{r-1}$ be the rational map associated to $D$. Then it holds:*

- *(i) if $d \leq 2g-2$ then $d \geq 2r-2$;*
- *(ii) if $\deg(\phi_{|\eta|}) = m$ then $d \geq m(r-1)$;*
- *(iii) if $\phi_{|\eta|} = 1$ then a) if $d \leq g-1$ then $d \geq 3r-4$, b) if $d \geq g$ then $d \geq \frac{1}{2}(3r+g-4)$.*

*Moreover if there exists a smooth curve $C$, a double cover $\sigma : F \to C$ and a divisor $\eta$ on $C$ such that $\mid \eta \mid = \zeta + \sigma^\star \mid \eta' \mid$, where $\zeta$ is the fixed part of $\mid \eta \mid$ then $d \geq 2(r-1+g(C))$.*

**Proof.** See [4][Lemme 5.1] □

Assume that $\mathcal{O}_S(H)$ is an invertible sheaf on $S$ such that $\mathcal{E} = f_\star\mathcal{O}_S(H)$ is a rank $g$, locally free sheaf on $B$. Let $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$ be its Harder-Narasimhan filtration and let $\eta \in \mathrm{Pic}(B)$ be a sufficiently ample sheaf such that $\mathcal{E}_i(\eta) = \mathcal{E}_i \otimes \eta$ is globally generated. The natural sheaf homomorphism $f^*\mathcal{E}_i \to f^*f_*\mathcal{O}_S(H) \to \mathcal{O}_S(H)$ induces a rational map $\rho_i : S \to I\!P(\mathcal{E}_i)$. Let $\sigma : \tilde{S} \to S$ be the elimination of the indeterminacy of $\rho_i$ for every $i$ and let $I\!P_i$

be the sublinear system of $\mid \sigma*(H+f^{\star}\eta) \mid$ such that $I\!P_i = I\!P(H^0(B, \mathcal{E}_i(\eta)))$ for $i = 1, ..., l$. Xiao defined the following divisors on $\tilde{S}$: $Z_i$ which is the fixed part of $I\!P_i$, $M_i = \sigma*H - Z_i$ which is the moving part of $I\!P_i$ and $N_i = M_i - \mu_i F$. By [12] for every $i$ $N_i$ is a nef $\mathbb{Q}$-divisor. We observe that the restriction $I\!P_{i_{|F}}$ is a sublinear system of $\mid H_{|F} \mid$ of dimension at least $r_i = \mathrm{rk}(\mathcal{E}_i)$, with fixed part $Z_{i_{|F}}$, moving part $M_{i_{|F}}$ and of degree $d_i = N_i F$. It is easy to see that these definitions do not depend on $\eta$. Thus we can give the following definition:

**Definition 1.5** We call $\{M_{i|F}, r_i, d_i\}$ the Xiao's data associated to the sheaf $\mathcal{E} = f_\star \mathcal{O}_S(H)$.

**Proposition 1.6** *Let $f : S \to B$ be a fibration with general fibre $F$. Let $H$ be a divisor on $S$ and suppose there are a sequence of effective divisors on a suitable blow up $\sigma : \tilde{S} \to S$, $Z_1 \geq Z_2 \cdots \geq Z_l \geq Z_{l+1} = 0$ and a sequence of rational numbers $\mu_1 > \mu_2 > \cdots > \mu_l \geq \mu_{l+1} = 0$ such that for every $i$, $N_i = \sigma^* H - Z_i - \mu_i F$ is a nef $\mathbb{Q}$-divisor then*

$$H^2 \geq \sum_{i=1}^{l} (d_i + d_{i+1})(\mu_i - \mu_{i+1})$$

*where $d_i = N_i F$.*

**Proof.** See [15][Lemma 2]. □

Xiao's method is to combine 1.6 and 1.4 when $H = K_{S/B}$ or when $H$ induces on the general fibre $F$ a sublinear system of $\mid K_F \mid$.

# 2 The slope of double covers

**Definition 2.1** Let $f : S \longrightarrow B$ be a relatively minimal fibration. We say that $f$ is a double cover fibration (double cover, for short) if there exists a relatively minimal fibration $\phi : V \longrightarrow B$ and a rational map $\pi : S - - - \to V$ over $B$ which is a generically two to one map. Otherwise we say that $f$ is a non double cover fibration (non double cover for short).

Roughly speaking, double cover fibrations correspond to the curves with an involution in the theory of curves. Of course, if $f$ is a double cover fibration then $F$ is a double cover. The converse is not true. In [1] Example 1.2 a bielliptic fibration is given for which a non trivial base change is needed in order to be a double cover fibration. In [3] is proved that if the general fibre $F$ is a double cover of a curve of fixed genus $\gamma$ in a unique way, then the corresponding involution glues to a global involution of $S$ and so $f$ itself is a double cover. It is easy to check that if $g \geq 4\gamma + 2$ such condition holds.

Let $f$ be a double cover fibration. We have

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\sigma"] & \widetilde{V} \arrow[d, "\eta"] \\ S \arrow[r, "\pi"] \arrow[d, "f"] & V \arrow[dl, "\phi"] \\ B & \end{tikzcd}$$

where $\phi$ and $\pi$ exits by definition, $\pi$ is a generically 2-to-1 rational map and $\phi$ is a relatively minimal fibration; $\eta : \widetilde{V} \longrightarrow V$ and $\sigma : \widetilde{S} \longrightarrow S$ are any birational maps such that the induced rational map $\widetilde{\pi} = \eta^{-1} \circ \pi \circ \sigma$ is a morphism. Let $\widetilde{f} = f \circ \sigma$ and $\widetilde{\phi} = \phi \circ \eta$. Note that at general $t \in B$ $f^{-1}(t) = \widetilde{f}^{-1}(t)$, $\phi^{-1}(t) = \widetilde{\phi}^{-1}(t)$. The map $\eta \circ \widetilde{\pi}$ factorizes by Stein Theorem as $\eta \circ \widetilde{\pi} = \pi_0 \circ u$, where $\pi_0$ is finite and $u$ is birational. Let $R$ be the branching divisor of $\pi_0$ and $\mathcal{L} \in Pic(V)$ such that $\mathcal{L}^{\otimes 2} = \mathcal{O}_V(R)$. By standard theory of cyclic coverings we have that

$$\begin{aligned} f_*\omega_{S/B} = \widetilde{f}_*\omega_{\widetilde{S}/B} = \phi_*((\eta \circ \widetilde{\pi})_*\omega_{\widetilde{S}/B}) = \phi_*(\omega_{V/B} \oplus (\omega_{V/B} \otimes \mathcal{L})) = \\ = \phi_*\omega_{V/B} \oplus \phi_*(\omega_{V/B} \otimes \mathcal{L}). \end{aligned}$$

**Remark 2.2** With the previous notation. Let $\mathcal{H} = \phi_*\omega_{V/B}, \mathcal{K} = \phi_*(\omega_{V/B} \otimes \mathcal{L})$. Let $h = q(S) - b, s_1 = q(V) - b$ and $s_2 = h - s_1$. According to Fujita's decomposition (see[7]) we have $f_*\omega_{S/B} = \mathcal{E} \oplus \mathcal{O}_B^{\oplus s}$. A simple computation shows that then we obtain $\mathcal{H} = \mathcal{F} \oplus \mathcal{O}_B^{\oplus s_1}$, $\mathcal{K} = \mathcal{G} \oplus \mathcal{O}_B^{\oplus s_2}$ and $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$.

We define $\chi_1 = \deg\mathcal{H}$ and $\chi_2 = \deg\mathcal{K}$.

Let $0 \subset \mathcal{H}_1 \subset \ldots \subset \mathcal{H}_{\ell_1 - 1} \subset \mathcal{H}_{\ell_1}$ and $0 \subset \mathcal{K}_1 \subset \ldots \subset \mathcal{K}_{l_2 - 1} \subset \mathcal{K}_{l_2}$ be respectively, the Harder Narasimhan filtration of $\mathcal{H}$ and $\mathcal{K}$. Let $\{(M^{\mathcal{H}}_{i|E}, r^{\mathcal{H}}_i, d^{\mathcal{H}}_i)\}_{i=1}^{\ell_1}$ and $\{(M^{\mathcal{K}}_{i|E}, r^{\mathcal{K}}_i, d^{\mathcal{K}}_i)\}_{i=1}^{l_2}$ be respectively the Xiao's data of $\mathcal{H}$ and $\mathcal{K}$.

In order to estimate a lower bound for the slope in the double cover case the following is the key technical result. The first part is due to Konno (cf. [10]). We reproduce here a proof for lack of a suitable reference.

