KAM aspects of the quasiperiodic Hamiltonian Hopf bifurcation

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October, 18th 2007

Abstract

In this work we consider a 1:-1 non semi-simple resonant periodic orbit of a three-degrees of freedom real analytic Hamiltonian system. From the formal analysis of the normal form, we prove the branching off a two-parameter family of two-dimensional invariant tori of the normalized system, whose normal behavior depends intrinsically on the coefficients of its low-order terms. Thus, only elliptic or elliptic together with parabolic and hyperbolic tori may detach from the resonant periodic orbit. Both patterns are mentioned in the literature as the direct and inverse, respectively, quasiperiodic Hopf bifurcation. In this paper we focus on the direct case, which has many applications in several fields of science. Our target is to prove, in the framework of KAM theory, the persistence of most of the (normally) elliptic tori of the normal form, when the whole Hamiltonian is taken into account, and to give a very precise characterization of the parameters labelling them, which can be selected with a very clear dynamical meaning. Furthermore, we give sharp quantitative estimates on the "density" of surviving tori, when the distance to the resonant periodic orbit goes to zero, and show that the 4-dimensional Cantor manifold holding them admits a Whitney-C^∞ extension. Due to the strong degeneracy of the problem, some standard KAM methods for elliptic low-dimensional tori of Hamiltonian systems do not apply directly, so one needs to suite properly these techniques to the context.

AMS Mathematics Subject Classification: 37J20, 37J40
Keywords: Hopf Bifurcation, KAM Theory, Low Dimensional Tori, Normal Forms

Contents

1 Introduction

This paper is related with the persistence of the quasiperiodic Hopf bifurcation scenario in the Hamiltonian context. In its more simple formulation, we shall consider a real analytic three-degrees of freedom Hamiltonian system with a one-parameter family of periodic orbits undergoing a 1:-1 resonance for some value of the parameter. By 1:-1 resonance we mean that, for the corresponding *resonant* or *critical* periodic orbit, a pairwise collision of its characteristic nontrivial multipliers (i. e., those different from 1) takes place at two conjugate points on the unit circle. When varying the parameter it turns out that, generically, prior to the collision the nontrivial multipliers are different and lay on the unit circle (by conjugate pairs) and, after that, they leave it to the complex plane so the periodic orbits of the family become unstable. This mechanism of instabilization is often referred in the literature as *complex instability* (see [21]), and has been also studied for families of four-dimensional symplectic maps, where an elliptic fixed point evolves to a complex saddle as the parameter of the family moves (see [14, 39]).

Under general conditions, the branching off two-dimensional quasiperiodic solutions (respectively, invariant curves for mappings) from the resonant periodic orbit (respectively, the fixed point) has been described both numerically (in [13, 22, 34, 37, 38, 40]) and analytically (in [6, 28, 35]). The analytic approach relies on the computation of the normal form around the critical orbit. Thus, a biparametric family of two-dimensional invariant tori follows at once from the dynamics of the (integrable) normal form. This bifurcation can be *direct* or *inverse*. In our context, the direct case means that only normally elliptic tori unfold, while in the inverse case normally parabolic and hyperbolic tori are present as well. The type of bifurcation is determined by the coefficients of a low-order normal form.

However, this bifurcation pattern cannot be directly stated for the complete Hamiltonian, since the normal form computed at all orders is, generically, divergent. If we stop the normalizing process up to some finite order, the initial Hamiltonian is then casted (by means of a canonical transformation) into the sum of an integrable part plus a non-integrable remainder. Hence, the question is whether some quasiperiodic solutions of the integrable part survive in the whole system, and we know there are chances for this to happen if the remainder is sufficiently small to be thought of as a perturbation (see [16] for a nonperturbative approach to KAM theory).

This work tackles with the persistence of the elliptic bifurcated tori in the direct case. In the inverse case, elliptic and hyperbolic tori can be dealt in a complete analogous way (see remark 4.4) whilst parabolic tori require a slightly different approach (we refer to [8, 19, 20] for works concerning the persistence of parabolic invariant tori).

For the direct case, in theorem 3.1 we prove that there exists a two-parameter Cantor family of two-dimensional elliptic tori branching off the resonant periodic orbit. Moreover, we also give quantitative estimates on the (Lebesgue) measure, in the parameters space, of the holes between invariant tori and prove the Whitney-C^∞ smoothness of the 4D (Cantor) manifold holding them. Amid the features of this theorem, here we stress two. One is the precise description of the parameter set of "basic frequencies" for which we have an invariant bifurcated torus, i. e., the "geometry of the bifurcation". The other one is the sharp asymptotic measure estimates for the size of these holes, when the distance to the periodic orbit goes to zero.

The mere existence of this "smooth" family can be also derived from [6] —where the quasiperiodic Hopf bifurcation is considered under a more general setting—, by adding external parameters and applying Broer-Huitema-Takens theory to the extended system (see [10]). However, neither asymptotic quantitative measure estimates nor a discussion on the dynamical characterization of the parameters set follow directly from their approach.

The existence of this bifurcated family has some straight applications, for instance in Celestial Mechanics. Indeed, let us consider the so-called *vertical family* of periodic orbits of the (Lagrange) equilibrium point L_4 in the (spatial) Restricted Three Body Problem, that is, the Lyapunov family associated with the vertical oscillations of L_4. It turns out that, for values of the mass parameter

greater than the Routh's value, there appear normally elliptic 2D-tori linked to the transition stable-complex unstable of the family, that constitutes a direct (quasiperiodic) Hamiltonian Hopf bifurcation. These invariant tori were computed numerically in [34]. For other applications, see [35] and references therein.

When computing the normal form of a Hamiltonian around maximal dimensional tori, elliptic fixed points or normally elliptic periodic orbits or tori, there are (standard) results providing exponentially small estimates for the size of the remainder as function of the distance, R, to the object (if the order of the normal form is chosen appropriately as function of R). These estimates can be translated into bounds for the relative measure of the complement of the Cantor set of parameters for which we have invariant tori (see [7, 17, 24, 25, 26] for papers dealing with exponentially small measure estimates in KAM theory). In the present context, the generic situation at the resonant periodic orbit is a non semi-simple structure for the Jordan blocks of the monodromy matrix associated with the colliding characteristic multipliers. This yields to homological equations in the normal form computations that cannot be reduced to diagonal form. When the homological equations are diagonal, it means that only one "small divisor" appears as a denominator of any coefficient of the solution. In the non semi-simple case, there are (at any order) some coefficients having as a denominator a small divisor raised up to the order of the corresponding monomial. This fact gives rise to very big "amplification factors" in the normal form computations, that do not allow to obtain exponentially small estimates for the remainder. In [36] it is proved that it decays with respect to R faster than any power of R, but with less sharp bounds than in the semi-simple case. This fact translates into poor measure estimates for the bifurcated tori.

To prove the persistence of these tori, we are faced with KAM methods for elliptic low-dimensional tori (see [9, 10, 18, 25, 26, 41]). More precisely, the proof resembles those on the existence of invariant tori when adding to a periodic orbit the excitations of its elliptic normal modes (compare [18, 25, 43]), but with the additional intricacies due to the present bifurcation scenario. The main difficulty in tackling this persistence problem has to do with the choice of suitable parameters to characterize the tori of the family along the iterative KAM process. In this case one has three frequencies to control, the two intrinsic (those of the quasiperiodic motion) and the normal one, but only two parameters (those of the family) to keep track of them. So, we are bound to deal with the so-called "lack of parameters" problem for low-dimensional tori (see [9, 33, 44]). However, some usual tricks for dealing with elliptic tori cannot be applied directly to the problem at hand, for the reasons shown below.

When applying KAM techniques for invariant tori of Hamiltonian systems, it is usual to set a diffeomorphism between the intrinsic frequencies and the "parameter space" of the family of tori (typically the actions). In this way, in the case of elliptic low-dimensional tori, the normal frequencies can be expressed as a function of the intrinsic ones. Under these assumptions, the standard non-degeneracy conditions on the normal frequencies require that the denominators of the KAM process, which depend on the normal and intrinsic frequencies, "move" as function of the latter ones. Assuming these transversality conditions, the Diophantine ones can be fulfilled at each step of the KAM iterative process. Unfortunately, in the current context these conditions are not defined at the critical orbit, due to the strong degeneracy of the problem. In few words, the elliptic invariant tori we study are too close to parabolic. This catch is worked out taking as vector of *basic frequencies* (those labelling the tori) not the intrinsic ones, say $\Omega = (\Omega_1, \Omega_2)$, but the vector $\Lambda = (\mu, \Omega_2)$, where μ is the normal frequency of the torus. Then, we put the other (intrinsic) frequency as a function of Λ, i. e., $\Omega_1 = \Omega_1(\Lambda)$. With this parametrization, the denominators of the KAM process move with Λ even if we are close to the resonant periodic orbit.

Another difficulty we have to face refers to the computation of the sequence of canonical transformations of the KAM scheme. At any step of this iterative process we compute the corresponding canonical transformation by means of the Lie method. Typically in the KAM context, the (homological) equations verified by the generating function of this transformation are coupled through a triangular structure, so we can solve them recursively. However, due to the forementioned proximity

to parabolic, in the present case some equations —corresponding to the average of the system with respect to the angles of the tori— become simultaneously coupled, and have to be solved all together. Then, the resolution of the homological equations becomes a little more tricky, specially for what refers to the verification of the nondegeneracy conditions needed to solve them.

This work is organized as follows. We begin fixing the notation and introducing several definitions in section 2. In section 3 we formulate theorem 3.1, which constitutes the main result of the paper. Section 4 is devoted to review some previous results about the normal form around a $1:-1$ resonant periodic orbit (both from the qualitative and quantitative point of view). The proof of theorem 3.1 is given in section 5, whilst in appendix A we compile some technical results used throughout the text.

2   Basic notation and definitions

Given a complex vector $u \in \mathbb{C}^n$, we denote by $|u|$ its supremum norm, $|u| = \sup_{1 \leq i \leq n}\{|u_i|\}$. We extend this notation to any matrix $A \in \mathbb{M}_{r,s}(\mathbb{C})$, so that $|A|$ means the induced matrix norm. Similarly, we write $|u|_1 = \sum_{i=1}^{n}|u_i|$ for the absolute norm of a vector and $|u|_2$ for its Euclidean norm. We denote by u^* and A^* the transpose vector and matrix, respectively. As usual, for any $u, v \in \mathbb{C}^n$, their bracket $\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i$ is the inner product of $\mathbb{C}^n$. Moreover, $\lfloor \cdot \rfloor$ stands for the integer part of a real number.

We deal with analytic functions $f = f(\theta, x, I, y)$ defined in the domain

$$\mathcal{D}_{r,s}(\rho, R) = \{(\theta, x, I, y) \in \mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^r \times \mathbb{C}^s : |\operatorname{Im}\theta| \leq \rho, |(x, y)| \leq R, |I| \leq R^2\}, \tag{1}$$

for some integers r, s and some $\rho > 0$, $R > 0$. These functions are 2π-periodic in θ and take values on $\mathbb{C}$, $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$. By expanding f in Taylor-Fourier series (we use multi-index notation through the paper),

$$f = \sum_{(k,l,m) \in \mathbb{Z}^r \times \mathbb{Z}_+^r \times \mathbb{Z}_+^{2s}} f_{k,l,m} \exp(\mathrm{i}\langle k, \theta \rangle) I^l z^m, \tag{2}$$

where $z = (x, y)$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we introduce the weighted norm

$$|f|_{\rho,R} = \sum_{k,l,m} |f_{k,l,m}| \exp(|k|_1 \rho) R^{2|l|_1 + |m|_1}. \tag{3}$$

We observe that $|f|_{\rho,R} < +\infty$ implies that f is analytic in the interior of $\mathcal{D}_{r,s}(\rho, R)$ and bounded up to the boundary. Conversely, if f is analytic in a neighbourhood of $\mathcal{D}_{r,s}(\rho, R)$, then $|f|_{\rho,R} < +\infty$. Moreover, we point out that $|f|_{\rho,R}$ is an upper bound for the supremum norm of f in $\mathcal{D}_{r,s}(\rho, R)$. Some of the properties of this norm have been surveyed in section A.1. These properties are very similar to the corresponding ones for the supremum norm. We work with weighted norms instead of the supremum norm because some estimates become simpler with them, specially those on small divisors. Several examples of the use of these norms can be found in [17, 25, 36]. Alternatively, one can work with the supremum norm and use the estimates of Rüssmann on small divisors (see [42]).

For a complex-valued function $f = f(\theta, x, I, y)$ we use Taylor expansions of the form

$$f = \mathrm{a}(\theta) + \langle \mathrm{b}(\theta), z \rangle + \langle \mathrm{c}(\theta), I \rangle + \frac{1}{2}\langle z, \mathrm{B}(\theta) z \rangle + \langle I, \mathrm{E}(\theta) z \rangle + \frac{1}{2}\langle I, \mathrm{C}(\theta) I \rangle + \mathrm{F}(\theta, x, I, y), \tag{4}$$

with $\mathrm{B}^* = \mathrm{B}$, $\mathrm{C}^* = \mathrm{C}$ and F holding the higher order terms with respect to (z, I). From (4) we introduce the notations $[f]_0 = \mathrm{a}$, $[f]_z = \mathrm{b}$, $[f]_I = \mathrm{c}$, $[f]_{z,z} = \mathrm{B}$, $[f]_{I,z} = \mathrm{E}$, $[f]_{I,I} = \mathrm{C}$ and $[f] = \mathrm{F}$.

The coordinates $(\theta, x, I, y) \in \mathcal{D}_{r,s}(\rho, R)$ are canonical through the symplectic form $d\theta \wedge dI + dx \wedge dy$. Hence, given scalar functions $f = f(\theta, x, I, y)$ and $g = g(\theta, x, I, y)$, we define their *Poisson bracket* by

$$\{f, g\} = (\nabla f)^* \mathcal{J}_{r+s} \nabla g,$$

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where ∇ is the gradient with respect to (θ, x, I, y) and $\mathcal{J}_n$ the standard symplectic $2n \times 2n$ matrix. If $\Psi = \Psi(\theta, x, I, y)$ is a canonical transformation, close to the identity, then we consider the following expression of Ψ (according to its natural vector-components),

$$\Psi = \mathrm{Id} + (\Theta, \mathcal{X}, \mathcal{I}, \mathcal{Y}), \qquad \mathcal{Z} = (\mathcal{X}, \mathcal{Y}). \tag{5}$$

To generate such canonical transformations we mainly use the *Lie series method*. Thus, given a Hamiltonian $H = H(\theta, x, I, y)$ we denote by Ψ_t^H the flow time t of the corresponding vector field, $\mathcal{J}_{r+s}\nabla H$. We observe that if $\mathcal{J}_{r+s}\nabla H$ is 2π-periodic in θ, then also is $\Psi_t^H - \mathrm{Id}$.

Let $f = f(\theta)$ be a 2π-periodic function defined in the r-dimensional complex strip

$$\Delta_r(\rho) = \{\theta \in \mathbb{C}^r : |\mathrm{Im}\,\theta| \leq \rho\}. \tag{6}$$

If we expand f in Fourier series, $f = \sum_{k \in \mathbb{Z}^r} f_k \exp(\mathrm{i}\langle k, \theta \rangle)$, we observe that $|f|_{\rho,0}$ gives the weighted norm of f in $\Delta_r(\rho)$. Moreover, given $N \in \mathbb{N}$, we consider the following truncated Fourier expansions,

$$f_{<N,\theta} = \sum_{|k|_1 < N} f_k \exp(\mathrm{i}\langle k, \theta \rangle), \qquad f_{\geq N,\theta} = f - f_{<N,\theta}. \tag{7}$$

Notation (7) can also be extended to $f = f(\theta, x, I, y)$. Furthermore, we also introduce

$$L_\Omega f = \sum_{j=1}^r \Omega_j \partial_{\theta_j} f, \qquad \langle f \rangle_\theta = \frac{1}{(2\pi)^r} \int_{\mathbb{T}^r} f(\theta)\,\mathrm{d}\theta, \qquad \{f\}_\theta = f - \langle f \rangle_\theta, \tag{8}$$

where $\Omega \in \mathbb{R}^r$ and $\mathbb{T}^r = (\mathbb{R}/2\pi\mathbb{Z})^r$. We refer to $\langle f \rangle_\theta$ as the *average* of f.

Given an analytic function $f = f(u)$ defined for $u \in \mathbb{C}^n$, $|u| \leq R$, we consider its Taylor expansion around the origin, $f(u) = \sum_{m \in \mathbb{Z}_+^n} f_m u^m$, and define $|f|_R = \sum_m |f_m| R^{|m|_1}$.

Let $f = f(\phi)$ be a function defined for $\phi \in \mathcal{A} \subset \mathbb{C}^n$. For this function we denote its supremum norm and its Lipschitz constant by

$$|f|_\mathcal{A} = \sup_{\phi \in \mathcal{A}} |f(\phi)|, \qquad \mathrm{Lip}_\mathcal{A}(f) = \sup\left\{ \frac{|f(\phi') - f(\phi)|}{|\phi' - \phi|} : \phi, \phi' \in \mathcal{A},\ \phi \neq \phi' \right\}$$

Moreover, if $f = f(\theta, x, I, y; \phi)$ is a family of functions defined in $\mathcal{D}_{r,s}(\rho, R)$, for any $\phi \in \mathcal{A}$, we denote by $|f|_{\mathcal{A},\rho,R} = \sup_{\phi \in \mathcal{A}} |f(\cdot; \phi)|_{\rho,R}$.

Finally, given $\sigma > 0$, one defines the *complex σ-widening* of the set $\mathcal{A}$ as

$$\mathcal{A} + \sigma = \bigcup_{z \in \mathcal{A}} \{z' \in \mathbb{C}^n : |z - z'| \leq \sigma\}, \tag{9}$$

i. e., $\mathcal{A} + \sigma$ is the union of all (complex) balls of radius σ (in the norm $|\cdot|$) centered at points of $\mathcal{A}$.

3 Formulation of the main result

Let us consider a three-degrees of freedom real analytic Hamiltonian system $\mathcal{H}$ with a 1:-1 resonant periodic orbit. We assume that we have a system of symplectic coordinates specially suited for this orbit, so that the phase space is described by $(\theta, x, I, y) \in \mathbb{T}^1 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, being $x = (x_1, x_2)$ and $y = (y_1, y_2)$, endowed with the 2-form $\mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}x \wedge \mathrm{d}y$. In this reference system we want the periodic orbit to be given by the circle $I = 0$, $x = y = 0$. Such (local) coordinates can always be found for a given periodic orbit (see [11, 12] and [27] for an explicit example). In addition, a (symplectic) Floquet transformation is performed to reduce to constant coefficients the quadratic part of the Hamiltonian with respect to the *normal directions* (x, y) (see [28]). If the resonant eigenvalues of the monodromy

matrix of the critical orbit are non semi-simple, the Hamiltonian expressed in the new variables can be written as [1]

$$\mathcal{H}(\theta, x, I, y) = \omega_1 I + \omega_2(y_1 x_2 - y_2 x_1) + \frac{1}{2}(y_1^2 + y_2^2) + \widehat{\mathcal{H}}(\theta, x, I, y), \tag{10}$$

where ω_1 is the angular frequency of the periodic orbit and ω_2 its (only) normal frequency, so that its nontrivial characteristic multipliers are $\{\lambda, \lambda, 1/\lambda, 1/\lambda\}$, with $\lambda = \exp(2\pi i \omega_2/\omega_1)$. The function $\widehat{\mathcal{H}}$ is 2π-periodic in θ, holds the higher order terms in (x, I, y) and can be analytically extended to a complex neighbourhood of the periodic orbit. From now on, we set $\mathcal{H}$ to be our initial Hamiltonian.

