Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity

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Summary

In this paper we study the exponentially small splitting of a heteroclinic orbit in some unfoldings of the central singularity also called Hopf-zero singularity.

The fields under consideration are of the form:

$$
\begin{aligned}
\frac{dx}{d\tau} &= -\delta xz - y\left(\alpha + c\delta z\right) + \delta^{p+1} f(\delta x, \delta y, \delta z, \delta) \\
\frac{dy}{d\tau} &= -\delta yz + x\left(\alpha + c\delta z\right) + \delta^{p+1} g(\delta x, \delta y, \delta z, \delta) \\
\frac{dz}{d\tau} &= \delta\left(-1 + b(x^2+y^2) + z^2\right) + \delta^{p+1} h(\delta x, \delta y, \delta z, \delta),
\end{aligned}
$$

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where f, g and h are real analytic functions, α, b and c are constants and δ is a small parameter.

When $f = g = h = 0$ the system has a heteroclinic orbit between the critical points $(0, 0, \pm 1)$ given by: $\{(x, y) = (0, 0)\ ; -1 < z < 1\}$.

Let $d^{\mathrm{s,u}}$ be the distance between the one dimensional stable and unstable manifold of the perturbed system measured at the plane $z = 0$. We prove that for any f, g such that $\hat{m}(\mathrm{i}\,\alpha) \neq 0$, where $\hat{m}$ is the Borel transform of the function $m(u) = u^{1+\mathrm{i}\,c}(f + \mathrm{i}\,g)(0, 0, u, 0)$

$$|d^{\mathrm{s,u}}| = 2\pi\,\mathrm{e}^{c\pi/2}\,|\hat{m}(\mathrm{i}\,\alpha)|\delta^p\,\mathrm{e}^{-\pi|\alpha|/(2\delta)}(1 + O(\delta^{p+2}|\log\delta|)), \quad p > -2.$$

Keywords: Exponentially small splitting, Hopf-zero bifurcation, Melnikov function, Borel transform.

1 Introduction

One of the most frequently studied problems in the last century was the existence of transversal intersections between stable and unstable manifolds of one or more critical points of a dynamical system. This phenomenon is also known as the problem of the *splitting of separatrices*. The interest of this problem was already noted by Poincaré who described it as the *fundamental problem of the mechanics* [Poi90]; it is one of the main causes of chaotic behavior. It is well known that the size of the splitting of separatrices gives a measure of the stochastic region of the system.

The most simple setting where this phenomenon occurs is in T-periodically perturbed integrable planar systems. In this regular perturbative context, Poincaré, and later Melnikov, constructed a method which allows computation of the splitting of invariant manifolds of hyperbolic critical points which coincide in the unperturbed integrable system. The Poincaré-Melnikov method provides a function whose non-degenerate zeros give rise to transversal homoclinic orbits in the perturbed system, see [Mel63, GH83]. Several authors have dealt with the problem of generalizing this method to higher dimensional systems. Specifically, for Hamiltonian systems, the (vectorial) Melnikov-Poincaré function turns out to be the gradient of a scalar function which is known as the Melnikov potential. See [Eli94, DG00] and references therein.

A difficult question arises when this Poincaré-Melnikov function turns out to be exponentially small with respect to the perturbative parameter and hence is not *a priori* the dominant term. This happens, for instance, in rapidly forced periodic or quasi-periodic perturbations of one degree of freedom Hamiltonian systems, in nearly integrable symplectic mappings which are close to the identity, and in Hamiltonian systems with two or more degrees of freedom which have hyperbolic tori with some fast frequencies, among others.

For some cases of near integrable time periodic or quasi-periodic Hamiltonian systems, several studies [Ang93, DS97, Gel97a, BF04, DGJS97] have validated the prediction of the Poincaré-Melnikov function for the splitting and give a rigorous proof of an asymptotic formula for different quantities related to this phenomenon. It is worth mentioning that the Poincaré-Melnikov function does not always give the correct prediction for the splitting, see [Tre97, Gel97b]. As we have pointed out elsewhere we can encounter this phenomenon in maps, see [Laz03, DRR98, GS01]. The methods developed in these works draw heavily on the Hamiltonian character of the system, especially its symplectic structure.

In this paper we deal with a different setting. We study the splitting of a heteroclinic orbit in a family X_δ (see (1)) of near integrable analytic vector fields of $\mathbb{R}^3$ introduced in Subsection 1.1.

The family X_δ under consideration becomes, when $\delta = 0$, the Hopf-zero singularity

(2) also called the central singularity in [GH83]. In fact, in Subsection 1.2, we show that any generic conservative unfolding of the Hopf-zero singularity can be expressed, after some changes of variables, in a form similar to the family X_δ considered in this paper.

In Subsection 1.3 we study the relevance of the splitting of the heteroclinic connection for the analytic unfoldings of the Hopf-zero bifurcation. We show that the breakdown of this heteroclinic orbit can lead to the birth of some homoclinic connection in the unfoldings producing what is known as a Shilnikov bifurcation. In this subsection we also present some results about the existence of Shilnikov bifurcations in the C^∞ case. Our final goal is to prove the existence of Shilnikov bifurcations in the analytic unfoldings of the Hopf-zero singularity. We give an asymptotic formula to measure the splitting of the heteroclinic connection for analytic families X_δ which turns out to be exponentially small with respect to δ.

Even though the family (1) under consideration can be seen as a perturbation of an integrable conservative vector field, we do not require the perturbation to be conservative at all. For this reason, our proof does not use any geometric structure of the system. The computation of the difference between the stable and unstable manifolds is estimated simply by using the idea in [Sau01, OSS03, Bal] that this difference satisfies some linear equation whose solutions can be controlled.

1.1 Set up

The fields under consideration in this study are of the form:

$$\begin{aligned}
\frac{dx}{d\tau} &= -\delta xz - y\left(\alpha + c\delta z\right) + \delta^{p+1} f(\delta x, \delta y, \delta z, \delta) \\
\frac{dy}{d\tau} &= -\delta yz + x\left(\alpha + c\delta z\right) + \delta^{p+1} g(\delta x, \delta y, \delta z, \delta) \\
\frac{dz}{d\tau} &= \delta\left(-1 + b(x^2+y^2) + z^2\right) + \delta^{p+1} h(\delta x, \delta y, \delta z, \delta),
\end{aligned} \tag{1}$$

where $p \geq -2$, α, c, b are given constants, $\delta > 0$ is a small parameter and f, g, h are real analytic functions in all their variables whose Taylor series begin at least with terms of degree three.

As we will see in Lemma 1.2, system (1) has a one dimensional heteroclinic connection, $\{(x,y) = (0,0);\ -1 < z < 1\}$, between two critical points $(0,0,\pm 1)$ of saddle-focus type when $f = g = h = 0$. The goal of this paper is to study the effects of any analytic perturbation (f,g,h) on the invariant stable and unstable manifolds of the critical points of the perturbed system. We will see that generically, if $p > -2$, this heteroclinic connection is destroyed and, moreover, we will compute the distance between the perturbed manifolds when they meet the plane $z = 0$. We will prove that such distance, $d^{\mathrm{s,u}}$, is an exponentially small quantity given by

$$d^{\mathrm{s,u}} = C\delta^p \,\mathrm{e}^{-\pi|\alpha|/(2\delta)}(1 + O(\delta^{p+2}|\log\delta|)),$$

for some constant C computed exactly in Theorem 1.6 in terms of the Borel transforms of f and g. In particular, if $C \neq 0$ and $\delta > 0$ is small enough, we obtain that the heteroclinic connection is destroyed. To state the above result properly, we present the necessary notation in Subsection 1.5. The rigorous statement is left until to Subsection 1.6.

Even though we have presented system (1) as a regular perturbation of the case $f = g = h = 0$, this system lies in the context of singular perturbation theory. Indeed, when the parameter $\delta = 0$, the family of vector fields we are working with becomes

$$\frac{dx}{d\tau} = -\alpha y, \qquad \frac{dy}{d\tau} = \alpha x, \qquad \frac{dz}{d\tau} = 0. \tag{2}$$

Thus family (1) is a perturbation of system (2). This system is known as the Hopf-zero singularity or the central singularity.

1.2 Analytic unfoldings of the central singularity

Let us consider a vector field in $\mathbb{R}^3$ which has the origin as a critical point and, for some positive α^*, the eigenvalues of the linear part at the origin are $0, \pm\alpha^*\mathrm{i}$. If we assume that the linear part of this vector field is in Jordan normal form it will be given by

$$\begin{pmatrix} 0 & -\alpha^* & 0 \\ \alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha^* > 0. \tag{3}$$

If we consider only the linear context, it is clear that this singularity can be met by a generic family of linear vector fields if it contains at least two parameters. So, it has codimension two. But the linear system we are studying has zero divergence; hence it can be also considered in the context of conservative vector fields. In this context it will occur in one parameter families, and will then have codimension one.

The unfoldings of this singularity has been studied by several authors [Tak73a, Tak74, Tak73b, Guc81, BV84, AMF+03, FGRLA02, DI98, GH83] looking at the different type of bifurcations that a two (or one) parameter family of vector fields unfolding this singularity can present.

Following [BV84] we present here a description of the singularity we are considering as well as the normal form for the unfoldings of this singularity. Let us then explain the process for obtaining a perturbative setting from the normal form procedure in the more general case.

We consider $X_{\mu,\nu}$, a family of vector fields on $\mathbb{R}^3$ such that $X_{0,0}$ has the origin as a critical point with linear part (3). After some normalization, if we perform the normal form procedure up to order two, we obtain that the vector field $X_{\mu,\nu}$ in the new coordinates $(\bar{x}, \bar{y}, \bar{z})$ takes the form :

$$
\begin{aligned}
\frac{d\bar{x}}{ds} &= \bar{x}\left(A_2(\mu,\nu) + A_4(\mu,\nu)\bar{z}\right) - \bar{y}\left(A_1(\mu,\nu) + A_3(\mu,\nu)\bar{z}\right) + O_3(\bar{x},\bar{y},\bar{z},\mu,\nu) \\
\frac{d\bar{y}}{ds} &= \bar{x}\left(A_1(\mu,\nu) + A_3(\mu,\nu)\bar{z}\right) + \bar{y}\left(A_2(\mu,\nu) + A_4(\mu,\nu)\bar{z}\right) + O_3(\bar{x},\bar{y},\bar{z},\mu,\nu) \\
\frac{d\bar{z}}{ds} &= B_1(\mu,\nu) + B_2(\mu,\nu)\bar{z} + B_3(\mu,\nu)(\bar{x}^2+\bar{y}^2) + B_4(\mu,\nu)\bar{z}^2 + O_3(\bar{x},\bar{y},\bar{z},\mu,\nu)
\end{aligned}
$$

where $A_1(0,0) = \alpha^*$, $A_2(0,0) = B_1(0,0) = B_2(0,0) = 0$. And moreover, after some scaling of the parameters we can assume that $\partial_\mu A_2(0,0) = \partial_\nu B_1(0,0) = 0$, $\partial_\nu A_2(0,0) = 1$ and $\partial_\mu B_1(0,0) = -1$.

The conservative case can be done analogously, considering only one parameter families with parameter $\mu > 0$, and using that, in this case $B_2(\mu) = -2A_2(\mu)$ and $B_4(\mu) = -A_4(\mu)$. When $\mu > 0$, after the scaling $\bar{x} = \delta\tilde{x}$, $\bar{y} = \delta\tilde{y}$, $\bar{z} = \delta\tilde{z}$, $\delta = \sqrt{\mu}$ and calling

$$a_j = A_j(0,0), \quad b_j = B_j(0,0), \quad \text{for } j = 3,4,$$

the system becomes:

$$
\begin{aligned}
\frac{d\tilde{x}}{ds} &= a_4\delta\tilde{x}\tilde{z} - \tilde{y}\left(\alpha^* + a_3\delta\tilde{z}\right) + \delta^{-1}\tilde{f}(\delta\tilde{x},\delta\tilde{y},\delta\tilde{z},\delta) \\
\frac{d\tilde{y}}{ds} &= \tilde{x}\left(\alpha^* + a_3\delta\tilde{z}\right) + a_4\delta\tilde{y}\tilde{z} + \delta^{-1}\tilde{g}(\delta\tilde{x},\delta\tilde{y},\delta\tilde{z},\delta) \\
\frac{d\tilde{z}}{ds} &= \delta\left(-1 + b_3(\tilde{x}^2+\tilde{y}^2) - a_4\tilde{z}^2\right) + \delta^{-1}\tilde{h}(\delta\tilde{x},\delta\tilde{y},\delta\tilde{z},\delta).
\end{aligned}
\tag{4}
$$

From now on, we will focus our study on the conservative case when $a_4 < 0$. In this case, in order to eliminate the parameter a_4, we perform the scaling $x = \tilde{x}\sqrt{-a_4}$, $y = \tilde{y}\sqrt{-a_4}$, $z = \tilde{z}\sqrt{-a_4}$, and, after the change of time $\tau = \sqrt{-a_4}s$, system (4) becomes:

$$
\begin{aligned}
\frac{dx}{d\tau} &= -\delta xz - (\alpha + c\delta z)y + \delta^{-1}f(\delta x,\delta y,\delta z,\delta) \\
\frac{dy}{d\tau} &= (\alpha + c\delta z)x - \delta yz + \delta^{-1}g(\delta x,\delta y,\delta z,\delta) \\
\frac{dz}{d\tau} &= -\delta\big(1 + b(x^2+y^2) + z^2\big) + \delta^{-1}h(\delta x,\delta y,\delta z,\delta),
\end{aligned}
$$

where $\alpha = \frac{\alpha^*}{\sqrt{-a_4}}$, $c = \frac{a_3}{\sqrt{-a_4}}$, and $b = \frac{b_3}{\sqrt{-a_4}}$.

So, if $p = -2$, system (1) under consideration corresponds to the versal unfoldings of the central singularity, also called Hopf-zero bifurcation, after some changes of variables.

1.3 The central singularity and the Shilnikov bifurcation

Even though this is not the subject of this paper, let us remember here that a Shilnikov bifurcation occurs when a critical point of saddle-focus type exists and its stable and unstable manifolds intersect, giving rise to the existence of a homoclinic orbit [Šil65, Šil70, SSTC01].

In 1984 Broer and Vegter presented, in [BV84], a complete proof of the existence of subordinate Shilnikov bifurcations in generic $\mathcal{C}^\infty$ unfoldings of the singularity (3), which have codimension one. The method used in [BV84] to prove the existence of homoclinic orbits is based on the following normal form theorem (see [Bro81a, Bro81b]):

Theorem 1.1 *Let $X = X_\mu(x, y, z)$ be a real $\mathcal{C}^\infty$ family of vector fields, where $(x, y, z) \in \mathbb{R}^3$ and $\mu \in \mathbb{R}^k$. Suppose that for $\mu = 0$ the vector field has in $(0, 0, 0)$ a critical point with linear part (3). Then, there exists a $\mathcal{C}^\infty$ μ-dependent change of variables, such that the new vector field becomes $X_\mu = \tilde{X}_\mu + P$, where $\tilde{X}_\mu$, when written in cylindrical coordinates (r, φ, z), is:*

$$
\begin{aligned}
\frac{d\varphi}{ds} &= \tilde{f}(r^2, z, \mu) \\
\frac{dr}{ds} &= r\tilde{g}(r^2, z, \mu) \\
\frac{dz}{ds} &= \tilde{h}(r^2, z, \mu)
\end{aligned}
$$

being $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ C^∞ functions verifying $\tilde{g}(0, 0, 0) = \tilde{h}(0, 0, 0) = \frac{\partial \tilde{h}}{\partial z}(0, 0, 0) = 0$ and $\tilde{f}(0, 0, 0) = \alpha^*$, and the function P is flat at $(x, y, z, \mu) = (0, 0, 0, 0)$.

In the conservative case, the change can be chosen to be conservative at any order, so $\tilde{X}_\mu$ and P have zero divergence. In particular: $\frac{\partial \tilde{f}}{\partial \varphi} + r\frac{\partial \tilde{g}}{\partial r} + \tilde{g} + \frac{\partial \tilde{h}}{\partial z} = 0$.

Analyzing the normal form $\tilde{X}_\mu$ one can see that it has two hyperbolic critical points of saddle-focus type with a one dimensional heteroclinic orbit between them. Moreover, for any value of μ in the conservative case and for a suitable curve in the parameter space in the dissipative case, these points also have a two dimensional heteroclinic manifold.

By Theorem 1.1 any vector field in the unfolding is given by $X_\mu = \tilde{X}_\mu + P$. The strategy followed in [BV84] to prove the existence of homoclinic connections is to choose suitable "flat" perturbations $\tilde{P}$ that break the heteroclinic manifolds (both the one dimensional and the two dimensional) giving rise to the existence of homoclinic orbits to one of the critical points, for a sequence $\{\mu_n\}_n$ going to 0 as n goes to infinity.

More recently, also in the $\mathcal{C}^\infty$ context but for reversible systems, a similar result is obtained in [LTW04].

Our final goal is to achieve the same kind of results for analytic unfoldings X_μ, where this phenomenon is what is known as a "beyond all orders" or exponentially small phenomenon.

Note that the normal form Theorem 1.1 is not true in the analytic case. For the $\mathcal{C}^\infty$ case, there exists not only a formal procedure that casts the system into a formal normal form up to flat terms, but a $\mathcal{C}^\infty$ change of variables. This comes from the fact that, even if the formal series obtained are divergent, a classical result in asymptotic series, the Borel-Ritt theorem, gives the existence of $\mathcal{C}^\infty$ functions having them as Taylor series.

Of course this reasoning fails in the analytic case, because the function obtained through the Borel-Ritt theorem can not be real analytic if the formal series is divergent. Moreover, for analytic X_μ we can not use "flat" perturbations to break the heteroclinic connections which exist in the normal form, because flat functions are not analytic. However, to prove that these heteroclinic connections are destroyed is a necessary step towards the possible birth of homoclinic orbits.

This paper in which we deal with the case $p > -2$ is a first contributions to the complete proof of the breakdown of heteroclinic connections which, in our view, is quite delicate and lies in the context of singular perturbation theory.

1.4 Some comments about the singular case $p = -2$

As is clear from the above discussions, the generic unfoldings of singularity (3) become system (1) with $p = -2$ after changes of variables and scalings. Nevertheless, it is worth to say that for some (non-generic) unfoldings we fit in the form (1) for $p > -2$. In fact, system (1) itself is a degenerate unfolding (for instance, among other degenerations, it does not contain second order terms in its Taylor expansion) of this singularity.

