THE STRUCTURE AND SLOPE OF BIELLIPTIC FIBRATIONS

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A la memoria de Fernando

0 Introduction

Let $\pi : S \longrightarrow B$ be a *fibration*, i.e. a surjective morphism with connected fibres, from a smooth surface S onto a smooth curve B. A fibration is said to be *relatively minimal* when it has no vertical (-1)-curve. Let g denote the genus of a general fibre and b the genus of B.

Let $\omega_{S/B} = \omega_S \otimes \pi^*(\omega_B^{-1})$ be the relative canonical bundle and let $\Delta(\pi) := \deg\, \pi_*(\omega_{S/B})$. It is known that $\Delta(\pi) \geq 0$ and that $\Delta(\pi) = 0$ if and only if π is locally trivial. Assume π is not locally trivial. Then we define the *slope* of π as

$$\lambda(\pi) := \omega_{S/B}^2 \Big/ \Delta(\pi)$$

(see [19]). There are several results on the lower slope of relatively minimal fibrations of genus $g \geq 2$. First of all we have $\lambda \geq 4 - \frac{4}{g}$ (see [8], [12], [13], [18] for the hyperelliptic case and [19] for the general case) and equality holds only in the hyperelliptic case ([9]). There are improvements in the non-hyperelliptic case for $g \leq 5$ (see [4], [7], [9], [11], [14]) but the presently known techniques seem to have some limitations to extend these results to higher genus.

Recently Konno is trying to find good bounds depending on some extra numerical invariants of the general fibre, such as the Clifford index. In [10], Konno finds better bounds for trigonal and plane quintic fibrations (so Clifford index 1), although they do not seem to be sharp. Also in [11] he gets general bounds depending on the Clifford index in some cases.

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In this paper we deal with the case of bielliptic fibrations (i.e., when the general fibre has a 2-to-1 map onto an elliptic curve). In the study of general fibrations it turns out to be relevant the properties of the canonical embeding of the general fibre with respect to quadrics. A first example is the trigonal case where Konno uses the fact that such a curve lies in a rational normal scroll which is the intersection of the quadrics containing it. In our case the role of the quadrics through the canonical bielliptic curve can not be used directly. A different approach using relative Brill-Noether loci shows

Theorem 2.4 *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus $g \geq 6$. Then S is, birationally, a double cover of an elliptic smooth surface V over B.*

In chapter 3 we prove that the conclusions of the above theorem hold for genus 5 bielliptic fibrations *after a base change* and we give an example where a nontrivial base change is needed.

As a by-product we obtain that every smooth bielliptic fibration of genus $g \geq 5$ is isotrivial (see proposition 2.6 and corollary 3.3).

Finally, using theorem 2.4 and canonical resolution of singularities for double covers we get the following sharp bound for the slope of bielliptic fibrations

Theorem 4.1 *Let $\pi : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in theorem 2.4. Then*

(a) $\lambda(\pi) \geq 4 + \dfrac{2(g-5)\mathcal{X}\mathcal{O}_V}{\Delta(\pi)} \geq 4.$

(b) $\lambda(\pi) = 4$ *if and only if S is the minimal desingularization of a double cover $S_0 \longrightarrow V$ of a smooth elliptic surface such that*

- *All the fibres of the elliptic fibration $\tau : V \longrightarrow B$ are smooth and isomorphic.*
- *The branch divisor of the double cover has only* negligeable *singularities.*

In particular, the bound is sharp.

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Notations and conventions

All throughout this paper we work over the field of complex numbers $\mathbb{C}$.

If X is a scheme and $\mathcal{G}$ is a sheaf of graded algebras and $\mathcal{E}$ is a locally free coherent sheaf on X we define Proj $\mathcal{G}$ and $\mathbb{P}(\mathcal{E})$ as in [5],II.7.

1 Some generalities on fibred surfaces

We recall some basic facts about fibred surfaces (see [9]). The following are some easy but useful results.

Lemma 1.1 *Let X be a smooth variety and $\varphi : X \longrightarrow B$ a morphism onto a curve. Let X_t be the fibre of φ over $t \in B$.*

For any coherent sheaf $\mathcal{F}$ on X and for any $\mathfrak{a} \in Pic\ (B)$ let $\mathcal{F}(\mathfrak{a}) = \mathcal{F} \otimes \varphi^*(\mathfrak{a})$.

Suppose that $\mathfrak{a}$ is ample enough and that $\mathcal{F}$ satisfies the following technical condition: for general $t \in B$ the sequence

$$0 \longrightarrow \mathcal{F}(-t) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_{|X_t} \longrightarrow 0$$

is exact.

Then, the natural morphism

$$H^0(X, \mathcal{F}(\mathfrak{a})) \longrightarrow H^0(X_t, \mathcal{F}_{|X_t})$$

is surjective for general $t \in B$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{F}(\mathfrak{a} - t) \longrightarrow \mathcal{F}(\mathfrak{a}) \longrightarrow \mathcal{F}_{|X_t} \longrightarrow 0$$

for general $t \in B$. Taking cohomology we get

$$0 \longrightarrow H^0(X, \mathcal{F}(\mathfrak{a} - t)) \longrightarrow H^0(X, \mathcal{F}(\mathfrak{a})) \xrightarrow{m_t} H^0(X_t, \mathcal{F}_{|X_t})$$

If $\mathfrak{a}$ is ample enough then $h^1(B, (\varphi_*\mathcal{F}) \otimes (\mathfrak{a} - t)) = 0$ and then $h^0(X, \mathcal{F}(\mathfrak{a} - t)) = h^0(B, (\varphi_*\mathcal{F}) \otimes (\mathfrak{a} - t))$ does not depend on t by the Hirzebruch-Riemann-Roch theorem for coherent sheaves on B.

Furthermore

$$\begin{aligned} \dim \mathrm{Im}\ (m_t) &= h^0(X, \mathcal{F}(\mathfrak{a})) - h^0(X, \mathcal{F}(\mathfrak{a} - t)) = \\ &= h^0(B, (\varphi_*\mathcal{F}) \otimes \mathfrak{a}) - h^0(B, (\varphi_*\mathcal{F}) \otimes (\mathfrak{a} - t)) = \\ &= d + r(a + 1 - b) - (d + r(a - 1 + 1 - b)) = r \end{aligned}$$

where

$$\begin{aligned} d &= \deg \varphi_*\mathcal{F} \\ a &= \deg \mathfrak{a} \\ r &= \mathrm{rank}\ (\varphi_*\mathcal{F}) = h^0(X_t, \mathcal{F}_{|X_t}) \quad \text{for } t \in B \text{ general} \end{aligned}$$ □

Lemma 1.2 *Let $\pi : S \longrightarrow B$ be a fibration. Let $\mathcal{L} \in Pic\ S$. If $\mathfrak{a} \in Pic\ B$ is ample enough, then the natural map*

$$h : \pi^*\pi_*\mathcal{L}(\mathfrak{a}) \longrightarrow \mathcal{L}(\mathfrak{a})$$

is an epimorphism just except at the base points of the linear system $|\mathcal{L}(\mathfrak{a})|$. Moreover, if for a general fibre F of π, the linear system $|\mathcal{L}_{|F}|$ is base-point free, then such base points are concentrated on a finite number of fibres.

Proof. From the sequence of maps $S \xrightarrow{\pi} B \xrightarrow{\rho} \operatorname{Spec} \mathbb{C}$ we can consider the following natural commutative diagram

$$\begin{array}{ccc} \pi^*\rho^*\rho_*\pi_*\mathcal{L}(\mathfrak{a}) = (\rho\circ\pi)^*(\rho\circ\pi)_*\mathcal{L}(\mathfrak{a}) = H^0(S,\mathcal{L}(\mathfrak{a}))\otimes_{\mathbb{C}}\mathcal{O}_S & \xrightarrow{e} & \mathcal{L}(\mathfrak{a}) \\ \downarrow k & \swarrow h & \\ \pi^*\pi_*\mathcal{L}(\mathfrak{a}) & & \end{array}$$

Since k is surjective for $\mathfrak{a}$ ample enough, it follows that surjectivity of h is equivalent to surjectivity of e. This fails to be an epimorphism precisely at the base points of $|\mathcal{L}(\mathfrak{a})|$. Finally, using lemma 1.1 one has that $|\mathcal{L}(\mathfrak{a})|$ has no base points on a general fibre F of π. $\square$

Then we can construct the relative canonical morphism of S. Consider $\mathcal{L} = \omega_{S/B} = \omega_S \otimes \pi^*\omega_B^{-1}$; then $\mathcal{L}_{|F} = \omega_F$ which has no base points for smooth F if $g \geq 2$. $\mathcal{L}(\mathfrak{a})$ has only base points on singular fibres of π, for $\mathfrak{a}$ ample enough by lemma 1.2 and the natural map

$$h : \pi^*\pi_*\mathcal{L}(\mathfrak{a}) \longrightarrow \mathcal{L}(\mathfrak{a}) \tag{1}$$

is an epimorphism away from such a base points.