To describe the dynamics of $\mathcal{H}$ around the critical orbit we use normal forms. A detailed analysis of the (formal) normal form for a 1:-1 resonant periodic orbit and of the (integrable) dynamics associated to it can be found in [35]. The only (generic) nonresonant condition required to carry out this normalization (at any order) is that $\omega_1/\omega_2 \notin \mathbb{Q}$, which is usually referred as *irrational collision*.

The normalized Hamiltonian of (10) up to "degree four" in (x, I, y) looks like

$$\mathcal{Z}_2(x, I, y) = \omega_1 I + \omega_2 L + \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(aq^2 + bI^2 + cL^2) + dqI + eqL + fIL, \tag{11}$$

where $q = (x_1^2 + x_2^2)/2$, $L = y_1 x_2 - y_2 x_1$ and $a, b, c, d, e, f \in \mathbb{R}$. As usual, the contribution of the action I to the degree is counted twice. Now, writing the Hamilton equations of $\mathcal{Z}_2$, it is easy to realize that the manifold $x = y = 0$ is foliated by a family of periodic orbits, parametrized by I, that contains the critical one. By assuming irrational collision, it is clear that —applying Lyapunov Center Theorem, see [45]— this family also exists (locally) for the full system (10). The (nondegeneracy) condition that determines the transition from stability to complex instability of this family is $d \neq 0$. Moreover, the direct or inverse character of the bifurcation is defined in terms of the sign of a and, for our concerns, $a > 0$ implies direct bifurcation. Hence, in the forthcoming we shall assume $d \neq 0$ and $a > 0$.

Once a direct quasiperiodic Hopf bifurcation is set, we can establish for the dynamics of $\mathcal{Z}_2$ and, in fact, for the dynamics of the truncated normal form up to an arbitrary order, the existence of a two-parameter family of two-dimensional elliptic tori branching off the resonant periodic orbit. Of course, due to the small divisors of the problem, it is not possible to expect full persistence of this family in the complete Hamiltonian system (10), but only a Cantor family of two-dimensional tori.

The precise result we have obtained about the persistence of this family is stated as follows, and constitutes the main result of the paper.

Theorem 3.1. *We assume that the real analytic Hamiltonian $\mathcal{H}$ in (10) is defined in the complex domain $\mathcal{D}_{1,2}(\rho_0, R_0)$, for some $\rho_0 > 0$, $R_0 > 0$, and that the weighted norm $|\mathcal{H}|_{\rho_0, R_0}$ is finite. Moreover, we also assume that the (real) coefficients a and d of its low-order normal form $\mathcal{Z}_2$ in (11), verify $a > 0$, $d \neq 0$, and that the vector $\omega = (\omega_1, \omega_2)$ satisfies the Diophantine condition[2]*

$$|\langle k, \omega \rangle| \geq \gamma |k|_1^{-\tau}, \qquad \forall\, k \in \mathbb{Z}^2 \setminus \{0\}, \tag{12}$$

for some $\gamma > 0$ and $\tau > 1$. Then, we have:

(i) The $1 : -1$ resonant periodic orbit $I = 0$, $x = y = 0$ of $\mathcal{H}$ is embedded into a one-parameter family of periodic orbits having a transition from stability to complex instability at this critical orbit.

(ii) There exists a Cantor set $\mathcal{E}^{(\infty)} \subset \mathbb{R}^+ \times \mathbb{R}$ such that, for any $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$, the Hamiltonian system $\mathcal{H}$ has an analytic two-dimensional elliptic invariant torus —with vector of intrinsic frequencies $\Omega(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$ and normal frequency μ— branching off the critical periodic orbit. However, for some values of Λ this torus is complex (i. e., a torus laying on the complex phase space but carrying out quasiperiodic motion for real time).

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[1] Nevertheless, to achieve this form, yet an involution in time may be necessary. See [35].

[2] The Lebesgue measure of the set of values $\omega \in \mathbb{R}^2$ for which condition (12) is not fulfilled is zero (see [30], appendix 4).

(iii) The "density" of the set $\mathcal{E}^{(\infty)}$ becomes almost one as we approach to the resonant periodic orbit. Indeed, there exist constants $c^* > 0$ and $\bar{c}^* > 0$ such that, if we define

$$\mathcal{V}(R) := \{\Lambda = (\mu, \Omega_2) \in \mathbb{R}^2 : 0 < \mu \leq c^* R, |\Omega_2 - \omega_2| \leq c^* R\}$$

and $\mathcal{E}^{(\infty)}(R) = \mathcal{E}^{(\infty)} \cap \mathcal{V}(R)$, then, for any given $0 < \alpha < 1/19$, there is $\breve{R}^* = \breve{R}^*(\alpha)$ such that

$$meas\left(\mathcal{V}(R) \setminus \mathcal{E}^{(\infty)}(R)\right) \leq \bar{c}^* (M^{(0)}(R))^{\alpha/4}, \tag{13}$$

for any $0 < R \leq \breve{R}^*$. Here, meas stands for the Lebesgue measure of $\mathbb{R}^2$ and the expression $M^{(0)}(R)$, which is defined precisely in the statement of theorem 4.1, goes to zero faster than any power of R (in spite of it is not exponentially small in R).

(iv) There exists a real analytic function $\widetilde{\Omega}_2$, with $\widetilde{\Omega}_2(0) = \omega_2$, such that the curves $\gamma_1(\eta) = (2\eta, \eta + \widetilde{\Omega}_2(\eta^2))$ and $\gamma_2(\eta) = (2\eta, -\eta + \widetilde{\Omega}_2(\eta^2))$, locally separate between the parameters $\Lambda \in \mathcal{E}^{(\infty)}$ giving rise to real or complex tori. Indeed, if $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$ and $\mu = 2\eta > 0$, then real tori are those with $-\eta + \widetilde{\Omega}_2(\eta^2) < \Omega_2 < \eta + \widetilde{\Omega}_2(\eta^2)$. The meaning of the curves γ_1 and γ_2 are that their graphs represent, in the Λ-space, the periodic orbits of the family (i), but only those in the stable side of the transition. For a given $\eta > 0$, the periodic orbit labelled by $\gamma_1(\eta)$ is identified by the one labelled by $\gamma_2(\eta)$, being $\eta + \widetilde{\Omega}_2(\eta^2)$ and $-\eta + \widetilde{\Omega}_2(\eta^2)$ the two normal frequencies of the orbit ($\eta = 0$ corresponds to the critical one).

(v) The function $\Omega_1^{(\infty)} : \mathcal{E}^{(\infty)} \to \mathbb{R}$ is C^∞ in the sense of Whitney. Moreover, for each $\Lambda \in \mathcal{E}^{(\infty)}$, the following Diophantine conditions are fulfilled by the intrinsic frequencies and the normal one of the corresponding torus:

$$|\langle k, \Omega^{(\infty)}(\Lambda)\rangle + \ell\mu| \geq (M^{(0)}(R))^{\alpha/2}|k|_1^{-\tau}, \qquad k \in \mathbb{Z}^2, \ell \in \{0, 1, 2\}, |k|_1 + \ell \neq 0.$$

(vi) Let $\breve{\mathcal{E}}^{(\infty)}$ be the subset of $\mathcal{E}^{(\infty)}$ corresponding to real tori. There is a function $\Phi^{(\infty)}(\theta, \Lambda)$, defined as $\Phi^{(\infty)} : \mathbb{T}^2 \times \breve{\mathcal{E}}^{(\infty)} \to \mathbb{T} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, analytic in θ and Whitney-C^∞ with respect to Λ, giving a parametrization of the Cantorian four-dimensional manifold defined by the real two-dimensional invariant tori of $\mathcal{H}$, branching off the critical periodic orbit. Precisely, for any $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, the function $\Phi^{(\infty)}(\cdot, \Lambda)$ gives a parametrization of the corresponding two-dimensional invariant torus of $\mathcal{H}$, in such a way the pull-back of the dynamics on the torus to the variable θ is a linear quasiperiodic flow. Thus, for any $\theta^{(0)} \in \mathbb{T}^2$, then $t \in \mathbb{R} \mapsto \Phi^{(\infty)}(\Omega(\Lambda) \cdot t + \theta^{(0)}, \Lambda)$ is a solution of the Hamilton equations of $\mathcal{H}$. Moreover, $\Phi^{(\infty)}$ can be extended to a smooth function of $\mathbb{T}^2 \times \mathbb{R}^2$ —analytic in θ and C^∞ with respect to Λ—.

The proof of theorem 3.1 extends till the end of the paper.

4   Previous results

In this section we review some previous results we use to carry out the proof of theorem 3.1. Concretely, in section 4.1 we discuss precisely how the normal form around a $1 : -1$ resonant periodic orbit looks like and give, as function of the distance to the critical orbit, quantitative estimates on the remainder of this normal form. In sections 4.2 and 4.3 we identify the family of 2D-bifurcated tori of the normal form, branching off the critical orbit, and its (linear) normal behaviour.

4.1 Quantitative normal form

Our first step is to compute the normal form of $\mathcal{H}$ in (10) up to a suitable order. This order is chosen to minimize (as much as possible) the size of the non-integrable remainder of the normal form. Hence, for any $R > 0$ (small enough), we consider a neighbourhood of "size" R around the critical periodic orbit (see (1)), and select the normalizing order, $r_{opt}(R)$, so that the remainder of the normal form of $\mathcal{H}$ up to degree $r_{opt}(R)$ becomes as small as possible in this neighbourhood. As we have pointed out before, for an elliptic nonresonant periodic orbit it is possible to select this order so that the remainder becomes exponentially small in R. In the present resonant setting, the non semi-simple character of the homological equations leads to poor estimates for the remainder. The following result, that can be derived from [36], states the normal form up to "optimal" order and the bounds for the corresponding remainder.

Theorem 4.1. *With the same hypotheses of theorem 3.1. Given any $\varepsilon > 0$ and $\sigma > 1$, both fixed, there exists $0 < R^* < 1$ such that, for any $0 < R \leq R^*$, there is a real analytic canonical diffeomorphism $\widehat{\Psi}^{(R)}$ verifying:*

(i) $\widehat{\Psi}^{(R)} : \mathcal{D}_{1,2}(\sigma^{-2}\rho_0/2, R) \to \mathcal{D}_{1,2}(\rho_0/2, \sigma R)$.

(ii) *If $\widehat{\Psi}^{(R)} - \mathrm{Id} = (\widehat{\Theta}^{(R)}, \widehat{\mathcal{X}}^{(R)}, \widehat{\mathcal{I}}^{(R)}, \widehat{\mathcal{Y}}^{(R)})$, then all the components are 2π-periodic in θ and satisfy*

$$|\widehat{\Theta}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (1 - \sigma^{-2})\rho_0/2, \qquad |\widehat{\mathcal{I}}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma^2 - 1)R^2,$$

$$|\widehat{\mathcal{X}}_j^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma - 1)R, \qquad |\widehat{\mathcal{Y}}_j^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma - 1)R, \quad j = 1, 2.$$

(iii) *The transformed Hamiltonian by the action of $\widehat{\Psi}^{(R)}$ takes the form:*

$$\mathcal{H} \circ \widehat{\Psi}^{(R)}(\theta, x, I, y) = \mathcal{Z}^{(R)}(x, I, y) + \mathcal{R}^{(R)}(\theta, x, I, y), \tag{14}$$

where $\mathcal{Z}^{(R)}$ (the normal form) is an integrable Hamiltonian system which looks like

$$\mathcal{Z}^{(R)}(x, I, y) = \mathcal{Z}_2(x, I, y) + \widetilde{Z}^{(R)}(x, I, y), \tag{15}$$

where $\mathcal{Z}_2$ is given by (11) and $\widetilde{Z}^{(R)}(x, I, y) = Z^{(R)}(q, I, L/2)$, with $q = (x_1^2 + x_2^2)/2$ and $L = y_1 x_2 - x_1 y_2$. The function $Z^{(R)}(u_1, u_2, u_3)$ is analytic around the origin, with Taylor expansion starting at degree three. More precisely, $Z^{(R)}(u_1, u_2, u_3)$ is a polynomial of degree less than or equal to $\lfloor r_{opt}(R)/2 \rfloor$, except by the affine part on u_1 and u_3, which allows generic dependence on u_2. The remainder $\mathcal{R}^{(R)}$ contains terms in (x, I, y) of higher order than "the polynomial part" of $\mathcal{Z}^{(R)}$, being all of them of $\mathrm{O}_3(x, y)$.

(iv) *The expression $r_{opt}(R)$ is given by*

$$r_{opt}(R) := 2 + \left\lfloor \exp\left(W\left(\log\left(\frac{1}{R^{1/(\tau+1+\varepsilon)}} \right) \right) \right) \right\rfloor, \tag{16}$$

with $W : (0, +\infty) \to (0, +\infty)$ defined from the equation $W(z) \exp(W(z)) = z$.

(v) *$\mathcal{R}^{(R)}$ satisfies the bound*

$$|\mathcal{R}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq M^{(0)}(R) := R^{r_{opt}(R)/2}. \tag{17}$$

In particular, $M^{(0)}(R)$ goes to zero with R faster than any algebraic order, that is

$$\lim_{R \to 0^+} \frac{M^{(0)}(R)}{R^n} = 0, \qquad \forall n \geq 1.$$

(*vi*) *There exists a constant $\tilde{c}$ independent of R (but depending on ε and σ) such that*

$$|\mathcal{Z}^{(R)}|_{0,R} \leq |\mathcal{H}|_{\rho_0,R_0}, \qquad |\widetilde{Z}^{(R)}|_{0,R} \leq \tilde{c}R^6. \tag{18}$$

Remark 4.1. *The function W corresponds to the principal branch of a special function $W : \mathbb{C} \to \mathbb{C}$ known as the Lambert W function. A detailed description of its properties can be found in [15].*

Actually, the full statement of theorem 4.1 is not explicitly contained in [36], but can be gleaned easily from the paper. Let us describe which are the new features we are talking about.

First, we have modified the action of the transformation $\widehat{\Psi}^{(R)}$ so that the family of periodic orbits of $\mathcal{H}$, in which the critical orbit is embedded, and its normal (Floquet) behavior, are fully described (locally) by the normal form $\mathcal{Z}^{(R)}$ of (15). Thus, the fact that the remainder $\mathcal{R}^{(R)}$ is of $O_3(x, y)$, implies that neither the family of periodic orbits nor its Floquet multipliers change in (14) from those of $\mathcal{Z}^{(R)}$ (see sections 4.2 and 4.3). To achieve this, we are forced to work not only with a polynomial expression for the normal form $\mathcal{Z}^{(R)}$ (as done in [36]), but to allow generic dependence on I for the coefficients of the affine part of the expansion of $\mathcal{Z}^{(R)}$ in powers of q and L. For this purpose, we have to extend the normal form criteria used in [36]. We do not plan to give here full details on these modifications, but we are going to summarize the main ideas below.

Let us consider the initial Hamiltonian $\mathcal{H}$ in (10). Then, we start by applying a *partial normal form* process to it in order to reduce the remainder to $O_3(x, y)$, and to arrange the affine part of the normal form in q and L. After this process, the family of periodic orbits of $\mathcal{H}$ and its Floquet behaviour remain the same if we compute them either in the complete transformed system or in the truncated one when removing the $O_3(x, y)$ remainder. We point out that the divisors appearing in this (partial) normal form are $k\omega_1 + l\omega_2$, with $k \in \mathbb{Z}$ and $l \in \{0, \pm 1, \pm 2\}$ (excluding the case $k = l = 0$). As we are assuming irrational collision, these divisors are not "small divisors" at all, because all of them are uniformly bounded from below and go to infinity with k. Hence, we can ensure convergence of this normalizing process in a neighborhood of the periodic orbit.

After we carry out this convergent (partial) normal form scheme on $\mathcal{H}$, we apply the result of [36] to the resulting system. In this way we establish the quantitative estimates, as function of R, for the normal form up to "optimal order". It is easy to realize that the normal form procedure of [36] does not "destroy" the $O_3(x, y)$ structure of the remainder $\mathcal{R}^{(R)}$.

However, we want to emphasize that the particular structure for the normal form $\mathcal{Z}^{(R)}$ stated in theorem 4.1 is not necessary to apply KAM methods. We can prove the existence of the (Cantor) bifurcated family of 2D-tori only by using the polynomial normal form of [36]. The reason motivating to modify the former normal form is only to characterize easily which bifurcated tori are real tori as stated in point (*iv*) of the statement of theorem 3.1 (for details, see remark 4.2 and section 5.13).

The second remark on theorem 4.1 refers to the bound on $\widetilde{Z}^{(R)}$ given in the last point of the statement, that neither is explicitly contained in [36]. Again, it can be easily gleaned from the paper. However, there is also the chance to derive it by hand form the bound on $\mathcal{Z}^{(R)}$ and its particular structure. This is done in section A.2.

4.2 Bifurcated family of 2D-tori of the normal form

It turns out that the normal form $\mathcal{Z}^{(R)}$ is integrable, but in this paper we are only concerned with the two-parameter family of bifurcated 2D-invariant tori associated with this Hopf scenario. See [35] for a full description of the dynamics. To easily identify this family, we introduce new (canonical) coordinates $(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}$, with the 2-form $d\phi \wedge dJ + dq \wedge dp$, defined through the change:

$$
\begin{aligned}
\theta &= \phi_1, & x_1 &= \sqrt{2q}\cos\phi_2, & y_1 &= -\frac{J_2}{\sqrt{2q}}\sin\phi_2 + p\sqrt{2q}\cos\phi_2, \\
I &= J_1, & x_2 &= -\sqrt{2q}\sin\phi_2, & y_2 &= -\frac{J_2}{\sqrt{2q}}\cos\phi_2 - p\sqrt{2q}\sin\phi_2,
\end{aligned}
\tag{19}
$$

that casts the Hamiltonian (14) into (dropping the superindex (R)):

$$\breve{\mathcal{H}}(\phi, q, J, p) = \breve{\mathcal{Z}}(q, J, p) + \breve{\mathcal{R}}(\phi, q, J, p), \tag{20}$$

where,

$$\breve{\mathcal{Z}}(q, J, p) = \langle \omega, J \rangle + qp^2 + \frac{J_2^2}{4q} + \frac{1}{2}(aq^2 + bJ_1^2 + cJ_2^2) + dqJ_1 + eqJ_2 + fJ_1J_2 + Z(q, J_1, J_2/2). \tag{21}$$

Let us consider the Hamilton equations of $\breve{\mathcal{Z}}$:

$$\begin{aligned}
\dot{\phi}_1 &= \omega_1 + bJ_1 + dq + fJ_2 + \partial_2 Z(q, J_1, J_2/2), & \dot{J}_1 &= 0, \\
\dot{\phi}_2 &= \omega_2 + \frac{J_2}{2q} + cJ_2 + eq + fJ_1 + \frac{1}{2}\partial_3 Z(q, J_1, J_2/2), & \dot{J}_2 &= 0, \\
\dot{p} &= -p^2 + \frac{J_2^2}{4q^2} - aq - dJ_1 - eJ_2 - \partial_1 Z(q, J_1, J_2/2), & \dot{q} &= 2qp.
\end{aligned}$$

Next result sets precisely the bifurcated family of 2D-tori of $\breve{\mathcal{Z}}$ (and hence of $\mathcal{Z}$).