Even if the results of this paper are only rigorously valid for $p > -2$, we hope that it will be possible to adapt some of the methods implemented here to a proof of the exponentially small phenomenon in the limit case $p = -2$.

This limit case, which corresponds to generic unfoldings, is what is known as a singular perturbation case. The reason is the following. The phenomenon we are going to study is the splitting of a heteroclinic orbit between two critical points of system (1). As we will see in Section 3, this splitting will be exponentially small with respect to the perturbative parameter (δ in our case).

As any expert in this field knows, to give a rigorous proof of an asymptotic formula for this exponentially small splitting it is necessary to obtain good approximations of the stable and unstable manifolds, not only in the real domain, where they are quite well approximated by the heteroclinic orbit of the unperturbed system, but also in some suitable complex domains.

The perturbation terms $\delta^{-1}f$, $\delta^{-1}g$, $\delta^{-1}h$, which in the real domain of the variables are of order δ^2, become very big (of order $1/\delta$) when one works in these complex domains where the size of the variables becomes $O(1/\delta)$. Hence the system is no longer perturbative and moreover, the manifolds are not close to the unperturbed heterolinic orbit. In this case, matching techniques in the complex plane between different approximations of the manifolds are required to achieve the result. Moreover, contrarily to what happens in the perturbative case $p > -2$, see Theorem 1.6, the final asymptotic formula will depend on the full jet of the functions f, g, h. Examples of rigorous studies of exponentially small phenomena in the singular case can be found in [Gel97a, Tre97, OSS03, RMT97].

To date, only one rigorous proof of the splitting of the one dimensional heteroclinic connection has been presented, for a special family called the Michelson system [Mic86]:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= c^2 - \frac{x^2}{2} - y \ . \end{aligned}$$

For this system, an unfolding of (3) which has been widely studied (see [Mic86, JTM92, KT76, RMT97]), there is a rigorous proof of an asymptotic formula of the heteroclinic splitting given in [RMT97]. The proof, which falls in the context of singular perturbation theory, draws heavily on the fact that Michelson system comes from a third order differential equation. For this reason it is not clear that the methods used can be adapted to generic analytic unfoldings X_μ of the central singularity.

1.5 Notation and preliminary results

Throughout this paper $|\cdot|$ denotes the maximum norm in $\mathbb{C}^n$:

$$|z| = \max_{i=1,\cdots,n} |z_i|, \qquad z = (z_1, \cdots, z_n) \in \mathbb{C}^n.$$

and $B(r_0) \subset \mathbb{C}$ the open ball of radius r_0. We will also use the notation $B^3(r_0) = B(r_0) \times B(r_0) \times B(r_0)$.

As usual, we will denote by $\pi^i : \mathbb{C}^3 \to \mathbb{C}$ the projection over the i-component for $i = 1, 2, 3$. Moreover, we will also write $\pi^{i,j} : \mathbb{C}^3 \to \mathbb{C}^2$ to indicate the projection on the i, j components with $i, j \in \{1, 2, 3\}$.

It will be more convenient for our purposes to write system (1) as:

$$\begin{aligned}
\frac{dx}{dt} &= -xz - \big(\frac{\alpha}{\delta} + cz\big)y + \delta^p f(\delta x, \delta y, \delta z, \delta) \\
\frac{dy}{dt} &= \big(\frac{\alpha}{\delta} + cz\big)x - yz + \delta^p g(\delta x, \delta y, \delta z, \delta) \\
\frac{dz}{dt} &= -1 + b(x^2+y^2) + z^2 + \delta^p h(\delta x, \delta y, \delta z, \delta),
\end{aligned} \tag{5}$$

where $t = \tau\delta$, $p > -2$, f, g and h are real analytic functions in $B(r_0)^3 \times B(\delta_0)$ and moreover, $f, g, h = O(|(x, y, z, \delta)|^3)$. We will assume in the sequel that r_0 is big enough but independent of δ.

Even though system (5) has no meaning for $\delta = 0$, when we speak about the unperturbed system we will refer to system (5) with $f = g = h = 0$.

The following lemma describes the more relevant geometric facts of the unperturbed system.

Lemma 1.2 *The unperturbed system (system (5) with $f = g = h = 0$), verifies, for any value of $\delta > 0$:*

1. *It possesses only two hyperbolic fixed points $S^0_\pm = (0, 0, \pm 1)$ which are of saddle-focus type with eigenvalues $\mp 1 + |\frac{\alpha}{\delta} + c|\mathrm{i}$, $\mp 1 - |\frac{\alpha}{\delta} + c|\mathrm{i}$ and ± 2.*

2. *The one-dimensional unstable manifold of S^0_+ and the one-dimensional stable manifold of S^0_- coincide along the heteroclinic connection $\{(x, y) = (0, 0); -1 < z < 1\}$. This heteroclinic orbit can be parameterized by*

   $$\sigma_0(t) = (0, 0, -\tanh t)$$

   if we require $\sigma_0(0) = (0, 0, 0)$.

3. *The polynomial $H(x, y, z) = \frac{x^2+y^2}{2}(z^2 + \frac{b}{2}(x^2 + y^2) - 1)$ is a first integral of the system.*

4. *If $b > 0$, the two-dimensional stable manifold of S^0_+ and the two-dimensional unstable manifold of S^0_- coincide, giving rise to a two-dimensional heteroclinic surface. Moreover, this heteroclinic surface is given by $z^2 + \frac{b}{2}(x^2 + y^2) - 1 = 0$.*

Lemma 1.2 describes system (5) as a perturbation of an integrable system. The following result ensures that system (5) has two fixed points of saddle focus type even when $f, g, h \neq 0$.

Lemma 1.3 *If $\delta > 0$ is small enough, system (5) has two fixed points $S_\pm(\delta)$ of saddle focus type such that $S_+(\delta)$ has a one-dimensional unstable manifold and $S_-(\delta)$ has a stable one. We call them $W^{\rm u,s}$ respectively.*

Moreover there are no other fixed points of (5) in the closed ball $B(\delta^{-1/3})$.

Proof. It is straightforward since we only need to consider the function

$$P(x,y,z,\delta) = \begin{pmatrix} -xz\delta - (\alpha+\delta cz)y + \delta^{p+1} f(\delta x,\delta y,\delta z,\delta) \\ -yz\delta + (\alpha+\delta cz)x + \delta^{p+1} g(\delta x,\delta y,\delta z,\delta) \\ -1 + b(x^2+y^2) + z^2 + \delta^p h(\delta x,\delta y,\delta z,\delta) \end{pmatrix}.$$

It is clear that (x_0,y_0,z_0) is a fixed point of system (5) if and only if there exists $\delta > 0$ such that $P(x_0,y_0,z_0,\delta) = 0$. Hence, since $P(0,0,\pm1,0) = 0$, applying the implicit function theorem we have that there exist neighborhoods of $(0,0,\pm1)$, $U_\pm$, $\delta_0 > 0$ and C^1 functions $S_\pm : B(\delta_0) \to U_\pm$ such that $S_\pm(0) = (0,0,\pm1)$ and $P(x,y,z,\delta) = 0$ if and only if $(x,y,z) = S_\pm(\delta)$. Moreover, one can easily check that $S_\pm(\delta)$ are of saddle-focus type with eigenvalues $\mp1 + |\frac{\alpha}{\delta} + c|{\rm i} + O(\delta)$, $\mp1 - |\frac{\alpha}{\delta} + c|{\rm i} + O(\delta)$ and $\pm2 + O(\delta)$.

To check the second part of the statement, let us assume that there exists a fixed point $(x,y,z) \in B(\delta^{-1/3})$ of (5). Then, $|f(\delta x,\delta y,\delta z,\delta)|, |g(\delta x,\delta y,\delta z,\delta)| \leq K\delta^2$ for some constant $K > 0$. Using the triangular inequality,

$$|\alpha x| \leq K\delta^{p+3} + (1+|c|)\delta^{1/3}, \quad |\alpha y| \leq K\delta^{p+3} + (1+|c|)\delta^{1/3}. \tag{6}$$

In addition, taking into account (6), we deduce $|-1+z^2| \leq K\delta^{p+2} + 2|b|C\delta^{2/3}$ provided that $|h(\delta x,\delta y,\delta z,\delta)| \leq K\delta^2$ and $p+3 > 1$. Henceforth, we have that $|x|, |y|, |-1+z^2| \leq K_0\delta^\nu$ with $\nu = \min\{p+2, 1/3\}$ and this implies that, taking δ small enough, $(x,y,z) \in U_\pm$ and therefore by the uniqueness of $S_\pm$, $(x,y,z) = S_\pm(\delta)$. ∎

1.6 Main result

By Lemma 1.3, system (5) has two critical points $S_\pm(\delta)$ having one-dimensional stable and unstable manifolds respectively. We are interested in measuring the distance between the stable manifold $W^{\rm s}$ and the unstable one $W^{\rm u}$ at the plane $z = 0$. We observe that, since system (5) is autonomous we can fix the origin of time at $t = 0$.

Theorem 1.4 *Let us consider system (5) with $p > -2$ and f, g and h real analytic functions in $B(r_0)^3 \times B(\delta_0)$. Moreover, $f,g,h = O(|(x,y,z,\delta)|^3)$. Then, if $\delta > 0$ is small enough we have:*

1. *The one-dimensional stable manifold of $S_-(\delta)$ and the one-dimensional unstable manifold of $S_+(\delta)$ can be parameterized by $\sigma^{\mathrm{s}}(t,\delta)$, $\sigma^{\mathrm{u}}(t,\delta)$ which are solutions of system (5) such that*

$$\lim_{t\to\infty}\sigma^{\mathrm{s}}(t,\delta)=S_-(\delta), \qquad \lim_{t\to-\infty}\sigma^{\mathrm{u}}(t,\delta)=S_+(\delta)$$
$$\pi^3\sigma^{\mathrm{u}}(0,\delta) = \pi^3\sigma^{\mathrm{s}}(0,\delta)=0.$$

2. *Let $m(u) = u^{1+\mathrm{i}\,c}(f(0,0,u,0)+\mathrm{i}\,g(0,0,u,0)) = \sum_{n\geq 3} m_n u^{n+1+\mathrm{i}\,c}$ and $\hat{m}(\zeta) = \sum_{n\geq 3} m_n \frac{\zeta^{n+\mathrm{i}\,c}}{\Gamma(n+1+\mathrm{i}\,c)}$ be its Borel transform.*

   The difference between the stable and unstable manifolds, $\Delta\sigma(t,\delta)=\sigma^{\mathrm{u}}(t,\delta)-\sigma^{\mathrm{s}}(t,\delta)$, at $t=0$ is given asymptotically by:

$$\Delta\sigma(0,\delta)=\Delta\sigma_1(0,\delta)+O(\delta^{p+2}|\log\delta|)\,\mathrm{e}^{-|\alpha|\pi/(2\delta)}$$

   with $\pi^3(\Delta\sigma_1(0,\delta))=0$ and

$$\begin{aligned}\pi^1(\Delta\sigma_1(0,\delta))+\mathrm{i}\,\pi^2(\Delta\sigma_1(0,\delta)) =& 2\pi\,\mathrm{e}^{c\pi/2}\,\hat{m}(\mathrm{i}\,\alpha)\,\mathrm{e}^{-\mathrm{i}\,c\log\delta}\,\delta^p\,\mathrm{e}^{-|\alpha|\pi/(2\delta)}\\ &+O(\delta^{p+1})\,\mathrm{e}^{-|\alpha|\pi/(2\delta)}.\end{aligned}$$

Remark 1.5 *Our context could be non conservative. That is, even if system (5) comes from a conservative context, we do not need this fact in our proof. We do not ask f, g and h to satisfy any additional condition to those stated previously.*

Remark 1.6 *Even though the distance between the stable and unstable manifold depends on f,g,h and all the parameters of the system, observe that the dominant term for the difference between the stable and unstable manifold depends neither on the function h nor on the parameter b.*

It is worth mentioning that given f and g such that the Borel transform $\hat{m}$ does not vanish at the point $\mathrm{i}\,\alpha$, we can state that the heteroclinic connection of system 5 is destroyed.

The proof of this Theorem is decomposed in two steps which are developed in Section 2 and Section 3. Section 2 is devoted to proving the existence of analytic parameterizations of $\sigma^{\mathrm{s,u}}$ in a suitable complex domain. It is worth mentioning that the parameter we will use is just the time t. After that, in Section 3, we will compute the difference between them.

2 A parameterization for the stable and unstable manifolds

The purpose of this section is to provide analytic parameterizations for the stable and unstable manifolds $\sigma^{\mathrm{s,u}}$, associated to the fixed points $S_-(\delta)$ and $S_+(\delta)$ respectively, of system (5). These parameterizations are of the form

$$\sigma^{\mathrm{s,u}}(t,\delta)=\sigma_0(t)+\tilde{\sigma}^{\mathrm{s,u}}(t,\delta) \tag{7}$$

and are defined in an appropriate complex domain which we will describe below. The parameter t we will use is just the time.

In order to shorten the notation, we introduce $z_0(t)=-\tanh t$ and note that the heteroclinic connection is $\sigma_0=(0,0,z_0)$.

In Subsection 2.1 we find the differential equation that $\tilde{\sigma}^{\mathrm{s,u}}$ have to satisfy and we perform a complex change of variables in order to put the linear part of this differential equation in diagonal form. We will use this new system throughout the remaining part of the paper. After that, in Subsection 2.2, we introduce some functional spaces with which we will work in this section. Finally, in Subsection 2.3, we prove the existence and some useful properties of the parameterizations of the stable and unstable manifolds $\sigma^{\mathrm{s,u}}$ by using a suitable version of the fixed point theorem.

2.1 A preliminary change of coordinates

In this subsection, we will write system (5) in a more appropriate way.

Since we are looking for parameterizations of the stable and unstable manifolds of the form (7), as usual, we perform the time dependent change of coordinates given by $(u,v,w,t)=C_0(x,y,z,t)=(x,y,z-z_0(t),t)$. For simplicity, we also perform the change of variables given by $(\xi,\bar{\xi},\eta)=C_1(u,v,w)=(u+\mathrm{i}\,v,u-\mathrm{i}\,v,w)$ in order to put the linear part of the new system in diagonal form. After these changes, system (5) becomes:

$$\begin{aligned}
\dot{\xi} &= -\Big(\frac{\alpha}{\delta}+cz_0(t)\Big)\mathrm{i}\,\xi-\xi z_0(t)-(1+\mathrm{i}\,c)\eta\xi+\delta^p F_1(\delta\xi,\delta\bar{\xi},\delta(z_0(t)+\eta),\delta)\\
\dot{\bar{\xi}} &= \Big(\frac{\alpha}{\delta}+cz_0(t)\Big)\mathrm{i}\,\bar{\xi}-\bar{\xi} z_0(t)-(1-\mathrm{i}\,c)\eta\bar{\xi}+\delta^p F_2(\delta\xi,\delta\bar{\xi},\delta(z_0(t)+\eta),\delta)\\
\dot{\eta} &= 2z_0(t)\eta+b\xi\bar{\xi}+\eta^2+\delta^p H(\delta\xi,\delta\bar{\xi},\delta(z_0(t)+\eta),\delta)
\end{aligned} \tag{8}$$

where $\cdot=\frac{d}{dt}$ and $F=(F_1,F_2)$ with

$$\begin{aligned}
F_1(\delta\xi,\delta\bar{\xi},\delta\eta,\delta) &= (f+\mathrm{i}\,g)(\delta C_1^{-1}(\xi,\bar{\xi},\eta),\delta),\\
F_2(\delta\xi,\delta\bar{\xi},\delta\eta,\delta) &= (f-\mathrm{i}\,g)(\delta C_1^{-1}(\xi,\bar{\xi},\eta),\delta),\\
H(\delta\xi,\delta\bar{\xi},\delta\eta,\delta) &= h(\delta C_1^{-1}(\xi,\bar{\xi},\eta),\delta).
\end{aligned} \tag{9}$$

We write $\zeta=(\xi,\bar\xi,\eta)$ and we define

$$\mathcal{R}=(\mathcal{M},\mathcal{N}), \tag{10}$$

$$\mathcal{M}(\zeta)=\begin{pmatrix}\xi\eta(-1-\mathrm{i}\,c)\\ \bar\xi\eta(-1+\mathrm{i}\,c)\end{pmatrix}+\delta^pF(\delta\xi,\delta\bar\xi,\delta(z_0(t)+\eta),\delta) \tag{11}$$

$$\mathcal{N}(\zeta)=b\xi\bar\xi+\eta^2+\delta^pH(\delta\xi,\delta\bar\xi,\delta(z_0(t)+\eta),\delta) \tag{12}$$

and the matrix

$$A(t)=\begin{pmatrix}-\left(\frac{\alpha}{\delta}+cz_0(t)\right)\mathrm{i}-z_0(t) & 0 & 0\\ 0 & \left(\frac{\alpha}{\delta}+cz_0(t)\right)\mathrm{i}-z_0(t) & 0\\ 0 & 0 & 2z_0(t)\end{pmatrix}.$$

It is clear that, with this notation, system (8) can be simply written as

$$\dot\zeta=A(t)\zeta+\mathcal{R}(\zeta). \tag{13}$$

Lemma 2.1 *The fundamental matrix, Φ, of system $\dot\zeta=A(t)\zeta$ satisfying that $\Phi(0)=Id$ is given by*

$$\Phi(t)=\begin{pmatrix}\cosh t\,\mathrm{e}^{-\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh t)} & 0 & 0\\ 0 & \cosh t\,\mathrm{e}^{\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cosh t)} & 0\\ 0 & 0 & \cosh^{-2}t\end{pmatrix}.$$

2.2 Domains and functional spaces

This subsection is mainly devoted to introducing the Banach spaces we will use throughout this section.