In fact we have

$$\widetilde{h} : (\pi^*\pi_*\mathcal{L}(\mathfrak{a})) \otimes \mathcal{L}(-\mathfrak{a}) \longrightarrow \mathcal{O}_S \tag{2}$$

with $\operatorname{Im} \widetilde{h} = \mathcal{I}_Z = \mathcal{O}_S(-R) \otimes \mathcal{I}_\Gamma$ being the ideal sheaf of such base points, endowed with some scheme strcture, where $\mathcal{O}_S(-R)$ is the ideal sheaf of its divisorial part and $\mathcal{I}_\Gamma$ is the ideal sheaf of its discrete part (see [14]).

Consider a sequence of blow-ups $\sigma : \widetilde{S} \longrightarrow S$ such that the base locus of $|\sigma^*\mathcal{L}(\mathfrak{a})|$ is of pure codimension 1. Then, in particular we have an epimorphism

$$\sigma^*\mathcal{I}_\Gamma \longrightarrow \sigma^{-1}\mathcal{I}_\Gamma \cdot \mathcal{O}_{\widetilde{S}} = \mathcal{O}_{\widetilde{S}}(-E)$$

where E is a divisor on $\widetilde{S}$ exceptional with respect to σ.

Thus from $\pi^*\pi_*\mathcal{L}(\mathfrak{a}) \longrightarrow \mathcal{L}(\mathfrak{a}) \otimes \mathcal{O}_S(-R) \otimes \mathcal{I}_\Gamma$ (epimorphism) we get

$$\sigma^*\pi^*(\pi_*\mathcal{L}(\mathfrak{a})) \longrightarrow \sigma^*\mathcal{L}(\mathfrak{a}) \otimes \sigma^*\mathcal{O}_S(-R) \otimes \sigma^*\mathcal{I}_\Gamma \longrightarrow \sigma^*\mathcal{L}(\mathfrak{a}) \otimes \sigma^*\mathcal{O}_S(-R) \otimes \mathcal{O}_{\widetilde{S}}(-E) \tag{3}$$

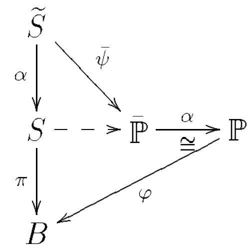
which is also an epimorphism.

We shall call

$$\begin{array}{ll} \mathcal{O}_{\widetilde{S}}(M(\mathfrak{a})) = \sigma^*\mathcal{L}(\mathfrak{a}) \otimes \sigma^*\mathcal{O}_S(-R) \otimes \mathcal{O}_{\widetilde{S}}(-E) & \text{the } \textit{moving part} \text{ of } \sigma^*\mathcal{L}(\mathfrak{a}) \\ \mathcal{O}_{\widetilde{S}}(Z(\mathfrak{a})) = \sigma^*\mathcal{O}_S(R) \otimes \mathcal{O}_{\widetilde{S}}(E) & \text{the } \textit{fixed part} \text{ of } \sigma^*\mathcal{L}(\mathfrak{a}) \end{array}$$

(as one can easily see $(\pi\circ\sigma)^*(\pi\circ\sigma)_*\mathcal{O}_{\widetilde{S}}(M(\mathfrak{a})) \longrightarrow \mathcal{O}_{\widetilde{S}}(M(\mathfrak{a}))$ is an epimorphism so, in view of lemma 1.2, $|M(\mathfrak{a})|$ has no base points).

Then (3) leads to



where

$$\begin{array}{lcl} \bar{\mathbb{P}} & = & \mathbb{P}_B(\pi_*\mathcal{L}(\mathfrak{a})) \\ \mathbb{P} & = & \mathbb{P}_B(\pi_*\mathcal{L}) \\ \alpha & & \text{is the natural isomorphism} \end{array}$$

such that $\bar{\psi}^*\mathcal{O}_{\bar{\mathbb{P}}}(1) = \mathcal{O}_{\widetilde{S}}(M(\mathfrak{a}))$. If we call $\psi = \alpha \circ \bar{\psi}$ we get

$$\psi^*(\mathcal{O}_{\mathbb{P}}(1) \otimes \varphi^*(\mathfrak{a})) = \mathcal{O}_{\widetilde{S}}(M(\mathfrak{a}))\,.$$

We shall call Σ the image of $\widetilde{S}$ by ψ. In fact Σ is the closure in $\mathbb{P}$ of the image of the birational map induced on S by h in (1), so Σ doesn't depend on $\mathfrak{a}$. Then we can change $\mathfrak{a} \in \operatorname{Pic} B$ if needed. We shall call Σ *the relative canonical image of* S and ψ *the relative canonical map.*

We remark that for $F \subseteq S$ a smooth fibre of π, $F \cong \sigma^*(F) \cong \psi(\sigma^*(F))$, so we denote it always by F if no confusion arises.

For $t \in B$ we call $\mathbb{P}_t = \varphi^{-1}(t) \cong \mathbb{P}^{g-1}$.

We shall call $\mathcal{O}_{\mathbb{P}}(T)$ the tautological line bundle on $\mathbb{P}$.

Remark 1.3 Consider the sheaf $\mathcal{F} = \mathcal{I}_{\Sigma,\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(2T) \otimes \varphi^*(2\mathfrak{a})$, where $\mathcal{I}_{\Sigma,\mathbb{P}}$ is the ideal sheaf of Σ in $\mathbb{P}$. For $t \in B$ we have $\mathcal{F}_{|\mathbb{P}_t} = \mathcal{I}_{F_t,\mathbb{P}_t}(2)$. We claim that $\mathcal{F}$ verifies the hypothesis of lemma 1.1 and so that for $\mathfrak{a}$ ample enough we have an epimorphism

$$H^0(\mathbb{P}, \mathcal{I}_{\Sigma,\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(2T) \otimes \varphi^*(2\mathfrak{a})) \longrightarrow H^0(\mathbb{P}^{g-1}, \mathcal{I}_{F_t,\mathbb{P}_t}(2))\,.$$

Indeed, just consider

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{\Sigma,\mathbb{P}} \otimes \varphi^*(-t) & \longrightarrow & \mathcal{O}_{\mathbb{P}}(\varphi^*(-t)) & \longrightarrow & \mathcal{O}_{\Sigma}(\varphi^*(-t)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{\Sigma,\mathbb{P}} & \longrightarrow & \mathcal{O}_{\mathbb{P}} & \longrightarrow & \mathcal{O}_{\Sigma} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{F_t,\mathbb{P}_t} & \longrightarrow & \mathcal{O}_{\mathbb{P}_t} & \longrightarrow & \mathcal{O}_{F_t} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with the three rows and the two right columns trivially exact. Then the snake lemma makes the left hand side column exact. Now just tensor with $\mathcal{O}_{\mathbb{P}}(2T)$, which is locally free.

2 Bielliptic fibrations of genus $g \geq 6$

From now on we consider bielliptic fibrations only, i.e. fibrations such that the general fibre admits a 2-to-1 morphism onto an elliptic curve. It seems reasonable to think that all those maps can be glued together to yield a global 2-to-1 map from S onto an elliptic surface over B.

A similar case, very much studied (see [8], [12], [13], [18]) is that of hyperelliptic fibrations, i.e. when fibres are hyperelliptic. Here the hyperelliptic involutions glue together to yield a birational double cover of a ruled surface in an easy way: just consider the relative canonical morphism of $\pi : S \longrightarrow B$. The case where the general fibres are trigonal also globalizes to a 3-to-1 map from S onto a ruled surface over B (this fact is implicit in [10]).

We prove in this chapter that such a double cover actually exists for any bielliptic fibration of genus at least 6.