Theorem 4.2. *With the same notations of theorem 4.1. If $d \neq 0$, there exists a real analytic function $\mathbb{I}(\xi, \eta)$ defined in $\Gamma \subset \mathbb{C}^2$, $(0,0) \in \Gamma$, determined implicitly by the equation*

$$\eta^2 = a\xi + d\mathbb{I}(\xi, \eta) + 2e\xi\eta + \partial_1 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta), \tag{22}$$

with $\mathbb{I}(0,0) = 0$ and such that, for any $\zeta = (\xi, \eta) \in \Gamma \cap \mathbb{R}^2$, the two-dimensional torus

$$\mathcal{T}_{\xi,\eta}^{(0)} = \{(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} : q = \xi, \ J_1 = \mathbb{I}(\xi, \eta), \ J_2 = 2\xi\eta, \ p = 0\}$$

is invariant under the flow of $\breve{\mathcal{Z}}$ with parallel dynamics for ϕ determined by the vector $\Omega = (\Omega_1, \Omega_2)$ of intrinsic frequencies:

$$\Omega_1(\xi, \eta) = \omega_1 + b\mathbb{I}(\xi, \eta) + d\xi + 2f\xi\eta + \partial_2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_1}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}, \tag{23}$$

$$\Omega_2(\xi, \eta) = \omega_2 + \eta + 2c\xi\eta + e\xi + f\mathbb{I}(\xi, \eta) + \frac{1}{2}\partial_3 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_2}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}. \tag{24}$$

Moreover, for $\xi > 0$, the corresponding tori of $\mathcal{Z}$ are real.

Remark 4.2. *If we set $\xi = 0$, then $\mathcal{T}_{0,\eta}^{(0)}$ corresponds to the family of periodic orbits of $\mathcal{Z}$ in which the critical one is embedded, but only those in the stable side of the bifurcation. These periodic orbits are parametrized by $q = p = J_2 = 0$ and $J_1 = \mathbb{I}(0, \eta) := \widetilde{\mathbb{I}}(\eta^2)$, and hence the periodic orbit given by η is the same given by $-\eta$. The angular frequency of the periodic orbit $\mathcal{T}_{0,\eta}^{(0)}$ is given by $\Omega_1(0, \eta) := \widetilde{\Omega}_1(\eta^2)$ and the two normal ones are $\Omega_2(0, \eta) := \eta + \widetilde{\Omega}_2(\eta^2)$ and $-\eta + \widetilde{\Omega}_2(\eta^2)$ (check it in the Hamiltonian equations of (15)). We observe that $\Omega_2(0, \eta)$ depends on the sign of η, but that to change η by $-\eta$ only switches both normal frequencies. Moreover, the functions $\widetilde{\mathbb{I}}$, $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are analytic around the origin and, as a consequence of the normal form criteria of theorem 4.1, they are independent of R and give the parametrization of the family of periodic orbits of (14) and of their intrinsic and normal frequencies. See figure 1.*

The proof of theorem 4.2 follows directly by substitution in the Hamilton equations of $\breve{\mathcal{Z}}$. Here we shall only stress that $d \neq 0$ is the only necessary hypothesis for the implicit function $\mathbb{I}$ to exist in a neighbourhood of $(0,0)$. On its turn, the reality condition follows at once writing the invariant tori in the former coordinates (θ, x, I, y) (see (19)). Explicitly, the corresponding quasiperiodic solutions are

$$\begin{aligned}
\theta &= \Omega_1(\zeta)t + \phi_1^{(0)}, & x_1 &= \sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}), & x_2 &= -\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}), \\
I &= \mathbb{I}(\zeta), & y_1 &= -\eta\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}), & y_2 &= -\eta\sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}).
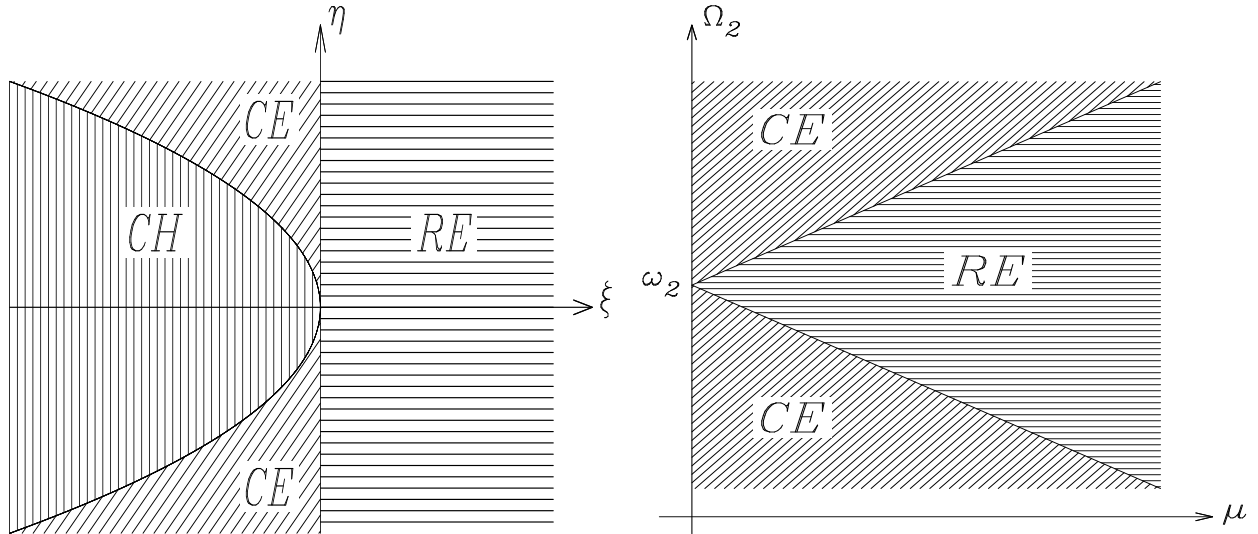\end{aligned}$$

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Figure 1: Qualitative plots of the distribution of invariant tori of the normal form, linked to the direct quasiperiodic Hopf bifurcation, in the parameter spaces (ξ, η) and (μ, Ω_2). The acronyms R, C, E and H indicate real, complex, elliptic and hyperbolic tori, respectively. In the left plot, the curve separating CE and CH (which is close to the parabola $\xi = -2\eta^2/a$) corresponds to complex parabolic tori, whilst the line $\xi = 0$ and the curves separating RE and CE in the right plot (which are close to the straight lines $\Omega_2 = \omega_2 \pm \mu/2$) correspond to stable periodic orbits.

Therefore, $\zeta = (\xi, \eta)$ are the parameters of the family of tori, so they "label" an specific invariant torus of $\mathcal{Z}$. Classically, when applying KAM methods, it is usual to require the *frequency map*, $\zeta \mapsto \Omega(\zeta)$, to be a diffeomorphism, so that we can label the tori in terms of its vector of intrinsic frequencies. This is (locally) achieved by means of the standard Kolmogorov nondegeneracy condition, $\det(\partial_\zeta \Omega) \neq 0$. In the present case, simple computations show that:

$$\mathbb{I}(\xi, \eta) = -\frac{a}{d}\xi + \cdots, \quad \Omega_1(\xi, \eta) = \omega_1 + \left(d - \frac{ab}{d}\right)\xi + \cdots, \quad \Omega_2(\xi, \eta) = \omega_2 + \left(e - \frac{af}{d}\right)\xi + \eta + \cdots \quad (25)$$

(for higher order terms see [35]). Then, Kolmogorov's condition computed at the resonant orbit reads as $d - ab/d \neq 0$. Although this is the classic approach, we shall be forced to choose a set of parameters on the family different from the intrinsic frequencies.

4.3 Normal behaviour of the bifurcated tori

Let us consider the variational equations of $\breve{\mathcal{Z}}$ around the family of (real) bifurcated tori $\mathcal{T}_{\xi,\eta}^{(0)}$ (with $\xi > 0$). The restriction of these equations to the normal directions (q, p) is given by a two dimensional linear system with constant coefficients, with matrix

$$M_{\xi,\eta} = \begin{pmatrix} 0 & 2\xi \\ -\dfrac{2\eta^2}{\xi} - a - \partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) & 0 \end{pmatrix}. \quad (26)$$

Then, the *characteristic exponents* (or *normal eigenvalues*) of this torus are

$$\lambda_{\pm}(\xi, \eta) = \pm\sqrt{-4\eta^2 - 2a\xi - 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta)}. \quad (27)$$

If $a > 0$, it is easy to realize that the eigenvalues $\lambda_{\pm}$ are purely imaginary if $\xi > 0$ and η are both small enough, and hence the family $\mathcal{T}_{\xi,\eta}^{(0)}$ holds only elliptic tori. If $a < 0$, then elliptic, hyperbolic and parabolic tori co-exist simultaneously in the family. In this paper we are only interested in the case

$a > 0$ (direct bifurcation), so from now on we shall only be concerned with elliptic tori. Hence, we denote by $\mu = \mu(\xi, \eta) > 0$ the only *normal frequency* of the torus $\mathcal{T}^{(0)}_{\xi,\eta}$, so that $\lambda_\pm = \pm i\mu$, with

$$\mu^2 := 2\xi \partial^2_{q,q} \breve{\mathcal{Z}}|_{\mathcal{T}^{(0)}_\zeta} = 4\eta^2 + 2a\xi + 2\xi \partial^2_{1,1} Z(\xi, \mathbb{I}(\xi,\eta), \xi\eta). \tag{28}$$

If we pick up the (stable) periodic orbit $(\pm\eta, 0)$, then $\mu = 2|\eta|$. Hence, it is clear that $\mu \to 0$ as we approach to the resonant orbit $\xi = \eta = 0$. Thus, the elliptic bifurcated tori of the normal form are very close to parabolic. This is the main source of problems when proving their persistence in the complete system.

Remark 4.3. *Besides those having $0 < \xi << 1$ and $|\eta| << 1$ we observe that, from formula (27), those tori having $\xi < 0$ but $4\eta^2 + 2a\xi + 2\xi\partial^2_{1,1}Z(\xi, \mathbb{I}(\xi,\eta), \xi\eta) > 0$ are elliptic too, albeit they are complex tori when written in the original variables (recall that $\xi = 0$ corresponds to the stable periodic orbits of the family, see remark 4.2). However, when performing the KAM scheme, we will work with them all together (real or complex tori), because they turn to be real when written in the "action–angle" variables introduced in (19). The discussion between real or complex tori of the original system (10) is carried on in section 5.13. See figure 1.*

Remark 4.4. *There is almost no difference in studying the persistence of elliptic tori in the inverse case using the approach of the paper for the direct case. For hyperbolic tori, the same methodology of the paper also works, only taking into account that now $\lambda_\pm = \pm\mu$, $\mu > 0$. Thus, in the hyperbolic case we can also use the iterative KAM scheme described in section 5.3, with the only difference that some of the divisors, appearing when solving the homological equations (eq1)–(eq2), are not "small divisors" at all, because their real part has an uniform lower bound in terms of μ (see (51), (54) and (55)). This fact simplifies a lot the measure estimates of the surviving tori. As pointed before in the introduction, the parabolic case, $\lambda_\pm = 0$, requires a different approach and it is not covered by this paper.*

5 Proof of theorem 3.1

We consider the initial Hamiltonian $\mathcal{H}$ in (10) and take $R > 0$, small enough, fixed from now on. Then, we compute the normal form of $\mathcal{H}$ up to a "suitable order", depending on R, as stated in theorem 4.1. As the normalizing transformation $\widehat{\Psi}^{(R)}$ depends on the selected R, it is clear that the transformed Hamiltonian $\mathcal{H} \circ \widehat{\Psi}^{(R)}$ also does. However, as R is fixed, in the forthcoming we drop the explicit dependence on R unless it is strictly necessary. Now, we introduce the canonical coordinates (19) and obtain the Hamiltonian $\breve{\mathcal{H}}$ in (20). Then, the keystone of the proof of theorem 3.1 is a KAM process applied to $\breve{\mathcal{H}}$.

To carry out this procedure, first in section 5.1 we discuss which is the vector Λ of basic frequencies we use to label the bifurcated tori. In section 5.2 we introduce Λ as a parameter on the Hamiltonian $\breve{\mathcal{H}}$. Moreover, the resulting system is complexified in order to simplify the resolution of the homological equations. The iterative KAM scheme we perform is explained in section 5.3. We also discuss the main difficulties we found when applying this process —in the present close-to-parabolic setting— with respect to the standard non-degenerate context. To justify the validity of our approach, the particular non-degeneracy condition linked to this construction is checked in section 5.4. In section 5.5 we explain how we carry out in the KAM process the ultra-violet cut-off with respect to the angles of the tori. This cut-off is performed in order to prove the Whitney-smoothness, with respect to the parameter Λ, of the surviving tori. After that, we begin with the quantitative part of the proof. To do that, first we have to select the initial set of basic frequencies in which we look for the corresponding invariant torus (section 5.6). Then, we have to control the bounds on the initial family of Hamiltonians (section 5.7), the quantitative estimates on the KAM iterative process introduced before (section 5.8) and the convergence of this procedure in a suitable set of basic frequencies (sections 5.9 and 5.10). To

discuss the measure of this set we use Lipschitz constants. In section 5.11 we assure that we have a suitable control on these constants, whilst in section 5.12 we properly control this measure. Finally, in section 5.13 we discuss which of the invariant tori we have obtained are real when expressed in the original coordinates and, in section 5.14, we establish the Whitney-C^∞ smoothness of the bifurcated family.

5.1   Lack of parameters

One of the problems intrinsically linked to the perturbation of elliptic invariant tori is the so-called "lack of parameters". In fact, this is a common difficulty in the theory of quasiperiodic motions in dynamical systems (see [9, 33, 44]). Basically, it implies that one cannot construct a perturbed torus with a fixed set of (Diophantine) intrinsic and normal frequencies, for the system does not contain enough internal parameters to control them all simultaneously. All that one can expect is to build perturbed tori with only a given subset of *basic frequencies* previously fixed (equal to the numbers of parameters one has). The remaining frequencies have to be dealt (when possible) as function of the prefixed ones.

Let us suppose for the moment that, in our case, the two intrinsic frequencies could be the basic ones and that the normal frequency is function of the intrinsic ones (this is the standard approach). These three frequencies are present on the (small) denominators of the KAM iterative scheme (see (29)). It means that to carry out the first step of this process, one has to restrict the parameter set to the intrinsic frequencies so that they, together with the corresponding normal one of the unperturbed torus, satisfy the required Diophantine conditions (see (69)). After this first step, we only can keep fixed the values of the intrinsic frequencies (assuming Kolmogorov nondegeneracy), but the function giving the normal frequency of the new approximation to the invariant torus has changed. Thus, we cannot guarantee *a priori* that the new normal frequency is nonresonant with the former intrinsic ones.

To succeed in the iterative application of the KAM process, it is usual to ask for the denominators corresponding to the unperturbed tori to move when the basic frequencies do. In our context, with only one frequency to control, this is guaranteed if we can add suitable nondegeneracy conditions on the function giving the normal frequency. These *transversality conditions* avoid the possibility that one of the denominators falls permanently inside a resonance, and allows to obtain estimates for the Lebesgue measure of the set of "good" basic frequencies at any step of the iterative process. For 2D-elliptic low-dimensional tori with only one normal frequency, the denominators to be taken into account are[3] (the so-called *Mel'nikov's second non-resonance condition*, see [31, 32])

$$\mathrm{i}\langle k, \Omega \rangle + \mathrm{i}\ell\mu, \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \quad \forall \ell \in \{0, \pm 1, \pm 2\}, \tag{29}$$

where $\Omega \in \mathbb{R}^2$ are the intrinsic frequencies and $\mu = \mu(\Omega) > 0$ the normal one. Now, we compute the gradient with respect to Ω of such divisors, and require them not to vanish. These transversality conditions are equivalent to $2\nabla_\Omega \mu(\Omega) \notin \mathbb{Z}^2 \setminus \{0\}$. For equivalent conditions in the "general" case see [25]. For nondegeneracy conditions of higher order see [9, 44].

This, however, does not work in the current situation. To realize, a glance at (25) shows that the first order expansion, at $\Omega = \omega$, of the inverse of the frequency map is

$$\xi = \frac{d}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots, \qquad \eta = \frac{af - ed}{d^2 - ab}(\Omega_1 - \omega_1) + \Omega_2 - \omega_2 + \cdots.$$

Now, substitution in the expression (27) gives for the normal frequency

$$\mu(\Omega) = \sqrt{\frac{2ad}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots},$$

---

[3]Bourgain showed in [4, 5] that conditions with $\ell = \pm 2$ can be omitted, but the proof becomes extremely involved.

so $\nabla_\Omega \mu(\Omega)$ is not well defined at the critical periodic orbit. Therefore we use different parameters on the family. From (25) and (27), it can be seen that ξ and η may be expressed as a function of μ and Ω_2,

$$\xi = \frac{\mu^2}{2a} - \frac{2}{a}(\Omega_2 - \omega_2)^2 + \cdots, \qquad \eta = \Omega_2 - \omega_2 + \left(\frac{f}{2d} - \frac{e}{2a}\right)\mu^2 + \left(\frac{2e}{a} - \frac{3f}{d}\right)(\Omega_2 - \omega_2)^2 + \cdots.$$

Now, let us denote $\Lambda = (\mu, \Omega_2)$ the new set of basic frequencies and write Ω_1 as function of them. Substitution in the expression for Ω_1 in (25) yields:

$$\Omega_1^{(0)}(\mu, \Omega_2) := \omega_1 + \left(\frac{d}{2a} - \frac{b}{2d}\right)\mu^2 + \left(\frac{3b}{d} - \frac{2d}{a}\right)(\Omega_2 - \omega_2)^2 + \cdots. \tag{30}$$

The derivatives with respect to Λ of the KAM denominators (29) are, at the critical periodic orbit,

$$\nabla_\Lambda \left(k_1 \Omega_1^{(0)}(\Lambda) + k_2 \Omega_2 + \ell\mu\right)\Big|_{\Lambda=(0,\omega_2)} = (\ell, k_2), \qquad k_1, k_2, \ell \in \mathbb{Z}, \text{ with } |\ell| \leq 2. \tag{31}$$

So, the divisors will change with Λ whenever the integer vector $(\ell, k_2) \neq (0,0)$. But if $\ell = k_2 = 0$ then $k_1 \neq 0$, and the modulus of the divisor $k_1 \Omega_1^{(0)}(\Lambda)$ will be bounded from below.

5.2   Expansion around the unperturbed tori and complexification of the system

Once we have selected the parameters on the family, the next step is to put the system (20) into a more suitable form. Concretely, we replace the Hamiltonian $\breve{\mathcal{H}}$ by a family of Hamiltonians, $H_\Lambda^{(0)}$, having as a parameter the vector of basic frequencies Λ. This is done by placing "at the origin" the invariant torus of the "unperturbed Hamiltonian" $\breve{\mathcal{Z}}$, corresponding to the parameter Λ, and then arranging the corresponding normal variational equations of $\breve{\mathcal{Z}}$ to diagonal form and uncoupling (up to first order) the "central" and normal terms around the torus. This means to remove the quadratic term $[\cdot]_{I,z}$ (see (4)) from the unperturbed part of (34).