Let us recall that $\sigma_0(t)$, the heteroclinic orbit of the unperturbed system, is given by $\sigma_0(t)=(0,0,z_0(t))$. The function $z_0(t)=-\tanh t$ is a real analytic function, which has poles in $t=\pm\frac{\pi}{2}\,\mathrm{i}+2k\pi\,\mathrm{i}$. In order to achieve the desired results, we will need to work in a suitable complex domain which reaches a small neighborhood of the first singularities $\pm\,\mathrm{i}\,\pi/2$. For any $\rho>0$, we define the complex domain:

$$\begin{aligned}D^{\mathrm{u}}_\rho=&\{t\in\mathbb{C}:|\operatorname{Im}t|\le a,\ \operatorname{Re}t\le-\delta\}\cup\{t\in\mathbb{C}:|\operatorname{Im}t|\le(a-\delta)\big(1-\frac{\operatorname{Re}t}{\rho}\big),\ \operatorname{Re}t\ge0\}\\ &\cup\{t\in\mathbb{C}:-\delta\le\operatorname{Re}t\le0\ \ |\operatorname{Im}t|\le a-\sqrt{\delta^2-(\operatorname{Re}t)^2}\}\end{aligned} \tag{14}$$

where $a=\frac{\pi}{2}$. We take $T>2\log 2$ and we decompose $D^{\mathrm{u}}_\rho=D^{\mathrm{u}}_1\cup D^{\mathrm{u}}_2\cup D^{\mathrm{u}}_3$ where D^{u}_i for $i=1,2,3$, are the sets defined by the figure:

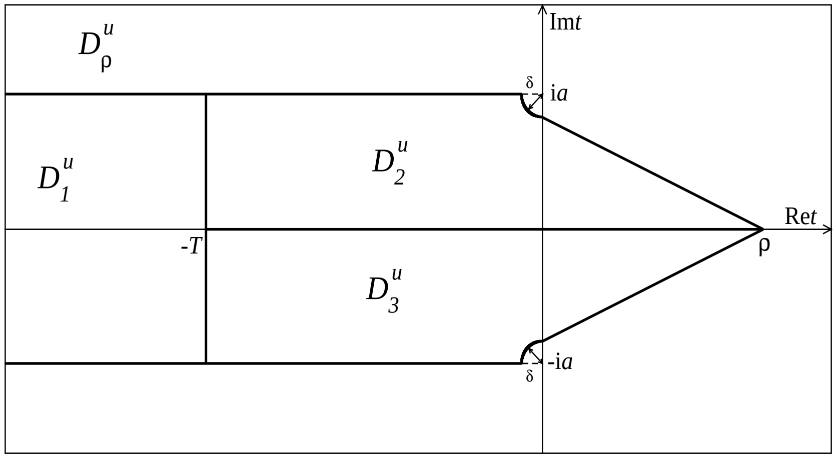

Figure 1

That is,

$$\begin{aligned}
D_1^{\mathrm{u}} &= \{t \in D_\rho^{\mathrm{u}} : \operatorname{Re} t \leq -T\}, \\
D_2^{\mathrm{u}} &= \{t \in D_\rho^{\mathrm{u}} : \operatorname{Re} t \geq -T \text{ and } \operatorname{Im} t \geq 0\}, \\
D_3^{\mathrm{u}} &= \{t \in D_\rho^{\mathrm{u}} : \operatorname{Re} t \geq -T \text{ and } \operatorname{Im} t \leq 0\}.
\end{aligned} \tag{15}$$

Analogously we denote

$$D_\rho^{\mathrm{s}} = \{t \in \mathbb{C} : -t \in D_\rho^{\mathrm{u}}\}, \qquad D_i^{\mathrm{s}} = \{t \in \mathbb{C} : -t \in D_i^{\mathrm{u}}\}, \quad \text{for } i = 1, 2, 3.$$

All the functions we will deal with will depend on δ as a parameter. If there is no danger of confusion we will skip this dependence in our notation, and we will take $|\delta| < \delta_0$.

For any $\nu \geq 0$, we introduce $\mathcal{X}_\nu^{\mathrm{u}}$, the space of analytic functions such that f belongs to $\mathcal{X}_\nu^{\mathrm{u}}$ if and only if:

1. $f : D_\rho^{\mathrm{u}} \to \mathbb{C}$ is continuous and analytic in $\mathring{D}_\rho^{\mathrm{u}}$.
2. f satisfies that

$$\sup_{t \in D_1^{\mathrm{u}}} |f(t)| + \sup_{t \in D_2^{\mathrm{u}}} |(t - \mathrm{i}\, a)^\nu f(t)| + \sup_{t \in D_3^{\mathrm{u}}} |(t + \mathrm{i}\, a)^\nu f(t)| < +\infty.$$

We endow $\mathcal{X}_\nu^{\mathrm{u}}$ with the norm:

$$\|f\|_\nu = \sup_{t \in D_1^{\mathrm{u}}} |f(t)| + \sup_{t \in D_2^{\mathrm{u}}} |(t - \mathrm{i}\, a)^\nu f(t)| + \sup_{t \in D_3^{\mathrm{u}}} |(t + \mathrm{i}\, a)^\nu f(t)|. \tag{16}$$

With this norm, $\mathcal{X}_\nu^{\mathrm{u}}$ becomes a Banach space. We also introduce

$$\mathcal{X}_\nu^{\mathrm{s}} = \{f : D_\rho^{\mathrm{s}} \to \mathbb{C} \text{ such that } g_f \text{ defined by } g_f(t) := f(-t) \text{ belongs to } \mathcal{X}_\nu^{\mathrm{u}}\}.$$

Trivially, $\mathcal{X}_\nu^{\mathrm{s}}$ is a Banach space with the norm $\|f\|_\nu^{\mathrm{s}} = \|g_f\|_\nu$.

From now on, if there is no danger of confusion we will also denote $\|\cdot\|_\nu^{\mathrm{s}}$ simply by $\|\cdot\|_\nu$.

Remark 2.2 *If $\nu_1 \leq \nu_2$, then $\mathcal{X}_{\nu_1}^{\mathrm{s,u}} \subset \mathcal{X}_{\nu_2}^{\mathrm{s,u}}$ and moreover, there exists a constant $K = K(T, a)$ such that for all $f \in \mathcal{X}_{\nu_1}^{\mathrm{s,u}}$,*

$$\|f\|_{\nu_2} \leq K\|f\|_{\nu_1}.$$

Proof. *We fix $f \in \mathcal{X}_{\nu_1}^{\mathrm{u}}$ (the case $f \in \mathcal{X}_{\nu_1}^{\mathrm{s}}$ is analogous). Let $t \in D_2^{\mathrm{u}}$. Then*

$$|t - \mathrm{i}\, a|^{\nu_2}|f(t)| \leq |t - \mathrm{i}\, a|^{\nu_2-\nu_1}\|f\|_{\nu_1} \leq ((\max\{\rho, T\})^2 + a^2)^{(\nu_2-\nu_1)/2}\|f\|_{\nu_1}.$$

We denote $C = ((\max\{\rho, T\})^2 + a^2)^{(\nu_2-\nu_1)/2}$. In the same way one can check that, if $t \in D_3^{\mathrm{u}}$, $|t + \mathrm{i}\, a|^{\nu_2}|f(t)| \leq C\|f\|_{\nu_1}$ and hence $f \in \mathcal{X}_{\nu_2}^{\mathrm{u}}$ and

$$\|f\|_{\nu_2} \leq (1 + 2C)\|f\|_{\nu_1}.$$

■

For technical reasons, we endow the product space $\mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$ with the norm

$$\|f\| = \|f_1\|_3 + \|f_2\|_3 + \delta|\log\delta|^{-1}\|f_3\|_2, \qquad f = (f_1, f_2, f_3) \in \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}. \quad (17)$$

We will also use the norm

$$\|f\|_{\times,\nu} = \|f_1\|_\nu + \|f_2\|_\nu, \qquad f = (f_1, f_2) \in \mathcal{X}_\nu^{\mathrm{s,u}} \times \mathcal{X}_\nu^{\mathrm{s,u}}. \quad (18)$$

2.3 Analytic parameterization of the invariant manifolds

In this subsection we prove that equation (13):

$$\dot{\zeta} = A(t)\zeta + \mathcal{R}(\zeta)$$

has solutions, $\varphi^{\mathrm{s,u}}$, defined in $D_\rho^{\mathrm{s,u}}$ satisfying that

$$\pi^3(\varphi^{\mathrm{s,u}}(0)) = 0 \quad \text{and} \quad \sup_{t \in D_\rho^{\mathrm{s,u}}} |\varphi^{\mathrm{s,u}}(t)| \leq M. \quad (19)$$

If $\varphi^{\mathrm{s,u}}$ satisfy these conditions, the parameterizations $\sigma^{\mathrm{s,u}}$ of the stable and unstable manifolds we are looking for are given by $\sigma^{\mathrm{s,u}} = \sigma_0 + C_1^{-1}(\varphi^{\mathrm{s,u}})$. Indeed, Lemma 1.3 says that system (5) has stable and unstable manifolds associated to the fixed points $S_+(\delta)$ and $S_-(\delta)$. Moreover, according to hyperbolic theory, the only bounded solutions in $B(\delta^{-1/3})$.

In order to prove the existence and properties of the solutions $\varphi^{\mathrm{s,u}}$ of system (13), our strategy will be to use a suitable version of the fixed point theorem in the Banach spaces $\mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$. To this end our first step will be to find a fixed point equation for $\varphi^{\mathrm{u,s}}$.

Let us consider the following linear operators, acting on functions $\phi : D_\rho^{\mathrm{s,u}} \to \mathbb{C}$:

$$\mathcal{L}_{\alpha,c}^{\mathrm{s,u}}(\phi)(t) = \cosh t \int_{\pm\infty}^{0} \frac{1}{\cosh(t+r)} \mathrm{e}^{\mathrm{i}\,\alpha r/\delta}\, \mathrm{e}^{\mathrm{i}\,c(\log\cosh t - \log\cosh(t+r))}\, \phi(t+r)\, dr \tag{20}$$

$$\mathcal{T}(\phi)(t) = \frac{1}{\cosh^2 t} \int_0^t \cosh^2 r \phi(r)\, dr. \tag{21}$$

Where in (20) + stands for s and − stands for u. We also define the linear operator $\mathcal{S}^{\mathrm{s,u}}$ given by

$$\mathcal{S}^{\mathrm{s,u}}(\psi) = (\mathcal{L}_{\alpha,c}^{\mathrm{s,u}}(\pi^1\psi), \mathcal{L}_{-\alpha,-c}^{\mathrm{s,u}}(\pi^2\psi)), \qquad \psi : D_\rho^{\mathrm{s,u}} \to \mathbb{C}^2 \tag{22}$$

and finally the operator $\mathcal{L}^{\mathrm{s,u}}$ by

$$\mathcal{L}^{\mathrm{s,u}}(\chi) = (\mathcal{S}^{\mathrm{s,u}} \circ \pi^{1,2}(\chi), \mathcal{T} \circ \pi^3(\chi)), \qquad \chi : D_\rho^{\mathrm{s,u}} \to \mathbb{C}^3. \tag{23}$$

Lemma 2.3 *With the above notation, if a bounded and continuous function* $\varphi^{\mathrm{s,u}} : D_\rho^{\mathrm{s,u}} \to \mathbb{C}^3$ *satisfies the fixed point equation*

$$\varphi^{\mathrm{s,u}} = \mathcal{L}^{\mathrm{s,u}} \circ \mathcal{R}(\varphi^{\mathrm{s,u}}), \tag{24}$$

then it is a solution of (13) satisfying (19).

Proof. The proof is straightforward; we only have to differentiate with respect to t equation (24). ■

Remark 2.4 *The choice of the linear operators* $\mathcal{L}^{\mathrm{s,u}}$ *is, in some sense, natural. Indeed, by Lemma 2.1, any solution* φ *of equation (13) must satisfy the integral equation*

$$\varphi(t) = \Phi(t)\Big[\varphi(0) + \int_0^t \Phi^{-1}(s)\mathcal{R}(\varphi(s))\, ds\Big].$$

(The fundamental matrix Φ was defined in Lemma 2.1.) Since we are looking for solutions satisfying the properties (19), $\varphi^{\mathrm{s,u}}$ must satisfy the integral equation:

$$\varphi^{\mathrm{s,u}}(t) = \begin{pmatrix} \cosh t \displaystyle\int_{\pm\infty}^{t} \frac{1}{\cosh r}\, \mathrm{e}^{-\mathrm{i}\,\alpha(t-r)/\delta}\, \mathrm{e}^{\mathrm{i}\,c(\log(\frac{\cosh t}{\cosh r}))}\, \pi^1(\mathcal{M}(\varphi^{\mathrm{s,u}}(r)))\, dr \\ \cosh t \displaystyle\int_{\pm\infty}^{t} \frac{1}{\cosh r}\, \mathrm{e}^{\mathrm{i}\,\alpha(t-r)/\delta}\, \mathrm{e}^{-\mathrm{i}\,c(\log(\frac{\cosh t}{\cosh r}))}\, \pi^2(\mathcal{M}(\varphi^{\mathrm{s,u}}(r)))\, dr \\ \displaystyle\frac{1}{\cosh^2 t}\int_{0}^{t} \cosh^2 r\, \mathcal{N}(\varphi^{\mathrm{s,u}}(r))\, dr \end{pmatrix}, \qquad (25)$$

taking the + sign for s *and − for* u. *We observe that the third component of (25) is identical to the third component of equation (24). Finally, one can easily check that, changing the integration path to $\gamma(t) = t + s$, $s \in (\pm\infty, 0]$ and using Cauchy's theorem, the first and second components of (25) are actually $\mathcal{S}^{\mathrm{s,u}} \circ \mathcal{M}(\varphi^{\mathrm{s,u}})$.*

The remaining part of this subsection is devoted to proving the following proposition:

Proposition 2.5 *If $p > -2$ and δ is small enough, system (13) has two solutions $\varphi^{\mathrm{s,u}}$ satisfying that $\varphi^{\mathrm{s,u}} = \varphi_1^{\mathrm{s,u}} + \varphi_2^{\mathrm{s,u}}$ with $\varphi_1^{\mathrm{s,u}}$ and $\varphi_2^{\mathrm{s,u}}$ having the following properties:*

1. $\varphi_1^{\mathrm{s,u}} = \mathcal{L}^{\mathrm{s,u}} \circ \mathcal{R}(0) \in \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$ *and* $\|\varphi_1^{\mathrm{s,u}}\| \leq K\delta^{p+4}$.
2. $\varphi_2^{\mathrm{s,u}} \in \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$ *and* $\|\varphi_2^{\mathrm{s,u}}\| \leq K\delta^{p+2}|\log\delta|\|\varphi_1^{\mathrm{s,u}}\|$

for some constant K independent of δ.

Remark 2.6 *We note that, by Lemma 2.3 we are allowed to use the fixed point equation (24) to prove Proposition 2.5.*

The proof of Proposition 2.5 is broken down into three steps which are developed in the subsections below. In Subsection 2.3.1, we prove that the linear operators $\mathcal{S}^{\mathrm{s,u}}$ and $\mathcal{T}$ are continuous in suitable Banach spaces. After that, in Subsection 2.3.2, we will give the properties of $\varphi_1^{\mathrm{s,u}}$ enunciated in Proposition 2.5. Finally, we complete the proof by using an appropriate version of the fixed point theorem. This last step is done in Subsection 2.3.3.

From now on we only deal with the unstable manifold φ^{u}. For this reason we will skip the -u- sign of our notation, writing, for instance, φ, $\mathcal{S}$, D_ρ and D_i instead of φ^{u}, $\mathcal{S}^{\mathrm{u}}$, D_ρ^{u} and D_i^{u} respectively.

In the remaining part of this section, we will make particular use of the geometry of the domain D_ρ defined in (14) and its decomposition $D_\rho = D_1 \cup D_2 \cup D_3$ given in (15). See also Figure 1.

2.3.1 The linear operators $\mathcal{S}$ and $\mathcal{T}$

Here we will study the linear operators $\mathcal{S}$ and $\mathcal{T}$. First of all we enunciate a technical lemma, see ([DS97]).

Lemma 2.7 *The following bounds hold:*

1. *Let $\nu \geq 1$. There exists a constant $C = C(\rho, T)$ such that, if $s \in D_2$ and $s_0 \in \mathbb{R}$,*

$$\left| \int_0^{s_0} |s + r - \mathrm{i}\, a|^{-\nu}\, dr \right| \leq C \begin{cases} \sup_{r \in [0,s_0]} |s + r - \mathrm{i}\, a|^{-\nu+1} & \text{if} \quad \nu > 1 \\ \sup_{r \in [0,s_0]} \log |s + r - \mathrm{i}\, a| & \text{if} \quad \nu = 1, \end{cases}$$

   moreover

$$\left| \int_0^{s} |r - \mathrm{i}\, a|^{-1}\, dr \right| \leq C |\log \delta|.$$

2. *There exist constants $K_1, K_2 > 0$ depending on T, ρ such that*

$$\begin{array}{ll} K_1 |t - \mathrm{i}\, a| \leq |\cosh t| \leq K_2 |t - \mathrm{i}\, a|, & t \in D_2 \\ K_1 |t + \mathrm{i}\, a| \leq |\cosh t| \leq K_2 |t + \mathrm{i}\, a|, & t \in D_3 \end{array}$$

3. *Let $T > 2 \log 2$. If $t \in D_1$ and $r < 0$, then*

$$|\cosh(t + r)| \geq |\sinh(\mathrm{Re}\, t + r)| \geq \frac{\mathrm{e}^{|\mathrm{Re}\, t + r|}}{4}.$$

   Moreover, we also have that for all $t \in \mathbb{C}$, $|\cosh t| \leq \cosh(\mathrm{Re}\, t) \leq \mathrm{e}^{|\mathrm{Re}\, t|}$.

The next lemma studies the linear operator $\mathcal{S}$.

Lemma 2.8 *For any $\nu > 1$ and T big enough, the operator $\mathcal{S} : \mathcal{X}_\nu \times \mathcal{X}_\nu \to \mathcal{X}_{\nu-1} \times \mathcal{X}_{\nu-1}$ given in (22) is well defined and there exists a constant K independent of δ such that*

$$\|\mathcal{S}(\psi)\|_{\times,\nu-1} \leq K \|\psi\|_{\times,\nu}, \qquad \text{for any } \psi \in \mathcal{X}_\nu \times \mathcal{X}_\nu.$$

Proof. First we observe that if $t \in D_\rho$, then $|\mathrm{Im}(\log(\cosh t))| \leq \pi/2$. Hence, since $t + r \in D_\rho$ if $t \in D_\rho$ and $r \leq 0$, we have that

$$|\mathrm{e}^{\mathrm{i}\, c(\log\cosh t - \log\cosh(t+r))}| \leq \mathrm{e}^{|c|\pi}. \tag{26}$$

Let $\psi \in \mathcal{X}_\nu \times \mathcal{X}_\nu$. By definition of $\mathcal{S}$ it is enough to check that $\|\mathcal{L}_{\alpha,c}(\pi^1\psi)\|_{\nu-1} \leq K\|\pi^1\psi\|_\nu$ and $\|\mathcal{L}_{-\alpha,-c}(\pi^2\psi)\|_{\nu-1} \leq K\|\pi^2\psi\|_\nu$ for some constant K.