To see this we need, first of all, a canonical way of constructing the double cover of an elliptic curve from F. Recall ([1] Ch.IV) that for any smooth curve F we have a variety $W^r_d(F)$ inside Pic $^d(F)$ parametrizing the complete linear series on F of degree d and dimension at least r.

Given a bielliptic involution $\sigma : F \longrightarrow E$ there is a natural way of embedding E in $W^1_4(F)$. This map is given by the composition of σ with the hyperelliptic involutions coming from E.

On the other hand $W^0_4(F)$, which has dimension four, is singular precisely along $W^1_4(F)$ if $g \geq 5$ (see [1] p.160). Take $L \in W^1_4(F)$; we denote $T_L = \mathbb{P}\mathcal{T}_L W^0_4(F)$ the projectivized tangent cone to $W^0_4(F)$ at L, which fits canonically in $\mathbb{P}(H^0(F,\omega_F)^*) \cong \mathbb{P}^{g-1}$. If $g \geq 5$ we have by the Kempf's Singularity Theorem ([1] p.241) that T_L is a threefold of degree $(g-3)$ in $\mathbb{P}^{g-1}$ containing F.

For any effective divisor D on F we denote by $<D>$ the linear subspace of $\mathbb{P}(H^0(F,\omega_F)^*)$ spanned by D, in the sense of [1] p.12.

Lemma 2.1 *Let $\sigma : F \longrightarrow E$ be a bielliptic cover. Then*

(a) For $g \geq 4$ the union of the lines $<\sigma^*(p)>$, with $p \in E$, is an elliptic normal cone R of degree $(g-1)$, containing F.

(b) For $g \geq 6$ the bielliptic involution $\sigma : F \longrightarrow E$ is unique, $W^1_4(F) \cong E$ and if $L_1, L_2 \in W^1_4(F)$ are two distinct points, we have $R = T_{L_1} \cap T_{L_2}$.

Proof. If $q_1, \ldots, q_i \in E$ are distinct general points, Riemann-Roch on E and F shows that

$$\dim <\sigma^*(q_1) + \ldots + \sigma^*(q_i)> = i \quad \text{for} \quad i+1 \leq g\,. \tag{4}$$

Then, for $i = 3$ and $g \geq 4$, it follows that all the lines $<\sigma^*(p)>$, for $p \in E$, meet at the same point $q \in \mathbb{P}^{g-1}$. So projection from q gives the elliptic curve E as an elliptic normal curve of $\mathbb{P}^{g-2}$, which proves (a).

Let $\sigma_1 : F \longrightarrow E$ be a bielliptic involution. Assume there exists a 4-to-1 map $\sigma_2 : F \longrightarrow \mathbb{P}^1$ distinct from those produced by σ_1. Then the map $\bar{\sigma} : F \longrightarrow E \times \mathbb{P}^1$ given by $\bar{\sigma}(p) = (\sigma_1(p), \sigma_2(p))$ is birational. If we denote $a = E \times \{p_2\}$, $b = \{p_1\} \times \mathbb{P}^1$ for $p_1 \in E$, $p_2 \in \mathbb{P}^1$ we have $\bar{\sigma}(F) \equiv 2a + 4b$ and hence, by adjuntion in $E \times \mathbb{P}^1$, we get $g(F) \leq p_a(\bar{\sigma}(F)) = 5$. So if $g \geq 6$ the bielliptic involution is unique and $W^1_4(F) \cong E$.

For $g \geq 6$ we have from (4) that if

$$L_j = |\sigma^*(q_1^j + q_2^j)| \quad q_k^i \in E, \quad j = 1, 2$$

are two different points in $W^1_4(F)$ then

$$\dim \Big(< \sigma^*(q_1^1 + q_2^1) > \cap < \sigma^*(q_1^2 + q_2^2) >\Big) = 0 \quad \text{or} \quad 1 \tag{5}$$

depending on whether $\{q_1^1, q_2^1\} \cap \{q_1^2, q_2^2\}$ is empty or has one point, respectively.

Then, again by Kempf's Singularity Theorem, we have the set-theoretic equality

$$T_{L_i} = \bigcup_{D \in L_i} < D >$$

and thus

$$T_{L_1} \cap T_{L_2} = \bigcup_{\substack{D \in L_1 \\ D' \in L_2}} (< D > \cap < D' >) = \bigcup_{p \in E} < \sigma^*(p) >= R$$

using (5), if $L_1 \neq L_2$. □

Theorem 2.2 *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus $g \geq 6$. Let $\Sigma \subseteq \mathbb{P} = \mathbb{P}(\pi_* \omega_{S/B})$ be the relative canonical image of S as in section 1. Then there exists a threefold W such that*

- *(a)* $\Sigma \subseteq W \subseteq \mathbb{P}$.
- *(b) For $t \in B$ such that the fibre F_t is smooth we have $W \cap \mathbb{P}_t = R_t$ (the elliptic normal cone containing F_t).*

Proof. We prove the theorem in several steps.

Step 1. We can assume that the fibration has enough sections.

Indeed, consider a very ample line bundle $\mathcal{L}$ on S and let $\bar{B}$ be a global smooth section of $\mathcal{L}$. Consider the diagramm

$$\begin{array}{ccccc} \bar{S} & \xrightarrow{\gamma} & \widehat{S} & \xrightarrow{\eta} & S \\ & \searrow^{\bar{\pi}} & \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\ & & \bar{B} & \xrightarrow{\delta} & B \end{array}$$

where $\delta = \pi_{|\bar{B}}$, $\widehat{\pi}$ is given by flat base change and $\gamma : \bar{S} \longrightarrow \widehat{S}$ is a minimal desingularization of $\widehat{S}$.

In this situation we remark that $\widehat{\pi}$, and hence $\bar{\pi}$, has a section. Indeed, this section is given by $\{(b, b) \in \bar{B} \times_B \bar{B} \subseteq S \times_B \bar{B} = \widehat{S}\}$ which is a component of $\bar{B} \times_B \bar{B}$. We can repeat this construction in order to get as many sections as we need.

Then we have

$$\begin{array}{ll} \widehat{\pi}_*(\eta^*(\omega_S)) = \delta^*(\pi_*(\omega_S)) & \text{by flat base change} \\ 0 \longrightarrow \gamma^*(\eta^*(\omega_S)) \longrightarrow \omega_{\bar{S}} & \text{by ramification formula} \\ \bar{\pi}_*(\gamma^*(\eta^*(\omega_S))) = \widehat{\pi}_*(\gamma_*\gamma^*\eta^*(\omega_S)) = \widehat{\pi}_*(\eta^*(\omega_S) \otimes \gamma_*\mathcal{O}_{\bar{S}}) & \text{by projection formula} \\ 0 \longrightarrow \mathcal{O}_{\widehat{S}} \longrightarrow \gamma_*\mathcal{O}_{\bar{S}} & \gamma \text{ dominant} \end{array}$$

and then

$$0 \longrightarrow \delta^*(\pi_*\omega_S) \longrightarrow \bar{\pi}_*(\omega_{\bar{S}})\,.$$

Being both locally free sheaves of the same rank we get a birational map given by a sequence of elementary transformation on certain fibres

$$\mathbb{P}_{\bar{B}}(\delta^*\pi_*\omega_S) \underset{\simeq}{\leftarrow - - -} \mathbb{P}_{\bar{B}}(\bar{\pi}_*\omega_{\bar{S}})$$

which is an isomorphism for a general fibre.

So finally we get a generically finite rational map

$$\beta : \mathbb{P}_{\bar{B}}(\bar{\pi}_*\omega_{\bar{S}/\bar{B}}) - - - \rightarrow \mathbb{P}_B(\pi_*\omega_{S/B})$$

given as the composite

$$\mathbb{P}_{\bar{B}}(\bar{\pi}_*\omega_{\bar{S}/\bar{B}}) \cong \mathbb{P}_{\bar{B}}(\bar{\pi}_*\omega_{\bar{S}}) \underset{\simeq}{- - - \rightarrow} \mathbb{P}_{\bar{B}}(\delta^*\pi_*\omega_S) \longrightarrow \mathbb{P}_B(\pi_*\omega_S) \cong \mathbb{P}_B(\pi_*\omega_{S/B})$$

which is linear on fibres and restricts to the natural map from the relative canonical image $\bar{\Sigma}$ of $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$ onto the relative canonical image Σ of $\pi : S \longrightarrow B$.