**Proposition 2.3** *Let $f : S \to B$ be a genus $g$, relatively minimal non isotrivial fibration, which is a double cover of a genus $\gamma \geq 1$ fibration $\phi : V \to B$. Let $\chi_f = \deg(f_* \omega_{S/B})$ and consider the previous notations. Assume $g \geq 2\gamma + 1$. Then*

*(i)* $K^2_{S/B} - 4\chi \geq -4(\mu_1^{\mathcal{H}} + \mu_l^{\mathcal{H}}) + 2(g - 2\gamma + 1)\max\{\frac{\mu_1^{\mathcal{H}}}{\gamma}, \mu_l^{\mathcal{H}}\}$. *In particular if* $g \geq 4\gamma + 1$ *then* $\lambda(f) \geq 4$.

*(ii)* $K^2_{S/B} \geq \frac{8g(g-1)}{g^2+g-1}\chi_1$.

*(iii) If* $g \geq 2\gamma + s_2$ *then*

$$K^2_{S/B} \geq 4\frac{(g-1)(g-s_2-1)}{(g-1)(g-\gamma) - s_2 g}\chi_2.$$

*If* $g \leq 2\gamma + s_2$ *then*

$$K^2_{S/B} \geq 8\frac{g(g-1)}{g^2+g-1)}\chi_2.$$

**Proof.** We have obtained $u = \eta \circ \widetilde{\pi} : \widetilde{S} \to V$ a generically 2-to-1 morphism from a blow-up of $S$ onto a relatively minimal genus $b$ pencil of genus $\gamma$. Now consider

$$\begin{tikzcd}
\widetilde{S} \arrow[d, "\sigma"] \arrow[drrr, "u"] & & & \\
& \overline{S} = S_k \arrow[r] \arrow[dl] \arrow[dd, "\pi_k"] & \cdots \arrow[r] & S_0 \arrow[dd, "\pi_0"] \\
S \arrow[dd, "f"] & & \cdots & \\
& \overline{V} = V_k \arrow[r] & \cdots \arrow[r] & V_0 = V \arrow[dlll] \\
B & & &
\end{tikzcd}$$

where:

• $\pi = \pi_0 \circ u$ is the Stein factorization of $\pi$, with $u$ birational, $\pi_0$ finite (so it is a double cover) and $S_0$ normal.

• $\pi_k : S_k \longrightarrow V_k$ is the canonical resolution of singularities of $\pi_0 : S_0 \longrightarrow V_0$.

• $\bar{\sigma} : S_k \longrightarrow S$ is the birational morphism defined by the relative minimality of $f$. The maps $\pi_0 : S_0 \longrightarrow V_0$ and $\pi_k : S_k \longrightarrow V_k$ are determined by divisors $R_0$ on $V_0$, $R_k$ on $V_k$ and line bundles $\mathcal{L}_0$, $\mathcal{L}_k$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(R_0)$, $\mathcal{L}_k^{\otimes 2} = \mathcal{O}_{V_k}(R_k)$. First of all we have

$$K_{S/B}^2 - 4\chi_f = (K_S^2 - 4\mathcal{X}\mathcal{O}_S) - 4(b-1)(g-1) \geq (K_{\overline{S}}^2 - 4\mathcal{X}\mathcal{O}_{\overline{S}}) - 4(b-1)(g-1)\,. \tag{1}$$

For smooth double covers $\pi_k : \overline{S} \longrightarrow \overline{V}$ we have

$$\begin{aligned} \mathcal{X}\mathcal{O}_{\overline{S}} &= 2\mathcal{X}\mathcal{O}_{\overline{V}} + \frac{1}{2}\mathcal{L}_k K_{\overline{V}} + \frac{1}{2}\mathcal{L}_k\mathcal{L}_k \\ K_{\overline{S}}^2 &= 2K_{\overline{V}}^2 + 4\mathcal{L}_k K_{\overline{V}} + 2\mathcal{L}_k\mathcal{L}_k \end{aligned}$$

so we have

$$K_{\overline{S}}^2 - 4\mathcal{X}\mathcal{O}_{\overline{S}} = 2[K_{V_k}^2 - 4\mathcal{X}\mathcal{O}_{V_k}] + 2\mathcal{L}_k K_{V_k}\,. \tag{2}$$

By the canonical resolution of singularities of $\pi_0 : S_0 \longrightarrow V_0$ we obtain

$$2[K_{V_k}^2 - 4\mathcal{X}\mathcal{O}_{V_k}] + 2\mathcal{L}_k K_{V_k} \geq 2[K_{V_0}^2 - 4\mathcal{X}\mathcal{O}_{V_o}] + 2\mathcal{L}_0 K_{V_0} \tag{3}$$

(i) By (1), (2) and (3) we have that

$$K_{S/B}^2 - 4\chi_f \geq 2(K_{V/B}^2 - 4\chi_1) + K_{V/B}R$$

where $R = R_0$ is the branch divisor of $S_0 \longrightarrow V_0 = V$.

By 1.6 we have a nef $\mathbb{Q}$-divisor $N_1$ and an effective divisor $Z_1$ in $V$ such that $K_{V/B} \equiv N_1 + \mu_1^{\mathcal{H}} E + Z_1$. Let $R = R_h + R_v$ the decomposition of $R$ in its horizontal and vertical part respectively. Let $R_h = C_1 + \ldots + C_m$ the decomposition in irreducible components (note that $R$ is reduced since $S_0$ is normal). Let $n_i$ the multiplicity of $C_i$ in $Z_1$. Then

$$\sum_{i=1}^{m} n_i C_i E \leq Z_1 E \leq 2(h-1) \tag{4}$$

since $E$ is nef and $Z_1 \leq K_{V/B}$.

Hurwitz formula yields

$$2(g-2h+1) = R_h E = \sum_{i=1}^{m} C_i E \tag{5}$$

By construction

$$(n_i+1)K_{V/B} - \mu_1^{\mathcal{H}} E \equiv n_i(K_{V/B}+C_i) + N_1 + (Z_1 - n_i C_i)$$

we have that

$$((n_i+1)K_{V/B} - \mu_1 E)C_i \geq 0 \tag{6}$$

since $(K_{V/B}+C_i)C_i \geq 0$ (Hurwitz formula), $N_1 C_i \geq 0$ ($N_i$ is nef) and $(Z_1 - n_i C_i)C_i \geq 0$ ($C_i$ is not a component of $Z_1 - n_i C_i$).

**Claim.** $K_{V/B}R \geq \frac{2(g-2\gamma+1)}{h}\mu_1^{\mathcal{H}}$

Proof of the Claim. We can assume $n_1 \geq n_2 \geq \ldots \geq n_m \geq 0$.

If $h-1 \geq n_1$ ($\geq n_i$ for all $i$) we have that $(hK_{V/B} - \mu_1 E)C_i \geq 0$ by (6) since $K_{V/B}$ is nef.

Assume $h \leq n_1$. Since $n_1 C_1 E \leq 2(h-1)$ we must have $C_1 E = 1$. Note that (4) gives $n_i \leq 2h-2-n_1$ for $i \geq 2$. Hence, using (5) and (6) we have

$$K_{V/B}R_h \geq \mu_1^{\mathcal{H}} \sum_{i=1}^{m} \frac{1}{n_i+1} C_i E \quad \geq \mu_1^{\mathcal{H}} \left(\frac{C_1 E}{n_1+1} + \frac{(R_h - C_1)E}{2h-1-n_1}\right) =$$

$$= {\mu_1}^{\mathcal{H}} \left(\frac{1}{n_1+1} + \frac{2g-4h+1}{2h-1-n_1}\right) \geq {\mu_1}^{\mathcal{H}} \frac{2(g-2h+1)}{h}$$

since $n_1 \geq h$. This proves the Claim.

