A NOTE ON A QUESTION OF FUJITA AND A CONJECTURE OF XIAO

Miguel A. BARJA

0 Introduction

Let Y be a smooth variety of dimension m and $\mathcal{M} = \mathcal{O}_y(D)$ an invertible sheaf. $\mathcal{M}$ is said to be *nef* (respectively *strictely nef*, *ample*, *semiample*) if for any curve $C \subseteq Y$, we have $DC \geq 0$ (respectively $DC > 0$, $D^k E > 0$ for any dimension k subvariety E, $\mathcal{O}_y(rD) = \mathcal{M}^{\otimes r}$ is generated by global sections for some $r > 0$). If $\mathcal{F}$ is a locally free sheaf on Y we say that $\mathcal{F}$ is nef, strictly wf, ample or semiample if $\mathcal{O}_{\mathbb{P}}(1)$ is, where $\mathbb{P} = \mathbb{P}_Y(\mathcal{F})$. We have the obvious relations ample $\Longrightarrow$ semiample $\Longrightarrow$ nef, ample $\Longrightarrow$ strictily nef $\Longrightarrow$ nef.

Let X, Y be smooth projective varieties of dimensions n and m respectively. A map $f : X \longrightarrow Y$ is said to be a *fiber space*, or a *fibration*, if f is surjective and has connected fibres. We will consider the relative dualizing sheaf $w_{x/y} = w_x \otimes f^*(w_y^{-1})$ and the relative canonical divisor $w_{x/y} = \mathcal{O}_x(K_{x/y})$.

There are several results of positivity for $f_x w_{x/y}^{\otimes r}$, Un general those sheaves are not locally free but only torsion free ([9], [10]). These results involve the notion of *weakly positiveness* of Viehweg (cf. [14]), but for our purposes, we need only to deal with the simplest case. The following proposition is a brief account of them.

Proposition 0.1 *Let $f : X \longrightarrow Y$ be a fiber space between smooth projective varieties of dimensions n and m respectively. Then*

(i) ([?], II. 2.6) $R^i f_* w_{x/y}$ and $R^j f_* \mathcal{O}_x$ are locally free.

(ii) (relative duality) For $0 \leq i \leq d = n - m$, $(R^i f_* w_{x/y})^v \cong R^{d-i} f_* \mathcal{O}_x$.

(iii) ([14]) For $k \geq 1$, $f_* w_{x/y}^{\otimes k}$ is nef.

(iv) ([?]) For $k \geq 2$, $m = 1$, $f_* w_{x/y}^{\otimes k}$ is ample.

(v) ([4], [5]) If $k = m = 1$, we have a decomposition $\varepsilon = f_* w_{x/y} = \mathcal{A} \otimes \varepsilon_1 \oplus \ldots \oplus \varepsilon_r$, where $\mathcal{A}$ is ample, for $i = 1 \div r$, ε_i are stable, degree zero and, if $s = q(X) - g(Y)$, $\varepsilon_1 = \ldots = \varepsilon_s = \mathcal{O}_Y$, $\varepsilon_j \neq \mathcal{O}_Y$ for $j = s+1 \div r$. Moreover, if $\mathcal{F}$ is a stable degree zero sheaf such that there exists a surjective map $\varepsilon \twoheadrightarrow \mathcal{F}$, then $\mathcal{F}$ is a direct summand of ε and hence $\mathcal{F} = \varepsilon_i$ for some $i \in \{1, \ldots, r\}$.

In [6] page 600, Fujita asks the following question: for any fiber space $f : X \longrightarrow Y$, is there a birational model $f' : X' \longrightarrow Y'$ such that $\varepsilon' = f' w_{x'/y'}$ is semiample?

As for the semiampleness, we remind some basic properties. A good reference is [8].

Proposition 0.2 *[8], [7]. Let $\mathcal{F}$ be a locally free sheaf on a smooth variety Y*

(i) If X is smooth and $g : X \longrightarrow Y$ dominating then $\mathcal{F}$ is semiample if and only if $g^*\mathcal{F}$ is semiample.

(ii) If $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_k$, $\mathcal{F}$ is semiample if and only if $\mathcal{F}_i$ is semiample for $i = 1 \div k$.