Then suppose there exists $\bar{W}$ as in the theorem for the bielliptic fibration $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$. We had that for smooth $\bar{F}_t$, $\bar{W} \cap \bar{\mathbb{P}}_t = \bar{R}_t$. Now just consider $W = \beta(\bar{W})$ which verifies the desired conditions. Indeed, for $t \in B$ such that F_t is smooth we have

$$W \cap \mathbb{P}_t = \bigcup_{\delta(t')=t} \beta(\bar{W} \cap \bar{\mathbb{P}}_{t'}) = \bigcup_{\delta(t')=t} \beta(\bar{R}_{t'}) = R_t$$

because $\beta(\bar{R}_{t'})$ is an elliptic normal cone containing $F_t = \beta(\bar{F}_{t'})$, which is unique by lemma 2.1.

Step 2. We can assume that π is smooth and has enough disjoint sections.

Once we have enough sections we can restrict the fibration to the nonempty Zariski open set where π is smooth and the sections do not meet (we only have to avoid a finite number of fibres). If there exists $W_U \subseteq \mathbb{P}_U(i^*(\pi_*\omega_{S/B}))$, where $i : U \longrightarrow B$ is the natural inclusion, verifying the theorem, then as W it is enough to take the closure of W_U inside $\mathbb{P} = \mathbb{P}_B(\pi_*\omega_{S/B})$.

Step 3. Existence of W.

From the previous steps we have $\pi : S \longrightarrow B$ a smooth fibration over a, possibly non complete, curve with enough disjoint sections. Under these assumptions (see 16) there exist schemes $W^r_d(\pi)$, Pic $^d(\pi)$ over B such that

(i) For every $t \in B$, $(W^r_d(\pi))_t = W^r_d(F_t)$, $(\text{Pic}\,^d(\pi))_t = \text{Pic}\,^d(F_t)$.

(ii) (Base change property) If $\delta : \bar{B} \longrightarrow B$ is a base change and $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$ satisfies the same good properties as π, then $W^r_d(\bar{\pi}) = \overline{W^r_d(\pi)}$ and $\mathrm{Pic}\ ^d(\bar{\pi}) = \overline{\mathrm{Pic}\ ^d(\pi)}$ (where, as always, $\bar{X}$ denotes the base change of $X \longrightarrow B$).

Then we can consider $W^1_4(\pi) \subseteq W^0_4(\pi) \subseteq \mathrm{Pic}\ ^4(\pi)$. Moreover we remark that $W^1_4(\pi) \longrightarrow B$ is an elliptic fibration which, up to base change, we can assume has at least two sections (by the base change property this would correspond to a base change for π that, as proven at step 1, can always be done).

We also remark that, if we set $J = \mathrm{Pic}\ ^4(\pi) \xrightarrow{f} B$, then the sheaf of relative differentials of f is just $\Omega^1_{J/B} = f^*(\pi_*\omega_{S/B})$ (see [17] p.2).

Then we can proceed as follows. Consider

$$\begin{array}{ccc} W^1_4(\pi) \subseteq W^0_4(\pi) \overset{i}{\subseteq} \mathrm{Pic}\ ^4(\pi) = & J \\ {\nwarrow}^{s} & \downarrow f \\ & B \end{array}$$

where s is a section of $f_{|W^1_4(\pi)}$ (we are assuming that such an s exists). Let $\widetilde{B}$ be the image of s. If we call $\mathcal{I}_1$, $\mathcal{I}_2$ the ideal sheaves of $\widetilde{B}$ in $W^0_4(\pi)$ and J respectively, the natural epimorphism $\mathcal{I}_2 \longrightarrow i_*\mathcal{I}_1$ induces the epimorphisms

$$\mathcal{S}\left(\mathcal{I}_2 / \mathcal{I}_2^2\right) \longrightarrow \mathcal{S}\left(i_*\mathcal{I}_1 / (i_*\mathcal{I}_1)^2\right) \cong \mathcal{S}\left(\mathcal{I}_1 / \mathcal{I}_1^2\right) \longrightarrow \bigoplus_i \mathcal{I}_1^i / \mathcal{I}_1^{i+1}$$

(where $\mathcal{S}$ denotes the symmetric algebra) and then also the inclusions

$$Z_1 = \mathrm{Proj}\ \left(\bigoplus_i \mathcal{I}_1^i / \mathcal{I}_1^{i+1}\right) \hookrightarrow Z_2 = \mathbb{P}_{\widetilde{B}}\left(\mathcal{I}_1 / \mathcal{I}_1^2\right) \hookrightarrow Z_3 = \mathbb{P}_{\widetilde{B}}\left(\mathcal{I}_2 / \mathcal{I}_2^2\right).$$

Since $W^0_4(\pi)$ is singular along $W^1_4(\pi)$, hence along $\widetilde{B}$, we have that Z_1 is the relative projectivized tangent cone of $W^0_4(\pi)$ along $\widetilde{B}$ which fits canonically into Z_2 and Z_3, the relative projectivized Zariski tangent spaces to $W^0_4(\pi)$ and J, respectively, along $\widetilde{B}$.

On the other hand, since $\widetilde{B} \subseteq J$ is smooth, we have an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_2 / \mathcal{I}_2^2 & \longrightarrow & \Omega^1_{J/B} \otimes \mathcal{O}_{\widetilde{B}} & \longrightarrow & \Omega^1_{\widetilde{B}/B} & \longrightarrow & 0 \\ & & & & & & \| & & \\ & & & & & & 0 & & \end{array}$$

that leads to $\mathcal{I}_2 / \mathcal{I}_2^2 \cong \Omega^1_{J/B} \otimes \mathcal{O}_{\widetilde{B}}$. Then

$$\begin{aligned} Z_3 &= \mathbb{P}_{\widetilde{B}}\left(\mathcal{I}^2 / \mathcal{I}_2^2\right) = \mathbb{P}_{\widetilde{B}}\left(\Omega^1_{J/B} \otimes \mathcal{O}_{\widetilde{B}}\right) \cong \mathbb{P}_B\left(s^*\Omega^1_{J/B}\right) = \\ &= \mathbb{P}_B\left(s^*f^*\left(\pi_*\omega_{S/B}\right)\right) = \mathbb{P}_B\left(\pi_*\omega_{S/B}\right) \end{aligned}$$

and so we have a variety $T_s := \mathrm{Im}\,(Z_1 \subseteq \mathbb{P}_B(\pi_*\omega_{S/B}))$ that by functoriality of the constructions verifies that, for $t \in B$ general

$$T_s \cap \mathbb{P}_t = \mathbb{P}\mathcal{T}_{s(t)} W^0_4(F_t) =: T_{s(t)} \quad \text{in the sense of lemma 2.1.}$$

Consider two such sections s_1, s_2 and set W the horizontal part of $T_{s_1} \cap T_{s_2}$. This is the variety we are seeking; for general $t \in B$

$$W \cap \mathbb{P}_t = T_{s_1(t)} \cap T_{s_2(t)} = R_t \quad \text{by lemma 2.1.}$$

□

Remark 2.3 Once we know the existence of W we can look at it from a different point of view. Consider a smooth fibre F_t of S (and Σ) and its respective cone R_t and elliptic base curve E_t. Since $E_t \subseteq \mathbb{P}^{g-2}$, and hence $R_t \subseteq \mathbb{P}^{g-1}$, are projectively normal we have

$$h^0\mathcal{I}_{E_t,\mathbb{P}^{g-2}}(2) = h^0\mathcal{I}_{R_t,\mathbb{P}^{g-1}}(2) = \frac{(g-1)(g-4)}{2}$$
$$h^0\mathcal{I}_{F_t,\mathbb{P}^{g-1}}(2) = \frac{(g-2)(g-3)}{2} = h^0\mathcal{I}_{R_t,\mathbb{P}^{g-1}}(2) + 1$$

Then we have a hyperplane

$$P = H^0\mathcal{I}_{R_t,\mathbb{P}^{g-1}}(2) \subseteq H^0\mathcal{I}_{F_t,\mathbb{P}^{g-1}}(2)\,.$$

We also know that E_t, and hence R_t, is an intersection of quadrics and then, that the quadrics containing R_t are all singular at the vertex q_t of the cone.