Finally, since $K_{V/B} - \mu_n E$ is nef we have by (5)

$$K_{V/B}R \geq 2(g-2h+1)\mu_{\ell_1}^{\mathcal{H}}$$

In [15] p. 460 Xiao gives the following bound for any fibration

$$K_{V/B}^2 \geq 4\chi_1 - 2(\mu_1^{\mathcal{H}} + \mu_{\ell_1}^{\mathcal{H}})$$

So

$$K_{V/B}^2 - 4\chi_f \geq -4(\mu_1^{\mathcal{H}} + \mu_{\ell_1}^{\mathcal{H}}) + 2(g - 2\gamma + 1)\max\{\frac{\mu_1^{\mathcal{H}}}{h}, \mu_{\ell_1}^{\mathcal{H}}\}.$$

(ii) We consider the Xiao's data $\{(M_{i|E}^{\mathcal{H}}, r_i^{\mathcal{H}}, d_i^{\mathcal{H}})\}_{i=1}^{\ell_1}$. Since $\mid M_{i|E}^{\mathcal{H}} \mid$ is a sublinear system of $\mid K_E \mid$ by Clifford's lemma $d_i^{\mathcal{H}} \geq 2(r_i^{\mathcal{H}} - 1)$. Then it induces on $F$ a linear system of degree $a_i \geq 4(r_i^{\mathcal{H}} - 1)$. Hence, for $1 \leq i \leq \ell_1 - 1$ we have $a_i + a_{i+1} \geq 8r_i^{\mathcal{H}} - 4$. For $i = \ell_1$, $a_{\ell_1} + a_{\ell_1+1} \geq 4r_{\ell_1} - 4 + 2g - 2 \geq 8h - 4$ since $g \geq 2h + 1$ by hypothesis. By 1.6 we obtain

$$K_{S/B}^2 \geq 8\sum_{i=1}^{\ell_1} r_i^{\mathcal{H}}(\mu_i^{\mathcal{H}} - \mu_{i+1}^{\mathcal{H}}) - 4\mu_1^{\mathcal{H}} = 8\chi_1 - 4\mu_1^{\mathcal{H}} \geq 8\chi_1 - 4\mu_1^{\mathcal{E}}$$

since $\mu_1^{\mathcal{E}} \geq \mu_1^{\mathcal{H}}$. Hence by (i) we have: $(1 + \frac{2g-1}{g(g-1)})K_{S/B}^2 \geq 8\chi_1$.

(iii) Now we want compare $K_{S/B}^2$ with $\chi_2$. Let $N_i^{\mathcal{K}}$ and $Z_i^{\mathcal{K}}$ be the divisor on $V$ (on a suitable blow up of $V$) associated to the Harder-Narasimhan decomposition of $\mathcal{K}$. We put $N_i = \pi^*(N_i^{\mathcal{K}})$, $Z_i = \pi^*(Z_i^{\mathcal{K}})$, $H = K_{S/B}$, $\mu_i = \mu_i^{\mathcal{K}}$ where $i = 1, \ldots, l_2$. Since $H_i = H - Z_i - \mu_i F = \pi^*(K_{V/B} + \mathcal{L} - Z_i^{\mathcal{K}} - \mu_i^{\mathcal{K}} E)$ then $H_i$ is nef and by 1.6 $H^2 \geq \sum_{i=1}^{l_2}(d_i + d_{i+1})(\mu_i - \mu_{i+1})$ where $d_i = N_i F = 2d_i^{\mathcal{K}}$, $i = 1, \cdots, l_2$. We consider Xiao's data for $\mathcal{K}$: $\{(M_{i|E}^{\mathcal{K}}, r_i^{\mathcal{K}}, d_i^{\mathcal{K}})\}_{i=1}^{l_2}$. Now the linear systems $\mid M_{i|E}^{\mathcal{K}} \mid$ are sub-linear systems of $\mid K_E + \mathcal{L}_{|E} \mid$, so not always they are special.

We put $r_i = r_i^{\mathcal{K}}$. We have: $d_i^{\mathcal{K}} \geq 2(r_i^{\mathcal{K}} - 1)$ if $r_i^{\mathcal{K}} \leq \gamma$ and $d_i^{\mathcal{K}} = r_i^{\mathcal{K}} + \gamma - 1$ if $r_i^{\mathcal{K}} \geq \gamma - 1$. We distinguish two cases: $g \geq 2\gamma + s_2$ or $g \leq 2\gamma + s_2$.

**First Case:**$g \geq 2\gamma + s_2$. If we consider the degree as a function of the rank we easily see that $d_i^{\mathcal{K}} \geq \frac{g-s_2-1}{g-s_2-\gamma-1}$. Thus

$$d_i^{\mathcal{K}} + d_{i+1}^{\mathcal{K}} \geq 2\frac{g - s_2 - 1}{g - s_2 - \gamma - 1}r_i - \frac{g - s_2 - 1}{g - s_2 - \gamma - 1}$$

if $i \leq l_2 - 1$ and $d_{l_2}^{\mathcal{K}} + d_{l_2+1}^{\mathcal{K}} \geq 2\frac{g-s_2-1}{g-s_2-\gamma-1}(g - s_2 - \gamma) - 2\frac{g-s_2-1}{g-s_2-\gamma-1}$. We put $A = \frac{g-s_2-1}{g-s_2-\gamma-1}$ and $B = 2\frac{g-s_2-1}{g-s_2-\gamma-1}$, then

$$K_{S/B}^2 \geq \sum_{i=1}^{l_2}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \geq \sum_{i=1}^{l_2}(4Ar_i - B)(\mu_i - \mu_{i+1}) - B\mu_{l_2},$$

that is: $K_{S/B}^2 \geq 4A\chi_2 - B(\mu_1 + \mu_{l_2})$. We recall that $K_{S/B}^2 \geq d_{l_2}(\mu_1 + \mu_{l_2})$; then

$$(1 + \frac{2g-2-2s_2}{(2g-2)(g-\gamma-s_2-1)} K_{S/B}^2 \geq 4A\chi_2$$

so: $K_{S/B}^2 \geq 4\frac{(g-1)(g-s_2-1)}{(g-1)(g-\gamma)-s_2 g}\chi_2$.

**Second Case**:$g \leq 2\gamma + s_2$.

If $g \leq 2\gamma + s_2$ then $d_i \geq 4(r_i - 1)$, so $K_{S/B}^2 \geq 8\chi_2 - 4\mu_1$ then by (i)

$$K_{S/B}^2 \geq 8\frac{g(g-1)}{g^2+g-1}\chi_2.$$

□

In the next two theorems we find lower bounds for $\lambda(f)$ in the case of double covers, considering or not the influence of the relative irregularity of $f$ (see §3). Our limiting functions $l = l(g, \gamma, s_1, s_2)$ or $\tilde{l} = \tilde{l}(g, \gamma)$ have rather complicate expressions to be able to check their sharpness. Nevertheless we can give examples to check its assimptotic good behaviour.

**Theorem 2.4** *Let $f : S \to B$ be a genus $g$, relatively minimal, non isotrivial fibration. Assume $f$ is a double cover fibration of a fibration of genus $\gamma$. We use the notations of 2.2.*

*(i) If $g \geq 2\gamma + s_2$ and $g > 4\gamma + 1$ then*

$$\lambda(f) \geq 4+4\frac{(g-4\gamma-1)[(g-1)(\gamma-1)+s_2]}{(g-4\gamma-1)[(g-1)(g-\gamma)-gs_2]+2(g-1)(g-s_2-1)(\gamma-s_1)\gamma}.$$

*(ii) If $4\gamma + 1 \leq g \leq 2\gamma + s_2$ then*

$$\lambda(f) \geq 4 + 8\frac{2(g-4\gamma-1)(g-3}{2(g-4\gamma-1)+8(g-1)(\gamma-s_1)\gamma}.$$

**Proof.** Looking independently to the cases $\mu_{\ell_1}^{\mathcal{H}} \geq \mu_1^{\mathcal{H}}/\gamma$ and $\mu_{\ell_1}^{\mathcal{H}} \leq \mu_1^{\mathcal{H}}/\gamma$ in 2.3 we always get

$$K_{S/B}^2 - 4\chi \geq \frac{2(g-4\gamma-1)}{\gamma}\mu_1^{\mathcal{H}}.$$

Since $\mu_1^{\mathcal{H}} \geq \frac{1}{\gamma - s_1}\chi_1$ and $g \geq 4\gamma + 1$ then

$$K_{S/B}^2 \geq y_1 = 4\chi + \frac{2(g-4\gamma-1)}{\gamma(\gamma-s_1)}\chi_1.$$

By Proposition 2.3 we know that if $g \geq 2\gamma + s_2$ then

$$K_{S/B}^2 \geq y_2 = 4\frac{(g-1)(g-s_2-1)}{(g-1)(g-\gamma)-s_2 g}\chi - 4\frac{(g-1)(g-s_2-1)}{(g-1)(g-\gamma)-s_2 g}\chi_1.$$

We consider the bounds given above and in Proposition 2.3 as functions of $\chi_1 = x$. We consider the region delimited by this three linear inequalities in the plane $(x, y)$. Since $y_2(x_0) = y_1(x_0)$ implies

$$x_0 = \frac{[4g(\gamma-1)+1-\gamma+s_2](\gamma-s_1)\gamma}{2[(g-1)(g-\gamma)-gs_2](g-4\gamma-1)+4(g-1)(g-s_2-1)(\gamma-s_1)\gamma}\chi$$

then we find

$$\lambda(f) \geq 4+4\frac{(g-4\gamma-1)[g(\gamma-1)+1-\gamma+s_2]}{(g-4\gamma-1)[(g-1)(g-\gamma)-gs_2]+2(g-1)(g-s_2-1)(\gamma-s_1)\gamma}.$$

If $g \leq 2\gamma + s_2$ the same argument shows the claim. □

**Theorem 2.5** *Let $f : S \to B$ be a genus $g$, relatively minimal, non isotrivial fibration. Assume $f$ is a double cover fibration of a fibration of genus $\gamma$. Then, if $g \geq 4\gamma + 1$ we have*

$$\lambda(f) \geq 4 + 4\frac{(\gamma-1)(g-4\gamma-1)}{(g-4\gamma-1)(g-\gamma)+2(g-1)\gamma^2}.$$

**Proof.** The same argument of 2.4 works. □

**Corollary 2.6** *With the above hypotheses, if $q(S) = q(V) = b + \gamma$ and $g \geq 2\gamma + 1$ then*

$$\lambda(f) \geq 4\frac{g-1}{g-\gamma}.$$

**Proof.** We have $s_2 = 0$, $s_1 = s = \gamma$. Then the claim follows from 2.4. $\square$

**Remark 2.7** We notice that if $\gamma = 1$ we find $\lambda(f) \geq 4$ for biellitic fibrations of genus $g \geq 5$; see [1].