We deal only with $\pi^1\psi$; the other case is analogous. We denote $\phi = \pi^1\psi \in \mathcal{X}_\nu$. Using inequality (26) and definition (20) of $\mathcal{L}_{\alpha,c}(\phi)$, we obtain that, for any $t \in D_\rho$,

$$|\mathcal{L}_{\alpha,c}(\phi)(t)| \leq \mathrm{e}^{|c|\pi} |\cosh t| \int_{-\infty}^0 \frac{|\phi(t+r)|}{|\cosh(t+r)|}\, dr := \mathrm{e}^{|c|\pi} I(t). \tag{27}$$

Now we will bound $I(t)$. For that purpose we will distinguish three cases according to the D_i where t belongs to (see Figure 1).

If $t \in D_1$, then $t + r \in D_1$ for all $r < 0$ and hence $|\phi(t+r)| \leq \|\phi\|_\nu$. Using 3. of Lemma 2.7 to bound $I(t)$ in (27), we get that

$$I(t) \leq 4\|\phi\|_\nu\, \mathrm{e}^{|\operatorname{Re} t|} \int_{-\infty}^0 \mathrm{e}^{-|\operatorname{Re} t + r|}\, dr = 4\|\phi\|_\nu. \tag{28}$$

Now we deal with the case when $t \in D_2$. Then $t + r \in D_2$ if $r \in [-T - \operatorname{Re} t, 0]$ and $t + r \in D_1$ if $r < -T - \operatorname{Re} t$. Hence, in order to bound $I(t)$, we have to decompose it in two parts, that is

$$\begin{aligned} I(t) &= I_1(t) + I_2(t) \\ &:= |\cosh t| \int_{-\infty}^{-T-\operatorname{Re} t} \frac{|\phi(t+r)|}{|\cosh(t+r)|}\, dr + |\cosh t| \int_{-T-\operatorname{Re} t}^0 \frac{|\phi(t+r)|}{|\cosh(t+r)|}\, dr. \end{aligned}$$

First we deal with $I_1(t)$. It is clear that

$$\begin{aligned} I_1(t) &= \frac{|\cosh t|}{|\cosh(-T + \mathrm{i}\operatorname{Im} t)|} |\cosh(-T + \mathrm{i}\operatorname{Im} t)| \int_{-\infty}^0 \frac{|\phi(-T + \mathrm{i}\operatorname{Im} t + r)|}{|\cosh(-T + \mathrm{i}\operatorname{Im} t + r)|}\, dr \\ &= \frac{|\cosh t|}{|\cosh(-T + \mathrm{i}\operatorname{Im} t)|} I(-T + \mathrm{i}\operatorname{Im} t). \end{aligned}$$

Hence, since $-T + \mathrm{i}\operatorname{Im} t \in D_1$ we can use (28) to bound $I(-T + \mathrm{i}\operatorname{Im} t)$. Moreover, again using 3. from Lemma 2.7 we have that

$$I_1(t) \leq 16\, \mathrm{e}^{|\operatorname{Re} t| - T} \|\phi\|_\nu \leq 16\, \mathrm{e}^{\max\{0, \rho - T\}} \|\phi\|_\nu \tag{29}$$

provided $|\operatorname{Re} t| \leq \max\{T, \rho\}$.

Next we deal with $I_2(t)$. As we pointed out before, since $t \in D_2$ and $r \in [-T - \operatorname{Re} t, 0]$, $t + r \in D_2$ and hence $|\phi(t+r)| \leq \|\phi\|_\nu |t + r - \mathrm{i}\, a|^{-\nu}$. Thus, using 1. and 2. of Lemma 2.7 to bound $I_2(t)$ we have that

$$\begin{aligned} I_2(t) &\leq K_2 K_1^{-1} \|\phi\|_\nu |t - \mathrm{i}\, a| \int_{-T-\operatorname{Re} t}^0 |t + r - \mathrm{i}\, a|^{-\nu-1}\, dr \\ &\leq C K_2 K_1^{-1} \|\phi\|_\nu |t - \mathrm{i}\, a| \sup_{r \in [-T - \operatorname{Re} t, 0]} |t + r - \mathrm{i}\, a|^{-\nu}. \end{aligned} \tag{30}$$

Let $C_\rho = \big(1 + \rho^2/(a-\delta)^2\big)^{1/2}$. First we note that if $t \in D_2$ and $\operatorname{Re} t > 0$, then $|t - \mathrm{i}\, a| \le C_\rho |\operatorname{Im} t - a|$. Next we observe that, if $t \in D_2$ and $r \in [-T - \operatorname{Re} t, 0]$,

$$
\begin{aligned}
&|t + r - \mathrm{i}\, a| \ge |t - \mathrm{i}\, a|, \qquad \text{if } \operatorname{Re} t \le 0,\\
&|t + r - \mathrm{i}\, a| \ge |\operatorname{Im} t - a| \ge C_\rho^{-1} |t - \mathrm{i}\, a|, \qquad \text{if } \operatorname{Re} t > 0.
\end{aligned}
$$

Therefore, since $C_\rho^{-1} \le 1$, we have that $|t + r - \mathrm{i}\, a| \ge C_\rho^{-1}|t - \mathrm{i}\, a|$ for all $t \in D_2$ and $r \in [-T - \operatorname{Re} t, 0]$. Thus we obtain

$$
I_2(t) \le C K_2 K_1^{-1} C_\rho^{\nu} \|\phi\|_\nu |t - \mathrm{i}\, a|^{-\nu+1}, \qquad \text{if } t \in D_2. \tag{31}
$$

Using bounds (29) and (31) we have that, if $t \in D_2$,

$$
\begin{aligned}
|t - \mathrm{i}\, a|^{\nu-1} I(t) &= |t - \mathrm{i}\, a|^{\nu-1}(I_1(t) + I_2(t))\\
&\le 16\, \mathrm{e}^{\max\{0,\rho-T\}}\, |t - \mathrm{i}\, a|^{\nu-1} \|\phi\|_\nu + C K_2 K_1^{-1} C_\rho^{\nu} \|\phi\|_\nu\\
&\le K \|\phi\|_\nu
\end{aligned} \tag{32}
$$

with $K = \max\{16\, \mathrm{e}^{\max\{0,\rho-T\}}((\max\{T,\rho\})^2 + a^2)^{(\nu-1)/2}, C K_2 K_1^{-1} C_\rho^{\nu}\}$.

In an analogous way one can see that

$$
|t + \mathrm{i}\, a|^{\nu-1} I(t) \le K \|\phi\|_\nu, \qquad \text{if } t \in D_3. \tag{33}
$$

Finally, using bounds (28), (32) and (33) we obtain that $I \in \mathcal{X}_{\nu-1}$ and $\|I\|_{\nu-1} \le (4 + 2K)\|\phi\|_\nu$. Using (27), we have that $\mathcal{L}_{\alpha,c}(\phi) \in \mathcal{X}_{\nu-1}$ and moreover

$$
\|\mathcal{L}_{\alpha,c}(\phi)\|_{\nu-1} \le \mathrm{e}^{|c|\pi} \big(4 + 2K\big) \|\phi\|_\nu.
$$

Proceeding analogously with $\mathcal{L}_{-\alpha,-c}$ and using that $\mathcal{S} = (\mathcal{L}_{\alpha,c} \circ \pi^1, \mathcal{L}_{-\alpha,-c} \circ \pi^2)$ we finish the proof of the lemma. ■

In Lemma 2.9 we enunciate the properties of $\mathcal{T}$ we will use in the sequel.

Lemma 2.9 *The operator $\mathcal{T} : \mathcal{X}_3 \to \mathcal{X}_2$ given by (21) is well defined and there exists a constant K, independent of δ, such that*

$$
\|\mathcal{T}(\phi)\|_2 \le K |\log \delta| \|\phi\|_3, \qquad \textit{for any } \phi \in \mathcal{X}_3.
$$

Proof. Let $t \in D_2$. First we claim that, by 1. and 2. of Lemma 2.7 and using that $|\phi(r)| \le |r - \mathrm{i}\, a|^{-3} \|\phi\|_3$, for all $r \in [0, t]$,

$$
|\mathcal{T}(\phi)(t)| \le K_1^{-2} K_2^2 |t - \mathrm{i}\, a|^{-2} \|\phi\|_3 \left| \int_0^t \frac{1}{|r - \mathrm{i}\, a|}\, dr \right| \le C K_1^{-2} K_2^2 |t - \mathrm{i}\, a|^{-2} \|\phi\|_3 |\log \delta|.
$$

Hence,

$$|\mathcal{T}(\phi)(t)(t-\mathrm{i}\,a)^2| \le CK_1^{-2}K_2^2|\log\delta|\|\phi\|_3, \qquad t\in D_2, \tag{34}$$

and analogously we prove that

$$|\mathcal{T}(\phi)(t)(t+\mathrm{i}\,a)^2| \le CK_1^{-2}K_2^2|\log\delta|\|\phi\|_3, \qquad t\in D_3. \tag{35}$$

Now we deal with the case $t\in D_1$. We define $t^* = -tT/\operatorname{Re} t$. We note that the segment $\overline{0t^*}\subset D_2$ and $\overline{t^*t}\subset D_1$. We write

$$\mathcal{T}(\phi)(t) = \frac{1}{\cosh^2 t}\int_0^{t*}\cosh^2 r\phi(r)\,dr + \frac{1}{\cosh^2 t}\int_{t^*}^{t}\cosh^2 r\phi(r)\,dr := I_1(t)+I_2(t). \tag{36}$$

We begin by bounding $I_1(t) = \cosh^{-2} t\cosh^2 t^*\mathcal{T}(\phi)(t^*)$. We note that, since $t\in D_1$, we have that $\operatorname{Re} t\ge T\ge 2\log 2$ and hence $|\cosh t|\ge 1$. Moreover, $|\cosh t^*|\le K_2|t^*-\mathrm{i}\,a|$ provided that $t^*\in D_2$. Using these properties and bound (34) for $t=t^*$, we get that

$$|I_1(t)|\le |\cosh^{-2} t|\frac{|\cosh^2 t^*|}{|t^*-\mathrm{i}\,a|^2}CK_1^{-2}K_2^2|\log\delta|\|\phi\|_3 \le CK_1^{-2}K_2^4|\log\delta|\|\phi\|_3. \tag{37}$$

Now we bound $I_2(t)$. We claim that

$$|I_2(t)|\le 16\big(1+a/T\big)\|\phi\|_3. \tag{38}$$

Indeed, first we recall that $\mathrm{e}^{|\operatorname{Re} r|}/4\le|\cosh r|\le \mathrm{e}^{|\operatorname{Re} r|}$ and that $|\phi(r)|\le\|\phi\|_3$ for all $r\in D_1$. Hence, since by definition of t^*, $\operatorname{Re} t^* = -T$. Parameterizing the integration path in I_2 by $\gamma(r) = tr+t^*(1-r)$, we have that

$$\begin{aligned}|I_2(t)| &\le 16\,\mathrm{e}^{-2|\operatorname{Re} t|}\,|t-t^*|\|\phi\|_3\int_0^1 \mathrm{e}^{2(r\operatorname{Re} t-(1-r)T)}\,dr\\ &= 16\,\mathrm{e}^{-2(|\operatorname{Re} t|-T)}\,\frac{|t-t^*|}{\operatorname{Re} t+T}(\mathrm{e}^{2|\operatorname{Re} t+T|}-1)\|\phi\|_3.\end{aligned}$$

Finally, using that $|t-t^*| = |t||\operatorname{Re} t+T|/|\operatorname{Re} t|$ we get (38).

Hence, decomposition (36) and bounds (37) and (38) give us the following bound of $\mathcal{T}(\phi)(t)$ for all $t\in D_1$:

$$|\mathcal{T}(\phi)(t)|\le CK_1^{-2}K_2^4|\log\delta|\|\phi\|_3 + 16(1+a/T)\|\phi\|_3 \le 2CK_1^{-2}K_2^4|\log\delta|\|\phi\|_3 \tag{39}$$

if δ is small enough. Finally, bounds (34), (35) and (39) imply that

$$\|\mathcal{T}(\phi)\|_2\le 2(1+K_2^2)CK_1^{-2}K_2^2|\log\delta|\|\phi\|_3$$

and the lemma is proved. ∎

2.3.2 The independent term

Now we prove that the first approximation of φ, $\varphi_1 = \mathcal{L}\circ\mathcal{R}(0)$ (see (10) for the definition of the function $\mathcal{R}$), satisfies the properties enunciated in Proposition 2.5. Concretely we will prove that:

Lemma 2.10 *The function $\mathcal{L}\circ\mathcal{R}(0)\in\mathcal{X}_3\times\mathcal{X}_3\times\mathcal{X}_2$ and moreover, there exists a constant K independent of δ such that*

$$\|\mathcal{L}\circ\mathcal{R}(0)\|\le K\delta^{p+4},$$

where the norm $\|\cdot\|$ was defined in (17).

Proof. We note that $\mathcal{R}(0)\in\mathcal{X}_0\times\mathcal{X}_0\times\mathcal{X}_0$. This is due to the fact that δz_0 is bounded in D_ρ. Hence, by Remark 2.2, $\mathcal{R}(0)\in\mathcal{X}_\nu\times\mathcal{X}_\nu\times\mathcal{X}_\nu$ for any $\nu>0$, in particular for $\nu=3$.

First we claim that $D\mathcal{M}(0)\in\mathcal{X}_4\times\mathcal{X}_4$ (here D denotes $\frac{d}{dt}$) and that there exists a constant $K_{\mathcal{M}}$ independent on δ such that

$$\|\mathcal{M}(0)\|_{\times,3}\le K_{\mathcal{M}}\delta^{p+3},\qquad \|D\mathcal{M}(0)\|_{\times,4}\le K_{\mathcal{M}}\delta^{p+3}. \tag{40}$$

(The norm $\|\cdot\|_{\times,3}$ was defined in (18)). Indeed, we recall that by definition (11), $\mathcal{M}(0)(t)=\delta^pF(0,0,\delta z_0(t),\delta)$ and $F=(F_1,F_2)$ is an analytic function in $B^3(r_0)\times B(\delta_0)$, such that $|F(0,0,z,\delta)|\le C_F|(z,\delta)|^3$. Henceforth, for $t\in D_\rho$

$$|\mathcal{M}(0)(t)|\le\delta^{p+3}C_F|(z_0(t),1)|^3$$

and since $z_0\in\mathcal{X}_1$ and $1\in\mathcal{X}_0\subset\mathcal{X}_1$, we obtain $\|\mathcal{M}(0)\|_{\times,3}\le K_{\mathcal{M}}\delta^{p+3}$.

Let now $(z,\delta)\in B(r_0/2)\times B(\delta_0)$. We note that, if δ_0 is small enough then $z+|(z,\delta)|\,\mathrm{e}^{\mathrm{i}\,\theta}/2\in B(r_0)$ for all $\theta\in[0,2\pi]$. Hence by Cauchy's theorem,

$$|\partial_zF(0,0,z,\delta)|\le\frac{1}{\pi|(z,\delta)|}\int_0^{2\pi}|F(0,0,z+|(z,\delta)|\,\mathrm{e}^{\mathrm{i}\,\theta}/2,\delta)|\,d\theta\le\frac{27}{4}C_F|(z,\delta)|^2. \tag{41}$$

As we pointed out in Section 2.2, we assume that r_0 is big enough to satisfy that $\delta z_0(t)\in B(r_0/2)$ for all $t\in D_\rho$. Therefore,

$$|D\mathcal{M}(0)(t)|=\delta^{p+1}|\partial_zF(0,0,\delta z_0(t),\delta)||Dz_0(t)|\le\frac{27}{4}C_F\delta^{p+3}|(z_0(t),1)|^2|Dz_0(t)|$$

and (40) is proved provided that $1\in\mathcal{X}_0\subset\mathcal{X}_1$, $z_0\in\mathcal{X}_1$ and $Dz_0\in\mathcal{X}_2$.

In addition we observe that, by integrating by parts the integral in the definition (20) of $\mathcal{L}_{\alpha,c}$, we have a more suitable expression for $\mathcal{L}_{\alpha,c}(\pi^1\mathcal{R}(0))=\mathcal{L}_{\alpha,c}(\pi^1\mathcal{M}(0))$:

$$\mathcal{L}_{\alpha,c}(\pi^1\mathcal{M}(0))(t)=-\frac{\delta\,\mathrm{i}}{\alpha}\left[\pi^1\mathcal{M}(0)(t)-\mathcal{L}_{\alpha,c}(g_1)(t)\right]$$

with

$$g_1(t) = D(\pi^1\mathcal{M}(0))(t) + z_0(t)(1+\mathrm{i}\,c)\cdot\pi^1\mathcal{M}(0)(t).$$

We obtain an analogous expression for $\mathcal{L}_{-\alpha,-c}(\pi^2\mathcal{M}(0))$ and we conclude that

$$\mathcal{S}(\mathcal{M}(0)) = -\frac{\delta\,\mathrm{i}}{\alpha}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left[\mathcal{M}(0) - \mathcal{S}(g)\right]$$

with $g = (g_1, g_2)$ and $g_2(t) = D(\pi^2\mathcal{M}(0))(t) + z_0(t)(1-\mathrm{i}\,c)\cdot\pi^2\mathcal{M}(0)(t)$. We note that by (40) and since $z_0 \in \mathcal{X}_1$, $g \in \mathcal{X}_4 \times \mathcal{X}_4$ and $\|g\|_{\times,4} \le K\delta^{p+3}$ for some constant K. Hence, by Lemma 2.8 and using again (40), $\mathcal{S}\circ\mathcal{M}(0) \in \mathcal{X}_3 \times \mathcal{X}_3$ and

$$\|\mathcal{S}\circ\mathcal{M}(0)\|_{\times,3} \le C_0\delta^{p+4}. \tag{42}$$

Now we deal with $\mathcal{T}\circ\mathcal{N}$. As in the previous case, one can check that $\mathcal{N}\in\mathcal{X}_3$ and that there exists a constant $K_{\mathcal{N}}$ such that $\|\mathcal{N}\|_3 \le K_{\mathcal{N}}\delta^{p+3}$. Hence by Lemma 2.9, $\mathcal{T}\circ\mathcal{N}\in\mathcal{X}_2$ and

$$\|\mathcal{T}\circ\mathcal{N}\|_2 \le K\delta^{p+3}|\log\delta| \tag{43}$$

for some constant K independent of δ. Finally by bounds (42) and (43) and definition (17) of the norm $\|\cdot\|$ we get the result. ■

2.3.3 End of the proof of Proposition 2.5

As we pointed out in Lemma 2.3, if φ is a solution of

$$\varphi = \mathcal{L}\circ\mathcal{R}(\varphi), \tag{44}$$

then φ is also a solution of system (13) satisfying that $\pi^3\varphi(0) = 0$. Moreover, it is clear that, if $\varphi\in\mathcal{X}_3\times\mathcal{X}_3\times\mathcal{X}_2$, then φ is bounded on $\overline{D}_\rho$.