If for the moment we set the perturbation $\breve{\mathcal{R}}$ to zero, then $H_\Lambda^{(0)}$ constitutes a family of analytic Hamiltonians so that, for a given $\Lambda = (\mu, \Omega_2)$, the corresponding member has at the origin a 2D-elliptic invariant torus with normal frequency μ and intrinsic frequencies $(\Omega_1^{(0)}(\Lambda), \Omega_2)$, where $\Omega_1^{(0)}(\Lambda)$ is defined through (30). Our target is to prove that if we take the perturbation $\breve{\mathcal{R}}$ into account then, for most of the values of Λ (in a Cantor set), the full system $H_\Lambda^{(0)}$ has an invariant 2D-elliptic torus close to the origin, with the same vector of basic frequencies Λ, but perhaps with a different Ω_1. Similar ideas have been used in [25, 26].

To introduce $H_\Lambda^{(0)}$ we consider the family of symplectic transformations $(\theta_1, \theta_2, x, I_1, I_2, y) \mapsto (\phi_1, \phi_2, q, J_1, J_2, p)$, defined for $\Lambda \in \Gamma$ (see theorem 4.2) and given by:

$$\begin{aligned}
\phi_1 &= \theta_1 - \frac{2\xi}{\mu^2}(\partial_{J_1,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right), & J_1 &= \mathbb{I}(\zeta) + I_1, \\
\phi_2 &= \theta_2 - \frac{2\xi}{\mu^2}(\partial_{J_2,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right), & J_2 &= 2\xi\eta + I_2, \\
q &= \xi + x - \frac{\xi}{\lambda_+}y - \frac{2\xi}{\mu^2}(\partial_{J_1,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})I_1 - \frac{2\xi}{\mu^2}(\partial_{J_2,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})I_2, & p &= \frac{\lambda_+}{2\xi}x + \frac{1}{2}y,
\end{aligned} \tag{32}$$

where $\lambda_+ = \mathrm{i}\mu$. Although it has not been written explicitly, the parameters $\zeta = (\xi, \eta)$ must be thought of as functions of the basic frequencies Λ, i. e., $\zeta = \zeta(\Lambda)$.

This transformation can be read as the composition of two changes. One is the symplectic "diagonalizing" change

$$Q = x - \frac{\xi}{\lambda_+}y, \qquad P = \frac{\lambda_+}{2\xi}x + \frac{1}{2}y, \tag{33}$$

that puts the normal variational equations —associated with the unperturbed part $\breve{\mathcal{Z}}$— of the torus into diagonal form. We point out that we choose (33) as a diagonalizing change because it skips any square root of ξ or μ. The other change moves the torus to the origin and gets rid of the contribution of $\breve{\mathcal{Z}}$ to the term $[\cdot]_{I,z}$ of the Taylor expansion of $H_\Lambda^{(0)}$ (recall (28)). To diagonalize the normal variationals and to kill this coupling term is not strictly necessary, but both operations simplify a lot the homological equations of the KAM process (see (eq1)–(eq5)).

Note that the linear change (33) is a complexification of the real Hamiltonian (20), i. e., the *real* values of the normal variables (q,p) correspond now to *complex* values of (x,y). Nevertheless, the invariant tori of (34) we finally obtain are real tori when expressed in coordinates $(\phi_1, \phi_2, q, J_1, J_2, p)$, and those having $q > 0$ are also real in the original variables (through change (19)). The real character of the tori of (34) can be verified in two ways. The first one is to overcome the complexification (33) and to perform the KAM process by using the real variables (Q, P) instead of (x, y). The price we paid for using this methodology is that the solvability of the homological equations —of the iterative KAM process— becomes more involved, because they are no longer diagonal. The other way to proceed is to observe that the complexified homological equations have a unique (complex) solution. Thus, as we are dealing with linear (differential) equations, it implies that the corresponding real homological equations, written in terms of the variables (Q, P), also have a unique (real) solution. Hence, as the complexification (33) is canonical, it means that if we express the generating function S (see (40)) obtained as solution of the homological equations in the real variables (Q, P), then we obtain a real generating function (see remark 5.1 for more details). Consequently, the symmetries introduced by the complexification are kept after any step of the iterative KAM process, and we can go back to a real Hamiltonian by means of the inverse transformation of (33). Thus, for simplicity, we have preferred to follow this second approach and to use complex variables.

In this way, the Hamiltonian $\breve{\mathcal{H}}$ in (20) casts into $H_\Lambda^{(0)} = H_\Lambda^{(0)}(\theta, x, I, y)$, with

$$H_\Lambda^{(0)} = \phi^{(0)}(\Lambda) + \langle \Omega^{(0)}(\Lambda), I \rangle + \frac{1}{2}\langle z, \mathcal{B}(\Lambda)z \rangle + \frac{1}{2}\langle I, \mathcal{C}^{(0)}(\Lambda)I \rangle + \widetilde{H}^{(0)}(x, I, y; \Lambda) + \widehat{H}^{(0)}(\theta, x, I, y; \Lambda). \quad (34)$$

Here, $\widetilde{H}^{(0)}$ holds the terms of order greater than two in (z, I), where $z = (x, y)$, coming from the normal form $\breve{\mathcal{Z}}$, i. e., $[\widetilde{H}^{(0)}] = \widetilde{H}^{(0)}$ (see (4)), and $\widehat{H}^{(0)}$ is the transformed of the remainder $\breve{\mathcal{R}}$, whereas

$$\phi^{(0)}(\Lambda) = \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}, \qquad \Omega_1^{(0)}(\Lambda) = \partial_{J_1}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}, \qquad \Omega_2^{(0)}(\Lambda) = \Omega_2, \qquad \mathcal{B}(\Lambda) = \begin{pmatrix} 0 & \lambda_+ \\ \lambda_+ & 0 \end{pmatrix}, \qquad (35)$$

(see (23), (24), (26), (27), (28) and (30)) and the symmetric matrix $\mathcal{C}^{(0)}$ is given by

$$\begin{aligned}
\mathcal{C}_{1,1}^{(0)}(\Lambda) &= \partial_{J_1,J_1}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} - \frac{2\xi}{\mu^2}(\partial_{J_1,q}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})^2 = b + \partial_{2,2}^2 Z - \frac{2\xi}{\mu^2}(d + \partial_{1,2}^2 Z)^2, \\
\mathcal{C}_{1,2}^{(0)}(\Lambda) &= \partial_{J_1,J_2}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} - \frac{2\xi}{\mu^2}(\partial_{J_1,q}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})(\partial_{J_2,q}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}) \\
&= f + \frac{1}{2}\partial_{2,3}^2 Z - \frac{2\xi}{\mu^2}(d + \partial_{1,2}^2 Z)\left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial_{1,3}^2 Z\right), \\
\mathcal{C}_{2,2}^{(0)}(\Lambda) &= \partial_{J_2,J_2}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} - \frac{2\xi}{\mu^2}(\partial_{J_2,q}^2\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})^2 = \frac{1}{2\xi} + c + \frac{1}{4}\partial_{3,3}^2 Z - \frac{2\xi}{\mu^2}\left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial_{1,3}^2 Z\right)^2,
\end{aligned} \quad (36)$$

where the partial derivatives of Z above are evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$. If one skips the remainder $\widehat{H}^{(0)}$ off, then $I = 0$, $z = 0$ corresponds to an invariant 2D-elliptic torus of $H_\Lambda^{(0)}$ with basic frequency vector Λ. The normal variational equations of this torus are given by the (complex) diagonal matrix $\mathcal{J}_1\mathcal{B}$.

5.3 The iterative scheme

Now, we proceed to describe (here only formally) the KAM iterative procedure we use to construct the elliptic two-dimensional tori. The underlying idea goes back to Kolmogorov in [29] and Arnol'd

in [1, 2]. In what concerns low-dimensional tori, see references quoted in the introduction.

We perform a sequence of canonical changes on $H_\Lambda^{(0)}$ (see (34)), depending on the parameter Λ, obtaining thus a sequence of Hamiltonians $\{H_\Lambda^{(n)}\}_{n\geq 0}$, with a limit Hamiltonian $H_\Lambda^{(\infty)}$ having at the origin a 2D-elliptic invariant torus, with $\Lambda = (\mu, \Omega_2)$ as vector of basic frequencies. Concretely, we want $H_\Lambda^{(\infty)}$ to be of the form

$$H_\Lambda^{(\infty)}(\theta, x, I, y) = \phi^{(\infty)}(\Lambda) + \langle \Omega^{(\infty)}(\Lambda), I \rangle + \frac{1}{2}\langle z, \mathcal{B}(\Lambda)z \rangle + \frac{1}{2}\langle I, \mathcal{C}^{(\infty)}(\theta; \Lambda)I \rangle + \widetilde{H}^{(\infty)}(\theta, x, I, y; \Lambda), \quad (37)$$

with $[\widetilde{H}^{(\infty)}] = \widetilde{H}^{(\infty)}$, the matrix $\mathcal{B}$ given by (35) and the function $\Omega^{(\infty)}(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$. This process is built as a Newton-like iterative method, yielding to "quadratic convergence" if we restrict to the values of Λ for which suitable Diophantine conditions hold at any step. We point out that, albeit $\mathcal{C}^{(0)}$ and $\widetilde{H}^{(0)}$ are independent of θ, this property is not kept by the iterative process.

To describe a generic step of this iterative scheme we consider a Hamiltonian of the form (see (4))

$$H = \mathrm{a}(\theta) + \langle \mathrm{b}(\theta), z \rangle + \langle \mathrm{c}(\theta), I \rangle + \frac{1}{2}\langle z, \mathrm{B}(\theta)z \rangle + \langle I, \mathrm{E}(\theta)z \rangle + \frac{1}{2}\langle I, \mathrm{C}(\theta)I \rangle + \Xi(\theta, x, I, y). \quad (38)$$

Although we do not write this dependence explicitly, we suppose that H depends also on Λ (recall that everything depends also on the prefixed R). Moreover, we also assume that if we replace the "complex" variables (x, y) by (Q, P) through (33), then H becomes a real analytic function. For any $\Lambda = (\mu, \Omega_2)$ we define from (38)

$$\overline{H} = \langle \mathrm{a} \rangle_\theta + \langle \Omega, I \rangle + \frac{1}{2}\langle z, \mathcal{B}z \rangle + \frac{1}{2}\langle I, \mathrm{C}(\theta)I \rangle + \Xi(\theta, x, I, y), \quad (39)$$

and suppose that $H - \overline{H}$ is "small". To fix ideas, of $\mathrm{O}(\varepsilon)$ with ε going to zero with the step. We point out that if we start the iterative process with $H^{(0)}$ in (34), then $\varepsilon = \mathrm{O}(\widehat{H}^{(0)})$. The Hamiltonian $\overline{H}$ looks like (37), which is the form we want for the limit Hamiltonian, with $\Omega = (\Omega_1, \Omega_2)$, for certain $\Omega_1 = \Omega_1(\Lambda)$ to be chosen iteratively (initially we take $\Omega_1 = \Omega_1^{(0)}$ of (30)), and $\mathcal{B}(\Lambda)$ defined by (35) is held fixed during the iterative process. Moreover, we also assume that the matrix C is close to $\mathcal{C}^{(0)}(\Lambda)$ defined by (36), but we do not ask for C to remain constant with the step.

Now, we perform a canonical change on H so that it squares the size of ε. Concretely, if we call $H^{(1)}$ the transformed Hamiltonian, expand $H^{(1)}$ as H in (38) and define $\overline{H}^{(1)}$ from $H^{(1)}$ as in (39), we want (roughly speaking) the norm of $H^{(1)} - \overline{H}^{(1)}$ to be of $\mathrm{O}(\varepsilon^2)$.

The canonical transformations we use are defined by the time-one flow of a suitable Hamiltonian $S = S_\Lambda$, the so-called *generating function* of the change, that we denote as $\Psi_{t=1}^S$ or simply Ψ_1^S (see section 2). Precisely, we look for S of the form (compare [3, 25, 26])

$$S(\theta, x, I, y) = \langle \chi, \theta \rangle + \mathrm{d}(\theta) + \langle \mathrm{e}(\theta), z \rangle + \langle \mathrm{f}(\theta), I \rangle + \frac{1}{2}\langle z, \mathrm{G}(\theta)z \rangle + \langle I, \mathrm{F}(\theta)z \rangle, \quad (40)$$

where $\chi \in \mathbb{C}^2$, $\langle \mathrm{d} \rangle_\theta = 0$, $\langle \mathrm{f} \rangle_\theta = 0$ and G is a symmetric matrix with $\langle \mathrm{G}_{1,2} \rangle_\theta = \langle \mathrm{G}_{2,1} \rangle_\theta = 0$.

Remark 5.1. *Conditions above guarantee the uniqueness of S as a solution of the homological equations (eq1)–(eq5). Furthermore, as we want to ensure that we have a real generating function after applying the inverse of (33) to S, we have to require that $\chi \in \mathbb{R}^2$ and that $\mathrm{d}(\theta)$, $\mathcal{S}^*\mathrm{e}(\theta)$, $\mathrm{f}(\theta)$, $\mathcal{S}^*\mathrm{G}(\theta)\mathcal{S}$ and $\mathrm{F}(\theta)\mathcal{S}$ are real functions, where $\mathcal{S}$ is the matrix of the inverse of the linear change (33). So, if we set $\mathcal{G}(\theta) = \mathcal{S}^*\mathrm{G}(\theta)\mathcal{S}$, condition $\langle \mathrm{G}_{1,2} \rangle_\theta = 0$ reads, for the real matrix $\mathcal{G}$, as $4\xi^2\langle \mathcal{G}_{1,1} \rangle_\theta + \mu^2\langle \mathcal{G}_{2,2} \rangle_\theta = 0$. If we assume that these $\mathcal{S}$-symmetries hold for H, then it is clear that they also hold for S.*

Then we have,

$$H^{(1)} := H \circ \Psi_1^S = H + \{H, S\} + \int_0^1 (1-t)\{\{H, S\}, S\} \circ \Psi_t^S dt.$$

By assuming *a priori* that S is small, of $\mathrm{O}(\varepsilon)$, we select S so that $H + \{\overline{H}, S\}$ takes the form:

$$H + \{\overline{H}, S\} = \phi^{(1)} + \langle \Omega^{(1)}, I \rangle + \frac{1}{2}\langle z, \mathcal{B}z \rangle + \frac{1}{2}\langle I, \mathcal{C}^{(1)}(\theta)I \rangle + \widetilde{H}^{(1)}(\theta, x, I, y),$$

being $\Omega^{(1)} = (\Omega_1^{(1)}, \Omega_2)$ with $\widetilde{H}^{(1)}$ holding the terms of higher degree, i. e., $\widetilde{H}^{(1)} = [H + \{\overline{H}, S\}]$. If we write these conditions in terms of H and the generating function S, this leads to the following *homological equations* (see (8)):

$$\{\mathrm{a}\}_\theta - L_\Omega \mathrm{d} = 0, \tag{eq1}$$

$$\mathrm{b} - L_\Omega \mathrm{e} + \mathcal{B}\mathcal{J}_1 \mathrm{e} = 0, \tag{eq2}$$

$$\mathrm{c} - \Omega^{(1)} - L_\Omega \mathrm{f} - \mathrm{C}\left(\chi + (\partial_\theta \mathrm{d})^*\right) = 0, \tag{eq3}$$

$$\widetilde{\mathrm{B}} - \mathcal{B} - L_\Omega \mathrm{G} + \mathcal{B}\mathcal{J}_1 \mathrm{G} - \mathrm{G}\mathcal{J}_1 \mathcal{B} = 0, \tag{eq4}$$

$$\widetilde{\mathrm{E}} - L_\Omega \mathrm{F} - \mathrm{F}\mathcal{J}_1 \mathcal{B} = 0, \tag{eq5}$$

where

$$\Omega_1^{(1)} \quad := \quad \langle c_1 \rangle_\theta - \langle \mathrm{C}_{1,1}\left(\chi_1 + \partial_{\theta_1}\mathrm{d}\right) \rangle_\theta - \langle \mathrm{C}_{1,2}\left(\chi_2 + \partial_{\theta_2}\mathrm{d}\right) \rangle_\theta, \tag{41}$$

$$\widetilde{\mathrm{B}} \quad := \quad \mathrm{B} - [\partial_I \Xi\left(\chi + (\partial_\theta \mathrm{d})^*\right) - \partial_z \Xi \mathcal{J}_1 \mathrm{e}]_{(z,z)}, \tag{42}$$

$$\widetilde{\mathrm{E}} \quad := \quad \mathrm{E} - \mathrm{C}\left(\partial_\theta \mathrm{e}\right)^* - [\partial_I \Xi\left(\chi + (\partial_\theta \mathrm{d})^*\right) - \partial_z \Xi \mathcal{J}_1 \mathrm{e}]_{(I,z)}. \tag{43}$$

Prior to solve completely these equations, we want to discuss the reason for the definition of $\Omega_1^{(1)}$ and how the constant vector χ is fixed, because these are the most involved issues when solving them. These quantities are used to adjust the average of some components of the homological equations, ensuring the compatibility of the full system when they are appropriately chosen. First, $\Omega_1^{(1)}$ is defined so that the average of the first component of the (vectorial) equation (eq3) is zero. Moreover, as one wants Ω_2 and μ not to change from one iterate to another, χ must satisfy the linear system formed by the second component of (eq3) and the first row second column component of the (matricial) equation (eq4) (or, by symmetry, the second row first column of this equation). One obtains the linear system,

$$\langle \mathcal{A} \rangle_\theta \chi = -\mathrm{h}, \tag{44}$$

where

$$\mathcal{A}(\theta) = \begin{pmatrix} \mathrm{C}_{2,1}(\theta) & \mathrm{C}_{2,2}(\theta) \\ \partial^3_{I_1, x, y}\Xi(\theta, 0) & \partial^3_{I_2, x, y}\Xi(\theta, 0) \end{pmatrix} \tag{45}$$

and the components of the right hand side term in (44) are

$$\mathrm{h}_1 \quad := \quad \Omega_2 - \langle c_2 \rangle_\theta + \langle \mathrm{C}_{2,1}\partial_{\theta_1}\mathrm{d} \rangle_\theta + \langle \mathrm{C}_{2,2}\partial_{\theta_2}\mathrm{d} \rangle_\theta, \tag{46}$$

$$\begin{aligned} \mathrm{h}_2 \quad := \quad & \lambda_+ - \langle \mathrm{B}_{1,2} \rangle_\theta + \langle \partial^3_{I_1, x, y}\Xi(\theta, 0)\partial_{\theta_1}\mathrm{d} \rangle_\theta + \langle \partial^3_{I_2, x, y}\Xi(\theta, 0)\partial_{\theta_2}\mathrm{d} \rangle_\theta \\ & + \langle \partial^3_{x, y, y}\Xi(\theta, 0)\mathrm{e}_1 \rangle_\theta - \langle \partial^3_{x, x, y}\Xi(\theta, 0)\mathrm{e}_2 \rangle_\theta. \end{aligned} \tag{47}$$

Hence, to ensure the compatibility of the homological equations, it is necessary to see that the matrix $\langle \mathcal{A} \rangle_\theta$ is not singular and (in order to bound the solutions of the system (44) later on) to derive suitable estimates for the norm of its inverse. This is the most important nondegeneracy condition to fulfill in order to ensure that we made a good selection of basic frequencies to label the tori. Thus, next section is devoted to the verification of this condition for the unperturbed tori of $H^{(0)}$ (see (34)).