We give now some examples that show that under extra assumptions, the bounds given are assimptotically sharp.

**Example 2.8** Let $A$ be an abelian surface with a base point free linear system $|C|$, $C^2 = 4$. Then g(C)=3. Take $C_1, C_2$ two smooth and transversal members and let $\sigma : \widetilde{A} \longrightarrow A$ be the blow-up at the 4 base points. We have then a fibration $\tau : \widetilde{A} \longrightarrow \mathbb{P}^1$ with general fibre $\widetilde{C}$ a curve of genus 3. Let $E = E_1 + E_2 + E_3 + E_4$ be the $\sigma$-exceptional reduced and irreducible divisor. Note that $\tau(E) = \mathbb{P}^1$. Let $n >> m >> 0$ and $\delta = n\widetilde{C} + mE$; then $|2\delta|$ has no base point.

We can take then a smooth member $R \in |2\delta|$ and consider the associated double cover $\pi : S \longrightarrow \widetilde{A}$. Let $f : S \longrightarrow \mathbb{P}^1$ be the induced fibration, and let $F$ be a general fibre. Then $F$ is a double cover of $\widetilde{C}$ and we have (note that $K_{\widetilde{A}} = \sigma^* K_A + E = E$)

$$\begin{aligned}
&h = g(\widetilde{C}) = 3 \\
&g = g(F) = 4m + 5 \\
&K_S^2 = 2(K_{\widetilde{A}} + \delta)^2 = 8(m+1)(2n - m - 1) \\
&\chi\mathcal{O}_S = 2\chi\mathcal{O}_{\widetilde{A}} + \frac{1}{2}\delta K_{\widetilde{A}} + \frac{1}{2}\delta^2 = 2m(2n - m) + 2(n - m)
\end{aligned}$$

Moreover, observe that $\delta$ is nef and big since $\delta^2 > 0$ and $|2\delta|$ moves without base points. Then we can apply Kawamata-Viehweg vanishing theorem and get that $h^1(\widetilde{A}, \mathcal{O}_{\widetilde{A}}(-\delta)) = 0$; hence

$$q(S) = q(\widetilde{A}) = 2$$

Finally we obtain

$$\lambda(f) = \frac{8(m+1)(2n-m-1)+32(m+1)}{2m(2n-m)+2(n-m)+8(m+1)}$$

Fixing $m$ and making $n$ as big as needed we obtain fibrations with $g = 4m+5, h = 3$ and the slope arbitrarily near to

$$\frac{16(m+1)}{4m+2} = 4 + \frac{8}{g-3} = \lambda'.$$

Now if in Theorem 3.4 we put the data of the above fibration: $s_2 = 0$, $s_1 = 2$, $\gamma = 3$, $g = 4m+5$ with $m \gg 0$ then we obtain

$$\lambda(f) \geq 4 + \frac{8}{(g-3) + \frac{6(g-1)}{g-13}} \simeq 4 + \frac{8}{g+3}.$$

**Remark 2.9** Let $Y$ be a smooth surface, let $B$ be a smooth curve and denote by $\pi_Y : Y \times B \to Y$, $\pi_B : Y \times B \to B$ the two natural projections. Let $C, E \in \mathrm{Div}(Y)$ where $E$ has genus $\gamma$ and $\beta, \eta \in \mathrm{Div}(B)$ such that there exist $W$, $V$ smooth divisors, $W \in \mid 2(\pi_Y^*(C) + \pi_B^*(\beta)) \mid$ and $V \in \mid \pi_Y^*(E) + \pi_B^*(\eta) \mid$. Let $\tau : Z \to Y \times B$ be the double cover branched on $W$ and $S = \tau^*(V)$. We put $\pi = \tau_{|S}$, $\phi = \pi_{B|V}$, $f = \phi \circ \pi$ and we assume that $W_{|V} = \Delta$ is a smooth divisor. Then $S$ is smooth, $f : S \to B$ is a fibration and if $m = \deg(\eta)$, $n = \deg(\beta)$ and $g$ is the genus of the general fibre $F$ of $f$ then

$$\lambda(f) = 6\frac{4(g-1)(n+m) + 2m(K_Y + C + E)^2}{(3n+6m)(g-1) + 12m\chi(\mathcal{O}_Y) + 6m(g(C)-1) + 3nEC}$$

If we consider a $K3$ surface $Y$ we find that for each smooth curve $B$ and for each $\gamma \geq 2$ there exists a double cover fibration $f$ of genus $g = 4\gamma - 3$ such that $\lambda(f) = \frac{(16n+32m)(\gamma-1)}{(3n+5m)(\gamma-1)+4}$. If $B = \mathbb{P}^1$ we can take $n = m = 1$ and obtain a genus 5 fibration $f : S \to \mathbb{P}^1$ with $\lambda(f) = 4$ which is a double cover of a genus 2 fibration. In particular the slope 4 can be achieved by double cover fibrations with $\gamma > 1$.

**Remark 2.10** The constant $4\frac{g-1}{g-\gamma}$ that appears in Corollary 2.6 plays a curious role in the study of double covers: it appears as a limit bound when adding fibres to the ramification locus. Indeed, let $F$ and $E$ be respectively, the general fibre of the fibrations $f : S \to B$, $\phi : V \to B$ where $f$ is assumed to be relatively minimal and non isotrivial. To simplify we assume (although it is not necessary) that $\pi : S \to V$ is a double covering branched on a smooth divisor $\Delta \in \mid 2\mathcal{L} \mid$, where $\mathcal{L} \in \mathrm{Pic}(V)$ such that $f = \pi \circ \phi$. Let $\Delta_n \in \mid 2(\mathcal{L} + \phi^*\eta_n) \mid$ where $\eta_n$ is a divisor on $B$ of degree $n > 0$, let $\pi_n : S_n \to V$ be the double covering branched on $\Delta_n$ and $f_n = \pi_n \circ \phi$. Then the sequence of the slopes $\{\lambda(f_n)\}_{n\geq 0}$ is monotonous and $\lim_{n\to\infty} \lambda(f_n) = 4\frac{g-1}{g-\gamma}$.

In particular, if $\gamma = 0$ or $1$ (hyperelliptic or bielliptic fibrations) we obtain $4 - \frac{4}{g}$ and $4$ recpectively, which are the exact lower bounds (cf. [15], [1]). Nevertheless is not true that $\lambda_{\mathrm{exp}} = 4\frac{g-1}{g-\gamma}$ is in general a lower bound for double cover fibrations, as the following example shows.