We notice that, by Lemma 2.10, $\varphi_1 = \mathcal{L}\circ\mathcal{R}(0)\in\mathcal{X}_3\times\mathcal{X}_3\times\mathcal{X}_2$, thus we denote

$$R = 8\|\varphi_1\| \le 8K\delta^{p+4}.$$

Lemma 2.11 *We define $\mathcal{F} = \mathcal{L}\circ\mathcal{R}$ and $B(R)$ the closed ball of $\mathcal{X}_3\times\mathcal{X}_3\times\mathcal{X}_2$ centered at the origin of radius $R>0$.*

The operator $\mathcal{F} : B(R) \to B(R/4)$ is well defined and moreover

$$\|\mathcal{F}(\varphi) - \varphi_1\| \le K\delta^{p+2}|\log\delta|\|\varphi\|, \qquad \text{for all } \varphi\in B(R). \tag{45}$$

Remark 2.12 *We claim that Lemma 2.11 implies Proposition 2.5. Indeed, since the operator $\mathcal{F}$ is analytic in $B(R)$ and $\mathcal{F}(B(R))\subset B(R/4)$ a suitable version of the fixed point theorem (see [Ang93]) implies that the fixed point equation (44) has one solution $\varphi\in B(R)$. Moreover, taking $\varphi_2 = \mathcal{F}(\varphi) - \varphi_1$, by (45), the statement* 2. *of Proposition 2.5 holds trivially. Item* 1. *of Proposition 2.5 is fulfilled by Lemma 2.10.*

Proof of Lemma 2.11. First we will prove bound (45). Let $\varphi \in B(R)$. Since $R = 8\|\varphi_1\| \le 8K\delta^{p+4}$, we have that $\|\varphi\| \le C\delta^{p+4}$, with $C = 8K$.

We denote $\varphi = (\xi, \overline{\xi}, \eta)$ and we notice that

$$\|(\xi, \overline{\xi})\|_{\times,3} \le \|\varphi\| \le C\delta^{p+4}, \qquad \|\eta\|_2 \le \|\varphi\|\delta^{-1}|\log\delta| \le C\delta^{p+3}|\log\delta|. \tag{46}$$

We also introduce

$$\begin{aligned}\Delta F(t) &= F(\delta\xi(t), \delta\overline{\xi}(t), \delta(z_0(t) + \eta(t)), \delta) - F(0, 0, \delta z_0(t), \delta)\\ \Delta H(t) &= H(\delta\xi(t), \delta\overline{\xi}(t), \delta(z_0(t) + \eta(t)), \delta) - H(0, 0, \delta z_0(t), \delta)\end{aligned}$$

and we observe that, since ΔF and ΔH are bounded in D_ρ, ΔF and ΔH belong to $\mathcal{X}_0$ and by Remark 2.2, $\Delta F \in \mathcal{X}_4 \times \mathcal{X}_4$ and $\Delta H \in \mathcal{X}_3$.

Next we will bound $\|\Delta F\|_{\times,4}$. Throughout this proof, we will denote $D = D_{x,y,z}$ the first derivative with respect to (x, y, z). We note that, since F is analytic on $B^3(r_0) \times B(\delta_0)$, we have that, if δ_0 is small enough and $(x, y, z, \delta) \in B^3(r_0/2) \times B(\delta_0)$

$$|DF(x, y, z, \delta)| \le C_F|(x, y, z, \delta)|^2. \tag{47}$$

To check (47) is straightforward using Cauchy's theorem in an analogous way as in (41). Then, using (47) and the mean's value theorem, we have that for all $t \in D_\rho$,

$$\begin{aligned}|\Delta F(t)| &\le \delta \int_0^1 \left|DF(\delta\xi(t)\lambda, \delta\overline{\xi}(t)\lambda, \delta(z_0(t) + \eta(t)\lambda), \delta)\right| d\lambda \cdot |\varphi(t)|\\ &\le \delta^3 C_F\big(\max\{|\xi(t)|, |\overline{\xi}(t)|, |z_0(t)| + |\eta(t)|, 1\}\big)^2|\varphi(t)|\\ &\le 4\delta^3 C_F|\varphi(t)|\big(\max\{|\varphi(t)|, |z_0(t)|, 1\}\big)^2.\end{aligned} \tag{48}$$

Let $t \in D_1$. By (46) we have that $|\xi(t)|, |\overline{\xi}(t)| \le \|\varphi\|$ and $|\eta(t)| \le \delta^{-1}|\log\delta|\|\varphi\|$. Hence $|\varphi(t)| \le \|\varphi\|\delta^{-1}|\log\delta|$. Moreover, since $p > -2$ and $\|\varphi\| \le C\delta^{p+4}$, we have that there exists a constant M independent of δ such that

$$\max\{|\varphi(t)|, |z_0(t)|, 1\} \le M$$

if δ is small enough. Using these bounds to bound (48) we obtain

$$|\Delta F(t)| \le 4\delta^2|\log\delta|C_F\|\varphi\|M^2, \qquad t \in D_1. \tag{49}$$

Now we fix $t \in D_2$. By (46) and since $|t - \mathrm{i}\,a| \ge \delta$, we have that

$$\begin{aligned}|\varphi(t)| &\le \|\varphi\| \max\{|t - \mathrm{i}\,a|^{-3}, |t - \mathrm{i}\,a|^{-2}\delta^{-1}|\log\delta|\}\\ &\le \delta^{-1}|\log\delta|\|\varphi\||t - \mathrm{i}\,a|^{-2}.\end{aligned}$$

Using this bound and the fact that $|(t-\mathrm{i}\,a)^{-1}z_0(t)|$ is bounded in D_2, increasing M if necessary, we have that

$$\begin{aligned}\max\{|\varphi(t)|,|z_0(t)|,1\} &\leq \max\{\|z_0\|_1|t-\mathrm{i}\,a|^{-1},\delta^{-1}|\log\delta|\|\varphi\||t-\mathrm{i}\,a|^{-2},1\}\\ &\leq M|t-\mathrm{i}\,a|^{-1}\end{aligned} \tag{50}$$

where we have used that δ is small enough and that $p>-2$. Therefore, bounding (48) we have that for all $t\in D_2$

$$|t-\mathrm{i}\,a|^4|\Delta F(t)| \leq 4\delta^2|\log\delta|C_F\|\varphi\|M^2. \tag{51}$$

In the same way one can check that

$$|t+\mathrm{i}\,a|^4|\Delta F(t)| \leq 4\delta^2|\log\delta|C_F\|\varphi\|M^2, \qquad \text{if } t\in D_3. \tag{52}$$

Thus using bounds (49), (51) and (52), we get

$$\|\Delta F\|_{\times,4} \leq 12\delta^2|\log\delta|C_F\|\varphi\|M^2. \tag{53}$$

We claim that $\xi\cdot\eta\in\mathcal{X}_4$ and

$$\|\xi\cdot\eta\|_4 \leq \|\varphi\|C\delta^{p+2}|\log\delta|. \tag{54}$$

Indeed, it is clear that

$$|\xi(t)\eta(t)| \leq \|\varphi\|^2\delta^{-1}|\log\delta| \leq C\delta^{p+3}|\log\delta|\|\varphi\|, \qquad \text{if } t\in D_1$$

and, since $|t\pm\mathrm{i}\,a|\geq\delta$, $\xi\in\mathcal{X}_3$ and $\eta\in\mathcal{X}_2$, we also have that

$$\begin{aligned}|(t-\mathrm{i}\,a)^4\xi(t)\eta(t)| &\leq |t-\mathrm{i}\,a|^{-1}\|\xi\|_3\|\eta\|_2\\ &\leq \|\varphi\|C\delta^{p+3}|\log\delta||t-\mathrm{i}\,a|^{-1} \leq \|\varphi\|C\delta^{p+2}|\log\delta|, &&\text{if } t\in D_2\\ |(t+\mathrm{i}\,a)^4\xi(t)\eta(t)| &\leq \|\varphi\|C\delta^{p+2}|\log\delta|, &&\text{if } t\in D_3.\end{aligned}$$

Hence the claim is proved.

Analogously, we can also prove that $\|\overline{\xi}\cdot\eta\|_4 \leq \|\varphi\|C\delta^{p+2}|\log\delta|$. Therefore, by definition (12) of $\mathcal{M}$ and bounds (53) and (54)

$$\|\mathcal{M}(\varphi)-\mathcal{M}(0)\|_{\times,4} \leq \|\varphi\|\delta^{p+2}|\log\delta|\big((1+|c|)C+12C_FM^2\big)$$

and Lemma 2.8 implies that $\mathcal{S}(\mathcal{M}(\varphi)-\mathcal{M}(0))\in\mathcal{X}_3\times\mathcal{X}_3$ and

$$\|\mathcal{S}(\mathcal{M}(\varphi)-\mathcal{M}(0))\|_{\times,3} \leq K\delta^{p+2}|\log\delta|\|\varphi\| \tag{55}$$

for some constant K independent of δ.

Finally we deal with $\mathcal{T}(\mathcal{N}(\varphi)-\mathcal{N}(0))$. We observe that ΔH can be studied in the same way of ΔF. Therefore we can conclude that

$$\|\Delta H\|_3 \leq \|\Delta H\|_4 \sup_{t\in D_\rho} |t - \mathrm{i}\, a|^{-1} \leq 12\delta|\log\delta| C_H \|\varphi\| M^2.$$

We also have that

$$\|\xi\cdot\overline{\xi}\|_3 \leq \|\varphi\| C\delta^{p+1}, \qquad \|\eta^2\|_3 \leq \|\varphi\| C\delta^{p+1}|\log\delta|^2$$

and hence, there exists a constant K such that $\|\mathcal{N}(\varphi)-\mathcal{N}(0)\|_3 \leq \|\varphi\| K\delta^{p+1}|\log\delta|^2$. By Lemma 2.9, this implies that

$$\|\mathcal{T}(\mathcal{N}(\varphi)-\mathcal{N}(0))\|_2 \leq K\delta^{p+1}|\log\delta|^2\|\varphi\| \tag{56}$$

and therefore, by definition (17) of the norm $\|\cdot\|$, and using bounds (55) and (56) of $\|\mathcal{S}(\mathcal{M}(\varphi)-\mathcal{M}(0))\|_{\times,3}$ and $\|\mathcal{T}(\mathcal{N}(\varphi)-\mathcal{N}(0))\|_2$, we get

$$\begin{aligned}\|\mathcal{F}(\varphi)-\varphi_1\| &= \|\mathcal{S}(\mathcal{M}(\varphi)-\mathcal{M}(0))\|_{\times,3} + \delta|\log\delta|^{-1}\|\mathcal{T}(\mathcal{N}(\varphi)-\mathcal{N}(0))\|_2\\ &\leq K\delta^{p+2}|\log\delta|\|\varphi\|\end{aligned}$$

and bound (45) is proved.

Now we are done since, by definition of R and the previous bound we have that

$$\|\mathcal{F}(\varphi)\| \leq \|\varphi_1\| + \|\mathcal{F}(\varphi)-\varphi_1\| \leq \frac{R}{8} + K\delta^{p+2}|\log\delta| R < \frac{R}{4}$$

provided that $p > -2$ and δ is small enough. ■

3 Exponentially small splitting of the heteroclinic orbit

Throughout this section we will assume that $\alpha > 0$.

Let $\Delta\varphi = \varphi^{\mathrm{u}} - \varphi^{\mathrm{s}}$ where φ^{u} and φ^{s} are the solutions of system (13) given in Proposition 2.5. These solutions are defined by $D_\rho^{\mathrm{u,s}}$ respectively and they satisfy the equation

$$\dot{\varphi}^{\mathrm{s,u}} = A(t)\varphi^{\mathrm{s,u}} + \mathcal{R}(\varphi^{\mathrm{s,u}}).$$

Subtracting equations for φ^{u} and φ^{s}, we obtain that $\Delta\varphi$ is defined in $D_\rho^{\mathrm{u}} \cap D_\rho^{\mathrm{s}}$ and it must satisfy the linear equation

$$\dot{\zeta} = A(t)\zeta + B(t)\zeta \tag{57}$$

whose coefficients depend on $\varphi^{\mathrm{s,u}}$ and they are given by

$$B(t) = \int_0^1 D\mathcal{R}((1-\lambda)\varphi^{\mathrm{s}}(t) + \lambda\varphi^{\mathrm{u}}(t))\, d\lambda. \tag{58}$$

Here $D = D_{\xi,\overline{\xi},\eta}$ denotes the first derivative with respect to $(\xi, \overline{\xi}, \eta)$. To prove Theorem 1.4 our strategy will be to exploit equation (57). The idea behind the proof is that if a solution of (57) is analytic and bounded in $D_\rho^{\mathrm{u}} \cap D_\rho^{\mathrm{s}}$, then it has to be exponentially small with respect to δ when $t \in \mathbb{R} \cap D_\rho^{\mathrm{u}} \cap D_\rho^{\mathrm{s}}$. This is clear if one considers equation (57) with $B \equiv 0$, due to the special form of the fundamental matrix Φ of $\dot{\zeta} = A(t)\zeta$ given in Lemma 2.1. The same idea can be adapted for the full equation (57) using the fact that B is a small perturbation of A for δ small enough.

As in this paper we are not dealing with upper bounds of $\Delta\varphi$ but with an asymptotic expression of it, we need to decompose $\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2$ in such a way that $\Delta\varphi_1$ is the dominant term. We observe that the obvious decomposition, suggested by Proposition 2.5, $\Delta\varphi_1 = \varphi_1^{\mathrm{u}} - \varphi_1^{\mathrm{s}}$, with $\varphi_1^{\mathrm{s,u}} = \mathcal{L}^{\mathrm{s,u}} \circ \mathcal{R}(0)$, is not the most appropriate. The reason is that the third component of $\varphi_1^{\mathrm{s,u}}$ is given by

$$\pi^3\varphi_1^{\mathrm{s,u}}(t) = \frac{1}{\cosh^2 t} \int_0^t \cosh^2 r \mathcal{N}(0)\, dr$$

and then $\pi^3(\varphi_1^{\mathrm{u}} - \varphi_1^{\mathrm{s}})(t)$ is identically zero.

For the full solution $\Delta\varphi$ this cancellation will no longer be true. As $\varphi^{\mathrm{u,s}}$ verify (19), we only can assume that $\pi^3\Delta\varphi(0) = 0$. This makes necessary to look for another first approximation with a third component different from zero. On the other hand we would like to keep the two first components of $\varphi_1^{\mathrm{u}} - \varphi_1^{\mathrm{s}}$ as the main term of $\pi^{1,2}\Delta\varphi$. We use the fact that $\Delta\varphi$ satisfies the homogeneous linear differential equation (57) and hence can be expressed as

$$\Delta\varphi(t) = \Phi(t)\left[\Delta\varphi(0) + \int_0^t \Phi^{-1}(r)B(r)\Delta\varphi(r)\, dr\right] := \mathcal{B}(\Delta\varphi)(t)$$

with Φ given in Lemma 2.1. With a Gauss-Seidel type argument, we can use the two first components of $\mathcal{B}(\Delta\varphi)(t)$ to compute the third one. So $\Delta\varphi$ can also be written as

$$\Delta\varphi(t) = \Phi(t)\left[\Delta\varphi(0) + \begin{pmatrix} \int_0^t \pi^{1,2}\big(\Phi^{-1}(r)B(r)\Delta\varphi(r)\big)\, dr \\ \int_0^t \pi^3\big[\Phi^{-1}(r)B(r)\big(\pi^{1,2}\mathcal{B}(\Delta\varphi)(r), \pi^3\Delta\varphi(r)\big)\big]\, dr \end{pmatrix}\right]$$

Once one has a suitable fixed point equation for $\Delta\varphi$, it is natural to define the dominant term as

$$\Phi(t)\left[\Delta\varphi(0) + \begin{pmatrix} 0 \\ \int_0^t \pi^3\big[\Phi^{-1}(r)B(r)\big(\pi^{1,2}(\Phi(r)\Delta\varphi(0)), 0\big)\big]\, dr \end{pmatrix}\right]. \tag{59}$$

Of course, there is no sense in using $\Delta\varphi(0)$ since we do not know it (in fact our goal is to find an asymptotic expression for it); for this reason we use $\varphi_1^{\mathrm{u}}(0)-\varphi_1^{\mathrm{s}}(0)$ instead of $\Delta\varphi(0)$ in the expression of the dominant term (59). We notice that the two first components of this dominant term are

$$\pi^{1,2}\big(\Phi(t)(\varphi_1^{\mathrm{u}}(0)-\varphi_1^{\mathrm{s}}(0))\big)=\mathcal{S}^{\mathrm{u}}\circ\mathcal{M}(0)(t)-\mathcal{S}^{\mathrm{s}}\circ\mathcal{M}(0)(t),$$

(we recall that $\pi^{1,2}\mathcal{R}(0)=\mathcal{M}(0)$) where this equality is a consequence of the fact that the function $\varphi_1^{\mathrm{s}}-\varphi_1^{\mathrm{u}}$ is a solution of the homogeneous linear equation $\dot\chi=A(t)\chi$. In this way we take $\Delta\varphi_1=(\Delta\psi_1,\Delta\eta_1)$ with

$$\begin{aligned}
\Delta\psi_1(t)&=\pi^{1,2}\big(\Phi(t)(\varphi_1^{\mathrm{u}}(0)-\varphi_1^{\mathrm{s}}(0))\big)=\mathcal{S}^{\mathrm{u}}\circ\mathcal{M}(0)(t)-\mathcal{S}^{\mathrm{s}}\circ\mathcal{M}(0)(t)\\
&=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh t)}\cosh t\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\cosh r}\,\mathrm{e}^{\mathrm{i}\,\alpha r/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cosh r)}\,\pi^1\mathcal{M}(0)(r)\,dr\\ \mathrm{e}^{\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cosh t)}\cosh t\displaystyle\int_{-\infty}^{+\infty}\frac{1}{\cosh r}\,\mathrm{e}^{-\mathrm{i}\,\alpha r/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh r)}\,\pi^2\mathcal{M}(0)(r)\,dr\end{pmatrix}\\
\Delta\eta_1(t)&=\frac{1}{\cosh^2 t}\int_0^t\cosh^2 r\,\pi^3\big[B(r)(\Delta\psi_1(r),0)\big]\,dr.
\end{aligned}\tag{60}$$

We denote

$$\begin{aligned}
c_1^0&=\pi^1(\varphi_1^{\mathrm{u}}(0)-\varphi_1^{\mathrm{s}}(0))=\int_{-\infty}^{+\infty}\frac{1}{\cosh r}\,\mathrm{e}^{\mathrm{i}\,\alpha r/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cosh r)}\,\pi^1\mathcal{M}(0)(r)\,dr\\
c_2^0&=\pi^2(\varphi_1^{\mathrm{u}}(0)-\varphi_1^{\mathrm{s}}(0))=\int_{-\infty}^{+\infty}\frac{1}{\cosh r}\,\mathrm{e}^{-\mathrm{i}\,\alpha r/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh r)}\,\pi^2\mathcal{M}(0)(r)\,dr
\end{aligned}\tag{61}$$

and therefore

$$\Delta\psi_1(t)=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh t)}\cosh t\;c_1^0\\ \mathrm{e}^{\mathrm{i}\,\alpha t/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cosh t)}\cosh t\;c_2^0\end{pmatrix}.\tag{62}$$

Lemma 3.1 *The constants c_1^0 and c_2^0 satisfy that $c_1^0=\overline{c_2^0}$ and that*

$$|c_1^0|=|c_2^0|\le K\delta^p\,\mathrm{e}^{-\alpha a/\delta}.\tag{63}$$

We notice that this implies that $|\Delta\psi_1(0)|\le K\delta^p\,\mathrm{e}^{-\alpha a/\delta}$.