5.4 The nondegeneracy condition of the basic frequencies

Let us compute the matrix $\mathcal{A}$ associated with the "unperturbed" terms of the Hamiltonian $H^{(0)}$, namely $\bar{\mathcal{A}}^{(0)}$. This matrix is defined by taking $\mathrm{C} = \mathcal{C}^{(0)}$ and $\Xi = \widetilde{H}^{(0)}$ in (45) (see (34) and (36)). We observe that $\bar{\mathcal{A}}^{(0)}$ does not depend on θ, but this property is not kept for the matrices $\mathcal{A}$ of the iterative process. For $\widetilde{H}^{(0)}$ we have (see (21) and (32)),

$$
\begin{aligned}
\partial^3_{I_1,x,y}\widetilde{H}^{(0)}(0,0,0) &= -\frac{\xi}{\lambda_+}\partial^3_{J_1,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} + \partial^2_{J_1,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}\left(\frac{1}{\lambda_+} - \frac{2\xi^2}{\lambda_+^3}\partial^3_{q,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}\right) \\
&= -\frac{\xi}{\lambda_+}\partial^3_{1,1,2}Z + (d + \partial^2_{1,2}Z)\left(\frac{1}{\lambda_+} + 12\frac{\eta^2}{\lambda_+^3} - 2\frac{\xi^2}{\lambda_+^3}\partial^3_{1,1,1}Z\right),
\end{aligned}
$$

$$
\begin{aligned}
\partial^3_{I_2,x,y}\widetilde{H}^{(0)}(0,0,0) &= -\frac{\xi}{\lambda_+}\partial^3_{J_2,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} + \partial^2_{J_2,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}\left(\frac{1}{\lambda_+} - \frac{2\xi^2}{\lambda_+^3}\partial^3_{q,q,q}\breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}\right) \\
&= -\frac{\xi}{\lambda_+}\left(2\frac{\eta}{\xi^2} + \frac{1}{2}\partial^3_{1,1,3}Z\right) + \left(-\frac{\eta}{\xi} + e + \frac{1}{2}\partial^2_{1,3}Z\right)\left(\frac{1}{\lambda_+} + 12\frac{\eta^2}{\lambda_+^3} - 2\frac{\xi^2}{\lambda_+^3}\partial^3_{1,1,1}Z\right),
\end{aligned}
$$

where the partial derivatives of Z are evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$, i. e., at the unperturbed torus. Then, simple (but tedious) computations show that

$$
\det \bar{\mathcal{A}}^{(0)} = \mathcal{C}^{(0)}_{2,1}\partial^3_{I_2,x,y}\widetilde{H}^{(0)}(0,0,0) - \mathcal{C}^{(0)}_{2,2}\partial^3_{I_1,x,y}\widetilde{H}^{(0)}(0,0,0) = \frac{1}{\lambda_+^3}(\widetilde{\mathcal{A}}^{(0)} + \hat{\mathcal{A}}^{(0)}),
$$

where

$$
\begin{aligned}
\widetilde{\mathcal{A}}^{(0)} &= \frac{1}{\xi}\left(-(d + \partial^2_{1,2}Z)\left(\frac{\lambda_+^2}{2} + 2\eta^2\right) - \left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\eta(3\lambda_+^2 + 12\eta^2)\right), \\
\hat{\mathcal{A}}^{(0)} &= (\xi\lambda_+^2\mathcal{C}^{(0)}_{2,2})\partial^3_{1,1,2}Z + \left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\left(2\xi\eta\partial^3_{1,1,1}Z - \frac{\xi\lambda_+^2}{2}\partial^3_{1,1,3}Z\right) \\
&\quad + (\lambda_+^2 + 12\eta^2 - 2\xi^2\partial^3_{1,1,1}Z)\left(\left(f + \frac{1}{2}\partial^2_{2,3}Z\right)\left(e + \frac{1}{2}\partial^2_{1,3}Z\right) - \left(c + \frac{1}{4}\partial^2_{3,3}Z\right)(d + \partial^2_{1,2}Z)\right) \\
&\quad + (d + \partial^2_{1,2}Z)\left(\xi\partial^3_{1,1,1}Z - 4\eta\left(e + \frac{1}{2}\partial^2_{1,3}Z\right) - \xi\partial^3_{1,1,3}Z\left(-\eta + e\xi + \frac{\xi}{2}\partial^2_{1,3}Z\right)\right).
\end{aligned}
$$

We remark that albeit $\mathcal{C}^{(0)}_{2,2}$ becomes singular when $\xi = \eta = 0$, the expression $\xi\lambda_+^2\mathcal{C}^{(0)}_{2,2}$ goes to zero when $\zeta = (\xi, \eta)$ does, and so does $\hat{\mathcal{A}}^{(0)}$. Now, taking into account definition (27), we replace

$$
\lambda_+^2 = -4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1}Z
$$

in the expression of $\widetilde{\mathcal{A}}^{(0)}$. Then, some (nice) cancellations lead to the following expression

$$
\widetilde{\mathcal{A}}^{(0)} = ad + d\partial^2_{1,1}Z + (6f\eta + 3\eta\partial^2_{2,3}Z + \partial^2_{1,2}Z)(a + \partial^2_{1,1}Z).
$$

As a summary, we have that $\det \bar{\mathcal{A}}^{(0)} = (ad + \cdots)/\lambda_+^3$, where the terms denoted by dots vanish at the critical periodic orbit $\xi = \eta = 0$. Then, as $ad \neq 0$, we have for small values of ζ that $\det \bar{\mathcal{A}}^{(0)} \neq 0$. See section 5.7 for bounds on $(\bar{\mathcal{A}}^{(0)})^{-1}$.

5.5 The ultra-violet cut-off

Once we have fixed the way to compute χ, we discuss the solvability of the remaining part of the homological equations (eq1)–(eq5). By expanding them in Fourier series, we compute the different terms of S in (40) as solution of small divisors equations. The divisors appearing are those specified

in (29), which are integer combinations of the intrinsic frequencies $\Omega = (\Omega_1, \Omega_2)$ and of the normal one μ. For such divisors it is natural to ask for the following Diophantine conditions

$$|\langle k, \Omega \rangle + \ell\mu| \geq \widetilde{\gamma}|k|_1^{-\tau}, \tag{48}$$

for all $k \in \mathbb{Z}^2 \setminus \{0\}$ and $\ell \in \mathbb{Z}$, with $|\ell| \leq 2$, where $\tau > 1$ is given by (12) and $\widetilde{\gamma} > 0$ (depending on R) will be precised later (see (69)). As Ω_1 will be dealt as function of Λ, we expect to have a Cantor set of values of Λ for which (48) holds. Moreover, as the function $\Omega_1 = \Omega_1(\Lambda)$ changes from one step to another, this Cantor set also changes (shrinks) with the step.

If at any step of the iterative scheme we restrict Λ to a Cantor set, then it makes difficult to control the regularity with respect to Λ of the sequence of Hamiltonians $H^{(n)} = H_\Lambda^{(n)}$, because the parameter set has empty interior. The Λ-regularity is important, because it is used to control the (Lebesgue) measure of the "bad" and "good" parameters Λ along the iterative process (see section 5.12). For measure purposes, it is enough to use Lipschitz dependence (see for instance [23, 24, 25, 26]). In this work we have preferred to follow the approach of Arnol'd in [1, 2] and to deal with analytic dependence with respect to Λ. This forces us to consider a KAM process with an *ultra-violet cut-off*. Concretely, we select a "big" integer N, depending on the step and going to infinity, and consider the values of Λ for which (48) hold for any $k \in \mathbb{Z}^2 \setminus \{0\}$ and $|\ell| \leq 2$, but with $0 < |k|_1 < 2N$. This finite number of conditions define an open set for Λ that only becomes Cantor at the limit. Hence, the limit Hamiltonian is no longer analytic on Λ, but only C^∞ in the sense of Whitney (see section A.3).

Let us resume the iterative scheme of section 5.3 and explain precisely how we introduce the ultra-violet cut-off. After N is fixed appropriately, we decompose the actual Hamiltonian H as (see (7)),

$$H = H_{<N,\theta} + H_{\geq N,\theta}. \tag{49}$$

Next to that, $H_{<N,\theta}$ is arranged as H in (38) and then we apply the iterative scheme described in section 5.3 to $H_{<N,\theta}$ instead of H. This means that we compute the generating function $S = S_{<2N,\theta}$ in (40), by solving the homological equations (eq1)–(eq5) with $H_{<N,\theta}$ playing the rôle of H in (38), and hence, with $\overline{H}_{<N,\theta}$ playing the rôle of $\overline{H}$ in (39). The solution of such equations is given by (44) and by the following explicit formulas:

$$\mathrm{d} = \sum_{0 < |k|_1 < N} \frac{\mathrm{a}_k}{\mathrm{i}\langle k, \Omega \rangle} \exp\left(\mathrm{i}\langle k, \theta \rangle\right), \tag{50}$$

$$\mathrm{e}_j = \sum_{|k|_1 < N} \frac{\mathrm{b}_{j;k}}{\mathrm{i}\langle k, \Omega \rangle + (-1)^{j+1}\lambda_+} \exp\left(\mathrm{i}\langle k, \theta \rangle\right), \tag{51}$$

$$\mathrm{f}_j = \sum_{0 < |k|_1 < 2N} \frac{\widetilde{\mathrm{c}}_{j;k}}{\mathrm{i}\langle k, \Omega \rangle} \exp\left(\mathrm{i}\langle k, \theta \rangle\right), \tag{52}$$

$$\{\mathrm{G}_{j,l}\}_\theta = \sum_{0 < |k|_1 < 2N} \frac{\widetilde{\mathrm{B}}_{j,l;k}}{\mathrm{i}\langle k, \Omega \rangle + 2(-1)^{j+1}\lambda_+\delta_{j,l}} \exp\left(\mathrm{i}\langle k, \theta \rangle\right), \tag{53}$$

$$\langle \mathrm{G}_{1,1}\rangle_\theta = \frac{\widetilde{\mathrm{B}}_{1,1;0}}{2\lambda_+}, \qquad \langle \mathrm{G}_{2,2}\rangle_\theta = -\frac{\widetilde{\mathrm{B}}_{2,2;0}}{2\lambda_+}, \tag{54}$$

$$F_{j,l} = \sum_{|k|_1 < 2N} \frac{\widetilde{\mathrm{E}}_{j,l;k}}{\mathrm{i}\langle k, \Omega \rangle + (-1)^{l+1}\lambda_+} \exp\left(\mathrm{i}\langle k, \theta \rangle\right), \tag{55}$$

for $j, l = 1, 2$, where $\delta_{j,l}$ is Kronecker's delta and

$$\widetilde{\mathrm{c}} = \{\mathrm{c}\}_\theta - \{C(\chi + (\partial_\theta \mathrm{d})^*)\}_\theta. \tag{56}$$

Then, we can expand the new Hamiltonian $H^{(1)} := H \circ \Psi_1^S$ as

$$H^{(1)} = \phi^{(1)} + \langle \Omega^{(1)}, I \rangle + \frac{1}{2} \langle z, \mathcal{B}z \rangle + \frac{1}{2} \langle I, \mathcal{C}^{(1)}(\theta)I \rangle + \widetilde{H}^{(1)}(\theta, x, I, y) + \widehat{H}^{(1)}(\theta, x, I, y),$$

where $\Omega_2^{(1)} = \Omega_2$ and $\Omega_1^{(1)}$ is given by (41). Moreover (see section 2 for notations)

$$\phi^{(1)} = \langle a \rangle_\theta - \langle \chi, \Omega \rangle, \qquad \mathcal{C}^{(1)} = C + [\{\overline{H}_{<N,\theta}, S\}]_{I,I}, \qquad \widetilde{H}^{(1)} = \Xi + [\{\overline{H}_{<N,\theta}, S\}], \quad (57)$$

$$\widehat{H}^{(1)} = \{H_{<N,\theta} - \overline{H}_{<N,\theta}, S\} + \int_0^1 (1-t)\{\{H_{<N,\theta}, S\}, S\} \circ \Psi_t^S dt + H_{\geq N,\theta} \circ \Psi_1^S. \quad (58)$$

In particular, using formulas above and $S = S_{<2N,\theta}$, we observe that $\mathcal{C}^{(1)}_{\geq 3N,\theta} = 0$ and $\widetilde{H}^{(1)}_{\geq 3N,\theta} = 0$.

5.6 The parameters domain

In this section we fix the initial set of basic frequencies, $\Lambda \in \overline{\mathcal{U}} = \overline{\mathcal{U}}(R)$ (see (64)), to which we wish to apply the first step of the KAM process. As we want to work iteratively with analytic dependence with respect to $\Lambda = (\mu, \Omega_2)$, we are forced to complexify μ, Ω_2 and hence, the corresponding ξ, η. The concrete set of real parameters in which we look for the persistence of 2D-tori, of the full system (34), is the set $\mathcal{V} = \mathcal{V}(R)$ on the statement of theorem 3.1 (see also (102)).

To do that, we require some quantitative information on the normal form $\widetilde{Z} = \widetilde{Z}^{(R)}(x, I, y)$ of theorem 4.1 (see (15)). From (18) we have that $|\widetilde{Z}^{(R)}|_{0,R} \leq \tilde{c}R^6$, for some $\tilde{c}$ independent of R (eventually it depends on ε and σ but they are kept fixed along the paper). By using lemma A.7, we translate this estimate into a bound for the function $Z = Z^{(R)}(u_1, u_2, u_3)$, defined by writing $\widetilde{Z}$ in terms of $(q, I, L/2)$, thus obtaining $|Z|_{R^2} \leq \tilde{c}R^6$ (see section 2 for the definition of this weighted norm). Then, we conclude that there exists $c_0 > 0$, independent of R, such that

$$|Z|_{R^2} \leq c_0 R^6, \quad |\partial_i Z|_{R^2/2} \leq c_0 R^4, \quad |\partial_{i,j}^2 Z|_{R^2/2} \leq c_0 R^2, \quad |\partial_{i,j,k}^3 Z|_{R^2/2} \leq c_0, \qquad i, j, k = 1, 2, 3. \quad (59)$$

To obtain these bounds we use Cauchy estimates over the norm $|\cdot|_{R^2}$. These estimates, together with other properties of the weighted norms used, are shown in section A.1. Since all these properties are completely analogous to those of the usual supremum norm, sometimes they are going to be used along the proof without explicit mention.

The first application of bounds (59) is to size up the domain Γ of theorem 4.2.

Lemma 5.1. *With the same hypotheses of theorem 4.2. Let $0 < c_1 < \min\{1, |d|/(8(1 + a + 2|e|))\}$. Then, for any $R > 0$ small enough, there is a real analytic function $\mathbb{I} = \mathbb{I}^{(R)}(\zeta)$, defined on the set*

$$\Gamma = \Gamma(R) := \left\{ \zeta = (\xi, \eta) \in \mathbb{C}^2 : |\xi| \leq c_1 R^2, |\eta| \leq c_1 R \right\}, \quad (60)$$

solving equation (22). Moreover, $|\mathbb{I}|_\Gamma \leq R^2/4$, $|\partial_\xi \mathbb{I}|_\Gamma \leq 2a/|d|$ and $|\partial_\eta \mathbb{I}|_\Gamma \leq 4c_1 R/|d|$.

Proof. For a fixed $\zeta \in \Gamma$, we consider the function

$$\mathcal{F}(I; \zeta) = \frac{1}{d}(\eta^2 - a\xi - 2e\xi\eta - \partial_1 Z(\xi, I, \xi\eta)),$$

where ζ is dealt as a parameter. We are going to show that, for any R small enough, the function $\mathcal{F}(\cdot; \zeta)$ is a contraction on the set $\{I \in \mathbb{C} : |I| \leq R^2/4\}$, uniformly for any $\zeta \in \Gamma$. Then, $\mathbb{I}(\zeta) = \mathcal{F}(\mathbb{I}(\zeta); \zeta)$ is the only fixed point of $\mathcal{F}(\cdot; \zeta)$, with analytic dependence on ζ. Indeed, using (59) we have

$$|\mathcal{F}(I; \zeta)| \leq \frac{1}{|d|} \left(c_1^2 R^2 + ac_1 R^2 + 2|e|c_1^2 R^3 + c_0 R^4 \right) \leq \left(\frac{1 + a + 2|e|}{|d|} c_1 + \frac{c_0}{|d|} R^2 \right) R^2 \leq \frac{1}{4} R^2.$$

To ensure the contractive character of $\mathcal{F}(\cdot;\zeta)$ we apply the mean value theorem. Thus, given $\zeta \in \Gamma$ and $|I|, |I'| \leq R^2/4$, we have

$$|\mathcal{F}(I';\zeta) - \mathcal{F}(I;\zeta)| = \frac{|\partial_1 Z_1(\xi, I', \xi\eta) - \partial_1 Z_1(\xi, I, \xi\eta)|}{|d|} \leq \frac{c_0}{|d|} R^2 |I' - I| \leq \frac{1}{2}|I' - I|.$$

To finish the proof we only have to control the partial derivatives of $\mathbb{I}$. From the fixed point equation verified by this function, we have

$$\partial_\xi \mathbb{I} = \frac{-a - 2e\eta - \partial_{1,1}^2 Z - \eta\partial_{1,3}^2 Z}{d + \partial_{1,2}^2 Z}, \qquad \partial_\eta \mathbb{I} = \frac{2\eta - 2e\xi - \xi\partial_{1,3}^2 Z}{d + \partial_{1,2}^2 Z}, \tag{61}$$

with all the partial derivatives of Z evaluated at $(\xi, \mathbb{I}(\zeta), \xi\eta)$. Then, the bounds on the derivatives follow straightforward. $\qquad\square$

Once we have parametrized the family of 2D-bifurcated tori of the normal form as function of $\zeta = (\xi, \eta)$, we control the corresponding set of basic frequencies $\Lambda = (\mu, \Omega_2)$ as follows.

Lemma 5.2. *With the same hypotheses of lemma 5.1. Let us also assume $0 < c_1 < 4a/17$ and consider $0 < c_2 < c_1/2$. Then, for any $R > 0$ small enough, there is a real analytic vector-function $h = h^{(R)}(\Lambda)$, $h = (h_1, h_2)$, defined on the set*

$$\mathcal{U} = \mathcal{U}(R) := \{\Lambda = (\mu, \Omega_2) \in \mathbb{C}^2 : |\mu| \leq c_2 R, |\Omega_2 - \omega_2| \leq c_2 R\}, \tag{62}$$

solving, with respect to $\zeta = (\xi, \eta)$, the equations $\Omega_2 = \Omega_2^{(R)}(\zeta)$ and $\mu = \mu^{(R)}(\zeta)$ defined through (24), (27) and (28), i. e., $\xi = h_1(\Lambda)$ and $\eta = h_2(\Lambda)$. Moreover, $|h_1|_{\mathcal{U}} \leq c_1 R^2$, $|h_2|_{\mathcal{U}} \leq c_1 R$, $|\partial_\mu h_1|_{\mathcal{U}} \leq 2c_2 R/a$, $|\partial_\mu h_2|_{\mathcal{U}} \leq 4(|e|/a + |f|/|d|)c_2 R$, $|\partial_{\Omega_2} h_1|_{\mathcal{U}} \leq 8c_1 R/a$ and $|\partial_{\Omega_2} h_2|_{\mathcal{U}} \leq 2$.