(iii) If $\mathcal{F}$ is semiample then $\det(\mathcal{F})$ is semiample.

(iv) If $\mathcal{F}$ is semiample and $k \circ d(\det(\mathcal{F})) = 0$ then there exist an étale cover $g : \tilde{Y} \longrightarrow Y$ such that $g^*\mathcal{F}$ is trivial.

So, in the case $m = \dim Y = 1$, Fujita's question is equivalent to ask whetter the degree zero summands in Fujita's decomposition (proposition 0.1, (v)) are semiample. In theorem 1.1 we prove that for every $i = 1 \div r$, $\det \varepsilon_i$ are semiample (they are torsion line bundles). In particular we have that the rank one summands are semiample and gives an evidence for the higher rank ones. In fact the result of theorem 1.1 is valid for arbitrary dimension of Y as far as f has a normal crossings branch locus. Hence Fujita's question has an affirmative answer for Y a curve of genus at most one.

As a by product we obtain some results on the slope of fibred surfaces as a conjecture of Xiao ([15], conjecture 4.2) in the non-hyperelliptic case (theorem 2.1). We also remark that the same kind of results on the slope can be obtained in the case of fibred threefolds over curves ([1]).

Conventions. We work over the field of complex numbers.

1 On a question of Fujita

Theorem 1.1 *Let X, Y be smooth projective varieties of dimension n and m respectively. Let $f : X \longrightarrow Y$ be a filtration with branch locus contained in a normal crossings divisor of Y.*

Let $\varepsilon = f_* w_{x/y}$ and $\mathcal{F}$ a locally free sheaf on Y such that $\det(\mathcal{F}) = g(\mathcal{F}) = \mathcal{L} \in$ Pic $^\circ(Y)$.

It there exists a non-trivial map $\varepsilon \longrightarrow \mathcal{F}$, then $\mathcal{L} = \det(\mathcal{F})$ is torsion.

Proof. First of all note that it is enough to prove the theorem for $m = 1$. Indeed, let $\mathcal{H}$ be a very ample line bundle on Y and let $Z \in |\mathcal{H}|_*$ be a general smooth member. By Bertin's theorem $T = f^*(Z) = Z \underset{Y}{\times} Z$ is again smooth. Let $g = f_{|T} : T \longrightarrow Z$. By adjunction we have that $g_* w_{T/Z} = i^*(f_* w_{X/Y})$, where $i : Z \hookrightarrow X$ is the natural inclusion. Since $\mathcal{F}$ is locally free, $\mathrm{Im}(\varepsilon \longrightarrow \mathcal{F})$ is torsion free and non trivial, hence the induced map $g_* w_{T/Z} = i^*(f_* w_{X/Y}) \longrightarrow i^*\mathcal{F}$ is nontrivial; we also have $\det(i^*\mathcal{F}) = i^*(\det \mathcal{F}) \in \mathrm{Pic}\ ^\circ(Z)$. By induction we have for some $r \in \mathbb{N}$ for some $r \in \mathbb{N}$ $(i^*\mathcal{L})^{\otimes r} = i^*(\mathcal{L}^{\otimes r}) = \mathcal{O}_Z$. By Kodaria's vanishing we have $h^0(Y, \mathcal{L}^{\otimes r}) = h^0(Z, i^*(\mathcal{L}^{\otimes r})) = 1$ and hence $\mathcal{L}^{\otimes r} = \mathcal{O}_Y$.

From now on we assume Y to be a smooth curve of genus b. Let $d = n - m = n - 1$ the relative dimension of f. Note that according to Fujita decomposition (proposition 0.1, (v)) we can assume $\mathcal{F}$ to be stable.

Moreover, when $\mathcal{F}$ is stable any non trivial map $\varepsilon \longrightarrow \mathcal{F}$ is necessarily surjective since the image has nonnegative degree, being ε nef and hence, being $\mathcal{F}$ stable at degree zero, it is $\mathcal{F}$. □

Claim. Let $\mathcal{F}$ be a stable, degree zero vector bundle on Y

(i) There exists a non zero map $\varepsilon = f_* w_{X/Y} \longrightarrow \mathcal{F}$ if and only if $h^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F}) \neq 0$

(ii) For any $1 = 1 \div d$ there exists a finite number of stable, degree zero vector bundles $\mathcal{F}$ on Y such that $h^0(Y, (R^i f_* \mathcal{O}_X) \otimes \mathcal{F}) \neq 0$.