So aplying remark 1.3 we have that if $\mathfrak{a}$ is ample enough we get an epimorphism

$$H^0(\mathbb{P}, \mathcal{I}_{\Sigma,\mathbb{P}}(2T) \otimes \varphi^*(2\mathfrak{a})) \xrightarrow{\omega} H^0(\mathcal{I}_{F_t,\mathbb{P}^{g-1}}(2))$$

and then $\mathcal{P} = \omega^{-1}(P)$ is a hyperplane. Now we know that W is just the horizontal part of the base locus of relative hyperquadrics $\mathcal{Q} \in \mathcal{P}$.

Moreover, if we call B' the curve of vertices of the cones R_t, we have that the relative hyperquadrics in $\mathcal{P}$ are just those that are singular at B'.

Then we are ready to prove the main result of the section.

Theorem 2.4 *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus $g \geq 6$. Then S is, birationally, a double cover of an elliptic smooth surface V over B.*

Proof. First of all we remark that the variety W is singular at least at points on B' (see remark 2.3). Nevertheless, since for general $t \in B$, R_t is a cone over a projectively normal curve, hence a normal variety, we have that W is a normal variety at a general fibre by a result of Hironaka (see [5] III, lemma 9.12).

Now consider

$$\widetilde{W} \xrightarrow{\mu_2} W_1 \xrightarrow{\mu_1} W \qquad \mu = \mu_1 \circ \mu_2$$

where μ_1 is just the blow-up of W along B', and μ_2 is a desingularization of W_1. Since blow-up is functorial we have that for general $t \in B$, $\mu_1^{-1}(R_t) = \widetilde{R}_t$ is the blow-up of R_t

at its vertex q_t and, hence, an elliptic ruled surface over E_t (see [5] V, ex. 2.11.1). Then μ_2 only modifies certain bad fibres of W_1.

Consider then

$$\begin{array}{ccc} \widetilde{\Sigma} \subseteq \widetilde{W} & \xrightarrow{\ \mu\ } & W \subseteq \mathbb{P} \\ \downarrow & \swarrow_{\varphi} & \\ B & & \end{array}$$

where $\widetilde{\Sigma}$ is the strict transform of Σ. Since $B' \nsubseteq \Sigma$ we have that, for general $t \in B$, fibres of Σ and $\widetilde{\Sigma}$ are isomorphic. We remark that $\widetilde{W}$ is fibred over B with general fibre an elliptic scroll.

With notation as in section 1, consider $\mathcal{O}_{\widetilde{W}}(H) = \mu^*(\mathcal{O}_{\mathbb{P}}(T) \otimes \varphi^*(\mathfrak{a}))$ with $\mathfrak{a} \in \operatorname{Pic} B$ ample enough. Then $|H|$ has no base points and, since $\widetilde{W}$ is smooth, Bertini's theorem allows us to take a smooth section $V \in |H|$. For general $t \in B$, and by construction of μ, the fibre of V over t corresponds to a hyperplane section of R_t and hence it is a smooth elliptic curve. Therefore we get an elliptic fibration $\tau : V \longrightarrow B$.

We only have to prove that $\widetilde{\Sigma}$ is, birationally, a double cover of V. Since for general fibre $\widetilde{R}_t$ of $\widetilde{W}$ we have that F_t is a double cover of E_t and the morphism is given by the ruling, we only have to prove that such morphisms $\widetilde{R}_t \longrightarrow E_t$ can be glued to a global rational map $\widetilde{W} - -- \to V$.

The argument is standard and it is esentially the same as in [2] p.160. We reproduce the main points.

Fix $\widetilde{R} = \widetilde{R}_t$ a smooth fibre of $\widetilde{W} \xrightarrow{\nu} B$ and let $\Gamma \cong \mathbb{P}^1$ be a fixed fibre of the map $\widetilde{R} \longrightarrow E := E_t$. From the exact sequence of normal bundles

$$0 \longrightarrow N_{\Gamma,\widetilde{R}} \longrightarrow N_{\Gamma,\widetilde{W}} \longrightarrow N_{\widetilde{R},\widetilde{W}} \otimes \mathcal{O}_\Gamma \longrightarrow 0$$

and $N_{\Gamma,\widetilde{R}} = \mathcal{O}_\Gamma(\Gamma) = \mathcal{O}_\Gamma$, $N_{\widetilde{R},\widetilde{W}} = \mathcal{O}_{\widetilde{R}}(\widetilde{R}) = \mathcal{O}_{\widetilde{R}}$ we get

$$h^0(N_{\Gamma,\widetilde{W}}) = 2\,; \qquad h^1(N_{\Gamma,\widetilde{W}}) = 0$$

and then we can conclude that $\operatorname{Hilb}(\Gamma, \widetilde{W})$ (the Hilbert scheme of 1-cycles of $\widetilde{W}$ algebraically equivalents to Γ) is a surface, smooth at the point m_0 representing Γ. Let M be the irreducible component of $\operatorname{Hilb}(\Gamma, \widetilde{W})$ containing m_0, and let $\mathcal{M}$ be the universal family of curves of $\widetilde{W}$ parametrized by M. Then we have

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{\ \ j\ \ } & \widetilde{W} \times M & & \\ & {\scriptstyle\theta}\searrow & \swarrow{\scriptstyle\pi_2} & \searrow{\scriptstyle\pi_1} & \\ & & M & & \widetilde{W} \end{array}$$

where θ is a flat morphism by definition. Then for $m \in M$, $\theta^{-1}(m) = (\Gamma_m, m)$; $\Gamma_m \in \widetilde{W}$, $\Gamma_m \underset{\text{alg}}{\sim} \Gamma$. If Γ_m is smooth, since θ is flat we have that Γ_m is a rational curve. Moreover, since $\Gamma\widetilde{R} = 0$ we have that $\Gamma_m \widetilde{R}_{\nu(\Gamma_m)} = 0$ and, hence, that $\Gamma_m \subseteq \widetilde{R}_{\nu(\Gamma_m)}$. Then for general smooth Γ_m, Γ_m is a fibre of the scroll $\widetilde{R}_{\nu(\Gamma_m)}$.

In order to finish the proof we need the following

Claim.

(i) $\pi_1 \circ j : \mathcal{M} \longrightarrow \widetilde{W}$ is a birational map.

(ii) $V \hookrightarrow \widetilde{W} \overset{(\pi_1 \circ j)^{-1}}{- - - \to} \mathcal{M} \overset{\theta}{\longrightarrow} M$ is birational.

Once the claim is proved, it is immediate that the composite

$$S \longrightarrow \tilde{\Sigma} \subseteq \widetilde{W} \simeq \mathcal{M} \overset{\theta}{\longrightarrow} M \simeq V$$

is the desired 2-to-1 map.

Proof of claim. If U_1, U_2 are open sets on B and M where the fibres of ν and θ, respectively, are smooth, we define

$$\mathcal{M}_0 := j^{-1}(\nu^{-1}(U_1) \times U_2)$$

and so we have that for $(\Gamma_m, m) \subseteq \mathcal{M}_0$, Γ_m is a smooth fibre of the smooth scroll $\widetilde{R}_{\nu(\Gamma_m)}$.

Then, if $(x_1, m_1), (x_2, m_2) \in \mathcal{M}_0$ and $\pi_1(x_1, m_1) = x_1 = x_2 = \pi_1(x_2, m_2)$ we have that $x = x_1 = x_2 \in \Gamma_{m_1} \cap \Gamma_{m_2} \subseteq \widetilde{R}_{\nu(x)}$ and therefore $\Gamma_{m_1} = \Gamma_{m_2}$ and so $m_1 = m_2$. Then $\pi_1 \circ j$ is one-to-one on the open set $\mathcal{M}_0 \subseteq \mathcal{M}$ and hence $\pi_1 \circ j$ is a birational map.

Finally if we consider

$$\partial : V \hookrightarrow \widetilde{W} \overset{(\pi_1 \circ j)^{-1}}{- - - \to} \mathcal{M} \overset{\theta}{\longrightarrow} M$$

for any $x \in V_0 := (\pi_1 \circ j)(\mathcal{M}_0) \cap V \subseteq V$ we have $\partial(x) = \theta(x, m) = m$ where $x \in \Gamma_m$. Then for $x_1, x_2 \in V_0$, $m = \partial(x_1) = \partial(x_2)$ and so $x_1, x_2 \in \Gamma_m \cap V_0$. In view of $\Gamma_m V = \Gamma H = 1$ it follows that $x_1 = x_2$. Then again ∂ is one-to-one on an open set, hence it is birational. □

Remark 2.5 If $\pi : S \longrightarrow B$ is a smooth bielliptic fibration (i.e., all the fibres are smooth bielliptic) of genus $g \geq 6$ then we can even conclude from the proofs of theorems 2.2 and 2.4 that S is a double cover (everywhere defined) of a smooth minimal elliptic fibration $\tau : V \longrightarrow B$ (i.e., all fibres of τ are smooth).