**Example 2.11** We start as in [4][2.6 Example 3]. Let $Y = A \times H$ where $A$ and $H$ are elliptic curves $\epsilon$ a point of order two on $A$. Let $X = Y/\langle\sigma\rangle$ where $\sigma$ is an involution defined on $Y$ by $\sigma(a, h) = (a + \epsilon, -h)$. We denote by $A'$ the quotient of $A$ by the group $\{0, \epsilon\}$ and by $h_1, ..., h_4$ the points of order 2 on $H$. Let $p : X \to A'$ and $q : X \to I\!P^1 = B$ the two natural elliptic fibrations on $X$. Clearly $K_X = p^*(\eta)$ where $\eta$ is the divisor on $\mathrm{Pic}^0(A')$ associated to the étale covering $A \to A'$. Let $A'_i = q^{-1}(h_i)$ for $i = 1, 2, 3, 4$ and $Q \in A'$. Since the divisor $\delta = p^*(\eta + dQ) + A'_1$ is 2-divisible on $X$ we consider the double covering $\mu : V \to X$ associated to $\delta$. We denote $p \circ \mu = \phi : V \to B$ and $q \circ \mu = l : X \to I\!P^1$. Let $C = \phi^*(P')$ and $E = l^*(P)$ where $P' \in A'$ and $P \in B$. Let $\mathcal{L} = nE + mC$, $\pi : S_{d,n,m} \to V$ the double covering associated to $\mathcal{L}$ and $f_{d,n,m} : S_{d,n,m} \to B$ the induced fibration on $B$ with fibre $F = \pi^*(E)$. By the standard theory of double covering we have :

$$\lambda(f_{d,n,m}) = 8\frac{2nm + 2nd + 5m + 5d}{4nm + 2nd + 5m + 6d}$$

and $g = g(F) = 4m + 4d + 1$, $\gamma = g(E) = 2d + 1$. In particular $\lambda_{\mathrm{exp}} = 4\frac{g-1}{g-\gamma} = 8\frac{m+d}{2m+d} = \lim_{n\to\infty} \lambda(f_{d,n,m})$ but $\lim_{d\to\infty} \lambda_{\mathrm{exp}} = 8 > 8\frac{2n+5}{2n+6} = \lim_{d\to\infty} \lambda(f_{d,n,m})$.

We remark that to obtain $\lambda(f) < \lambda_{\mathrm{exp}}$ we have $g \sim 2\gamma$, and that $F$ is a double cover of a double cover.

Our techniques requires the assumption $g \geq 4\gamma + 1$. However the following theorem shows that for a double cover fibration $\lambda(f) \geq 4$ holds with a few exceptions.

**Theorem 2.12** *Let $f : S \to B$ be a relatively minimal, non isotrivial double cover fibration of $\sigma : V \to B$. Let $F$ and $E$ be the fibres of $f$ and $\sigma$ respectively and let $g = g(F)$, $\gamma = g(E)$. Assume $F$ is not hyperelliptic or tetragonal, $\gamma \geq 1$ and $g \geq 2\gamma + 11$. Then $\lambda(f) \geq 4$.*

**Proof.** By [1](Theorem 2.1) we can assume $F$ is not bielliptic since $2 + 11 = 13 \geq 6$. We can also assume $F$ is not trigonal otherwise $\lambda(f) \geq \frac{14(g-1)}{3g+1} \geq 4$ if $g \geq 9$ using [9] (Main theorem).
Consider the Harder Narasimhan filtration of $\mathcal{E} = f_*\omega_{S/B}$: $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_l = \mathcal{E}$ with slopes $\mu_1 > ... > \mu_l \geq 0$ and Xiao's data $\{(M_{i|F}, r_i, d_i)\}_{i=1}^{l}$ . Note that if $\mid M_{i|F} \mid$ induces a map $\phi_i$ we have :

If $\deg(\phi_i) = 1$ $d_i \geq 3r_i - 4$ (if $d_i \leq g-1$), $d_i \geq \frac{3r_i+g-4}{2}$ (otherwise);

If $\deg(\phi_i) = 2$ $d_i \geq 2r_i + 2$ (since $F$ is not hyperelliptic nor bielliptic)

If $\deg(\phi_i) = 3$ $d_i \geq 3r_i$ (since $F$ is not trigonal)

If $\deg(\phi_i) \geq 4$ $d_i \geq 4(r_i - 1)$ .

Observe that, since $M_i \leq M_{i+1}$, the map $\phi_i$ factorizes through $\phi_{i+1}$ and then $d_{i+1}|d_i$. Note also that $r_{i+1} \geq r_i + 1$ and $d_{i+1} \geq d_i$. Then we can prove $d_i + d_{i+1} \geq 4r_i + 1$ with a few exceptions. Indeed $\mid M_i \mid$ does not define any map only if $(r_1, d_1) = (1, 0)$. Then $d_2 \geq 5 = 4r_1 + 1$ except if $d_2 = 2, 3, 4$. All these possibilities imply $r_2 = 2$ according to the previous inequalities and hence $F$ would be hyperelliptic, trigonal or tetragonal, all of these being impossible by hypothesis. From now on we assume $r_i \geq 2$.

If $\deg(\phi_i) \geq 2$ then $d_i \geq 2r_i$ and hence $d_i + d_{i+1} \geq 2d_i + 1 \geq 4r_i + 1$, if $d_i < d_{i+1}$; if $d_i = d_{i+1}$, then $\phi_i = \phi_{i+1}$ and hence $d_i + d_{i+1} \geq 4r_i + 2$.

If $\deg(\phi_i) = 1$ then also $\deg(\phi_{i+1}) = 1$. If $d_i, d_{i+1} \leq g - 1$ then $d_i + d_{i+1} \geq 3r_i - 4 + 3r_{i+1} - 4 \geq 6r_i - 5 \geq 4r_i + 1$ since $r_i \geq 3$ ($\phi_i$ is birational).

If $d_i \leq g-1$, $d_{i+1} \geq g$ then $d_i + d_{i+1} \geq 2d_i + 1 \geq 6r_i - 7 \geq 4r_i + 1$ except if $r_i = 3$. But then $d_i + d_{i+1} \geq (3.3 - 4) + g = 5 + g \geq 13 \geq 4r_i + 1$ since $g \geq 11$ by hypothesis.

Finally assume $d_i, d_{i+1} \geq g$, being $\phi_i$ and $\phi_{i+1}$ birational maps. Then

$$d_i + d_{i+1} \geq \frac{3r_i + g - 4}{2} + \frac{3r_{i+1} + g - 4}{2} \geq 3r_i + g - 4 + \frac{3}{2} \geq 4r_i + 1$$

if $r_i \leq g - 3$ (the case $r_i = g - 3$ needs a bit care).

Assume $r_i = g - 2$. If $r_{i+1} = g$ then $d_{i+1} = 2g - 2$ and we are done. If $r_{i+1} = g-1$ then the only case to check is $d_i = 2g-5$, $d_{i+1} = 2g-3$. Note that then $h^0(F, K_F - M_{i_{|F}}) = h^0(F, K_F - M_{i+1_{|F}}) = 1$ since $F$ is not hyperelliptic. By Riemann-Roch $r_i = h^0(F, M_{i_{|F}}) = 1 + d_i + 1 - g = g - 3$ which is impossible.

Assume $r_i = g - 1$. Then $d_i = 2g - 3$ , $(M_{i+1_{|F}}, d_{i+1}) = (r_l, d_l) = (g, 2g - 2)$ and $d_i + d_{i+1} = 4g - 5 = 4r_i - 1$.

For $r_i = g = r_l$ we have $d_l + d_{l+1} = 2d_l = 4g - 4 = 4r_l - 4$. By 1.6 we conclude

$$\begin{array}{rcc} K^2_{S/B} & \geq & \sum_{i=1}^{l}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \geq \\ & \geq & \sum_{i=1}^{l}(4r_i + 1)(\mu_i - \mu_{i+1}) - 2(\mu_{l-1} - \mu_l) - 5\mu_l \\ & = & 4\chi + \mu_1 - 2\mu_{l-1} - 3\mu_l \end{array} \tag{7}$$

if $r_{l-1} = g - 1$, $d_{l-1} = 2g - 3$; otherwise

$$K^2_{S/B} \geq \sum_{i=1} l(4r_i + 1)(\mu_i - \mu_{i+1}) - 5\mu_l = 4\chi + \mu_1 - 5\mu_l.$$

Let us consider first the general case. If $\mu_1 \geq 5\mu_l$ we are done. Assume $\mu_1 \leq 5\mu_l$. Let $\mathcal{H} = \phi_*(\omega_{V|B})$ and $\mathcal{K} = \phi_*(\omega_{V|B} \otimes \mathcal{L})$. By 1.2, $\mathcal{E}_i = \mathcal{H}_{\psi(i)} \oplus \mathcal{K}_{\phi(i)}$ where $\mu_i = \mu^{\mathcal{H}}_{\psi(i)} = \mu^{\mathcal{K}}_{\phi(i)}$. By 1.3 we have that $\mu_l = \min\{\mu^{\mathcal{H}}_{\ell_1}, \mu^{\mathcal{K}}_{l_2}\} \leq \mu^{\mathcal{H}}_{\ell_1}$, $\mu_1 = \max\{\mu^{\mathcal{H}}_1, \mu^{\mathcal{K}}_1\} \geq \mu^{\mathcal{H}}_1$. Hence $\mu^{\mathcal{H}}_1 \leq \mu_1 < 5\mu_l \leq 5\mu^{\mathcal{H}}_{\ell_1}$. By 2.3 we have

$$K^2_{S/B} \geq 4\chi - 24\mu^{\mathcal{H}}_{\ell_1} + 2(g - 2\gamma + 1)\max\{\frac{\mu^{\mathcal{H}}_1}{\gamma}, \mu^{\mathcal{H}}_{\ell_1}\}.$$

If $\gamma \geq 5$ or $\frac{\mu^{\mathcal{H}}_1}{\gamma} \leq \mu^{\mathcal{H}}_{\ell_1}$ we have $\max\{\frac{\mu^{\mathcal{H}}_1}{\gamma}, \mu^{\mathcal{H}}_{\ell_1}\} = \mu^{\mathcal{H}}_{\ell_1}$ and hence

$$K^2_{S/B} \geq 4\chi + 2(g - 2\gamma - 11)\mu^{\mathcal{H}}_{\ell_1} \geq 4\chi$$

when $g \geq 2\gamma + 11$.