Proof. (i) Clearly $\hom(\varepsilon, \mathcal{F}) \cong H^0(Y, \varepsilon^V \otimes \mathcal{F}) = H^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F})$ be relative duality.

(ii) If $i = d$ the result is clear from (i) and proposition 0.1, (v). When $i \leq d-1$, $H^0(Y, (R^i f_* \mathcal{O}_X) \otimes \mathcal{F}) = \hom((R^i f_* \mathcal{O}_X)^V, \mathcal{F}) = \hom(R^{d-i} f_* w_{X/Y}, \mathcal{F})$ by proposition 0.1, (ii).

Finally, note that by a result of Kollar ([10], 2.34), $R^i f_* w_{X/Y}$ is a direct summand of $g_* w_{Z/Y}$ for some smooth Z, and fibration $g : Z \longrightarrow Y$ and hence we can apply the argument of the case $i = d$. □

Consider now, from Leray's spectral sequence

$$0 \longrightarrow H^1(Y, (R^{d-1} f_* \mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow H^{n-1}(X, f^*(\mathcal{F})) \longrightarrow H^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow 0 \quad (1)$$

Since, by the claim, $h^0(Y, (R^{d-1} f_* \mathcal{O}_X) \otimes \mathcal{F}) = h^0(Y, (R^d f_* \mathcal{O}_X) \otimes \mathcal{F}) = 0$ except for a finite number of such $\mathcal{F}$, we have that $h^{n-1}(X, f^*(\mathcal{F}))$ is constant, say a, except for a finite number of $\mathcal{F}$.

If rand $\mathcal{F} = s = 1$ then the claim and)1) $\mathcal{F} \in A = \{\mathcal{M} \in \text{Pic}\ ^\circ(Y) \mid h^{n-1}(X, f^*(\mathcal{M})) \geq a+1\}$ which is finite and hence, by a result of Simpson ([13]) is torsion.

Assume $s \geq 2$. Following Viehweg ([14], [12] 4.11) we can consider $X^{(s)}$ a resolution of the component of $X^s = X \underset{Y}{\times} \overset{s)}{\cdots} \underset{Y}{\times} X$ dominating Y and $f^{(s)} : X^{(s)} \longrightarrow Y$ the induced fibration. We have then an inclusion

$$\left(f_*^{(s)} w_{X^{(s)}/Y}\right)^{VV} \hookrightarrow \left(\overset{s}{\otimes} f_* w_{X/Y}\right)^{VV}$$

Note that since $\dim Y = 1$ both are locally free sheaves and in fact we have an inclusion of vector bundles at the same rank

$$j : f_*^{(s)} w_{X^{(s)}/Y} \hookrightarrow \overset{s}{\otimes} f_* w_{X/Y}$$

Now consider the projection π

$$\overset{s}{\otimes} f_* w_{X/Y} \twoheadrightarrow \overset{s}{\otimes} \mathcal{F} r \qquad \det \mathcal{F}$$

Note that $\pi \circ j$ is non trivial since both vector bundles have the same rank, and then we can apply the rank one argument. □

Corollary 1.2 *Let $f : X \longrightarrow B$ be a fibration of a smooth projective variety X onto an smooth curve of genus b.*

If $b \leq 1$ then $f_* w_{X/B}$ is semiample.

Proof. We just need apply that on an elliptic curve any stable degree zero sheaf has rank one. □

Corollary 1.3 *Let $f : X \longrightarrow B$ be as above. Let $\mathcal{F}$ be a stable degree zero vector bundle on B.*

If there exists a nontrivial map $\varepsilon = f_* w_{X/B} \longrightarrow \mathcal{F}$ then there is a base change $\sigma : \widetilde{B} \longrightarrow B$ such that $\sigma^+\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{O}_{\widetilde{B}}^{\oplus r}$, where $\mathcal{F}_0$ is a strictly nef vector bundle with trivial determinant.