Proof. The equality $c_1^0=\overline{c_2^0}$ comes from definition (61) of c_1^0,c_2^0 and from the fact that $\pi^1\mathcal{R}(0)=\overline{\pi^2\mathcal{R}(0)}$ (see definition (10) of $\mathcal{R}$). Moreover, since by Lemma 2.10, $\|\varphi_1^{\mathrm{s,u}}\|\le K\delta^{p+4}$, we have that $|\pi^1\Delta\psi_1(t)(t-\mathrm{i}\,a)^3|\le\|\varphi_1^{\mathrm{u}}\|+\|\varphi_1^{\mathrm{s}}\|\le K\delta^{p+4}$ for all $t\in D_\rho^{\mathrm{u}}\cap D_\rho^{\mathrm{s}}$. Then, since

$$|\pi^1\Delta\psi_1(\mathrm{i}(a-\delta))|=|\,\mathrm{e}^{\alpha(a-\delta)/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cosh\mathrm{i}(a-\delta))}\cosh\mathrm{i}(a-\delta)c_1^0|\le K\delta^{p+1}$$

and $a = \pi/2$, we have that $|c_1^0| \leq K\delta^p \, \mathrm{e}^{-\alpha a/\delta}$. Here we have used the fact that, taking the main determination of the logarithm, $|\,\mathrm{e}^{-\mathrm{i}\, c \log(\cosh \mathrm{i}(a-\delta))}\,| \leq \mathrm{e}^{|c|\pi/2}$ and that by statement 2. of Lemma 2.7, $|\cosh s| \geq K_1 |s - \mathrm{i}\, a| \geq K_1 \delta$. ■

Remark 3.2 *In fact, c_1^0 and c_2^0 will be computed more explicitly in Subsection 3.4 to get the asymptotic expression for $\Delta\varphi$ in Proposition 3.4.*

The remaining part of this section is devoted to prove Theorem 1.4 which will be a direct consequence of Proposition 3.3 and Proposition 3.4 enunciated below. We postpone their proofs to the following subsections.

In order to see that $\Delta\varphi(0)$ is given asymptotically by $\Delta\varphi_1(0)$, our next goal is to obtain an exponentially small bound for the difference $\Delta\varphi(0) - \Delta\varphi_1(0)$.

Proposition 3.3 *If $p > -2$,*

$$\Delta\varphi(0) = \Delta\varphi_1(0) + O(\delta^{2p+2}|\log\delta|)\, \mathrm{e}^{-|\alpha|\pi/(2\delta)} \, .$$

Finally we check that $\Delta\varphi_1$ is actually the dominant term of $\Delta\varphi$.

Proposition 3.4 *If $p > -2$ and δ is small enough,*

$$\begin{aligned}\pi^{1,2}\Delta\varphi_1(0) =& 2\pi\delta^p\, \mathrm{e}^{-\pi|\alpha|/(2\delta)} \sum_{n\geq 3} \alpha^n \begin{pmatrix} \mathrm{i}^n \left(\dfrac{|\alpha|}{\delta}\right)^{\mathrm{i}\, c} \dfrac{m_n}{\Gamma(n+1+\mathrm{i}\, c)} \\ (-\,\mathrm{i})^n \left(\dfrac{|\alpha|}{\delta}\right)^{-\,\mathrm{i}\, c} \dfrac{\overline{m_n}}{\Gamma(n+1-\mathrm{i}\, c)} \end{pmatrix} \\ & + O(\delta^{p+1})\, \mathrm{e}^{-\pi|\alpha|/(2\delta)}\end{aligned}$$

where the coefficients m_n were defined in Theorem 1.6.

End of the proof of Theorem 1.6. We point out that Proposition 3.3 and Proposition 3.4 imply, trivially, Theorem 1.4 if we observe that

$$\pi^1\Delta\sigma_1(0) + \mathrm{i}\,\pi^2\Delta\sigma_1(0) = \pi^1\Delta\varphi_1(0),$$

and

$$\begin{aligned}\pi^1\Delta\varphi_1(0) =& 2\pi\delta^p\, \mathrm{e}^{-\pi|\alpha|/(2\delta)} \left(\frac{|\alpha|}{\delta}\right)^{\mathrm{i}\, c} (\alpha\, \mathrm{i})^{-\,\mathrm{i}\, c} \sum_{n\geq 3} \frac{(\alpha\, \mathrm{i})^{n+\mathrm{i}\, c} m_n}{\Gamma(n+1+\mathrm{i}\, c)} + O(\delta^{p+1})\, \mathrm{e}^{-\pi|\alpha|/(2\delta)} \\ =& 2\pi\, \mathrm{e}^{c\pi/2}\, \hat{m}(\mathrm{i}\,\alpha)\delta^p\, \mathrm{e}^{-\pi|\alpha|/(2\delta)}\, \mathrm{e}^{-\,\mathrm{i}\, c\log\delta} + O(\delta^{p+1})\, \mathrm{e}^{-\pi|\alpha|/(2\delta)} \, .\end{aligned}$$

■

The remaining part of the paper is devoted to proving these results.

Since we are only interested in computing $\Delta\varphi(0)$, it will be enough to study the behavior of $\Delta\varphi(s)$ for $s = \mathrm{i}\, t$, purely imaginary. We restrict our definition domain to

$$E := \{\mathrm{i}\, t \in \mathrm{i}\,\mathbb{R} : |t| \leq a - \delta\} \subset D^{\mathrm{s}}_\rho \cap D^{\mathrm{u}}_\rho$$

Obviously, $\Delta\varphi$ and $\Delta\varphi_1$ are both defined in E.

3.1 The solutions of equation (57)

In this subsection we find a suitable expression of $\Delta\varphi$ by means of a linear operator.

Since $\varphi^{\mathrm{s,u}} \in \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$, it is natural to consider the following normed space $\mathcal{Y}$ where $\Delta\varphi = \varphi^{\mathrm{u}} - \varphi^{\mathrm{s}}$ belongs to.

$$\mathcal{Y} = \{f : E \to \mathbb{C}^3 : f \text{ is continuous}$$
$$\|f\|_{\mathcal{Y}} := \sup_{\mathrm{i}\, t\in E} |\pi^1 f(\mathrm{i}\, t)\cos^3 t| + \sup_{\mathrm{i}\, t\in E} |\pi^2 f(\mathrm{i}\, t)\cos^3 t| + \frac{\delta}{|\log\delta|}\sup_{\mathrm{i}\, t\in E} |\pi^3 f(\mathrm{i}\, t)\cos^2 t| < \infty\}.$$

We introduce the operator $\mathcal{F}_0(f) = (\mathcal{G}(f), \mathcal{H}(f))$ where

$$\mathcal{G}(f)(\mathrm{i}\, t) = \begin{pmatrix} \mathrm{i}\, \mathrm{e}^{\alpha t/\delta}\, \mathrm{e}^{\mathrm{i}\, c\log(\cos t)} \cos t \displaystyle\int_{a-\delta}^{t} \frac{\mathrm{e}^{-\alpha r/\delta}\, \mathrm{e}^{-\mathrm{i}\, c\log(\cos r)}}{\cos r} \pi^1\big(B(\mathrm{i}\, r) f(\mathrm{i}\, r)\big)\, dr \\ \mathrm{i}\, \mathrm{e}^{-\alpha t/\delta}\, \mathrm{e}^{-\mathrm{i}\, c\log(\cos t)} \cos t \displaystyle\int_{-(a-\delta)}^{t} \frac{\mathrm{e}^{\alpha r/\delta}\, \mathrm{e}^{\mathrm{i}\, c\log(\cos r)}}{\cos r} \pi^2\big(B(\mathrm{i}\, r) f(\mathrm{i}\, r)\big)\, dr \end{pmatrix}$$
$$\mathcal{H}(f)(\mathrm{i}\, t) = \frac{\mathrm{i}}{\cos^2 t}\int_0^t \cos^2 r\, \pi^3\big(B(\mathrm{i}\, r)(\mathcal{G}(f)(\mathrm{i}\, r), \pi^3 f(\mathrm{i}\, r))\big)\, dr. \tag{64}$$

In order to shorten the notation we also define for all $k_1, k_2 \in \mathbb{C}$

$$I(k_1,k_2)(t) = \begin{pmatrix} k_1\, \mathrm{e}^{\alpha t/\delta}\, \mathrm{e}^{\mathrm{i}\, c\log(\cos t)} \cos t \\ k_2\, \mathrm{e}^{-\alpha t/\delta}\, \mathrm{e}^{-\mathrm{i}\, c\log(\cos t)} \cos t \\ \displaystyle\frac{\mathrm{i}}{\cos^2 t}\int_0^t \cos^3 r\, \pi^3\big(B(\mathrm{i}\, r)(k_1\, \mathrm{e}^{\alpha r/\delta}\, \mathrm{e}^{\mathrm{i}\, c\log(\cos r)}, k_2\, \mathrm{e}^{\alpha r/\delta}\, \mathrm{e}^{-\mathrm{i}\, c\log(\cos r)}, 0)\big)\, dr \end{pmatrix}. \tag{65}$$

Lemma 3.5 *$\Delta\varphi \in \mathcal{Y}$ and $\|\Delta\varphi\|_{\mathcal{Y}} \le K\delta^{p+4}$. Moreover, there exist $c_1, c_2 \in \mathbb{C}$ such that*

$$\Delta\varphi(\mathrm{i}\, t) = I(c_1,c_2)(t) + \mathcal{F}_0(\Delta\varphi)(\mathrm{i}\, t) \tag{66}$$

and $|c_1|, |c_2| \le K\delta^p\, \mathrm{e}^{-\alpha a/\delta}$.

Proof. Since $\varphi^{\mathrm{s,u}} \in \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_3^{\mathrm{s,u}} \times \mathcal{X}_2^{\mathrm{s,u}}$, $\Delta\varphi \in \mathcal{Y}$ obviously. Moreover, by Proposition 2.5, $\|\Delta\varphi\|_{\mathcal{Y}} \le \|\varphi^{\mathrm{u}}\|_{\mathcal{Y}} + \|\varphi^{\mathrm{s}}\|_{\mathcal{Y}} \le K\delta^{p+4}$.

Now we check (66). Since $\Delta\varphi$ is a solution of the linear homogeneous equation (57) it can be written as

$$\Delta\varphi(s) = \begin{pmatrix} e^{-i\alpha s/\delta} e^{i c\log(\cosh s)} \cosh t\Big[c_1 + \displaystyle\int_{s_1}^{s} \frac{e^{i\alpha r/\delta} e^{-i c\log(\cosh r)}}{\cosh r} \pi^1\big(B(r)\Delta\varphi(r)\big)\,dr\Big] \\ e^{i\alpha s/\delta} e^{-i c\log(\cosh s)} \cosh t\Big[c_2 + \displaystyle\int_{s_2}^{s} \frac{e^{-i\alpha r/\delta} e^{i c\log(\cosh r)}}{\cosh r} \pi^2\big(B(r)\Delta\varphi(r)\big)\,dr\Big] \\ \dfrac{1}{\cosh^2 s}\Big[c_3 + \displaystyle\int_{s_3}^{s} \cosh^2 r \pi^3\big(B(r)\Delta\varphi(r)\big)\,dr\Big] \end{pmatrix}, \tag{67}$$

for suitable $c_1, c_2, c_3 \in \mathbb{C}$. We take $s_3 = 0$, $s_1 = i(a-\delta)$ and $s_2 = -i(a-\delta)$. We observe that, since we choose $\pi^3\varphi^{s}(0) = \pi^3\varphi^{u}(0) = 0$, we have that $\pi^3\Delta\varphi(0) = 0$ and then $c_3 = 0$. We perform the change of variables $r = i\,u$ in all the integrals of (67) and we obtain, taking $s = i\,t$ in (67),

$$\Delta\varphi(i\,t) = \begin{pmatrix} c_1 e^{\alpha t/\delta} e^{i c\log(\cos t)} \cos t \\ c_2 e^{-\alpha t/\delta} e^{-i c\log(\cos t)} \cos t \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{G}(\Delta\varphi)(i\,t) \\ \dfrac{i}{\cos^2 t}\displaystyle\int_0^t \cos^2 u \pi^3\big(B(i\,u)\Delta\varphi(i\,u)\big)\,du \end{pmatrix}. \tag{68}$$

Substituting the two first components of $\Delta\varphi$, $\pi^{1,2}\Delta\varphi$, by the right hand side of (68) which is $(c_1 e^{\alpha t/\delta} e^{i c\log\cos t}\cos t, c_2 e^{-\alpha t/\delta} e^{-i c\log\cos t}\cos t)^{\mathrm{T}} + \mathcal{G}(\Delta\varphi)$, in the third component of the expression (68) of $\Delta\varphi(i\,t)$, we get the form (66) stated in the lemma.

Now we bound $|c_1|, |c_2|$. Since $\Delta\varphi \in \mathcal{Y}$, we have that $|\pi^1(\Delta\varphi(i\,t))\cos^3 t| \le \|\Delta\varphi\|_{\mathcal{Y}}$. Then, since $s_1 = i(a-\delta)$,

$$|\pi^1(\Delta\varphi(s_1))\cos^3(a-\delta)| = |\, e^{\alpha(a-\delta)/\delta} e^{i c\log(cos(a-\delta))} \cos^4(a-\delta) c_1| \le \|\Delta\varphi\|_{\mathcal{Y}}$$

and thus, since $a = \pi/2$, $|c_1| \le \|\Delta\varphi\|_{\mathcal{Y}} e^{-\alpha(a-\delta)/\delta} |\, e^{-i c\log(\cos(a-\delta))}\,|\delta^{-4}$. In the same way one can check that $|c_2| \le \|\Delta\varphi\|_{\mathcal{Y}} e^{-\alpha(a-\delta)/\delta} |\, e^{i c\log(\cos(a-\delta))}\,|\delta^{-4}$.

Finally we observe that, $|\, e^{\pm i c\log(\cos(a-\delta))}\,| \le e^{|c|\pi/2}$ taking the main determination of the logarithm. Therefore, since $\|\Delta\varphi\|_{\mathcal{Y}} \le K\delta^{p+4}$ the lemma is proved. ■

3.2 The equation for $\Delta\varphi_2 := \Delta\varphi - \Delta\varphi_1$

We denote $\Delta\varphi_2 = \Delta\varphi - \Delta\varphi_1$ and we decompose $\Delta\varphi = (\Delta\psi, \Delta\eta)$ with $\Delta\psi = \pi^{1,2}\Delta\varphi$ and $\Delta\eta = \pi^3\Delta\varphi$. Analogously, we will write $\Delta\varphi_2 = (\Delta\psi_2, \Delta\eta_2)$.

We recall that by (62), (60) and definition (65) of I, we have that $\Delta\varphi_1 = I(c_1^0, c_2^0)$ and by (63) in Lemma 3.1, $|c_1^0| = |c_2^0| \le K\delta^p e^{-\alpha a/\delta}$.

The following lemma expresses $\Delta\varphi_2$ in a more appropriate way in terms of $I(k_1, k_2)$ and of the linear operator $\mathcal{F}_0$. We also provide useful bounds of k_1 and k_2.

Lemma 3.6 $\Delta\varphi_2 = \Delta\varphi - \Delta\varphi_1$ *satisfies the fixed point equation given by*

$$\Delta\varphi_2(\mathrm{i}\,t) = I(c_1 - c_1^0, c_2 - c_2^0)(t) + \mathcal{F}_0(\Delta\varphi_1)(\mathrm{i}\,t) + \mathcal{F}_0(\Delta\varphi_2)(\mathrm{i}\,t). \tag{69}$$

Moreover, there exists a constant K such that

$$|c_1 - c_1^0|, |c_2 - c_2^0| \leq K\delta^{2p+2}\,\mathrm{e}^{-\alpha a/\delta}\,. \tag{70}$$

Proof. To prove (69), we only have to take into account expression (66) of $\Delta\varphi$ in Lemma 3.5, that $\Delta\varphi_1(\mathrm{i}\,t) = I(c_1^0, c_2^0)(t)$ and the fact that $\mathcal{F}_0$ is linear.