Indeed, from section 1 we get that in this case $\widetilde{S} = S$ (because $\omega_{S/B} \otimes \pi^*(\mathfrak{a})$ has no base point) and that $S \cong \Sigma$. Moreover, since all fibres are smooth we have $\widetilde{W} = W_1$ and $\tilde{\Sigma} \cong \Sigma$ in the proof of theorem 2.4. Finally we claim that M is a smooth elliptic fibration and $\widetilde{W} \longrightarrow M$ is everywhere defined, so that $S \cong \tilde{\Sigma} \subseteq \widetilde{W} \longrightarrow M$ is the double cover we are seeking. This follows immediately from the proof of the claim in theorem 2.4. In fact we have that $V \underset{\cong}{\longrightarrow} M$ is the relative minimalization of V. (V is not minimal precisely at fibres E such that $\mu(E) \cap B' \neq \emptyset$ as one can easily see). We take M as elliptic base surface.

Then we can conclude the following

Proposition 2.6 *Let $\pi : S \longrightarrow B$ be a fibration of genus $g \geq 6$ such that all fibres are smooth and bielliptic. Then π is isotrivial, i.e. all fibres are isomorphic.*

Proof. In the last remark we have proven that there is a finite 2-to-1 morphism $f : S \longrightarrow V$ such that $\tau : V \longrightarrow B$ is an smooth elliptic fibration and that $\pi = \tau \circ f$.

Denote $E_t = \tau^{-1}(t)$ for $t \in B$ and let Z be the branch locus of f. Then for any $t \in B$ we have that $Z \cap E_t$ is the branch divisor of the 2-to-1 map $F_t \longrightarrow E_t$ between smooth curves. This implies that $\tau_{|Z} : Z \longrightarrow B$ is an étale morphism of degree $2g - 2$ by Hurwitz formula.

On the other hand, V is isomorphic to a quotient $(D \times E)/G$ where D and E are smooth curves, G is a finite group acting on both D and E, and so it is acting diagonally on $D \times E$. The map τ corresponds to the natural projection $(D \times E)/G \longrightarrow (D/G) \cong B$. In particular, all fibres of τ are isomorphic to E and the map $D \longrightarrow (D/G)$ is unramified. Base change yields

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow{\scriptstyle f'} & & {\scriptstyle f}\downarrow \\ D \times E & \xrightarrow{p} & V \\ \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} \\ D & \longrightarrow & B \end{array}$$

Note that p is étale. The branch locus $Z' = p^{-1}(Z)$ of f' maps onto D without ramification, so all its connected components are smooth. Let D' be one such component. Base change again yields

$$\begin{array}{ccc} S'' & \longrightarrow & S' \\ \downarrow{\scriptstyle f''} & & {\scriptstyle f'}\downarrow \\ D' \times E & \xrightarrow{q} & D \times E \\ \downarrow{\scriptstyle \tau''} & & \downarrow{\scriptstyle \tau'} \\ D' & \longrightarrow & D \end{array}$$

The branch locus of f'' is $Z'' = q^{-1}(Z')$, which maps onto D' without ramification and has a connected component which is a fibre of the projection $v : D' \times E \longrightarrow E$. It follows that Z'' is completely contained in fibres of v, and thus Z'' is a finite union of such fibres. Therefore, all fibres of $\pi'' = \tau'' \circ f'' : S'' \longrightarrow D'$ admit a 2-to-1 map onto E with the same branch locus. This implies that $\pi'' : S'' \longrightarrow D'$ is isotrivial, and so π is isotrivial, too. □

Remark 2.7 A similar result holds for hyperelliptic fibrations of genus $g \geq 2$, as shown by Xiao in [17].

3 Bielliptic fibrations of genus 5

A bielliptic curve of genus $g \leq 5$ can have more than one bielliptic involution. We give an example which proves that these involutions do not glue independently for a general fibration.

Example 3.1 Recall ([1] p.272) that a bielliptic curve F of genus 5 can have between one and five bielliptic structures. Such bielliptic involutions are in correspondence with the elliptic components of $W_4^1(F)$.

Take a genus five curve F with *exactly* two bielliptic involutions $\sigma_i : F \longrightarrow E_i$ such that $E_1 \not\cong E_2$, with E_i having no exceptional automorphisms (a count of constants shows that such an F can be chosen). Then we have that $\sigma_1 \times \sigma_2 : F \longrightarrow E_1 \times E_2$ embeds F as a smooth curve, $F \in |\ell_1^*(2p_1) \otimes \ell_2^*(2p_2)|$, being $\ell_i : E_1 \times E_2 \longrightarrow E_i$ the projections and $(p_1, p_2) \in E_1 \times E_2$. Since Aut $(E_1 \times E_2)$ acts transitively on $E_1 \times E_2$ we have that for every $(q_1, q_2) \in E_1 \times E_2$ there exists $\widetilde{F} \in |\ell_1^*(2q_1) \otimes \ell_2^*(2q_2)|$, $\widetilde{F} \cong F$.

Let B be any smooth curve having an involution ι and let $g : B \longrightarrow \bar{B} = B_{/<\iota>}$. Consider a morphism $\kappa : B \longrightarrow \mathbb{P}^1$ with no factorization through $\bar{B}$. Take a fixed $\bar{t} \in \bar{B}$ such that if $g^{-1}(\bar{t}) = \{t_1, t_2\}$ then $\kappa(t_1) \neq \kappa(t_2)$. After an automorphism of $\mathbb{P}^1$ we can suppose that $\kappa(t_i)$ is the modular invariant of E_i in $\mathbb{C} \subseteq \mathbb{P}^1$.

Then, by [3] p.160, there exists an elliptic fibration $\tau : V \longrightarrow B$ with a section, such that $\tau^{-1}(t_i) \cong E_i$. Let B' be the image in V of the section of τ. Consider the following pull-back

$$\begin{array}{ccc} Z := V \times_B V & \xrightarrow{\xi_2} & V \\ \downarrow{\scriptstyle \xi_1} & \searrow{\scriptstyle \xi} & \downarrow{\scriptstyle \tau} \\ V & \xrightarrow[\iota\circ\tau]{} & B \end{array}$$

Then, for $t \in B$ we have $Z_t = \xi^{-1}(t) = E_{\iota(t)} \times E_t$, where $E_m = \tau^{-1}(m)$. The natural involution on $V \times_{\mathbb{C}} V$ induces a commutative diagramm

$$\begin{array}{ccc} Z & \xrightarrow{\bar{\iota}} & Z \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ B & \xrightarrow{\iota} & B \end{array}$$

and then

$$\begin{array}{ccc} Z & \xrightarrow{\bar{g}} & \bar{Z} := Z_{/<\iota>} \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \bar{\xi}} \\ B & \xrightarrow{g} & \bar{B} \end{array}$$

Note that $\bar{Z}$ is a threefold fibred over $\bar{B}$ and the fibre over $g(t) \in \bar{B}$ general is $E_{\iota(t)} \times E_t$. We can assume $\bar{Z}$ is already smooth.