Proof. Assume $\mathcal{F}$ is not strictly nef. Then there exists a smooth curve C and a map $\sigma : C \longrightarrow B$ such that $\sigma^+\mathcal{F} \twoheadrightarrow \mathcal{L}$, deg $\mathcal{L} = 0$, where $\mathcal{L} = \widehat{\sigma}^*\mathcal{O}_{\mathbb{P}}(1)$ and $\widehat{\sigma}$ is a map $\widehat{\sigma} : C \longrightarrow \mathbb{P} = \mathbb{P}_B(\mathcal{F})$.

Since σ is flat we have ([12], 4.10)

$$0 \longrightarrow \widetilde{f}_* w_{\widetilde{S}/C} \longrightarrow \sigma^+ f_* w_{S/B}$$

where $\widetilde{S}$ is a desingularization of $S \underset{B}{\times} C$.

Since $\sigma^+\varepsilon \twoheadrightarrow \sigma^+\mathcal{F} \twoheadrightarrow \mathcal{L}$ and the induced map $\widetilde{f}_* w_{\widetilde{S}/C} \longrightarrow \mathcal{L}$ can not be zero ($\widetilde{f}_* w_{\widetilde{S}/C}$ and $\sigma^+ f_* w_{S/B}$ are of the same rank), by therorem 1.1 $\mathcal{L}$ is torsion and hence up to a new base change we can assume is trivial.

Then we argue by induction on the rank of $\mathcal{F}$. □

Remark 1.4 If $\mathcal{F}_0$ is as in the previous result, note that for every $\widehat{\sigma} : C \longrightarrow \mathbb{P}_B(\mathcal{F}_0)$, $\mathcal{O}_{\mathbb{P}}(1)\widehat{\sigma}(C) > 0$ but $\mathcal{F}_0$ is not ample. By [3] $\mathcal{F}_0$ is ample if and only if sld $(\mathcal{F}_0) = \inf \left\{ \mathcal{O}_{\mathbb{P}}(1)\widehat{\sigma}(C) \big/ \deg_{(C \to B)} \right\} > 0$. In our case sld $(\mathcal{F}_0) = 0$ but we can not achieve this infimum.

2 On a conjecture of Xiao

Let $f : S \longrightarrow B$ be a relatively minimal fibred surface. It is known (cf. [2]) that deg $f_* w_{S/B} = 0$ if and only if f is isotrivial. When f is non isotrivial the slope of f is defined as

$$\lambda(f) = \frac{w_{S/B}^2}{\deg f_* w_{S/B}} = \frac{K_S^2 - 8(b-1)(g-1)}{\mathcal{X}\mathcal{O}_S - (b-1)(g-1)}$$

where $b = g(B)$ and g is the genus of a generic fibre F of f, assuming $g \geq 2$.

In [15] Xiao proves that $\lambda(f) \geq 4 - \frac{4}{g}$ and that in fact if $q(S) > b$, $\lambda(f) \geq 4$ and $\lambda(f) = 4$ only if $f_* w_{S/B}$ is a direct sum of a semiestable sheaf of rank $(g-1)$ and $\mathcal{O}_B$ (in particular $q(S) = b + 1$). He also proves that when $\lambda(f) < 4$, $f_* w_{S/B}$ has no locally free quotient of degree zero and rank ≥ 2 when F is non-hyperelliptic. Then the natural conjecture ([15], 4. Conjecture 2) according to Fujita's decomposition (proposition 0.1, (v)) is that $\varepsilon = f_* w_{S/B}$ is ample whenever $\lambda(f) < 4$.

Using the results of the previous paragraph we can state

Theorem 2.1 *Let $f : S \longrightarrow B$ as above. Assume f is not isotrivial.*

(i) If $\lambda(f) < 4$ then $\varepsilon = f_* w_{S/B}$ is ample provided one of the following conditions hold

(a) F is non hyperelliptic.

(b) $b \leq 1$.

(c) $g(F) \leq 3$.

(ii) If $\lambda(f) = 4$ then ε has at most one degree zero rank one quotient $\mathcal{L}$. Moreover, in this case $\varepsilon = \mathcal{A} \oplus \mathcal{L}$ with $\mathcal{A}$ ample ans semiestable.