Now we deal with (70). As we pointed out in Remark 3.1, $\Delta\psi_1 = \pi^{1,2}(\varphi_1^{\mathrm{u}} - \varphi_1^{\mathrm{s}})$, thus we have that $\Delta\psi_2 = \pi^{1,2}(\varphi_2^{\mathrm{u}} - \varphi_2^{\mathrm{s}})$, where we recall that $\varphi_2^{\mathrm{s,u}}$ were defined in Proposition 2.5. Then, it is clear that, by item 2. in Proposition 2.5,

$$|\pi^{1,2}\Delta\psi_2(\mathrm{i}\,t)\cos^3 t| \leq \|\varphi_2^{\mathrm{s}}\| + \|\varphi_2^{\mathrm{u}}\| \leq K\delta^{2p+6}. \tag{71}$$

In addition, we observe that, taking into account expression (62) of $\Delta\psi_1$,

$$\begin{aligned} c_1^0 &= \Delta\psi_1(\mathrm{i}(a-\delta))\,\mathrm{e}^{-\alpha(a-\delta)}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta) \\ c_1^0 &= \Delta\psi_1(-\mathrm{i}(a-\delta))\,\mathrm{e}^{-\alpha(a-\delta)}\,\mathrm{e}^{\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta) \end{aligned}$$

(an analogous formula can be deduced for c_1 and c_2 using $\Delta\psi$ instead of $\Delta\psi_1$). Therefore,

$$\begin{aligned} c_1 - c_1^0 &= \mathrm{e}^{-\alpha(a-\delta)/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta)\pi^1\big(\Delta\psi(\mathrm{i}(a-\delta)) - \Delta\psi_1(\mathrm{i}(a-\delta))\big) \\ &= \mathrm{e}^{-\alpha(a-\delta)/\delta}\,\mathrm{e}^{\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta)\pi^1\Delta\psi_2(\mathrm{i}(a-\delta)) \qquad (72) \\ c_2 - c_2^0 &= \mathrm{e}^{-\alpha(a-\delta)/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta)\pi^2\big(\Delta\psi(-\mathrm{i}(a-\delta)) - \Delta\psi_1(-\mathrm{i}(a-\delta))\big) \\ &= \mathrm{e}^{-\alpha(a-\delta)/\delta}\,\mathrm{e}^{-\mathrm{i}\,c\log(\cos(a-\delta))}\cos^{-1}(a-\delta)\pi^2\Delta\psi_2(-\mathrm{i}(a-\delta)). \qquad (73) \end{aligned}$$

Hence, using (71) to bound $\pi^{1,2}\Delta\psi_2(\pm\,\mathrm{i}(a-\delta))$ in expressions (72) and (73), we obtain

$$|c_1 - c_1^0|, |c_2 - c_2^0| \leq K\delta^{2p+2}\,\mathrm{e}^{-\alpha a/\delta}\,.$$

■

3.3 Exponentially smallness of $\Delta\varphi_2$

Let us introduce the functional spaces

$$\begin{aligned} \mathcal{Z}_1 &= \{f : E \to \mathbb{C} : f \text{ is continuous and } \sup_{\mathrm{i}\,t\in E} |\,\mathrm{e}^{\alpha(a-|t|)/\delta}\cos^{-1} t\; f(\mathrm{i}\,t)| < +\infty\} \\ \mathcal{Z}_2 &= \{f : E \to \mathbb{C} : f \text{ is continuous and } \sup_{\mathrm{i}\,t\in E} |\,\mathrm{e}^{\alpha(a-|t|)/\delta}\cos t\, f(\mathrm{i}\,t)| < +\infty\}. \end{aligned}$$

We endow $\mathcal{Z}_1$ and $\mathcal{Z}_2$ with the norms

$$\|f_1\|_{\mathcal{Z}_1} = \sup_{\mathrm{i}\,t\in E} |\,\mathrm{e}^{\alpha(a-|t|)/\delta}\,\frac{1}{\cos t} f_1(\mathrm{i}\,t)|$$
$$\|f_2\|_{\mathcal{Z}_2} = \sup_{\mathrm{i}\,t\in E} |\,\mathrm{e}^{\alpha(a-|t|)/\delta}\cos t\, f_2(\mathrm{i}\,t)|.$$

We also consider the product space $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_1 \times \mathcal{Z}_2$ with the norm

$$\|g\|_{\mathcal{Z}} = \|\pi^1 g\|_{\mathcal{Z}_1} + \|\pi^2 g\|_{\mathcal{Z}_1} + |\log\delta|^{-1}\delta^{-p-4}\|\pi^3 g\|_{\mathcal{Z}_2}. \tag{74}$$

We notice that, if $g \in \mathcal{Z}$

$$|\pi^1 g(0)|, |\pi^2 g(0)| \le \|g\|_{\mathcal{Z}}\,\mathrm{e}^{-\alpha a/\delta} \qquad \text{and} \qquad |\pi^3 g(0)| \le \|g\|_{\mathcal{Z}}\delta^{p+4}|\log\delta|\,\mathrm{e}^{-\alpha a/\delta}.$$

Hence, in order to prove Proposition 3.3 we have to check that $\Delta\varphi_2$ expressed as (69) belongs to $\mathcal{Z}$ and $\|\Delta\varphi_2\|_{\mathcal{Z}} \le K\delta^{2p+2}$. Our method to prove these properties will be to check that both $I(c_1 - c_1^0, c_2 - c_2^0)$ and $\mathcal{F}_0(\Delta\varphi_1)$ belong to $\mathcal{Z}$ and that the operator $\mathrm{Id} - \mathcal{F}_0$ is invertible in $\mathcal{Z}$. Moreover, since we have to bound $\|\Delta\varphi_2\|_{\mathcal{Z}}$, we also provide bounds of $\|\mathcal{F}_0\|_{\mathcal{Z}} := \max\{\|\mathcal{F}_0(f)\|,\ \|f\|_{\mathcal{Z}} = 1\}$.

All the properties related to the operator $\mathcal{F}_0$ are enunciated and proved below in Subsection 3.3.1. The remaining part of the proof of the exponentially smallness of $\Delta\varphi_2$ is given in Subsection 3.3.2.

3.3.1 The operator $\mathcal{F}_0$

First we state a technical lemma related to matrix B defined in (58).

Lemma 3.7 *The matrix $B = (b_{i,j})$ satisfies that there exists a constant K independent of δ such that for all $i, j \in \{1, 2, 3\}$,*

$$\sup_{\mathrm{i}\,t\in E} |\cos^2 t\, b_{i,j}(\mathrm{i}\,t)| \le K\delta^{p+3}|\log\delta|, \qquad \text{for } \mathrm{i}\,t \in E.$$

Proof. We denote $\psi_\lambda = \varphi^{\mathrm{s}} + \lambda\Delta\varphi = (\xi_\lambda, \overline{\xi}_\lambda, \eta_\lambda)$. It is clear that, by Lemma 3.5 and Proposition 2.5, $\psi_\lambda \in \mathcal{Y}$ and $\|\psi_\lambda\|_{\mathcal{Y}} \le K\delta^{p+4}$.

Let $b_{i,j}(t)$, $i, j = 1, 2, 3$ be the coefficients of the matrix B defined in (58). We have that there exists a constant C such that for all $\mathrm{i}\,t \in E$

$$\begin{aligned} |b_{1,j}(\mathrm{i}\,t)| &\le |\pi^1 D\mathcal{R}(\psi_\lambda(\mathrm{i}\,t))| \\ &\le |\delta^{p+1} DF_1(\delta\xi_\lambda(\mathrm{i}\,t), \delta\overline{\xi}_\lambda(\mathrm{i}\,t), \delta(z_0(\mathrm{i}\,t) + \eta_\lambda(\mathrm{i}\,t)), \delta)| + C|\psi_\lambda(\mathrm{i}\,t)|. \end{aligned} \tag{75}$$

We recall that $\mathcal{R}$ was defined in (10), $|\cdot|$ denotes the maximum norm in $\mathbb{C}^3$ and $D = D_{\xi,\overline{\xi},\eta}$ denotes the first derivative with respect to $(\xi, \overline{\xi}, \eta)$. Then, using bound (47) of DF, that $|\cos^2 t\, \eta_\lambda(\mathrm{i}\, t)| \leq K\delta^{p+3}|\log\delta|$ and that $|\cos^2 t\, \xi_\lambda(\mathrm{i}\, t)|, |\cos^2 t\, \overline{\xi}_\lambda(\mathrm{i}\, t)| \leq K\delta^{p+4}(\cos t)^{-1} \leq K\delta^{p+3}$, we can bound (75) obtaining

$$|\cos^2 t b_{i,j}(\mathrm{i}\, t)| \leq 4\delta^{p+3}C_F|\cos^2 t|\big(\max\{|\psi_\lambda(\mathrm{i}\, t)|, |z_0(t)|, 1\}\big)^2 + K\delta^{p+3}|\log\delta|$$

if $t \in D_2 \cap E$. Finally, using an analogous bound as the one given in (50) of the quantity $\max\{|\psi_\lambda(\mathrm{i}\, t)|, |z_0(t)|, 1\}$, we obtain the result for $\mathrm{i}\, t \in D_2 \cap E$ provided that $|\cos t| \leq K_1|t - \mathrm{i}\, a|$ for some K_1. If $\mathrm{i}\, t \in D_3 \cap E$, we proceed in a similar way. We also can bound $|b_{2,j}|$ and $|b_{3,j}|$ and the lemma holds true. ■

The next lemma provides all the properties of $\mathcal{F}_0$ that we will use later on. We recall that $\mathcal{F}_0$ was defined in (64).

Lemma 3.8 *If $p > -2$, the linear operator $Id - \mathcal{F}_0$ is invertible in $\mathcal{Z}$. Moreover $\|\mathcal{F}_0\|_{\mathcal{Z}} \leq K\delta^{p+2}|\log\delta|$ and hence $\|(Id - \mathcal{F}_0)^{-1}\|_{\mathcal{Z}} \leq \big(1 - \|\mathcal{F}_0\|_{\mathcal{Z}}\big)^{-1} \leq 1 + K\delta^{p+2}|\log\delta|$.*

Proof. During the proof of this lemma, we will denote by K any constant independent of δ.

Since $\mathcal{F}_0$ is a linear operator, to prove this lemma we only have to check that $\|\mathcal{F}_0\|_{\mathcal{Z}} < K\delta^{p+2}|\log\delta|^2 < 1$, provided $p > -2$.

Let $h \in \mathcal{Z}$. We note that, by Lemma 3.7 and definition (74) of norm $\|\cdot\|_{\mathcal{Z}}$,

$$\begin{aligned}
|\pi^j(B(\mathrm{i}\, t)h(\mathrm{i}\, t))| &\leq K\delta^{p+3}|\log\delta|\frac{1}{\cos^2 t}\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \|h\|_{\mathcal{Z}}\left(2|\cos t| + |\log\delta|\delta^{p+4}\frac{1}{|\cos t|}\right)\\
&\leq K\delta^{p+3}|\log\delta|\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \|h\|_{\mathcal{Z}}\frac{1}{|\cos t|}\left(2 + |\log\delta|\delta^{p+4}\frac{1}{|\cos^2 t|}\right)\\
&\leq K\delta^{p+3}|\log\delta|\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \|h\|_{\mathcal{Z}}\frac{1}{|\cos t|} \qquad (76)
\end{aligned}$$

for $j = 1, 2, 3$. In the last inequality we have used that $p > -2$ and that δ is small enough. Therefore, we have that, for all $\mathrm{i}\, t \in E$, that is $|t| \leq a - \delta$:

$$\begin{aligned}
|\pi^1\mathcal{G}(h)(\mathrm{i}\, t)| &\leq \mathrm{e}^{\alpha t/\delta}\, \mathrm{e}^{|c|\pi/2} \cos t \int_t^{a-\delta} \frac{\mathrm{e}^{|c|\pi/2}}{\cos r}\, \mathrm{e}^{-\alpha r/\delta}\, |\pi^1(B(\mathrm{i}\, r)h(\mathrm{i}\, r))|\, dr\\
&\leq K\delta^{p+3}|\log\delta|\, \mathrm{e}^{-\alpha a/\delta}\, \mathrm{e}^{\alpha t/\delta}\, \mathrm{e}^{|c|\pi} \cos t\|h\|_{\mathcal{Z}} \int_t^{a-\delta} \frac{1}{\cos^2 r}\, \mathrm{e}^{-\alpha(r-|r|)/\delta}\, dr \qquad (77)
\end{aligned}$$

It is not difficult to check that there exists a constant C independent of δ such that, for any $t \in [-(a-\delta), (a-\delta)]$,

$$\mathrm{e}^{\alpha t/\delta} \int_t^{a-\delta} \frac{1}{\cos^2 r}\, \mathrm{e}^{-\alpha(r-|r|)/\delta}\, dr \leq C\, \mathrm{e}^{\alpha|t|/\delta}\, \delta^{-1}.$$

Using the previous bound in (77) we have that

$$\|\pi^1\mathcal{G}(h)\|_{\mathcal{Z}_1} = \sup_{\mathrm{i}\,t\in E} |\pi^1\mathcal{G}(h)(t)\, \mathrm{e}^{\alpha(a-|t|)/\delta} \cos^{-1} t| \leq K\delta^{p+2}|\log\delta|\|h\|_{\mathcal{Z}}. \tag{78}$$

In the same way one can check that

$$\|\pi^2\mathcal{G}(h)\|_{\mathcal{Z}_1} = \sup_{\mathrm{i}\,t\in E} |\pi^2\mathcal{G}(h)(t)\, \mathrm{e}^{\alpha(a-|t|)/\delta} \cos^{-1} t| \leq K\delta^{p+2}|\log\delta|\|h\|_{\mathcal{Z}}. \tag{79}$$

Now we deal with the operator $\mathcal{H}$. First we note that, by bounds (78) and (79) using again Lemma 3.7 to bound $b_{i,j}(t)$, we have that, since $|\cos t| \geq \delta$,

$$\begin{aligned}
|\pi^3\big(B(\mathrm{i}\,t)(\mathcal{G}(h)(t), \pi^3 h(\mathrm{i}\,t))\big)| \leq & K\delta^{p+3}|\log\delta|\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \frac{1}{\cos^2 t} \\
& \cdot\Big(|\cos t|\big(\|\pi^1\mathcal{G}(h)\|_{\mathcal{Z}_1} + \|\pi^2\mathcal{G}(h)\|_{\mathcal{Z}_1}\big) + \frac{\delta^{p+4}|\log\delta|}{|\cos t|}\|h\|_{\mathcal{Z}_2}\Big) \\
\leq & K\delta^{2p+5}|\log\delta|^2\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \|h\|_{\mathcal{Z}} \frac{1}{|\cos t|}.
\end{aligned}$$

Then,

$$|\mathcal{H}(h)(\mathrm{i}\,t)| \leq K\delta^{2p+5}|\log\delta|^2\, \mathrm{e}^{-\alpha a/\delta}\, \|h\|_{\mathcal{Z}} \frac{1}{\cos^2 t}\left|\int_0^t \cos r\, \mathrm{e}^{\alpha|r|/\delta}\, dr\right| \tag{80}$$

and using that

$$\begin{aligned}
\left|\int_0^t \cos r\, \mathrm{e}^{\alpha|r|/\delta}\, dr\right| &= \mathrm{e}^{\alpha|t|/\delta}\, \frac{\delta}{\alpha^2+\delta^2}\big(\alpha\cos t + \delta\sin|t|\big) - \alpha\frac{\delta}{\alpha^2+\delta^2} \\
&\leq C\frac{\delta}{\alpha}\, \mathrm{e}^{\alpha|t|/\delta}\cos t \qquad (81)
\end{aligned}$$

for some constant $C > 0$, we can prove that

$$|\mathcal{H}(h)(\mathrm{i}\,t)| \leq K\delta^{2p+6}|\log\delta|^2\, \mathrm{e}^{-\alpha(a-|t|)/\delta}\, \|h\|_{\mathcal{Z}} \frac{1}{|\cos t|}. \tag{82}$$

Finally, using bounds (78), (79) and (82) and the definitions of $\mathcal{F}_0$ and $\|\cdot\|_{\mathcal{Z}}$ we have that

$$\begin{aligned}
\|\mathcal{F}_0(h)\|_{\mathcal{Z}} &= \|\pi^1\mathcal{G}(h)\|_{\mathcal{Z}_1} + \|\pi^2\mathcal{G}(h)\|_{\mathcal{Z}_1} + \delta^{-p-4}|\log\delta|^{-1}\|\mathcal{H}(h)(t)\|_{\mathcal{Z}_2} \\
&\leq K\delta^{p+2}|\log\delta|\|h\|_{\mathcal{Z}}.
\end{aligned}$$

Therefore, $\|\mathcal{F}_0\|_{\mathcal{Z}} < 1$ provided that $p > -2$ and δ is small enough; this implies that the linear operator $\mathrm{Id} - \mathcal{F}_0$ is invertible and moreover $\|\big(\mathrm{Id} - \mathcal{F}_0\big)^{-1}\|_{\mathcal{Z}} \leq (1 - \|\mathcal{F}_0\|_{\mathcal{Z}})^{-1} \leq 1 + K\delta^{p+2}|\log\delta|$. ■

3.3.2 End of the proof of Proposition 3.3

At the beginning of Subsection 3.3 we explained our strategy to prove Proposition 3.3. The first step, the study of the linear operator $\mathcal{F}_0$, has been done in the previous subsection.

We recall that $\Delta\varphi_1 = I(c_1^0, c_2^0)$ where I was defined in (65). Taking into account this expression, we also notice that, by (69) in Lemma 3.6 we have that

$$(\mathrm{Id} - \mathcal{F}_0)\Delta\varphi_2 = I(c_1 - c_1^0, c_2 - c_2^0) + \mathcal{F}_0(I(c_1^0, c_2^0)).$$

It only remains to check that both $I(c_1 - c_1^0, c_2, c_2^0)$ and $I(c_1^0, c_2^0)$ belongs to $\mathcal{Z}$. This is done in the lemma below.

Lemma 3.9 *Given $k_1, k_2 \in \mathbb{C}$, $I(k_1, k_2) \in \mathcal{Z}$ and*

$$\|I(k_1, k_2)\|_{\mathcal{Z}} \leq K(|k_1| + |k_2|)\, \mathrm{e}^{\alpha a/\delta}.$$

Proof. Throughout this proof K will denote any constant independent of δ.