Proof. One proceeds analogously as in the proof of lemma 5.1. By taking into account formulas (24) and (27) we define

$$\mathcal{G}(\zeta; \Lambda) = \left(\frac{\mu^2}{2a} - \frac{2\eta^2}{a} - \frac{\xi}{a}\partial_{1,1}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta), \Omega_2 - \omega_2 - 2c\xi\eta - e\xi - f\mathbb{I}(\zeta) - \frac{1}{2}\partial_3 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right).$$

Unfortunately, the function $\mathcal{G}(\cdot; \Lambda)$ is not contractive in general. Then, we introduce

$$\mathcal{F}(\zeta; \Lambda) = (\mathcal{G}_1(\zeta; \Lambda), \mathcal{G}_2(\mathcal{G}_1(\zeta; \Lambda), \eta; \Lambda)).$$

We are going to verify that the function $\mathcal{F}(\cdot; \Lambda)$ is a contraction on $\Gamma = \Gamma(R)$ (see (60)), uniformly on $\Lambda \in \mathcal{U}$. Then, $h(\Lambda) = \mathcal{F}(h(\Lambda); \Lambda)$ is the only fixed point of $\mathcal{F}(\cdot; \Lambda)$. To do that, we make R as small as necessary and use the bounds on Z, $\mathbb{I}$ and their partial derivatives given by (59) and lemma 5.1. First, we have

$$|\mathcal{F}_1| \leq \frac{1}{a}\left(\frac{c_2^2}{2} + 2c_1^2 + c_0 c_1 R^2\right) R^2 \leq c_1 R^2.$$

Next to that we apply this bound to $\mathcal{F}_2$, thus obtaining

$$|\mathcal{F}_2| \leq \left(c_2 + 2|c|c_1^2 R^2 + |e|c_1 R + \frac{|f|}{4}R + \frac{c_0}{2}R^3\right) R \leq c_1 R.$$

Now, we check that $\mathcal{F}(\cdot; \Lambda)$ is a contraction on Γ. For $\mathcal{F}_1$ we have

$$
\begin{aligned}
|\mathcal{F}_1(\zeta'; \Lambda) - \mathcal{F}_1(\zeta; \Lambda)| &\leq \frac{2}{a}|(\eta')^2 - \eta^2| + \frac{|\xi' - \xi|}{a}|\partial_{1,1}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)| \\
&\quad + \frac{|\xi|}{a}|\partial_{1,1}^2 Z(\xi', \mathbb{I}(\zeta'), \xi'\eta') - \partial_{1,1}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)| \\
&\leq \frac{4c_1}{a} R|\eta' - \eta| + \frac{c_0}{a}(1 + c_1)R^2|\xi' - \xi| + \frac{c_0 c_1}{a} R^2(|\mathbb{I}(\zeta') - \mathbb{I}(\zeta)| + |\xi'\eta' - \xi\eta|) \\
&\leq \left(\frac{4c_1}{a} R + \frac{c_0 c_1}{a} R^3 \left(\frac{4c_1}{|d|} + c_1 R \right) \right) |\eta' - \eta| \\
&\quad + \left(\frac{c_0}{a} R^2 + \frac{c_0 c_1}{a} R^2 \left(1 + \frac{2a}{|d|} + c_1 R \right) \right) |\xi' - \xi| \\
&\leq \min \left\{ \frac{1}{2}, \frac{|d|}{8(|e||d| + 2a|f|)} \right\} |\zeta' - \zeta|.
\end{aligned} \tag{63}
$$

Similarly, we obtain the next bound for $\mathcal{G}_2$:

$$
\begin{aligned}
|\mathcal{G}_2(\zeta'; \Lambda) - \mathcal{G}_2(\zeta; \Lambda)| &\leq \left(2|c|c_1 R^2 + \frac{4|f|c_1}{|d|} R + \frac{c_0}{2} R^3 \left(\frac{4c_1}{|d|} + c_1 R \right) \right) |\eta' - \eta| \\
&\quad + \left(2|c|c_1 R + |e| + \frac{2a|f|}{|d|} + \frac{c_0}{2} R^2 \left(1 + \frac{4a}{|d|} + c_1 R \right) \right) |\xi' - \xi|.
\end{aligned}
$$

Hence, going back to the definition of $\mathcal{F}_2$ and using the inequality (63), we have

$$
|\mathcal{F}_2(\zeta'; \Lambda) - \mathcal{F}_2(\zeta; \Lambda)| \leq \frac{1}{2}|\zeta' - \zeta|.
$$

Finally, the bounds on the partial derivatives of $h(\Lambda)$ follow at once by computing the derivatives of the fixed point equation $h(\Lambda) = \mathcal{G}(h(\Lambda); \Lambda)$ (compare (61)). $\qquad \square$

At this point we can express the remaining intrinsic frequency Ω_1 in terms of Λ. Next lemma accounts for this dependence.

Lemma 5.3. *With the same hypotheses of lemma 5.2. Let us define $\Omega_1^{(0)} = \Omega_1^{(0,R)}(\Lambda)$ as the function $\Omega_1 = \Omega_1^{(R)}(\zeta)$ of (23), when expressed in terms of Λ through the change $\zeta = h^{(R)}(\Lambda)$ given by lemma 5.2 (see (30)). There exists a constant $c_3 > 0$, independent of R, such that if $R > 0$ is small enough then:*

$$
|\Omega_1^{(0)} - \omega_1|_{\mathcal{U}} \leq c_3 R^2, \qquad |\partial_\mu \Omega_1^{(0)}|_{\mathcal{U}} \leq c_3 R, \qquad |\partial_{\Omega_2} \Omega_1^{(0)}|_{\mathcal{U}} \leq c_3 R, \qquad \mathrm{Lip}_{\mathcal{U}}(\Omega_1^{(0)}) \leq 2c_3 R,
$$

where the set $\mathcal{U} = \mathcal{U}^{(R)}$ is defined in (62).

Proof. These bounds are straightforward from those of lemma 5.1 and lemma 5.2. We leave the details for the reader. In particular, to derive the Lipschitz estimate we note that $\mathcal{U}$ is a convex set. $\qquad \square$

Now we introduce the initial set of (complex) parameters $\Lambda = (\mu, \Omega_2) \in \overline{\mathcal{U}} = \overline{\mathcal{U}}(R)$ we are going to consider for the KAM scheme of sections 5.3 and 5.5. We define

$$
\begin{aligned}
\overline{\mathcal{U}} := \big\{ \Lambda \in \mathbb{C}^2 \ : \ &(M^{(0)})^{\alpha/2} \leq \mathrm{Re}\,\mu, \ |\mathrm{Im}\,\mu| \leq 2(M^{(0)})^{\alpha/2}, \ |\mu| \leq c_2 R, \\
&|\Omega_2 - \omega_2| \leq c_2 R, \ |\mathrm{Im}\,\Omega_2| \leq 2(M^{(0)})^{\alpha/2}, \ |h_1(\Lambda)| \geq (M^{(0)})^{\alpha/2} \big\},
\end{aligned} \tag{64}
$$

where α, $M^{(0)} = M^{(0)}(R)$ are thus on the statement of theorem 3.1 and c_2, $h = h^{(R)} = (h_1^{(R)}, h_2^{(R)})$ are given by lemma 5.2.

Remark 5.2. *The set $\overline{\mathcal{U}}$ is the restriction to $\mathbb{R}^+ \times \mathbb{R}$ of the set $\mathcal{U}$ provided by lemma 5.2, —i. e., the set $\mathcal{V}$ of (102)— plus a small complex widening and small technical restrictions. We point out that $\overline{\mathcal{U}} \subset \mathcal{U}$, for small R. In particular definition (64) implies that, if $\Lambda \in \overline{\mathcal{U}}$, then $|\mu| \geq (M^{(0)})^{\alpha/2}$ —i. e., we are "far" from parabolic tori given by $\mu = 0$— and $\zeta = (\xi, \eta) = h(\Lambda)$ verifies $(M^{(0)})^{\alpha/2} \leq |\xi| \leq c_1 R^2$ —i. e., we are "far" from the stable periodic orbits of the family given by $\xi = 0$— and $|\eta| \leq c_1 R$ (see lemma 5.1). See figure 1.*

5.7 Domain of definition of the Λ-family of Hamiltonians

Next to that, we consider the "initial" family of Hamiltonian systems $H_\Lambda^{(0)}$ (see (34)) and fix up their domain of definition.

Taking into account the symplectic changes (19) and (32), one may write the normal form coordinates $(\theta, x_1, x_2, I, y_1, y_2)$ of (14) as a function of $(\theta_1, \theta_2, x, I_1, I_2, y)$, depending on the prefixed R and the parameter Λ. Writing them up explicitly, we have

$$
\begin{aligned}
\theta &= \theta_1 + \frac{\xi}{\lambda_+}(d + \partial_{1,2}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta))\left(\frac{x}{\xi} + \frac{y}{\lambda_+}\right), & I &= \mathbb{I}(\zeta) + I_1, \\
y_1 &= \sqrt{2\xi(1+\hat{q})}\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right)\cos\phi_2 - \frac{2\xi\eta + I_2}{\sqrt{2\xi(1+\hat{q})}}\sin\phi_2, & x_1 &= \sqrt{2\xi(1+\hat{q})}\cos\phi_2, \\
y_2 &= -\sqrt{2\xi(1+\hat{q})}\left(\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right)\sin\phi_2 - \frac{2\xi\eta + I_2}{\sqrt{2\xi(1+\hat{q})}}\cos\phi_2, & x_2 &= -\sqrt{2\xi(1+\hat{q})}\sin\phi_2,
\end{aligned}
$$

being

$$
\begin{aligned}
\phi_2 &= \theta_2 + \frac{1}{\lambda_+}\left(-\eta + e\xi + \frac{\xi}{2}\partial_{1,3}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right)\left(\frac{x}{\xi} + \frac{y}{\lambda_+}\right), \\
\hat{q} &= \frac{x}{\xi} - \frac{y}{\lambda_+} + \frac{2}{\lambda_+^2}(d + \partial_{1,2}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta))I_1 + \frac{2}{\xi\lambda_+^2}\left(-\eta + e\xi + \frac{\xi}{2}\partial_{1,3}^2 Z(\xi, \mathbb{I}(\zeta), \xi\eta)\right)I_2,
\end{aligned}
$$

where $\lambda_+ = \mathrm{i}\mu$ and $\zeta = (\xi, \eta)$ are function of Λ through lemma 5.2. See also theorem 4.2 and lemma 5.1 for the definition and bounds on $\mathbb{I}$.

In view of this coordinate transformation, we select (see theorem 4.1),

$$
\rho^{(0)} := \min\{\sigma^{-2}\rho_0/4, \log(2)/2\}, \qquad R^{(0)} = R^{(0)}(R) := (M^{(0)}(R))^\alpha, \tag{65}
$$

and we want to show that the change $(\theta_1, \theta_2, x, I_1, I_2, y) \mapsto (\theta, x_1, x_2, I, y_1, y_2)$ is well defined from $\mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$ to $\mathcal{D}_{1,2}(\sigma^{-2}\rho_0/2, R)$ (see (1)), for any $\Lambda \in \overline{\mathcal{U}}$, and controlled in terms of the weighted norm $|\cdot|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}}$ —computed by expanding it in $(\theta_1, \theta_2, x, I_1, I_2, y)$—.

To do that, first of all we combine the definition of $\overline{\mathcal{U}}$ in (64) (see also remark 5.2) with the bounds (59) on the partial derivatives of $Z = Z^{(R)}$, to obtain, for any R small enough, the following estimates (we recall that $M^{(0)}(R)$ goes to zero faster than any power of R)

$$
\begin{aligned}
\left|\frac{x}{\xi} + \frac{y}{\lambda_+}\right|_{\overline{\mathcal{U}}, 0, R^{(0)}} &\leq 2(M^{(0)})^{\alpha/2}, & \left|\frac{\lambda_+}{2\xi}x + \frac{1}{2}y\right|_{\overline{\mathcal{U}}, 0, R^{(0)}} &\leq c_2 R\frac{(M^{(0)})^{3\alpha/4}}{\sqrt{|\xi|}}, \\
|\theta - \theta_1|_{\overline{\mathcal{U}}, 0, R^{(0)}} &\leq 2c_1 R^2(|d| + c_0 R^2) < \sigma^{-2}\rho_0/4, \\
|\phi_2 - \theta_2|_{\overline{\mathcal{U}}, 0, R^{(0)}} &\leq 2c_1 R + 2|e|c_1 R^2 + c_0 c_1 R^4 < \log(2)/2, \\
|\hat{q}|_{\overline{\mathcal{U}}, 0, R^{(0)}} &\leq 2(|d| + c_0 R^2)(M^{(0)})^\alpha + (2 + 2c_1 R + 2|e|c_1 R^2 + c_0 c_1 R^4)(M^{(0)})^{\alpha/2} \leq 3(M^{(0)})^{\alpha/2}.
\end{aligned}
$$

By assuming $3(M^{(0)})^{\alpha/2} \leq 1/2$, we use the estimate on $\hat{q}$ and lemma A.6 to obtain

$$
|\sqrt{1+\hat{q}}|_{\overline{\mathcal{U}}, 0, R^{(0)}} \leq (4 - \sqrt{2})/2, \qquad |(\sqrt{1+\hat{q}})^{-1}|_{\overline{\mathcal{U}}, 0, R^{(0)}} \leq \sqrt{2}.
$$

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The estimate on ϕ_2, combined with lemma A.2, gives

$$|\sin \phi_2|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}} \le e^{\rho^{(0)} + |\phi_2 - \theta_2|_{\overline{\mathcal{U}}, 0, R^{(0)}}} \le e^{\log(2)} \le 2,$$

and the same holds for $\cos \phi_2$. Moreover we also have, for small R, $|2\xi\eta + I_2|_{\overline{\mathcal{U}}, 0, R^{(0)}} \le 3c_1\sqrt{c_1}R^2\sqrt{|\xi|}$.

If we put these bounds all together we obtain, for $j = 1, 2$,

$$|I - I_1|_{\overline{\mathcal{U}}, 0, R^{(0)}} \le \frac{R^2}{4}, \qquad |x_j|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}} \le 2(2\sqrt{2} - 1)\sqrt{c_1}R, \qquad |y_j|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}} \le 7c_1\sqrt{c_1}R^2.$$

Consequently, if R is small enough and c_1 is such that $2(2\sqrt{2} - 1)\sqrt{c_1} < 1$, then we can ensure that the transformation is controlled as we claimed.

Now, we are in conditions to bound the different elements of the initial system $H^{(0)} = H_\Lambda^{(0)}$ in (34) in the domain $\mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$. To do that we introduce a constant $\kappa > 0$, independent of R, defined so that we achieve conditions below. Moreover, we take strong advantage on the use of weighted norms in order to control any term of the decomposition (34) by the norm of the full system. We have (see theorem 4.1):

$$|H^{(0)}|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}} \le \kappa, \qquad |\widetilde{H}^{(0)}|_{\overline{\mathcal{U}}, 0, R^{(0)}} \le \kappa, \qquad |\widehat{H}^{(0)}|_{\overline{\mathcal{U}}, \rho^{(0)}, R^{(0)}} \le M^{(0)}. \tag{66}$$

By using formulas (36) we also have

$$|\mathcal{C}_{1,1}^{(0)}|_{\overline{\mathcal{U}}} \le \frac{\kappa}{(M^{(0)})^\alpha}, \quad |\mathcal{C}_{1,2}^{(0)}|_{\overline{\mathcal{U}}} \le \frac{\kappa}{(M^{(0)})^\alpha}, \quad |\mathcal{C}_{2,2}^{(0)}|_{\overline{\mathcal{U}}} \le \frac{\kappa}{(M^{(0)})^{3\alpha/2}}, \quad |\mathcal{C}^{(0)}|_{\overline{\mathcal{U}}} \le \frac{\kappa}{(M^{(0)})^{3\alpha/2}}. \tag{67}$$

Finally, we consider the matrix $\bar{\mathcal{A}}^{(0)}$ discussed in section 5.4, whose determinant defines the non-degenerate character of the selected set of basic frequencies. For this matrix we have proved that $|\det \bar{\mathcal{A}}^{(0)}|_{\mathcal{U}} \ge |ad|/(2|\mu|^3)$, for any R small enough. Then, using again the closed formulas (36) for $\mathcal{C}_{1,2}^{(0)}, \mathcal{C}_{2,2}^{(0)}$ and those on the partial derivatives on $\widetilde{H}^{(0)}$ given in section 5.4, we have

$$|(\bar{\mathcal{A}}^{(0)})^{-1}|_{\overline{\mathcal{U}}} \le \frac{\kappa}{(M^{(0)})^{\alpha/2}}. \tag{68}$$

5.8 The Iterative Lemma

The purpose of this section is to give quantitative estimates on the effect of one step of the iterative process described in sections 5.3 and 5.5. The result controlling this process is stated as follows.

Lemma 5.4 (Iterative Lemma). *We consider a family of Hamiltonian systems $H = H_\Lambda(\theta, x, I, y)$ defined in $\mathcal{D}_{2,1}(\bar{\rho}, \bar{R})$ for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}} \subset \mathbb{C}^2$, for some $\bar{\rho}, \bar{R} \in (0, 1)$, with analytic dependence in all variables and parameters. The Hamiltonian H takes the form (with everything depending on Λ),*

$$H = \phi + \langle \Omega, I \rangle + \frac{1}{2}\langle z, \mathcal{B}z \rangle + \frac{1}{2}\langle I, \mathcal{C}(\theta)I \rangle + \widetilde{H}(\theta, x, I, y) + \widehat{H}(\theta, x, I, y),$$

where $\mathcal{B}$ is defined in (35), being $\lambda_+ = i\mu$, the function $\widetilde{H}$ contains "higher order terms", i. e., $\widetilde{H} = [\widetilde{H}]$ (see (4)) and (abusing notation) $\Omega(\Lambda) = (\Omega_1(\Lambda), \Omega_2)$. We suppose that there is an integer $\bar{N} \ge 1$, and real quantities $\tau > 1$, $0 < \alpha < 1$, $\kappa > 0$, $0 < \bar{M} \le M^{(0)} < 1$ so that $(M^{(0)})^\alpha/2 \le \bar{R} \le (M^{(0)})^\alpha$ and, for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}}$, we have $(H - \widehat{H})_{\ge \bar{N}, \theta} = 0$, $|\mu| \ge (M^{(0)})^{\alpha/2}$,

$$|\langle k, \Omega \rangle + \ell\mu| \ge (M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \qquad k \in \mathbb{Z}^2, 0 < |k|_1 < 2\bar{N}, \ell \in \{0, 1, 2\}, \tag{69}$$

and the following bounds:

$$|H|_{\bar{\mathcal{E}}, \bar{\rho}, \bar{R}} \le \kappa, \quad |\widetilde{H}|_{\bar{\mathcal{E}}, \bar{\rho}, \bar{R}} \le 2\kappa, \quad |\widehat{H}|_{\bar{\mathcal{E}}, \bar{\rho}, \bar{R}} \le \bar{M}, \quad |\mathcal{C}|_{\bar{\mathcal{E}}, \bar{\rho}, 0} \le \frac{2\kappa}{(M^{(0)})^{3\alpha/2}}, \quad |(\langle \bar{\mathcal{A}} \rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \le \frac{2\kappa}{(M^{(0)})^{\alpha/2}},$$

where $\bar{\mathcal{A}}$ denotes the matrix $\mathcal{A}$ defined in (45) by setting $\Xi = \widetilde{H}$ and $C = \mathcal{C}$.