Proof. (i) (a) According to the previous results of Xiao if only remains open the case of a degree zero, rank one, quotient $\mathcal{L}$ of ε. By theorem 1.1 such a $\mathcal{L}$ is torsion an then, after an étale base change

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S \\ \downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f} \\ \widetilde{B} & \xrightarrow{\sigma} & B \end{array}$$

we have $\widetilde{f}_* w_{\widetilde{S}/\widetilde{B}} = \sigma^+(f_* w_{S/B}) \longrightarrow\!\!\!\!\rightarrow \sigma^+\mathcal{L} = \mathcal{O}_{\widetilde{S}}$ and hence $\lambda(\widetilde{f}) \geq 4$ by the above refered result of Xiao. Finally note that $\lambda(f) = \lambda(\widetilde{f})$ ([15]).

(b) The same argument works.

(c) If $g = 2$ then $\varepsilon = \mathcal{A} \oplus \mathcal{L}$ or $\varepsilon = \mathcal{A}$ with $\mathcal{A}$ ample and $\mathcal{L}$ torsion and we are done. If $g = 3$ the only non trivial case is $\varepsilon = \mathcal{A} \oplus \mathcal{F}$ with $\mathcal{A}$ an ample line bundle and $\mathcal{F}$ a stable, degree zero, rank two vector bundle. But hence $w^2_{S/B} \geq (2g-2)\deg \mathcal{A} = 4 \deg \varepsilon$ by [15] theorem 2.

(ii) Any locally free invertible degree zero quotient $\mathcal{L}$ of ε becomes trivial after an étale base change σ. Since $\lambda(\widetilde{f}) = 4$, [15] theorem 3.3, proves that $\widetilde{\varepsilon}$ has at most one of this trivial quotients. Moreover if $\varepsilon = \mathcal{A} \oplus \mathcal{L}$, then $\widetilde{\varepsilon} = \sigma^+\mathcal{A} \oplus \mathcal{O}_B$ and then $\sigma^+\mathcal{A}$ is semiestable by the same result of Xiao. Then, by [11], proposition 3.2, $\mathcal{A}$ is semiestable and hence, by Fujita's decomposition, $\mathcal{A}$ is ample. □

References

[1] Barja, M.A.: *On the slope of fibred threefolds.* Preprint (1998).

[2] Beauville, A.: *L'inégalité $p_g \geq 2q-4$ pour les surfaces de type général.* Bull. Soc. Math. France, 110 (1982), 343–346.

[3] Esnalt, H.-Viehweg, E.: *Effective bounds for semi positive sheaves and the height of points on curves over complex function fields.* Compositio math., 76 (1990), 69–85.

[4] Fujita, T.: *On Kähler fibre spaces over curves.* J. Math. Soc. Japan, 30 (1978), 779–794.

[5] Fujita, T.: *The sheaf of relative canonical forms of a Kähler fiber space over a curve.* Proc. Japan. Acad., 54, Ser.A (1978), 183–184.

[6] Fujita, T.: Katata Symposium, Progress in Math., 39 (1983). Birkhäuser, page 600. Problem 5.

[7] Fujita, T.: *Semipositive live bundles.* J. Fac. Sci. Univ. Tokyo, 30 (1983), 353–378.

[8] Fujiwara, T.: *Varieties of small Kodaira dimension whose cotangent bundles are semiample.* Compositio Math., 84 (1992), 43–52.

[9] Kollar, J.: *Higher direct images of dualizing sheaves I.* Annals of Mathematics, 123 (1986), 11–42.

[10] Kollar, J.: *Higher direct images of dualizing sheaves II.* Annals of Mathematics, 124 (1986), 171–202.

[11] Miyaoka, Y.: *The Chern clases and Kodaira dimension of a minimal variety.* Adv. Studies in P. and. Appl. Math., 10 (1987), Sendai 1985, 449–476.

[12] Mori, S.: *Classification of higher-dimensional varieties.* Proc. Symp. Pure Math., 46 (1987).

[13] Simpson, C.: *Subspaces of moduli spaces of rank one local systems.* Ann. Scient. Éc. Norm. Sup., 4e série, t.26 (1993), 361–401.

[14] Viehweg, E.: *Quasi-projective moduli for polarized manifolds.* Berlin, Heidelberg, New York, Springer 1995.

[15] Xiao, G.: *Fibered algebraic surfaces with low slope.* Math. Ann. **276** (1987), 449–466.