Let $B'' = \bar{g}(\xi_2^{-1}(B'))$ and $\mathcal{L} = \mathcal{O}_{\bar{Z}}(2B'')$. We have that $\mathcal{L}_{|\bar{Z}_{\bar{t}}} \cong \ell_1^*(2q_1) \otimes \ell_2^*(2q_2)$ for some $(q_1, q_2) \in E_1 \times E_2$. Hence, by lemma 1.1, if $\mathfrak{a} \in$ Pic $\bar{B}$ is ample enough we have an epimorphism

$$H^0(\bar{Z}, \mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})) \longrightarrow H^0(E_1 \times E_2, \mathcal{L}_{|\bar{Z}_{\bar{t}}}) \,.$$

Since by hypothesis there exists $F \in |\mathcal{L}_{|\bar{Z}_{\bar{t}}}|$ we get $\bar{S} \in |\mathcal{L} \otimes \bar{\xi}^*(\mathfrak{a})|$ a surface fibred over $\bar{B}$, smooth at a general fibre and such that $\bar{S}_{\bar{t}} = F$. Again, we can suppose $\bar{S}$ is already smooth. Let $\bar{\pi} : \bar{S} \longrightarrow \bar{B}$ and $F_{\bar{m}} = \bar{\pi}^{-1}(\bar{m})$. For $\bar{m} \in \bar{B}$ general we have that $F_{\bar{m}}$ is an smooth curve of genus 5 having *at least* two bielliptic involutions given by the inclusion $F_{\bar{m}} \subseteq E_{\iota(m)} \times E_m$ (if $g(m) = \bar{m}$) as a $(2,2)$-divisor. We claim that for general $\bar{m} \in \bar{B}$, $F_{\bar{m}}$ has exactly two bielliptic involutions. Since this is the case for $F = F_{\bar{t}}$ we only have to prove that having *at most* two of them is an open condition. Consider $W^1_4(\bar{\pi}) \longrightarrow \bar{B}$ (after a base change if necessary). The number of bielliptic involutions of $F_{\bar{m}}$ is given by the number of elliptic components of $W^1_4(F_{\bar{m}}) \cong W^1_4(\bar{\pi})_{\bar{m}}$. Then, having at most two of such components is obviously an open condition.

We claim that $\bar{S}$ is not a (birational) double cover of any elliptic fibration $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$. Indeed, assume we have a double cover $\bar{f} : \bar{S} \longrightarrow \bar{V}$ (we can suppose $\bar{f}$ everywhere defined after some blow-ups). Consider the base change diagram

$$\begin{array}{ccc}
Z & \longrightarrow & \bar{Z} \\
\uparrow & & \uparrow \\
S & \longrightarrow & \bar{S} \\
{\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\
\widetilde{V} & \longrightarrow & \bar{V} \\
{\scriptstyle \widetilde{\tau}}\downarrow & & \downarrow{\scriptstyle \bar{\tau}} \\
B & \longrightarrow & \bar{B}
\end{array}$$

For S we have three double covers of elliptic fibrations over B:

$$\begin{array}{ll}
\widetilde{f} : S \longrightarrow \widetilde{V} & \\
f_i : S \longrightarrow V & \qquad f_i = \xi_{i|S} \quad i = 1,2
\end{array}$$

Set $U = \{m \in B \mid E_m \ncong E_{\iota(m)};\ E_m, E_{\iota(m)}$ and $\widetilde{E}_m$ are smooth and F_m has exactly two bielliptic involutions$\}$ (where $\widetilde{E}_m = \widetilde{\tau}^{-1}(m)$). We have that U is a non-empty open set of B. Since $f_{1|F_m}$, $f_{2|F_m}$, $\widetilde{f}_{|F_m}$ are double covers of $E_{\iota(m)}$, E_m and $\widetilde{E}_m$ respectively we have that for every $m \in U$, $\widetilde{E}_m \cong E_{\iota(m)}$ or $\widetilde{E}_m \cong E_m$.

If $g_1 = g \circ \iota_{|U} : U \longrightarrow \mathbb{P}^1$, $g_2 = g_{|U} : U \longrightarrow \mathbb{P}^1$ and $\widetilde{g} : U \longrightarrow \mathbb{P}^1$ are the modular morphisms induced by $\iota \circ \tau$, τ and $\widetilde{\tau}$ over U respectively we have that $\widetilde{g} = g_1$ or $\widetilde{g} = g_2$. Assume $\widetilde{g} = g_2$.

As we have $t_1, t_2 \in U$ and $\iota(t_1) = t_2$ we get

$$E_{t_1} = \tau^{-1}(t_1) = \widetilde{\tau}^{-1}(t_1) \cong \widetilde{\tau}^{-1}(t_2) = \tau^{-1}(t_2) = E_{t_2}$$

since $\widetilde{\tau}$ is induced by $\bar{\tau} : \bar{V} \longrightarrow \bar{B}$ and then $\widetilde{\tau}^{-1}(m) \cong \widetilde{\tau}^{-1}(\iota(m))$ for all $m \in B$. But this is imposible since by hypothesis $E_{t_1} = E_1 \ncong E_2 = E_{t_2}$. $\square$

Although this example shows that theorem 2.4 is not true for genus 5 bielliptic fibrations we have

Proposition 3.2 *Let $\pi : S \longrightarrow B$ be a bielliptic fibration of genus 5. Then, there exists a base change $\tilde{\pi} : \widetilde{S} \longrightarrow \widetilde{B}$ of π such that $\widetilde{S}$ is, birationally, a double cover of an elliptic fibration $\tilde{\tau} : \widetilde{V} \longrightarrow \widetilde{B}$.*

If the general fibre of π has only one bielliptic involution, then we can take $\tilde{\pi} = \pi$.

Proof. As in step 3 of theorem 2.2 we have after a base change $W_4^1(\hat{\pi}) \longrightarrow \widehat{B}$ (at least over a non-empty open subset of $\widehat{B}$). Let $\widehat{L}$ be an irreducible component of $W_4^1(\hat{\pi})$ such that for general $t \in \widehat{B}$; $\widehat{L}_t$ has only elliptic components. Let L be a desingularization of L. After a base change we can get

$$\begin{array}{ccc} \widetilde{L} & \longrightarrow & L \\ \downarrow & & \downarrow \\ \widetilde{B} & \longrightarrow & \widehat{B} \end{array}$$

such that $\widetilde{L} \longrightarrow \widetilde{B}$ has irreducible general fibres and at least two sections. If $\tilde{\pi} : \widetilde{S} \longrightarrow \widetilde{B}$ is the induced fibration we have that $\widetilde{L}$ is an irreducible component of $W_4^1(\tilde{\pi}) \longrightarrow \widetilde{B}$. Then step 3 in theorem 2.2 works and we get $\widetilde{S} - - > \widetilde{\Sigma} \subseteq \widetilde{W} \subseteq \mathbb{P}_{\widetilde{B}}(\tilde{\pi}_* \omega_{\widetilde{S}/\widetilde{B}})$. The existence of $\widetilde{W}$ is all what we need to use the same proof of theorem 2.4.

If the general fibre of π has only one bielliptic involution (hence there exists only one elliptic cone containing $F \subseteq \mathbb{P}^4$) then from the existence of $\widetilde{\Sigma} \subseteq \widetilde{W}$ we can deduce the existence of $\Sigma \subseteq W$ in the same way as step 1 in theorem 2.2 (uniqueness of elliptic cones containing the general fibre is all what we need). □

This result is enough to extend proposition 2.6 to the genus 5 case.

Corollary 3.3 *Let $\pi : S \longrightarrow B$ be a genus 5 fibration such that all fibres are smooth and bielliptic. Then π is isotrivial.*

Proof. We can check isotriviallity after a base change. The result follows then from proposition 3.2 and the proof of proposition 2.6. □

4 Double covers and the slope of bielliptic fibrations

We recall some basic facts about double covers (see [6], [3]).

By a double cover we mean a finite, degree two map between surfaces, $f_0 : S_0 \longrightarrow V_0$. This map is determined by a divisor Z_0 on V_0 (the branch divisor) and a line bundle $\mathcal{L}_0$ such that $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$. If V_0 is smooth, S_0 is normal (respectively smooth) if and only if Z_0 is reduced (respectively smooth).

Consider a double cover as above with S_0 normal and V_0 smooth. Then there exists a *canonical resolution of singularities* for S_0 which consists on a finite sequence of maps

$$\begin{array}{ccccccccc}
S_k & \xrightarrow{\sigma_k} & S_{k-1} & \longrightarrow & \dots & \longrightarrow & S_1 & \xrightarrow{\sigma_1} & S_0 \\
f_k \downarrow & & f_{k-1} \downarrow & & \dots & & \downarrow f_1 & & \downarrow f_0 \\
V_k & \xrightarrow[\alpha_k]{} & V_{k-1} & \longrightarrow & \dots & \longrightarrow & V_1 & \xrightarrow[\alpha_1]{} & V_0
\end{array}$$

satisfying:

(i) α_j is the blow-up of V_{j-1} at a singular point p_{j-1} of Z_{j-1} (the branching divisor of f_{j-1}).