If $\gamma = 2, 3, 4$ and $\frac{\mu^{\mathcal{H}}_1}{\gamma} \geq \mu^{\mathcal{H}}_{\ell_1}$ then

$$K^2_{S/B} \geq 4\chi - 24\mu^{\mathcal{H}}_{\ell_1} + 2(g - 2\gamma + 1)\frac{\mu^{\mathcal{H}}_1}{\gamma} \geq 4\chi + 2(g - 2\gamma - 11)\frac{\mu^{\mathcal{H}}_1}{\gamma} \geq 4\chi$$

when $g \geq 2\gamma + 11$.

Consider finally the special case $r_{l-1} = g - 1$, $d_{l-2} = 2g - 3$. By 1.6, with the notations of 1.6 where $H = \omega_{S/B}$ it follows that $Z_{l-1}$ is a section of $f$ such that $Z_{l-1_{|F}} \equiv K_F - M_{l-1_{|F}}$. We recall that $M_{l-1_{|F}}$ is the base point free linear system induced on the general fibre by the piece $\mathcal{E}_{l-1}$. By 1.2

$\mathcal{E}_{l-1} = \mathcal{H}_{\psi(l-1)} \oplus \mathcal{K}_{\phi(l-1)}$. Since $r_{l-1} = r_l - 1$ we only have two possibilities: either $\mathcal{H}_{\psi(l-1)} = \mathcal{H}_{\ell_1}$, $\mathcal{K}_{\phi(l-1)} = \mathcal{K}_{l_2-1}$ and $r^{\mathcal{K}}_{l_2-1} = g-\gamma-1$ or $\mathcal{H}_{\psi(l-1)} = \mathcal{H}_{\ell_1-1}$, $\mathcal{K}_{\phi(l-1)} = \mathcal{K}_{l_2}$ and $r^{\mathcal{K}}_{\ell_1-1} = \gamma-1$. We claim the second possibility can not occur. Indeed consider the double cover $\pi_{|F} : F \to E$. We have that

$$H^0(F,\omega_F) \simeq H^0(E,\omega_E) \oplus H^0(E,\omega_E \otimes \mathcal{L}_{|E})$$

This decomposition means that if $D$ is the ramification divisor on $F$ and $t \in H^0(F,\mathcal{O}_F(D))$ then for every $\omega \in H^0(F,\omega_F)$, $\omega = t\pi^*_{|F}(\omega_1) + \pi^*_{|F}(\omega_2)$ where $\omega_1 \in H^0(E,\omega_E)$ and $\omega_2 \in H^0(E,\omega_E \otimes \mathcal{L}_{|E})$.

We have $V \in H^0(F,\omega_F)$ a codimension one subspace which produces, after taking out the base point, the linear series $\mid M_{l-1_{|F}} \mid$. The second possibility asserts that $V = \pi^*_{|F}V_1 \oplus \pi^*_{|F}V_2$ where $V_2 = H^0(E,\omega_E \otimes \mathcal{L}_{|E})$ and $V_1$ is a codimension one subspace of $H^0(E,\omega_E)$. Since $\deg(\omega_E \otimes \mathcal{L}_{|E}) \geq 2\gamma+10$, $V_2$ is base point free. Hence $\pi^*_{|F}V_2$ is base point free: a contradiction since $V$ has a base point.

So we have the following decompositions

$$\mathcal{E}_l = \mathcal{H}_{\ell_1} \oplus \mathcal{K}_{l_2},\ \mathcal{E}_{l-1} = \mathcal{H}_{\ell_1} \oplus \mathcal{K}_{l_2-1}$$

where $r^{\mathcal{K}}_{l_2-1} = g-\gamma-1$. If $\mathcal{E}_{l-2} = \mathcal{H}_j \oplus \mathcal{K}_k$ we have several possibilities according to 1.2.

If $j = \ell_1$, $k = l_2 - 2$ then $\mu_{l-1} = \mu(\mathcal{E}_{l-1}/\mathcal{E}_{l-2}) = \mu(\mathcal{K}_{l_2-1}/\mathcal{K}_{l_2-2}) = \mu^{\mathcal{K}}_{l_2-1}$ and $\mu^{\mathcal{H}}_{\ell_1} > \mu_{l-1}$.

If $j = \ell_1 - 1$, $k = l_2 - 1$ then $\mu_{l-1} = \mu(\mathcal{H}_{\ell_1}/\mathcal{H}_{\ell_1-1}) = \mu^{\mathcal{H}}_{\ell_1}$.

If $j = \ell_1 - 1$, $k = l_2 - 2$ then $\mu_{l-1} = \mu^{\mathcal{H}}_{l_2} = \mu^{\mathcal{K}}_{l_2-1}$.

In any case we get $\mu_{l-1} = \mu^{\mathcal{H}}_{\ell_1}$. Since always happens that $\mu_1 \geq \mu^{\mathcal{H}}_1$ and $\mu_l \leq \mu^{\mathcal{H}}_{\ell_1}$, (7) reads:

$$K^2_{S/B} \geq 4\chi + \mu_1 - 2\mu_{l-1} - 3\mu_l \geq 4\chi + \mu^{\mathcal{H}}_1 - 5\mu^{\mathcal{H}}_{\ell_1}.$$

If $\mu^{\mathcal{H}}_1 \geq 5\mu^{\mathcal{H}}_{\ell_1}$ we are done. If $\mu^{\mathcal{H}}_1 < 5\mu^{\mathcal{H}}_{\ell_1}$ then we can repeat the argument of the general case. $\square$

# 3 The slope of non-Albanese fibrations

In this section we consider the problem of the influence of the relative irregularity $h = q(S) - b$ on the lower bound of the slope. In case $f$ is a double cover fibration this problem has been considered in the previous section so we will deal only with non double cover fibrations. The nice fact is that we find a lower bound which is an increasing function of the genus $g$ and of $h$.

When $h = 0$ then the general bound $\lambda(f) \geq 4 - \frac{4}{g}$ holds and is sharp. So we will consider fibrations with $h = q(S) - b > 0$. Those are precisely the fibrations for which the Albanese map of $S$ does not factorize through $f$ (i.e., $b = 0$ and $q > 0$ or $S$ is of Albanese general type). We call such fibrations *non-Albanese fibrations.*

Let us first recall the two basic known results in this area:

**Theorem 3.1** *Let $f : S \to B$ be a relatively minimal, non locally trivial, genus $b$ pencil of curves of genus $g$.*

*(i) If $h = q(S) - b > 0$, then $\lambda \geq 4$*

*(ii) Let $\mu_1 = \deg\mathcal{E}_1/\mathrm{rank}(\mathcal{E}_1)$ where $\mathcal{E}_1$ is the maximal semistable subbundles of $f_*\omega_{S/B}$. If $g \geq 2$ and $g > q - b$ then $K^2_{S/B} \geq \frac{4g(g-1)}{(2g-1)}\mu_1$. In particular $\lambda(f) \geq \frac{4g(g-1)}{(2g-1)(g-h)}$.*

**Proof.** $(i)$ is [15][Theorem 2.4]. $(ii)$ is [9][Lemma 2.7] □

**Theorem 3.2** *Let $f : S \longrightarrow B$ be a relatively minimal fibration which is not a double cover fibration. Assume $g = g(F) \geq 5$ and that $f$ is not locally trivial. Let $h = q(S) - b \geq 1$.*

*Then*

*(i) If $h \geq 2$ and $g \geq \frac{3}{2}h + 2$ then*

$$\lambda(f) \geq \frac{8g(g-1)(4g-3h-10)}{8g(g-1)(g-h-2)+3(h-2)(2g-1)} \quad \text{if } F \text{ is not trigonal}$$

$$\lambda(f) \geq \frac{4g(g-1)(4g-3h-10)}{4g(g-1)(g-h-2)+(g-4)(2g-1)} \quad \text{if } F \text{ is trigonal}$$