Under these conditions, given $0 < \rho^{(\infty)} < \bar{\rho}$, there is a constant $\bar{\kappa} \geq 1$, depending only on κ, τ and $\rho^{(\infty)}$, such that if for certain $0 < \bar{\delta} < 1/2$ we have $\bar{\rho}^{(1)} := \bar{\rho} - 6\bar{\delta} \geq \rho^{(\infty)}$ and

$$\frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{14\alpha}} \leq 1, \tag{70}$$

then for any $\Lambda \in \bar{\mathcal{E}}$ there exists a canonical transformation $\Psi = \Psi_{\Lambda}(\theta, x, I, y)$, with analytic dependence in all variables and parameters, acting as $\Psi : \mathcal{D}_{2,1}(\bar{\rho}^{(1)}, \bar{R}^{(1)}) \to \mathcal{D}_{2,1}(\bar{\rho} - 5\bar{\delta}, \bar{R}\exp(-2\bar{\delta}))$, being $\bar{R}^{(1)} := \bar{R}\exp(-3\bar{\delta})$. If $\Psi - \mathrm{Id} = (\Theta, \mathcal{X}, \mathcal{I}, \mathcal{Y})$, then all the components are 2π-periodic in θ and verify

$$|\Theta|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \quad \leq \quad \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}} \leq \bar{\delta}, \tag{71}$$

$$|\mathcal{I}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \quad \leq \quad \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+2}(M^{(0)})^{15\alpha/2}} \leq (\bar{R}\exp(-2\bar{\delta}))^2 - (\bar{R}\exp(-3\bar{\delta}))^2, \tag{72}$$

$$|\mathcal{Z}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \quad \leq \quad \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{17\alpha/2}} \leq \bar{R}\exp(-2\bar{\delta}) - \bar{R}\exp(-3\bar{\delta}), \tag{73}$$

being $\mathcal{Z} = (\mathcal{X}, \mathcal{Y})$. This canonical transformation is defined so that we can expand the transformed Hamiltonian H by the action of Ψ, $H^{(1)} = H_{\Lambda}^{(1)}(\theta, x, I, y)$, as

$$H^{(1)} := H \circ \Psi = \phi^{(1)} + \langle \Omega^{(1)}, I \rangle + \frac{1}{2}\langle z, \mathcal{B}z \rangle + \frac{1}{2}\langle I, \mathcal{C}^{(1)}(\theta)I \rangle + \widetilde{H}^{(1)}(\theta, x, I, y) + \widehat{H}^{(1)}(\theta, x, I, y),$$

with everything depending on Λ, where $\Omega_2^{(1)}(\Lambda) = \Omega_2$, $[\widetilde{H}^{(1)}] = \widetilde{H}^{(1)}$, $(H^{(1)} - \widehat{H}^{(1)})_{\geq 3\bar{N}, \theta} = 0$ and

$$|\Omega_1^{(1)} - \Omega_1|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau}(M^{(0)})^{13\alpha/2}}, \qquad\qquad |\widetilde{H}^{(1)} - \widetilde{H}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}},$$

$$|\mathcal{C}^{(1)} - \mathcal{C}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, 0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{27\alpha/2}}, \qquad |(\langle \bar{\mathcal{A}}^{(1)} \rangle_{\theta})^{-1} - (\langle \bar{\mathcal{A}} \rangle_{\theta})^{-1}|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{29\alpha/2}},$$

$$|H^{(1)}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \leq \kappa, \qquad\qquad\qquad |\widehat{H}^{(1)}|_{\bar{\mathcal{E}}, \bar{\rho}^{(1)}, \bar{R}^{(1)}} \leq \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{4\tau+6}(M^{(0)})^{19\alpha}} + \bar{M}\exp(-\bar{\delta}\bar{N}),$$

where $\bar{\mathcal{A}}^{(1)}$ is defined analogously as $\bar{\mathcal{A}}$.

Remark 5.3. *It is not difficult to realize that if the (complex) analytic Hamiltonian H of the statement verifies the symmetries due to the complexification (33), then the same holds for $H^{(1)}$ (see section 5.2 and remark 5.1 for more details).*

Proof. Our plan is to give only a sketch of the proof. Full details can be easily developed by hand by the interested reader. During the proof, and abusing notation, the constant $\bar{\kappa}$ will be re-defined several times in order to meet a finite number of conditions. The constant $\bar{\kappa}$ of the statement is the final one. Moreover, we will use some technical lemmas given in section A.1 in order to control the weighted norms of the derivatives (Cauchy estimates), composition of functions and solutions of small divisors equations, without explicit mention. Finally, the analytic dependence of Ψ_{Λ} on Λ, and so of $H_{\Lambda}^{(1)}$, follows straightforward from the way in which this canonical transformation is generated.

We start by decomposing $H = H_{<\bar{N}, \theta} + H_{\geq \bar{N}, \theta}$ as in (49), expanding $H_{<\bar{N}, \theta}$ as in (38) and defining $\overline{H}_{<\bar{N}, \theta}$ from $H_{<\bar{N}, \theta}$ as in (39). After that, we compute the generating function $S = S_{\Lambda}(\theta, x, I, y)$ of (40), defined by solving the homological equations (eq1)–(eq5). Then, we define the canonical transformation $\Psi = \Psi_{\Lambda}$ as the time one flow of the Hamiltonian system S, i. e., $\Psi = \Psi_{t=1}^S$. We recall that the condition $(H^{(1)} - \widehat{H}^{(1)})_{\geq 3\bar{N}, \theta} = 0$ follows at once using (57), (58) and $S = S_{<2N, \theta}$.

Next to that, we perform the quantitative part of the lemma. First of all, we have the following bounds for the terms of the decomposition (38) of $H_{<\bar{N},\theta}$ and for $H_{\geq\bar{N},\theta} = \widehat{H}_{\geq\bar{N},\theta}$,

$$|a - \phi|_{\bar{\mathcal{E}},\bar{\rho},0} \leq \bar{M}, \qquad |b|_{\bar{\mathcal{E}},\bar{\rho},0} \leq \frac{\bar{M}}{\bar{R}}, \qquad |c - \Omega|_{\bar{\mathcal{E}},\bar{\rho},0} \leq \frac{\bar{M}}{\bar{R}^2}, \qquad |B - \mathcal{B}|_{\bar{\mathcal{E}},\bar{\rho},0} \leq 8\frac{\bar{M}}{\bar{R}^2},$$

$$|C - \mathcal{C}|_{\bar{\mathcal{E}},\bar{\rho},0} \leq 8\frac{\bar{M}}{\bar{R}^4}, \qquad |E|_{\bar{\mathcal{E}},\bar{\rho},0} \leq 2\frac{\bar{M}}{\bar{R}^3}, \qquad |\Xi - \widetilde{H}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq \bar{M}, \qquad |H_{\geq\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho}-\bar{\delta},\bar{R}} \leq \bar{M}\exp(-\bar{\delta}\bar{N}).$$

By assuming $\bar{\kappa}\bar{M}/(M^{(0)})^{5\alpha/2} \leq 1$ we also have

$$|C|_{\bar{\mathcal{E}},\bar{\rho},0} \leq \frac{4\kappa}{(M^{(0)})^{3\alpha/2}}, \qquad |\Xi|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq 4\kappa.$$

Furthermore, we need to control the norm of $(\langle\mathcal{A}\rangle_\theta)^{-1}$, where $\mathcal{A}$ is defined in (45). We observe that

$$(\langle\mathcal{A}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1} = -\left(\mathrm{Id} + (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}(\langle\mathcal{A}\rangle_\theta - \langle\bar{\mathcal{A}}\rangle_\theta)\right)^{-1}(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}(\langle\mathcal{A}\rangle_\theta - \langle\bar{\mathcal{A}}\rangle_\theta)(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}. \quad (74)$$

If we also assume $\bar{\kappa}\bar{M}/(M^{(0)})^{9\alpha/2} \leq 1$, then we have

$$|\mathcal{A} - \bar{\mathcal{A}}|_{\bar{\mathcal{E}},\bar{\rho},0} \leq 16\frac{\bar{M}}{\bar{R}^4}, \qquad\qquad |(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}}\,|\langle\mathcal{A}\rangle_\theta - \langle\bar{\mathcal{A}}\rangle_\theta|_{\bar{\mathcal{E}}} \leq \frac{1}{2},$$

$$|(\langle\mathcal{A}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{(M^{(0)})^{5\alpha}}, \qquad |(\langle\mathcal{A}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leq \frac{4\kappa}{(M^{(0)})^{\alpha/2}}. \quad (75)$$

Remark 5.4. *To estimate the difference $\mathcal{A} - \bar{\mathcal{A}}$, we take into account that the partial derivatives of $\Xi - \widetilde{H}$ are evaluated at $z = 0$ and $I = 0$ when bounding them by means of Cauchy estimates.*

Now we bound the solutions of the homological equations, which are displayed explicitly from (50) to (55) (see also the compatibility equation (44)). For instance, we have the following bound for d (see (69) and lemma A.4):

$$|d|_{\bar{\mathcal{E}},\bar{\rho}-\bar{\delta},0} \leq \left(\frac{\tau}{\bar{\delta}\exp(1)}\right)^\tau \frac{|\{a\}_\theta|_{\bar{\mathcal{E}},\bar{\rho},0}}{(M^{(0)})^{\alpha/2}} \leq \left(\frac{\tau}{\bar{\delta}\exp(1)}\right)^\tau \frac{|a - \phi|_{\bar{\mathcal{E}},\bar{\rho},0}}{(M^{(0)})^{\alpha/2}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{\alpha/2}}.$$

Similarly, we bound (recursively) the remaining ingredients involved in the resolution of these equations (see (41), (42), (43), (46), (47) and (56)), thus obtaining

$$|e|_{\bar{\mathcal{E}},\bar{\rho}-\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{3\alpha/2}}, \qquad |h_1|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{2\alpha}}, \qquad |h_2|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{9\alpha/2}},$$

$$|\chi|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{5\alpha}}, \qquad |\Omega_1^{(1)} - \Omega_1|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^\tau(M^{(0)})^{13\alpha/2}}, \qquad |\widetilde{c}|_{\bar{\mathcal{E}},\bar{\rho}-2\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{13\alpha/2}},$$

$$|f|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{7\alpha}}, \qquad |\widetilde{B} - \mathcal{B}|_{\bar{\mathcal{E}},\bar{\rho}-2\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{9\alpha}}, \qquad |\langle G\rangle_\theta|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{19\alpha/2}},$$

$$|G|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}}, \qquad |\widetilde{E}|_{\bar{\mathcal{E}},\bar{\rho}-2\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{\tau+1}(M^{(0)})^{10\alpha}}, \qquad |F|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{21\alpha/2}}.$$

Remark 5.5. *Besides the trick pointed in remark 5.4, we have also used a similar idea to bound the average $\langle\cdot\rangle_\theta$ of any expression containing derivatives with respect to θ, i. e., $|\langle\partial_\theta(\cdot)\rangle_\theta| \leq |\cdot|_{\rho,0}/(\rho\exp(1))$.*

From here we have (see (40)),

$$|\nabla_\theta S|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+2}(M^{(0)})^{15\alpha/2}}, \quad |\nabla_I S|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},\bar{R}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{19\alpha/2}},$$

$$|\nabla_z S|_{\bar{\mathcal{E}},\bar{\rho}-3\bar{\delta},\bar{R}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+1}(M^{(0)})^{17\alpha/2}}.$$

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Now, we apply lemma A.3 to obtain the leftmost part of estimates (71), (72) and (73) on the components of the canonical change $\Psi = \Psi_1^S$. Then, the rightmost part of these estimates follows at once. For instance, for (72) we have

$$\frac{|\mathcal{I}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}}}{(\bar{R}\exp(-2\bar{\delta}))^2 - (\bar{R}\exp(-3\bar{\delta}))^2} = \frac{\exp(4\bar{\delta})|\mathcal{I}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},\bar{R}^{(1)}}}{\bar{R}^2(1 - \exp(-2\bar{\delta}))} \leq \frac{4\exp(2)\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}} \leq 1, \qquad (76)$$

which is guaranteed by (70). Here, we have used that $\bar{R} \geq (M^{(0)})^\alpha/2$, $0 < \bar{\delta} \leq 1/2$ and the bound $(1 - \exp(-x))^{-1} \leq 2/x$, whenever $0 < x \leq 1$. Similarly we obtain the rightmost part of (71) and (73).

To finish the proof it only remains to bound the transformed system $H^{(1)}$. Concretely, we have to focus on formulas (57) and (58). First we observe that,

$$|\overline{H}_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq |H_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} = |H - \widehat{H}_{\geq\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq \kappa + \bar{M} \leq 2\kappa, \quad |H_{<\bar{N},\theta} - \overline{H}_{<\bar{N},\theta}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq |\widehat{H}|_{\bar{\mathcal{E}},\bar{\rho},\bar{R}} \leq \bar{M}$$

and hence

$$|\{\overline{H}_{<\bar{N},\theta}, S\}|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}\exp(-\bar{\delta})} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}},$$

$$|\{\{\overline{H}_{<\bar{N},\theta}, S\}, S\}|_{\bar{\mathcal{E}},\bar{\rho}-5\bar{\delta},\bar{R}\exp(-2\bar{\delta})} \leq \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{4\tau+6}(M^{(0)})^{19\alpha}},$$

$$|\{H_{<\bar{N},\theta} - \overline{H}_{<\bar{N},\theta}, S\}|_{\bar{\mathcal{E}},\bar{\rho}-4\bar{\delta},\bar{R}\exp(-\bar{\delta})} \leq \frac{\bar{\kappa}\bar{M}^2}{\bar{\delta}^{2\tau+3}(M^{(0)})^{19\alpha/2}}.$$

From these bounds we easily derive the estimates for $H^{(1)}$ on the statement, for a suitable $\bar{\kappa}$. The only one that is not immediate is thus on $(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}$. To obtain this bound we proceed as in (74) and (75). Indeed,

$$|\bar{\mathcal{A}}^{(1)} - \bar{\mathcal{A}}|_{\bar{\mathcal{E}},\bar{\rho}^{(1)},0} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{27\alpha/2}}, \qquad\qquad |(\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \, |\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta - \langle\bar{\mathcal{A}}\rangle_\theta|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{14\alpha}},$$

$$|\left(\mathrm{Id} + (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta - \langle\bar{\mathcal{A}}\rangle_\theta)\right)^{-1}|_{\bar{\mathcal{E}}} \leq 2, \qquad |(\langle\bar{\mathcal{A}}^{(1)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}} \leq \frac{\bar{\kappa}\bar{M}}{\bar{\delta}^{2\tau+3}(M^{(0)})^{29\alpha/2}}.$$

The control of the expressions above induces the the strongest restriction when defining condition (70). $\qquad\square$

5.9 Convergence of the iterative scheme

Now, we have all the ingredients needed to prove the convergence of the iterative (KAM) scheme of sections 5.3 and 5.5. Concretely, we consider the sequence of transformed Hamiltonians $H^{(n)} = H_\Lambda^{(n)}$ —starting with $H_\Lambda^{(0)}$ of (34)— and we want this sequence to converge to the "normalized" Hamiltonian $H^{(\infty)} = H_\Lambda^{(\infty)}$ of (37) if $\Lambda = (\mu, \Omega_2)$ belongs to a suitable (Cantor) set $\bar{\mathcal{E}}^{(\infty)}$ (see (82)). This limit Hamiltonian has, for any $\Lambda \in \bar{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2$, an invariant 2D-torus with vector of basic frequencies Λ.

To construct this sequence we apply iteratively lemma 5.4, so that we define $H^{(n+1)} = H^{(n)} \circ \Psi^{(n)}$, where $\Psi^{(n)} = \Psi_\Lambda^{(n)}$ is the canonical transformation provided by the lemma. Of course, all this process depends on the value of R we have fixed at the beginning of section 5 and, at any step, everything is analytic on Λ, in a (complex) set $\bar{\mathcal{E}}^{(n)}$ shrinking with n (see (81)). Therefore, to ensure the inductive applicability of lemma 5.4 we have to control, at every step, the conditions of the statement.

First of all we observe that the constants τ, α and the function $M^{(0)} = M^{(0)}(R)$ (see (17)) have been clearly set during the paper, whilst the constant κ (independent of R) is the one introduced at the end of section 5.7.

Now we select a fixed $0 < \bar{\delta}^{(0)} < 1/2$ (independent of R) and introduce (see (65))

$$\rho^{(\infty)} := \rho^{(0)} - 13\bar{\delta}^{(0)}, \qquad R^{(\infty)} = R^{(\infty)}(R) := R^{(0)}(R)\exp(-7\bar{\delta}^{(0)}). \tag{77}$$

We also assume $\bar{\delta}^{(0)}$ small enough so that $\rho^{(\infty)} > 0$ and $\exp(7\bar{\delta}^{(0)}) \leq 2$. Hence, $R^{(\infty)} \geq (M^{(0)})^{\alpha}/2$. We use $\bar{\delta}^{(0)}$ to define, recursively,

$$\bar{\delta}^{(n)} := \bar{\delta}^{(0)}/2^n, \qquad \bar{\rho}^{(n+1)} := \bar{\rho}^{(n)} - 6\bar{\delta}^{(n)}, \qquad \bar{R}^{(n+1)} := \bar{R}^{(n)}\exp(-3\bar{\delta}^{(n)}), \qquad n \geq 0, \tag{78}$$

starting with $\bar{\rho}^{(0)} := \rho^{(0)}$ and $\bar{R}^{(0)} := R^{(0)}$. Hence, $\bar{R}^{(n)}$ depends on the prefixed R. Our purpose is to apply lemma 5.4 to $H^{(n)}$ with $\bar{\rho} = \bar{\rho}^{(n)}$ and $\bar{\delta} = \bar{\delta}^{(n)}$.

Next to that we set the value of $\bar{N}$ at the n step of the iterative process. To do that, we consider the bound for the size of the "error term" $\widehat{H}^{(1)}$ of the transformed Hamiltonian provided by the iterative lemma. Then, this expression suggests to select $\bar{N} = \bar{N}^{(n)}(R) \in \mathbb{N}$ so that

$$\bar{M}^{(n)}\exp(-\bar{\delta}^{(n)}\bar{N}^{(n)}) \leq \frac{\bar{\kappa}(\bar{M}^{(n)})^2}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}. \tag{79}$$

This implies that we can define after this n-stage

$$\bar{M}^{(n+1)} = \frac{2\bar{\kappa}(\bar{M}^{(n)})^2}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}, \tag{80}$$

so that $|\widehat{H}^{(n+1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}^{(n+1)}, \bar{R}^{(n+1)}} \leq \bar{M}^{(n+1)}$, starting with $\bar{M}^{(0)} := M^{(0)}$.

Moreover, we also notice that to define $\bar{N}^{(n)}$ we have to take care of the inductive condition $(H^{(n)} - \widehat{H}^{(n)})_{\geq \bar{N}^{(n)}, \theta} = 0$. Assuming it true at the n-step and using that the transformed Hamiltonian verifies $(H^{(n+1)} - \widehat{H}^{(n+1)})_{\geq 3\bar{N}^{(n)}, \theta} = 0$, then, to keep track of it, we only need to ensure that $\bar{N}^{(n+1)} > 3\bar{N}^{(n)}$ (see (85) and comments below).

Finally, we introduce the set $\bar{\mathcal{E}} = \bar{\mathcal{E}}^{(n)}(R)$ we dealt with at any step. This set is defined recursively from $\overline{\mathcal{U}}(R)$ (see (64)), by taking into account the Diophantine conditions (69). Concretely, we first introduce, for convenience, $\bar{\mathcal{E}}^{(-1)} = \overline{\mathcal{U}}$ and, for each $n \geq 0$,

$$\bar{\mathcal{E}}^{(n)} := \{\Lambda \in \bar{\mathcal{E}}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geq a_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \ 0 < |k|_1 < 2\bar{N}^{(n)}, \ell \in \{0,1,2\}\}, \tag{81}$$

being $a_n = 1 + 2^{-n}$, where $\Omega_1^{(n)}(\Lambda)$ is defined recursively (starting with $\Omega_1^{(0)}(\Lambda)$ given by lemma 5.3) and $\Omega_2^{(n)}(\Lambda) = \Omega_2$. In particular, we recall that $\Lambda \in \overline{\mathcal{U}}$ implies $|\mu| \geq (M^{(0)})^{\alpha/2}$. Moreover, we also point out that $a_n \geq 1$, so that conditions (69) are fulfilled for any n.