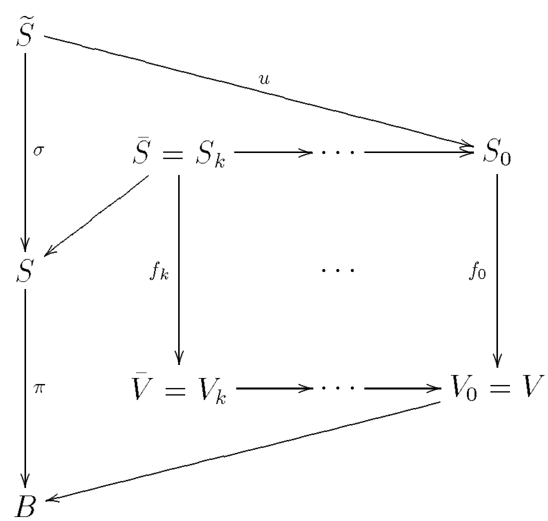
(ii) f_j is the double cover of V_j defined by $\mathcal{L}_j^{\otimes 2} \cong \mathcal{O}(Z_j)$, with $Z_j = \alpha_j^*(Z_{j-1}) - 2m_{j-1}E_j$, $\mathcal{L}_j = \alpha_j^*(\mathcal{L}_{j-1}) \otimes \mathcal{O}_{V_j}(-m_{j-1}E_j)$, where E_j is the exceptional divisor of α_j and p_{j-1} is a singular point of Z_{j-1} of multiplicity $2m_{j-1}$ or $2m_{j-1}+1$.

(iii) σ_j is a birational morphism induced by the cartesian diagram of α_j and f_{j-1}.

(iv) Z_k is smooth and, hence, S_k is a smooth surface.

Now we can use this as follows. Recall from section 2 that we have obtained $f : \widetilde{S} \longrightarrow V$ a generically 2-to-1 morphism (we can suppose that f is everywhere defined up to blow-ups) from a blow-up of S onto an elliptic fibration V over B which we can suppose relatively minimal after some blow-downs. Suppose that π is relatively minimal.

Now consider



where:

- $f = f_0 \circ u$ is the Stein factorization of f, with u birational, f_0 finite (so it is a double cover) and S_0 normal.
- $f_k : S_k \longrightarrow V_k$ is the canonical resolution of singularities of $f_0 : S_0 \longrightarrow V_0$.
- $\bar{\sigma} : S_k \longrightarrow S$ is the birational morphism defined by the relative minimality of π.

Theorem 4.1 *Let $\pi : S \longrightarrow B$ be a relatively minimal bielliptic fibration of genus $g \geq 6$. Let V be the relative minimal model of the elliptic fibration obtained in theorem 2.4. Then* *(a)* $\omega^2_{S/B} - 4\Delta(\pi) \geq 2(g-5)\mathcal{X}\mathcal{O}_V$. *In particular, if* π *is not locally trivial*

$$\lambda(\pi) \geq 4 + \frac{2(g-5)\mathcal{X}\mathcal{O}_V}{\Delta(\pi)} \geq 4$$

(b) $\lambda(\pi) = 4$ *if and only if* S *is the minimal desingularization of a double cover* $S_0 \longrightarrow V$ *of a smooth elliptic surface such that*

- *All the fibres of the elliptic fibration* $\tau : V \longrightarrow B$ *are smooth and isomorphic.*
- *The branch divisor of the double cover has only* negligeable *singularities (i.e., all the multiplicities* m_j *in the above process are* 2 *or* 3 *(see [13], [17])).*

In particular, the bound is sharp.

Proof.
(a) First of all we have

$$\omega^2_{S/B} - 4\Delta(\pi) = (K_S^2 - 4\mathcal{X}\mathcal{O}_S) - 4(b-1)(g-1) \geq (K_{\bar{S}}^2 - 4\mathcal{X}\mathcal{O}_{\bar{S}}) - 4(b-1)(g-1)\,. \quad (6)$$

For smooth double covers $f_k : \bar{S} \longrightarrow \bar{V}$ we have (see [3] p.183):

$$\begin{aligned} \mathcal{X}\mathcal{O}_{\bar{S}} &= 2\mathcal{X}\mathcal{O}_{\bar{V}} + \frac{1}{2}\mathcal{L}_k K_{\bar{V}} + \frac{1}{2}\mathcal{L}_k\mathcal{L}_k \\ K_{\bar{S}}^2 &= 2K_{\bar{V}}^2 + 4\mathcal{L}_k K_{\bar{V}} + 2\mathcal{L}_k\mathcal{L}_k \end{aligned}$$

so we have

$$K_{\bar{S}}^2 - 4\mathcal{X}\mathcal{O}_{\bar{S}} = 2[K_{V_k}^2 - 4\mathcal{X}\mathcal{O}_{V_k}] + 2\mathcal{L}_k K_{V_k}\,. \quad (7)$$

Moreover, in each blow-up $\alpha_j : V_j \longrightarrow V_{j-1}$ we get

$$\mathcal{X}\mathcal{O}_{V_j} = \mathcal{X}\mathcal{O}_{V_{j-1}}; \quad K_{V_j} = \alpha_j^* K_{V_{j-1}} + E_j; \quad \mathcal{L}_j = \alpha_j^*\mathcal{L}_{j-1} - m_{j-1}E_j\,.$$

Then

$$\begin{aligned} 2[K_{V_j}^2 - 4\mathcal{X}\mathcal{O}_{V_j}] + 2\mathcal{L}_j K_{V_j} &= 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + \\ 2\mathcal{L}_{j-1}K_{V_{j-1}} + 2(m_{j-1}-1) &\geq 2[K_{V_{j-1}}^2 - 4\mathcal{X}\mathcal{O}_{V_{j-1}}] + 2\mathcal{L}_{j-1}K_{V_{j-1}}\,. \end{aligned} \quad (8)$$

Finally as $\tau : V \longrightarrow B$ is an elliptic minimal fibration, numerically we have $K_V \equiv \left[2(b-1) + \mathcal{X}\mathcal{O}_V + \sum_i \frac{(n_i-1)}{n_i}\right] E$ ([3] p.162) where E denotes a smooth fibre of τ and $\{n_i\}$ are the multiplicities of singular fibres of τ. In particular $K_V^2 \equiv 0$.

As $\mathcal{L}_0^{\otimes 2} = \mathcal{O}_{V_0}(Z_0)$ and Z_0 is the branch divisor of f_0 we get $\mathcal{L}_0 E = (g-1)$ by Hurwitz formula. So

$$\begin{aligned} 2[K_{V_0}^2 - 4\mathcal{X}\mathcal{O}_{V_0}] + 2\mathcal{L}_0 K_{V_0} &= -8\mathcal{X}\mathcal{O}_{V_0} + \\ +2\mathcal{L}_0 E\left[2(b-1) + \mathcal{X}\mathcal{O}_{V_0} + \sum_i \frac{(n_i-1)}{n_i}\right] &\geq 4(b-1)(g-1) + 2(g-5)\mathcal{X}\mathcal{O}_V\,. \end{aligned} \quad (9)$$

Then (a) follows from (6), (7), (8) and (9) and from the fact that $\mathcal{X}\mathcal{O}_V \geq 0$ for elliptic fibrations.

(b) Looking at the proof of (a) we see that $\lambda = 4$ iff $\mathcal{X}\mathcal{O}_V = 0$ and equality holds in (6), (7), (8) and (9). So we have $\lambda = 4$ iff S is the minimal desingularization of a double cover of an elliptic, relatively minimal, fibration $\tau : V \longrightarrow B$ such that:

- τ has no multiple fibres ($\forall i \quad n_i = 1$).
- $\mathcal{X}\mathcal{O}_V = 0$.
- The branch divisor Z_0 of the double cover has only *negligeable singularities* (see [13], [17]), i.e. all the multiplicites of the singularities of the branch divisors in the process of canonical resolution are 2 or 3.

But the first two conditions are equivalent to the fact that τ is smooth and isotrivial (see 15 thms. 6,7 Ch.IV). This allows us to construct examples with $\lambda(\pi) = 4$ which are esentially the same as in [19] example 4.3. So the bound is sharp. □

Remark 4.2 Although we cannot use double covers for the case of bielliptic fibrations of genus 5 we already know that $\lambda \geq 4$ also holds for such fibrations (see [9] thm.5.1, [11]).

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