*(ii) If $g < \frac{3}{2}h + 2$ then*

$$\lambda(f) \geq \frac{4g(g-1)(2g-7)}{\frac{4}{3}g(g-1)(g-3)+(g-4)(2g-1)}.$$

**Proof.**

(i), (ii) Consider Fujita's decomposition $f_*\omega_{S/B} = \mathcal{A} \oplus \mathcal{Z}$ with $\mathcal{Z} = \mathcal{O}_B^{\oplus h}$. Consider the Harder-Narasimhan filtration of $\mathcal{A}$:

$$0 = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_\ell = \mathcal{A}$$

As in §1 we produce nef $\mathbb{Q}$-divisors $N_i$, and effective divisors $Z_i$ in a suitable blow-up of $S$ $\quad \sigma : \widetilde{S} \longrightarrow S$ such that

$$N_i + \mu_i F + Z_i \equiv N_j + \mu_j F + Z_j \equiv \sigma^* K_{S/B}$$

where $\{\mu_i\}$ are the Harder-Narasimhan slopes of $\mathcal{A}$. Note that we can define $N_{\ell+1} = \sigma^* K_{S/B}$, $Z_{\ell+1} = 0$, $\mu_{\ell+1} = 0$. Observe also that, if $r_i = \mathrm{rk}\mathcal{A}_i$, $\sum\limits_{i=1}^{\ell} r_i(\mu_i - \mu_{i+1}) = \deg\mathcal{A} = \chi_f$.

Each $N_i$ induces on $F$ a base point free linear system of degree $d_i$ and (projective) dimension greater or equal than $r_i - 1$. Note that $N_i + \mu_i F = H_i$ is induced by a map $\varphi_i : S \longrightarrow \mathbb{P}_B(\mathcal{A}_i)$ which restricted to fibres induces the above linear system. By hypothesis $\varphi_i$ is never a double cover onto the image and so the induced map $\psi_i$ on $F$ is not a double cover. Hence we have

$$\begin{aligned} d_i &\geq 3(r_i - 1) && \text{if } \deg\psi_i \geq 3 \\ d_i &\geq 3r_i - 4 && \text{if } \deg\psi_i = 1 \text{ and } d_i \leq g - 1 \\ d_i &\geq \tfrac{3r_i + g - 4}{2} && \text{if } \deg\psi_i = 1 \text{ and } d_i \geq g \end{aligned}$$

the latest two inequalities being "Clifford plus" Lemma. Considering the above inequalities in the $(r, d)$-plane, we have the following two possibilities (note that the lines $d = 3r - 4$ and $d = \frac{3r+g-4}{2}$ meet exactly at the point $(r = \frac{1}{3}(g+4), d = g)$) depending on $\mathrm{rank}\mathcal{A} = g - h$.

**Case 1.-** $g - h \geq \frac{1}{3}(g + 4)$

In this case note that for every $1 \leq i \leq \ell$, $d_i \geq \frac{2g - \frac{3}{2}h - 5}{g - h - 2} r - \frac{g-4}{g-h-2}$ (this border line joining the point (2,3) and the point $(g - h, 2g - \frac{3}{2}h - 2)$) except if $(r_1, d_1) = (1, 0)$. Note that $g - h - 2 > 0$ since $g \geq \frac{3}{2}h + 2$.

Note also that by definition we have $d_{\ell+1} = 2g - 2$. So for $1 \leq i \leq \ell$ we get (since $r_{i+1} \geq r_i + 1$)

$$d_i + d_{i+1} \geq \frac{4g - 3h - 10}{g - h - 2} r_i - \frac{3(h-2)}{2(g - h - 2)} =: Ar_i + B$$

except if $(r_1,d_1)=(1,0)$ and $(r_2,d_2)=(2,3)$. In this exceptional case we get

$$d_1+d_2-Ar_1-B=3-A-B=-\frac{g-\frac{3}{2}h-1}{g-h-2}$$

If this happens $F$ is trigonal since has a linear system of degree 3 and dimension 1.

Applying Xiao's formula we get, in the general case,

$$\begin{aligned} K^2_{S/B} &\geq \sum_{i=1}^{\ell}(d_i+d_{i+1})(\mu_i-\mu_{i+1}) \geq \sum_{i=1}^{\ell} Ar_i(\mu_i-\mu_{i+1})+\sum_{i=1}^{\ell} B(\mu_i-\mu_{i+1}) = \\ &= A\chi_f+B\mu_1=\tfrac{4g-3h-10}{g-h-2}\chi_f-\tfrac{3(h-2)}{2(g-h-2)}\mu_1 \end{aligned}$$

Applying again Konno's bound:

$$K^2_{S/B}\geq \frac{4g(g-1)}{2g-1}\mu_1$$

we can eliminate $\mu_1$ and get

$$K^2_{S/B}\geq \frac{8g(g-1)(4g-3h-10)}{8g(g-1)(g-h-2)+3(h-2)(2g-1)}\chi_f$$

Note that this bound is a strictly increasing function of $h$ and that $K^2_{S/B}\geq 4\chi_f$ if $h\geq 2$.

In the exceptional case (when $F$ is trigonal) we get

$$K^2_{S/B}\geq A\chi_f+B\mu_1-\frac{g-\frac{3}{2}h-1}{g-h-2}(\mu_1-\mu_2)\geq A\chi_f+\left(B-\frac{g-\frac{3}{2}h-1}{g-h-2}\right)\mu_1$$

The same argument using $K^2_{S/B}\geq\frac{4g(g-1)}{2g-1}\mu_1$ yields

$$K^2_{S/B}\geq \frac{4g(g-1)(4g-3h-10)}{4g(g-1)(g-h-2)+(g-4)(2g-1)}\chi_f$$

which is also a strictly increasing function of $h$. In this case we need $h\geq 4$ to get $K^2_{S/B}\geq 4\chi_f$.

**Case 2.-** $g-h\leq\frac{1}{3}(g+4)$

Let $\overline{h} = \left[\frac{2}{3}g - \frac{4}{3}\right]$. Under our hypotheses $h \geq \overline{h}$, so we can take $\overline{\mathcal{A}} = \mathcal{A} \oplus \mathcal{O}_B^{\oplus(h-\overline{h})}$ instead of $\mathcal{A}$. Hence we get according to whether we are in the general or in the special case

$$K_{S/B}^2 \geq \frac{8g(g-1)(4g-3\overline{h}-10)}{8g(g-1)(g-\overline{h}-2)+3(\overline{h}-2)(2g-1)}\chi_f \geq \frac{8g(g-1)(2g-7)}{\frac{8}{3}g(g-1)(g-3)+(2g-9)(2g-1)}\chi_f$$

$$K_{S/B}^2 \geq \frac{4g(g-1)(4g-3\overline{h}-10)}{4g(g-1)(g-\overline{h}-2)+(g-4)(2g-1)}\chi_f \geq \frac{4g(g-1)(2g-7)}{\frac{4}{3}g(g-1)(g-3)+(g-4)(2g-1)}\chi_f$$

since both expressions are increasing functions of $h$ and $\overline{h} \geq \frac{2}{3}g - 1$. Note that the second bound is slightly smaller than the first one. $\square$

**Remark 3.3** In the case (iii) of the theorem we could consider that for $1 \leq i \leq \ell$, $d_i \geq 3r_i - 4$ and hence $d_i + d_{i+1} \geq 6r_i - 5$ for $1 \leq i \leq \ell - 1$. But for $i = \ell$ we would have $d_\ell + d_{\ell+1} \geq 2d_\ell + 1 \geq 6r_\ell - 7$ which produces

$$K_{S/B}^2 \geq 6\chi_f - (5\mu_1 + 2\mu_\ell)$$

Hence using Xiao's inequality with indexes $\{1, \ell\}$

$$K_{S/B}^2 \geq (d_1 + d_\ell)(\mu_1 - \mu_\ell) + (d_\ell + d_{\ell+1})\mu_\ell \geq d_\ell(\mu_1 + \mu_\ell) \geq (3g - 3h - 4)(\mu_1 + \mu_\ell)$$

we get

$$K_{S/B}^2 \geq 6\frac{3g - 3h - 4}{3g - 3h + 1}\chi_f$$

which depends on $h$ and is better than (iii) for some special values of $(g, h)$ but is a *decreasing* function of $h$.

Nevertheless we must have in mind that case (iii) of the theorem is doubtfull to happen. Indeed, by a conjecture of Xiao (cf. [16]) the following inequality should hold: $h = q_f \leq \frac{1}{2}(g + 1)$. This inequality is true when $b = 0$ but is known to be false in general (cf. [13]) although it seems that only the constant term should be modified.