We fix $k_1, k_2 \in \mathbb{C}$. It is clear that

$$|k_1|\, \mathrm{e}^{\alpha a/\delta} \sup_{\mathrm{i}\, t\in E} \mathrm{e}^{\alpha t/\delta}\, \mathrm{e}^{-\alpha|t|/\delta} + |k_2|\, \mathrm{e}^{\alpha a/\delta} \sup_{\mathrm{i}\, t\in E} \mathrm{e}^{-\alpha t/\delta}\, \mathrm{e}^{-\alpha|t|/\delta} = \mathrm{e}^{\alpha a/\delta}(|k_1| + |k_2|).$$

In order to bound the third component of $I(k_1, k_2)$, we use bound (76) with $h = (\pi^1 I(k_1, k_2), \pi^2 I(k_1, k_2), 0)$ and we obtain that, since $\|h\|_{\mathcal{Z}} \leq \mathrm{e}^{\alpha a/\delta}\, \mathrm{e}^{c\pi/2}(|k_1| + |k_2|)$,

$$|\pi^3\big(B(\mathrm{i}\, t)(\pi^1 I(k_1, k_2)(t), \pi^2 I(k_1, k_2)(t), 0)\big)| \leq K\delta^{p+3}|\log\delta|\, \mathrm{e}^{\alpha|t|/\delta}(|k_1| + |k_2|)\frac{1}{|\cos t|}.$$

Hence,

$$|\pi^3 I(k_1, k_2)(t)| \leq K\delta^{p+3}|\log\delta|(|k_1| + |k_2|)\frac{1}{\cos^2 t}\left|\int_0^t \cos r\, \mathrm{e}^{\alpha|r|/\delta}\, dr\right|$$

and finally using (81) to bounding the last integral, we get

$$|\pi^3 I(k_1, k_2)(t)| \leq K\delta^{p+4}|\log\delta|(|k_1| + |k_2|)\frac{1}{|\cos t|}\, \mathrm{e}^{\alpha|t|/\delta}$$

and the result is proved since

$$\begin{aligned}\|I(k_1, k_2)\|_{\mathcal{Z}} &= \|\pi^1 I(k_1, k_2)\|_{\mathcal{Z}_1} + \|\pi^2 I(k_1, k_2)\|_{\mathcal{Z}_1} + |\log\delta|^{-1}\delta^{-p-4}\|\pi^3 I(k_1, k_2)\|_{\mathcal{Z}_2}\\ &\leq K(|k_1| + |k_2|)\, \mathrm{e}^{\alpha a/\delta}.\end{aligned}$$

■

Lemma 3.10 *$\Delta\varphi_2 \in \mathcal{Z}$ and it is determined by*

$$\Delta\varphi_2 = (Id - \mathcal{F}_0)^{-1}(I(c_1 - c_1^0, c_2 - c_2^0)) + (Id - \mathcal{F}_0)^{-1}(\mathcal{F}_0(\Delta\varphi_1)). \tag{83}$$

Moreover,

$$\|\Delta\varphi_2\|_{\mathcal{Z}} \leq K\delta^{2p+2}|\log\delta|$$

Proof. We recall that $\Delta\varphi_2$ satisfies equation (69) given in Lemma 3.6, that is we have that

$$(\mathrm{Id} - \mathcal{F}_0)(\Delta\varphi_2) = I(c_1 - c_1^0, c_2 - c_2^0) + \mathcal{F}_0(\Delta\varphi_1).$$

Hence, since by Lemma 3.8 the operator $\mathrm{Id} - \mathcal{F}_0$ is invertible in $\mathcal{Z}$, to prove (83) is equivalent to check that both $I(c_1 - c_1^0, c_2 - c_2^0)$ and $\mathcal{F}_0(\Delta\varphi_1)$ belong to $\mathcal{Z}$. As in Lemma 3.9 we have proved that $I(k_1, k_2) \in \mathcal{Z}$ for all $k_1, k_2 \in \mathbb{C}$, the functions $I(c_1 - c_1^0, c_2 - c_2^0)$ and $\Delta\varphi_1 = I(c_1^0, c_2^0)$ belong to $\mathcal{Z}$. Moreover, in Lemma 3.8, we saw that $\mathcal{F}_0(\mathcal{Z}) \subset \mathcal{Z}$ and hence $\mathcal{F}_0(\Delta\varphi_1) \in \mathcal{Z}$ which implies that $\Delta\varphi_2 \in \mathcal{Z}$.

Now we will prove the second part of the lemma: the bound of $\|\Delta\varphi_2\|_{\mathcal{Z}}$. Since $\Delta\varphi_2$ satisfies the identity (83),

$$\|\Delta\varphi_2\|_{\mathcal{Z}} \leq \|\big(\mathrm{Id} - \mathcal{F}_0\big)^{-1}\|_{\mathcal{Z}}\big(\|I(c_1 - c_1^0, c_2 - c_2^0)\|_{\mathcal{Z}} + \|\mathcal{F}_0\|_{\mathcal{Z}}\|\Delta\varphi_1\|_{\mathcal{Z}}\big).$$

Taking into account bound (63) of c_1^0, c_2^0 and bound (70) of Lemma 3.6 to estimate $c_1 - c_1^0, c_2 - c_2^0$, we are able to bound $\Delta\varphi_1 = I(c_1^0, c_2^0)$ and $I(c_1 - c_1^0, c_2 - c_2^0)$ by using Lemma 3.9. We also bound $\|\mathcal{F}_0\|_{\mathcal{Z}}$ and $\|\big(\mathrm{Id} - \mathcal{F}_0\big)^{-1}\|_{\mathcal{Z}}$ by using Lemma 3.8 and finally we obtain that, if δ is small enough,

$$\begin{aligned}\|\Delta\varphi_2\|_{\mathcal{Z}} &\leq 2K\,\mathrm{e}^{\alpha a/\delta}(|c_1 - c_1^0| + |c_2 - c_2^0|) + K\delta^{p+2}|\log\delta|\,\mathrm{e}^{\alpha a/\delta}(|c_1^0| + |c_2^0|)\\ &\leq K\delta^{2p+2}|\log\delta|\end{aligned}$$

and the lemma is proved. ■

End of the proof of Proposition 3.3. We note that, since $\pi^3\Delta\varphi(0) = \pi^3\Delta\varphi_1(0) = 0$, then $\pi^3\Delta\varphi_2(0) = 0$. Moreover, by Lemma 3.10 and definition (74) of $\|\cdot\|_{\mathcal{Z}}$, it is clear that

$$|\pi^{1,2}\Delta\varphi_2(0)| \leq K\delta^{2p+2}|\log\delta|\,\mathrm{e}^{-\alpha a/\delta}$$

and Proposition 3.3 is proved. ■

3.4 Proof of Proposition 3.4

We recall that $\pi^3\varphi_1(0) = 0$ and hence we only have to compute $\Delta\psi_1(0) = \pi^{1,2}\Delta\varphi_1(0)$. We also notice that

$$\Delta\psi_1(0) = (c_1^0, c_2^0)^{\mathrm{T}} \tag{84}$$

where c_1^0 and c_2^0 were defined in (61). Moreover, by Lemma 3.1, $c_1^0 = \overline{c_2^0}$; hence we only have to calculate c_1^0.

We recall that by definition (11) of $\mathcal{M}$, we have $\pi^1\mathcal{M}(0) = \delta^p F_1(0,0,\delta z_0(t),\delta) = \delta^p(f + \mathrm{i}\, g)(0,0,\delta z_0(t),\delta)$. Moreover, since $(f,g)(0,0,\delta z,\delta) = O(|(\delta z,\delta)|^3)$ one has:

$$(f + \mathrm{i}\, g)(0,0,\delta z,\delta) = \delta^{p+3}a_0(\delta) + \delta^{p+3}a_1(\delta)z + \delta^{p+3}a_2(\delta)z^2 + \sum_{n\geq 3}\delta^{p+n}a_n(\delta)z^n \tag{85}$$

where a_n are bounded and analytic functions in $B(\delta_0)$. Consequently:

$$\begin{aligned}\mathcal{M}(0)(t) =& \delta^{p+3}a_0(\delta) - \delta^{p+3}a_1(\delta)\tanh t + \delta^{p+3}a_2(\delta)\tanh^2 t\\ &+ \sum_{n\geq 3}\delta^{p+n}(-1)^n a_n(\delta)\tanh^n t.\end{aligned} \tag{86}$$

We denote

$$I_n := I_n(\alpha,c) = \int_{-\infty}^{+\infty}\frac{\sinh^n r}{(\cosh r)^{n+1+\mathrm{i}\, c}}\,\mathrm{e}^{\mathrm{i}\,\alpha r/\delta}\,dr \tag{87}$$

and we observe that, by expression (86) of $\mathcal{M}(0)$ and definition (61) of c_1^0 we have that

$$c_1^0 = \delta^{p+3}a_0(\delta)I_0 - \delta^{p+3}a_1(\delta)I_1 + \delta^{p+3}a_2(\delta)I_2 + \sum_{n\geq 3}\delta^{p+n}(-1)^n a_n(\delta)I_n. \tag{88}$$

To get the asymptotic expression of Proposition 3.4 we have to estimate I_n. First we claim that I_n satisfies the recurrence relation

$$I_n = \frac{1}{n+\mathrm{i}\, c}\frac{\mathrm{i}\,\alpha}{\delta}I_{n-1} + \frac{n-1}{n+\mathrm{i}\, c}I_{n-2}.$$

The claim follows easily doing parts in definition (87) of I_n. We define the sequence $\{J_n\}_{n\geq 0}$ by

$$J_n = \frac{1}{n+\mathrm{i}\, c}\frac{\mathrm{i}\,\alpha}{\delta}J_{n-1}, \qquad J_0 = I_0$$

and we claim that $I_n = J_n(1+\overline{I}_n)$ with $\overline{I}_n$ satisfying that $\overline{I}_0 = \overline{I}_1 = 0$ and $|\overline{I}_n| \leq \delta|\Gamma(n+1+\mathrm{i}\, c)|$. Indeed, it is clear that $I_1 = J_1$ and that $\overline{I}_n$ satisfies the recurrence relation given by

$$\overline{I}_n = \overline{I}_{n-1} + (n-1)(n-1+\mathrm{i}\, c)\left(\frac{\delta}{\alpha\,\mathrm{i}}\right)^2(1+\overline{I}_{n-2}), \qquad \overline{I}_0 = \overline{I}_1 = 0.$$

Now we proceed by induction. Let us assume that $|\overline{I}_k| \leq \delta|\Gamma(k+1+\mathrm{i}\,c)|$ for all $k<n$. Henceforth,

$$\begin{aligned}
\overline{I}_n &\leq \delta|\Gamma(n+\mathrm{i}\,c)| + (n-1)|n-1+\mathrm{i}\,c|\frac{\delta^2}{\alpha^2}(1+\delta|\Gamma(n-1+\mathrm{i}\,c)|) \\
&= \delta|\Gamma(n+\mathrm{i}\,c)|\Big(1+\frac{\delta^2}{\alpha^2}(n-1)\Big) + (n-1)|n-1+\mathrm{i}\,c|\frac{\delta^2}{\alpha^2} \\
&= \delta|\Gamma(n+\mathrm{i}\,c)|\Big(1+\frac{\delta^2}{\alpha^2}(n-1) + \frac{n-1}{|\Gamma(n-1+\mathrm{i}\,c)|}\frac{\delta}{\alpha^2}\Big) \\
&\leq \delta|\Gamma(n+\mathrm{i}\,c)|n \leq \delta|\Gamma(n+1+\mathrm{i}\,c)|
\end{aligned}$$

if δ is small enough, but independent of n. Here we have used that $\Gamma(z+1) = z\Gamma(z)$ and that $n/|\Gamma(n+\mathrm{i}\,c)|$ is bounded for all n.

Now we are going to estimate J_n. We have that

$$J_n = \prod_{k=1}^{n} \frac{1}{k+\mathrm{i}\,c}\left(\frac{\alpha\,\mathrm{i}}{\delta}\right)^n J_0 = \left(\frac{\alpha\,\mathrm{i}}{\delta}\right)^n \frac{\Gamma(1+\mathrm{i}\,c)}{\Gamma(n+1+\mathrm{i}\,c)} J_0, \quad n\geq 1. \tag{89}$$

Performing the change of variables $s=\tanh r$ we get that

$$J_0 = \int_{-\infty}^{+\infty} \frac{1}{(\cosh r)^{1+\mathrm{i}\,c}}\, \mathrm{e}^{\alpha\,\mathrm{i}\,r/\delta}\, dr = \int_{-1}^{1} (1+s)^{(-1+\mathrm{i}\,c)/2+\alpha\,\mathrm{i}/(2\delta)}(1-s)^{(-1+\mathrm{i}\,c)/2-\alpha\,\mathrm{i}/(2\delta)}\, ds.$$

This integral can be expressed as a confluent hypergeometric function (see pag 505, [AS92] for the definition):

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} 2^{1-b}\,\mathrm{e}^{z/2}\int_{-1}^{1} \mathrm{e}^{-zt/2}(1+t)^{b-a-1}(1-t)^{a-1}\,dt, \quad \operatorname{Re} b > \operatorname{Re} a > 0,$$

taking $a = (1+\mathrm{i}\,c)/2 - \mathrm{i}\,\alpha/(2\delta)$, $b = 1+\mathrm{i}\,c$ and $z=0$. It is well known, that $M(a,b,0)=1$. Hence we have that

$$J_0 = 2^{\mathrm{i}\,c}\frac{1}{\Gamma(1+\mathrm{i}\,c)}\Gamma\left(\frac{1+\mathrm{i}\,c}{2}+\frac{\alpha\,\mathrm{i}}{2\delta}\right)\Gamma\left(\frac{1+\mathrm{i}\,c}{2}-\frac{\alpha\,\mathrm{i}}{2\delta}\right)$$

and thus, substituting the previous expression of J_0 in (89) we have that

$$J_n = \left(\frac{\mathrm{i}\,\alpha}{\delta}\right)^n \frac{1}{\Gamma(n+1+\mathrm{i}\,c)} 2^{\mathrm{i}\,c}\,\Gamma\left(\frac{1+\mathrm{i}\,c}{2}+\frac{\alpha\,\mathrm{i}}{2\delta}\right)\Gamma\left(\frac{1+\mathrm{i}\,c}{2}-\frac{\alpha\,\mathrm{i}}{2\delta}\right). \tag{90}$$

Now we use that for any $z\in\mathbb{C}$ such that $|\arg(z)|<\pi$, $\Gamma(z+a) = \Gamma(z+b)z^{a-b}(1+O(z^{-1}))$ for arbitrary $a,b\in\mathbb{C}$ (see [EMOT53], pag 47) to obtain that

$$\Gamma\left(\frac{1+\mathrm{i}\,c}{2}\pm\frac{\alpha\,\mathrm{i}}{2\delta}\right) = \Gamma\left(\frac{1}{2}\pm\frac{\alpha\,\mathrm{i}}{2\delta}\right)\left(\pm\frac{\alpha\,\mathrm{i}}{2\delta}\right)^{\mathrm{i}\,c/2}(1+O(\delta))$$

(taking $z = \pm\alpha\,\mathrm{i}/(2\delta)$, $a = (1+\mathrm{i}\,c)/2$ and $b = 1/2$) and thus, using that $\Gamma(z)\Gamma(\overline{z}) = |\Gamma(z)|^2$, from (90) we have that

$$
\begin{aligned}
J_n &= \left(\frac{\mathrm{i}\,\alpha}{\delta}\right)^n \frac{1}{\Gamma(n+1+\mathrm{i}\,c)} \left|\Gamma\left(\frac{1}{2}+\frac{\mathrm{i}\,|\alpha|}{2\delta}\right)\right|^2 2^{\mathrm{i}\,c} \left|\frac{\mathrm{i}\,|\alpha|}{2\delta}\right|^{\mathrm{i}\,c} (1+O(\delta)) \\
&= \mathrm{i}^n \left(\frac{\alpha}{\delta}\right)^n \left(\frac{|\alpha|}{\delta}\right)^{\mathrm{i}\,c} \frac{1}{\Gamma(n+1+\mathrm{i}\,c)} \left|\Gamma\left(\frac{1}{2}+\frac{\mathrm{i}\,|\alpha|}{2\delta}\right)\right|^2 (1+O(\delta)). \qquad (91)
\end{aligned}
$$

Where $O(\delta)$ is independent of n. Finally, using that $\Gamma(z) = \mathrm{e}^{-z}\,\mathrm{e}^{(z-1/2)\log z}(2\pi)^{1/2}(1+O(z^{-1}))$ for any $z \in \mathbb{C}$ with $|\arg z| < \pi$, we have that, if δ is small enough,

$$
\left|\Gamma\left(\frac{1}{2}+\frac{\mathrm{i}\,|\alpha|}{2\delta}\right)\right|^2 = 2\pi\,\mathrm{e}^{-\pi|\alpha|/(2\delta)}(1+O(\delta)). \qquad (92)
$$

Therefore, using (92) in (91) we obtain that for any α, c J_n can be expressed as

$$
J_n = \mathrm{i}^n \left(\frac{\alpha}{\delta}\right)^n \left(\frac{|\alpha|}{\delta}\right)^{\mathrm{i}\,c} \frac{2\pi}{\Gamma(n+1+\mathrm{i}\,c)}\,\mathrm{e}^{-\pi|\alpha|/(2\delta)}(1+O(\delta)), \qquad (93)
$$

therefore

$$
\begin{aligned}
I_n(\alpha,c) &= \mathrm{i}^n \left(\frac{\alpha}{\delta}\right)^n \left(\frac{|\alpha|}{\delta}\right)^{\mathrm{i}\,c} \frac{2\pi}{\Gamma(n+1+\mathrm{i}\,c)}\,\mathrm{e}^{-\pi|\alpha|/(2\delta)}(1+O(\delta))(1+\overline{I}_n(\alpha,c)) \\
&= \mathrm{i}^n \left(\frac{\alpha}{\delta}\right)^n \left(\frac{|\alpha|}{\delta}\right)^{\mathrm{i}\,c} \frac{2\pi}{\Gamma(n+1+\mathrm{i}\,c)}\,\mathrm{e}^{-\pi|\alpha|/(2\delta)} + O(\delta^{1-n})\,\mathrm{e}^{-\pi|\alpha|/(2\delta)},
\end{aligned}
$$

provided that $\overline{I}_n(\alpha,c)$ satisfy that $|\overline{I}_n(\alpha,c)| \leq \delta|\Gamma(n+1+\mathrm{i}\,c)|$.

To finish the proof of Proposition 3.4 we introduce $m_n = (-1)^n a_n(0)$, for $n \geq 3$. If we substitute the above asymptotic expression in equality (88) and we take into account that $I_n = O(\delta^{-n}\,\mathrm{e}^{-\pi|\alpha|/(2\delta)})$ we obtain an asymptotic formula for c_1^0. Then, we only need to use the fact that, by (84), $\pi^{1,2}\varphi_1(0) = \Delta\psi_1(0) = (c_1^0, c_2^0)^{\mathrm{T}}$.

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