**Remark 3.4** In the above theorem we worked with $\mathcal{Z} = \mathcal{O}_B^{\oplus(q(S)-b)}$ and $h = \mathrm{rank}\mathcal{Z}$. In most parts of the proof we only use that $\deg\mathcal{Z} = 0$. Hence, we get the same bounds in (ii) if we define $h$ to be the rank of the degree zero part in Fujita's decomposition of $\mathcal{E} = f_*\omega_{S/B}$ ($h \geq q(S) - b$). Note that then the argument of Theorem 3.2 (ii) does not work since we do not know whether $\mathcal{Z}$ can be cut in pieces of the length we need. In any case the bound of the previous remark holds for this new definition of $h$.

**Remark 3.5** Remember that if $F$ is trigonal we have (cf. [9] and [14]):

$$\lambda(f) \geq \frac{14(g-1)}{3g+1}$$

which is better that Theorem 4.19 (ii) (special case) for $g \gg h = q - b$ and that gives $\lambda(f) \geq 4$ if $g \geq 9$.

**Remark 3.6** As a function on $g$ (fixing $h$) the bounds of 3.2 tend to be 4 when $g$ grows (compare Theorem 3.1 (ii) where this limit is 2).

**Example 3.7** Let $Y$ be a smooth surface, let $B$ be a smooth curve of genus $b$, $Z = Y \times B$ and let $\pi_Y : Z \to Y$, $\pi_B : Z \to B$ be the natural projections. If $F \in \mathrm{Div}(Y)$ is smooth of genus $g$, $\eta \in \mathrm{Div}(B)$ and there exists an ample and smooth divisor $S \in \mid \pi_Y^*(F) + \pi_B^*(\eta) \mid$ then the slope of the fibration $f : S \to B$ induced on $S$ by $\pi_B$ is

$$\lambda(f) = \frac{6g - 6 + K_Y^2 + K_Y F}{\chi(\mathcal{O}_Y) + g - 1}.$$

Now if $\rho : Y = I\!P(\mathcal{E}) \to C$ is a ruled surface, $H$ is a section such that $H^2 = \deg(\mathcal{E})$ and $F \equiv 3H$ the fibration $f_m : S_m \to B$ ($m = \deg\eta$) has slope:

$$\lambda(f_m) = \frac{15m + 16(g(C) - 1))}{3m + 2(g(C) - 1))}$$

and verifies that $h = q(S_m) - b = g(C)$. In particular $\lambda(f_m) \geq 5$ and $\lim_{m\to\infty} \lambda(f_m) = 5$. This result indicates that for any $h$, a general lower bound of $\lambda(f)$ is below 5.

¿From Theorem 3.2 we obtain that $\lambda(f)$ controls the existence of other fibrations on $S$:

**Theorem 3.8** *Let $f : S \longrightarrow B$ be a relatively minimal, non locally trivial fibration. Let $F$ be a fibre of $f$, $g = g(F)$ and $q = q(S)$. Assume $f$ is not a double cover fibration and that $h = q - b \geq 1$ (i.e., $f$ is not an Albanese* *fibration). Let* $\mathcal{C} = \{\pi_i : S \longrightarrow C_i$ *fibrations*$, c_i = g(C_i) \geq 2, \pi_i \neq f\}_{i\in I}$. *Assume* $\mathcal{C} \neq \emptyset$ *and let* $c = max\{c_i | i \in I\}$. *Then*

*(i)* $\lambda(f) \geq 4 + \frac{c-1}{g-c}$

*(ii) If, moreover,* $\dim alb(S) = 1$ *(then necessarily* $b = 0$*) we have*

$$\lambda(f) \geq 4 + \frac{q-1}{g-q}$$

**Proof.** Remember that if $f$ is not an Albanese fibration then either $\dim alb(S) = 2$ or $b = 0$ (provided $q(S) \neq 0$).

Let $\pi : S \longrightarrow C$ be the fibration with maximal base genus $c \geq 2$ (if $\dim alb(S) = 1$, then $c = q$ and $\pi = alb$).

Since in any case $f^*Pic^0(B)$ does not include $\pi^*Pic^0(C)$ we can choose for $n \gg 0$, a $n$-torsion element $\mathcal{L} \in Pic^0(C)$ such that $\pi^*\mathcal{L}^{\otimes i} \notin f^*Pic^0(B)$ for $1 \leq i \leq n-1$. Consider the base change

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{\alpha}"] & \widetilde{C} \arrow[d, "\alpha"] \\ S \arrow[r, "\pi"] \arrow[d, "f"] & C \\ B & \end{tikzcd}$$

and let $\widetilde{f} = f \circ \widetilde{\alpha}$. Since $\mathcal{L}^{\otimes i}_{|F} \neq \mathcal{O}_F$ for $1 \leq i \leq n-1$, $\widetilde{f}$ has connected fibres and so $\widetilde{f}$ is again a fibration over $B$. Let $\widetilde{F}$ be the fibre of $\widetilde{f}$. Then if $\widetilde{g} = g(\widetilde{F})$,

$$\widetilde{g} - 1 = n(g-1)$$

Moreover we have

$$q(\widetilde{S}) = h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}) = h^1(S, \mathcal{O}_S) + \sum_{i=1}^{n-1} h^1(S, (\pi^*\mathcal{L}^{-i}))$$

¿From the exact sequence

$$0 \longrightarrow H^1(B, \mathcal{L}^{-i}) \longrightarrow H^1(S, \pi^*\mathcal{L}^{-i}) \longrightarrow H^0(B, (R^1\pi_*\mathcal{O}_S) \otimes \mathcal{L}^{-i}) \longrightarrow 0$$

and using that $h^0(B,(R^1\pi_*\mathcal{O}_S)\otimes\mathcal{L}^{-i})=0$ except for a finite number of sheaves $\mathcal{L}^{-i}\in Pic^0(C)$ (which can be avoided with the election of $\mathcal{L}$ (see [5][Lemme 3.1] and [2] §3)) we get

$$\widetilde{h}=q(\widetilde{S})-b=q(S)-b+(n-1)(c-1)=h+(n-1)(c-1)$$

since $h^1(B,\mathcal{L}^{-i})=c-1$ by Riemann-Roch. In particular, $\widetilde{h}\geq 2$ if $n\geq 2$.

It is easy to check that if $F$ is trigonal then $\widetilde{F}$ is not if $n\gg 0$ (see for example [2], Lemma 5.12). On the other hand

$$\lim_{n\to\infty}\frac{\widetilde{g}}{\widetilde{h}}=\frac{g-1}{c-1}\geq 2$$

since the map $\pi_{|F}:F\longrightarrow C$ is at least of degree two (if it were of degree 1 clearly $F\cong C$ and $S=B\times C$). Hence if $n\gg 0$ the case $\widetilde{g}<\frac{3}{2}\widetilde{h}+2$ can not occur.

So if $n\gg 0$ we are under the hypotheses of Theorem 3.2 (ii) (non trigonal case). Using that the slope is invariant under étale changes of $S$ (cf. [15]) we get

$$\lambda(f)=\lambda(\widetilde{f})\geq\frac{8\widetilde{g}(\widetilde{g}-1)(4\widetilde{g}-3\widetilde{h}-14)}{8\widetilde{g}(\widetilde{g}-1)(\widetilde{g}-\widetilde{h}-3)+5(\widetilde{h}-2)(2\widetilde{g}-1)}$$

for $\widetilde{g}=n(g-1)+1$, $\widetilde{h}=h+(n-1)(c-1)$ and $n\in\mathbb{N}$ arbitrarily large. So we can take limit as $n$ grows and get

$$\lambda(f)\geq 4+\frac{c-1}{g-c}$$

In case $\dim alb(S)=1$ then clearly $c=q$. Note that if this happens and $b\geq 1$, then $alb(S)=B$ by the universal property of Albanese variety. $\square$

**Corollary 3.9** *Let $f:S\longrightarrow B$ be as in Theorem 4.19. Assume $\lambda(f)<4+\frac{1}{g-2}$. Then $S$ has no other fibration onto a curve of genus greater or equal than two.*

**Corollary 3.10** *Let $S$ be a minimal surface with $q(S)\geq 2$ and $F\subseteq S$ an irreducible curve of geometric genus $g$. Assume $h^0(S,\mathcal{O}_S(F))\geq 2$ and let $f:\widetilde{S}\longrightarrow\mathbb{P}^1$ be a relatively minimal fibration with fibre $F$. If $F$ is not a double cover and $\lambda(f)<4+\frac{g-1}{g-q}$ then $S$ is of Albanese general type.*

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