As we are dealing with a finite number of Diophantine conditions, then $\bar{\mathcal{E}}^{(n)}(R)$ is a set with non-empty interior, for each $n \geq 0$. Therefore, at the limit $n \to +\infty$ it becomes a Cantor set,

$$\bar{\mathcal{E}}^{(\infty)} := \bigcap_{n\geq 0}\bar{\mathcal{E}}^{(n)}. \tag{82}$$

We point out that, a priori, we cannot guarantee that $\bar{\mathcal{E}}^{(\infty)}$ is non-empty. Moreover, we also recall that we are only interested in real basic frequencies, but that $\bar{\mathcal{E}}^{(\infty)}$ can be a complex set. These two topics are discussed in section 5.12.

Our purpose now is to ensure that if $M^{(0)}$ is small enough —i. e., if R is small enough— then all the requirements needed to apply lemma 5.4 to $H^{(n)}$ are fulfilled for any $\Lambda \in \bar{\mathcal{E}}^{(n)}$ and $n \geq 0$. We begin by assuming a priori that we can iterate indefinitely. If this were possible, using (78) and (80) we establish the following expression for $\bar{M}^{(n)}$:

$$\bar{M}^{(n)} = \frac{(\bar{\delta}^{(0)})^{4\tau+6}(M^{(0)})^{19\alpha}}{2\bar{\kappa}}2^{-(n+1)(4\tau+6)}(\bar{\kappa}^{(0)})^{2^n}, \tag{83}$$

where

$$\bar{\kappa}^{(0)} := \frac{2^{4\tau+7}\bar{\kappa}(M^{(0)})^{1-19\alpha}}{(\bar{\delta}^{(0)})^{4\tau+6}}. \tag{84}$$

We point out that, as $0 < \alpha < 1/19$, we can make $\bar{\kappa}^{(0)}$ as small as required by simply taking R small enough. In particular, if we suppose $\bar{\kappa}^{(0)} \le 1/2$, then the size of the "error term" goes to zero with the step. Next consequence is that the inductive condition (70), formulated at the n-step, reads now as

$$2^{-(n+2)(2\tau+3)-1}(\bar{\delta}^{(0)})^{2\tau+3}(M^{(0)})^{5\alpha}(\bar{\kappa}^{(0)})^{2^n} \le 1,$$

and clearly holds if $\bar{\kappa}^{(0)} \le 1/2$ and $M^{(0)}$ is small enough.

Moreover, using (83) we can also give the explicit expression of the value $\bar{N} = \bar{N}^{(n)}(R)$ we select for the ultra-violet cut-off. Thus, from condition (79) it is natural to take $\bar{N}^{(n)} := \lfloor \hat{N}^{(n)} \rfloor + 1$, with

$$\hat{N}^{(n)} := -\frac{1}{\delta^{(n)}}\log\left(\frac{\bar{\kappa}(\bar{M}^{(n)})}{(\bar{\delta}^{(n)})^{4\tau+6}(M^{(0)})^{19\alpha}}\right) = \frac{2^{2n}}{\bar{\delta}^{(0)}}\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right) + \frac{2^n(4\tau+7)}{\bar{\delta}^{(0)}}\log(2). \tag{85}$$

From this definition one clearly conclude $\lim_{n\to+\infty} \bar{N}^{(n)} = +\infty$. In addition, if we also assume $\bar{\kappa}^{(0)} \le 2^{-4\tau-7}\mathrm{e}^{-\bar{\delta}^{(0)}}$, then $\hat{N}^{(n+1)} \ge 3\hat{N}^{(n)} + 3$, and hence the iterative condition $\bar{N}^{(n+1)} > 3\bar{N}^{(n)}$ is also fulfilled. Finally, to simplify the control of $\bar{N}^{(n)}$ we observe that

$$\frac{2^{2n}}{\bar{\delta}^{(0)}}\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right) \le \bar{N}^{(n)} \le \frac{2^{2n+1}}{\bar{\delta}^{(0)}}\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right). \tag{86}$$

To finish ensuring the inductive applicability of the iterative lemma, we have to guarantee that the size of $\Omega_1^{(n)}, \mathcal{C}^{(n)}, \widetilde{H}^{(n)}$ and $(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1}$ are controlled, for each $n \ge 0$, as required in the statement. We do not plan to give full details and we only illustrate this process in terms of $(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1}$, which turns out to be the term giving worst estimates. First, we recall that we have bound (68) for $(\langle\bar{\mathcal{A}}^{(0)}\rangle_\theta)^{-1}$. Moreover, the iterative application of the lemma gives

$$|(\langle\bar{\mathcal{A}}^{(n+1)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}^{(n)}} \le \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+3}(M^{(0)})^{29\alpha/2}}.$$

Then, it is natural to study the convergence of the following sum,

$$\sum_{n=0}^{\infty} \frac{\bar{\kappa}\bar{M}^{(n)}}{(\bar{\delta}^{(n)})^{2\tau+3}(M^{(0)})^{29\alpha/2}} = \sum_{n=0}^{\infty} (\bar{\delta}^{(0)})^{2\tau+3}(M^{(0)})^{9\alpha/2}\frac{(\bar{\kappa}^{(0)})^{2^n}}{2^{(n+2)(2\tau+3)+1}} \le 2\bar{\kappa}\frac{(M^{(0)})^{1-29\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \tag{87}$$

where we have used that $2^n \ge n+1$ to bound the sum in terms of a geometrical progression of ratio $2^{-(2\tau+3)}\bar{\kappa}^{(0)} \le 1/2$. By performing similar computations for the other terms, we obtain, for any $n \ge 1$,

$$|(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1} - (\langle\bar{\mathcal{A}}^{(0)}\rangle_\theta)^{-1}|_{\bar{\mathcal{E}}^{(n-1)}} \le 2\bar{\kappa}\frac{(M^{(0)})^{1-29\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \quad |\Omega_1^{(n)} - \Omega_1^{(0)}|_{\bar{\mathcal{E}}^{(n-1)}} \le 2\bar{\kappa}\frac{(M^{(0)})^{1-13\alpha/2}}{(\bar{\delta}^{(0)})^\tau},$$

$$|\widetilde{H}^{(n)} - \widetilde{H}^{(0)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},\bar{R}^{(n)}} \le 2\bar{\kappa}\frac{(M^{(0)})^{1-19\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}, \quad |\mathcal{C}^{(n)} - \mathcal{C}^{(0)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},0} \le 2\bar{\kappa}\frac{(M^{(0)})^{1-27\alpha/2}}{(\bar{\delta}^{(0)})^{2\tau+3}}.$$

We also have the direct bound $|H^{(n)}|_{\bar{\mathcal{E}}^{(n-1)},\bar{\rho}^{(n)},\bar{R}^{(n)}} \le \kappa$ (whenever it only involves compositions of functions). Then, the estimates on $\widetilde{H}^{(n)}$, $(\langle\bar{\mathcal{A}}^{(n)}\rangle_\theta)^{-1}$ and $\mathcal{C}^{(n)}$ needed on the statement of lemma 5.4 hold if $\bar{\kappa}^{(0)} \le \kappa$ (use definition of $\bar{\kappa}^{(0)}$ and bounds on the zero stage in (66), (67) and (68)). Of course, if we take $n \to +\infty$, then the same bounds hold for the limit Hamiltonian $H^{(\infty)}$ in (37).

5.10 Convergence of the change of variables

To finish the proof of the convergence of the iterative scheme, it only remains to check the convergence of the composition of the sequence of canonical transformations $\{\Psi_\Lambda^{(n)}\}_{n\geq 0}$. Concretely, we introduce $\widetilde{\Psi}^{(n)} = \widetilde{\Psi}_\Lambda^{(n)}$ defined as

$$\widetilde{\Psi}^{(n)} := \Psi^{(0)} \circ \cdots \circ \Psi^{(n)}, \tag{88}$$

and we are going to prove that, for any $\Lambda \in \bar{\mathcal{E}}^{(\infty)}$, there exists $\widetilde{\Psi}^{(\infty)} = \lim_{n\to+\infty} \widetilde{\Psi}^{(n)}$, giving an analytic canonical transformation defined as $\widetilde{\Psi}^{(\infty)} : \mathcal{D}_{2,1}(\rho^{(\infty)}, R^{(\infty)}) \to \mathcal{D}_{2,1}(\rho^{(0)}, R^{(0)})$.

Remark 5.6. *Of course, the dependence of $\widetilde{\Psi}^{(\infty)}$ on $\Lambda \in \bar{\mathcal{E}}^{(\infty)}$ is no longer analytic but, as we are going to discuss in section 5.14, this transformation admits a Whitney-C^∞ extension. Moreover, $\widetilde{\Psi}^{(\infty)}$ is not real analytic but, as discussed in remark 5.3, it can be* realified *(see section 5.13 for details).*

To prove convergence of $\widetilde{\Psi}^{(\infty)}$ and to bound it we use lemma A.5. First, we note that from the iterative application of lemma 5.4 we have that $\Psi^{(n)} : \mathcal{D}_{2,1}(\bar\rho^{(n+1)}, \bar{R}^{(n+1)}) \to \mathcal{D}_{2,1}(\bar\rho^{(n)}, \bar{R}^{(n)})$, with

$$|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}} \leq \frac{\bar\kappa \bar{M}^{(n)}}{(\bar\delta^{(n)})^{2\tau+1}(M^{(0)})^{19\alpha/2}} \leq (\bar\delta^{(0)})^{2\tau+5}(M^{(0)})^{19\alpha/2} 2^{-(n+2)(2\tau+5)+3}(\bar\kappa^{(0)})^{2^n},$$

$$|\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}} \leq \frac{\bar\kappa \bar{M}^{(n)}}{(\bar\delta^{(n)})^{2\tau+2}(M^{(0)})^{15\alpha/2}} \leq (\bar\delta^{(0)})^{2\tau+4}(M^{(0)})^{23\alpha/2} 2^{-(n+2)(2\tau+4)+1}(\bar\kappa^{(0)})^{2^n},$$

$$|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}} \leq \frac{\bar\kappa \bar{M}^{(n)}}{(\bar\delta^{(n)})^{2\tau+5}(M^{(0)})^{21\alpha/2}} \leq (\bar\delta^{(0)})^{2\tau+5}(M^{(0)})^{21\alpha/2} 2^{-(n+2)(2\tau+5)+3}(\bar\kappa^{(0)})^{2^n}.$$

Then, according to lemma A.5, we have to consider the sum with respect to n of each of these bounds. Indeed (compare (87)),

$$\sum_{n=0}^{+\infty} |\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}} \leq \frac{2\bar\kappa}{(\bar\delta^{(0)})^{2\tau+1}}(M^{(0)})^{1-19\alpha/2} := A. \tag{89}$$

Similarly, from the sum of the bounds on $\mathcal{I}^{(n)}$ and $\mathcal{Z}^{(n)}$ we can define, respectively,

$$B := \frac{2\bar\kappa}{(\bar\delta^{(0)})^{2\tau+2}}(M^{(0)})^{1-15\alpha/2}, \qquad C := \frac{2\bar\kappa}{(\bar\delta^{(0)})^{2\tau+1}}(M^{(0)})^{1-17\alpha/2}. \tag{90}$$

Next, we introduce $\bar\rho_-^{(n)} := \bar\rho^{(n)} - \bar\delta^{(0)}$ and $\bar{R}_-^{(n)} := \bar{R}^{(n)} \exp(-\bar\delta^{(0)})$. It is clear that $\lim_{n\to+\infty}(\bar\rho_-^{(n)}, \bar{R}_-^{(n)}) = (\rho^{(\infty)}, R^{(\infty)})$ (see (77) and (78)). Moreover, if we proceed analogously as in (76) and use condition $\bar\kappa^{(0)} \leq 1/2$ (recall also that $R^{(n)} \geq R^{(\infty)} \geq (M^{(0)})^\alpha/2$), we also have

$$|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho_-^{(n+1)},\bar{R}_-^{(n+1)}} \leq \bar\rho_-^{(n)} - \bar\rho_-^{(n+1)}, \quad |\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho_-^{(n+1)},\bar{R}_-^{(n+1)}} \leq (\bar{R}_-^{(n)})^2 - (\bar{R}_-^{(n+1)})^2,$$

$$|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho_-^{(n+1)},\bar{R}_-^{(n+1)}} \leq \bar{R}_-^{(n)} - \bar{R}_-^{(n+1)}.$$

Then, under conditions above we guarantee the applicability of lemma A.5. Consequently, from the point (ii) of the lemma, we have the following bounds for the components of $\widetilde{\Psi}^{(\infty)} - \text{Id}$ (see (5))

$$|\widetilde{\Theta}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leq A, \qquad |\widetilde{\mathcal{I}}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leq B, \qquad |\widetilde{\mathcal{Z}}^{(\infty)}|_{\bar{\mathcal{E}}^{(\infty)},\rho^{(\infty)},R^{(\infty)}} \leq C. \tag{91}$$

Finally, we use point (iii) of lemma A.5 to bound the difference between the components of $\widetilde{\Psi}^{(n)}$ and $\widetilde{\Psi}^{(n-1)}$. Thus, we define for each $n \geq 1$,

$$\begin{aligned}
\Pi_n &:= \frac{1}{\bar\delta^{(0)}}\left(\frac{2|\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}}}{\exp(1)} + \frac{2|\mathcal{I}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}}}{(R^{(\infty)})^2} + \frac{4|\mathcal{Z}^{(n)}|_{\bar{\mathcal{E}}^{(n)},\bar\rho^{(n+1)},\bar{R}^{(n+1)}}}{R^{(\infty)}}\right) \\
&\leq (\bar\delta^{(0)})^{2\tau+3}(M^{(0)})^{19\alpha/2} 2^{-(n+2)(2\tau+4)+5}(\bar\kappa^{(0)})^{2^n},
\end{aligned}$$

and we have, for R small enough,

$$
\begin{aligned}
|\widetilde{\Theta}^{(n)} - \widetilde{\Theta}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, \bar{R}_-^{(n+1)}} &\leq |\Theta^{(n)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, \bar{R}_-^{(n+1)}} + A\Pi_n \\
&\leq (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{19\alpha/2}2^{-(n+2)(2\tau+4)+1}(\bar{\kappa}^{(0)})^{2^n}.
\end{aligned}
$$

Similarly, we establish similar bounds for the other components:

$$
\begin{aligned}
|\widetilde{\mathcal{I}}^{(n)} - \widetilde{\mathcal{I}}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, \bar{R}_-^{(n+1)}} &\leq (\bar{\delta}^{(0)})^{2\tau+4}(M^{(0)})^{23\alpha/2}2^{-(n+2)(2\tau+4)+2}(\bar{\kappa}^{(0)})^{2^n}, \\
|\widetilde{\mathcal{Z}}^{(n)} - \widetilde{\mathcal{Z}}^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, \bar{R}_-^{(n+1)}} &\leq (\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{21\alpha/2}2^{-(n+2)(2\tau+4)+1}(\bar{\kappa}^{(0)})^{2^n}.
\end{aligned}
$$

5.11   Bound on the Lipschitz constant of $\Omega_1^{(n)}$

Once we have proved full convergence of the KAM iterative process, actually we have that, for any $\Lambda = (\mu, \Omega_2) \in \bar{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2$ (see (82)), there is an invariant torus of the non-integrable Hamiltonian system $\mathcal{H}$ in (10), with normal frequency μ and intrinsic frequencies $\Omega^{(\infty)}(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$, where $\Omega_1^{(\infty)} = \lim_{n \to +\infty} \Omega_1^{(n)}$. However, the mere convergence of the sequence $\Omega_1^{(n)}$ is not enough in order to build measure estimates along the iterative process. We also require some additional information about the Lipschitz constant of $\Omega_1^{(n)}$, which can be derived from the control of their partial derivatives. As we know, by construction, that $\Omega_1^{(n)}(\Lambda)$ depends analytically on $\Lambda \in \bar{\mathcal{E}}^{(n-1)}$ (see (81)), this process can be done by means of Cauchy estimates. To do that, we need to control the distance to the boundary of the points inside the set $\bar{\mathcal{E}}^{(n)}$ of "admissible" basic frequencies at the n-step.

For this purpose, we introduce the following sequence of sets. First, we define $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(R)$ as (compare $\overline{\mathcal{U}}$ in (64))

$$
\begin{aligned}
\widehat{\mathcal{U}} := \big\{ \Lambda \in \mathbb{C}^2 \quad : \quad &2(M^{(0)})^{\alpha/2} \leq \operatorname{Re}\mu, \; |\operatorname{Im}\mu| \leq (M^{(0)})^{\alpha/2}, \; |\mu| \leq c_2 R - (M^{(0)})^{\alpha/2}, \\
&|\Omega_2 - \omega_2| \leq c_2 R - (M^{(0)})^{\alpha/2}, \; |\operatorname{Im}\Omega_2| \leq (M^{(0)})^{\alpha/2}, \; |h_1(\Lambda)| \geq 2(M^{(0)})^{\alpha/2} \big\},
\end{aligned} \tag{92}
$$

and thus, in analogy with (81), we set $\hat{\mathcal{E}}^{(-1)} = \widehat{\mathcal{U}}$ and, for any $n \geq 0$,

$$
\hat{\mathcal{E}}^{(n)} := \{\Lambda \in \hat{\mathcal{E}}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geq b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \; 0 < |k|_1 < 2\bar{N}^{(n)}, \ell \in \{0, 1, 2\}\}, \tag{93}
$$

being now $b_n = 1 + 2^{-n+1}$. It is clear that, by construction, we always have $\hat{\mathcal{E}}^{(n)} \subset \bar{\mathcal{E}}^{(n)}$ (observe that $b_n > a_n$). Next to that, we introduce the sequence of positive numbers $\nu^{(n)} = \nu^{(n)}(R) > 0$ given by $\nu^{(-1)} := (M^{(0)})^{\alpha/2}/3$ and

$$
\nu^{(n)} := 2^{-n-\tau-3}\frac{(M^{(0)})^{\alpha/2}}{(\bar{N}^{(n)})^{\tau+1}}, \qquad n \geq 0. \tag{94}
$$

For further uses, we observe that (86) implies the lower bound

$$
\nu^{(n)} \geq \frac{(\bar{\delta}^{(0)})^{\tau+1}(M^{(0)})^{\alpha/2}}{2^{2\tau+4}}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{-\tau-1}2^{-n(2\tau+3)}, \qquad n \geq 0. \tag{95}
$$

Our objective is to show that, if R is small enough, then $\hat{\mathcal{E}}^{(n)} + 3\nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$, for each $n \geq -1$ (see (9)). Using this inclusion we can control the partial derivatives of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$ and its Lipschitz constant in $\hat{\mathcal{E}}^{(n)} + \nu^{(n)}$.

We start with $n = -1$. In this case we have to prove that $\widehat{\mathcal{U}} + (M^{(0)})^{\alpha/2} \subset \overline{\mathcal{U}}$. This means that, if we take an arbitrary $\Lambda \in \widehat{\mathcal{U}}$ and Λ' is such that $|\Lambda' - \Lambda| \leq (M^{(0)})^{\alpha/2}$, then $\Lambda' \in \overline{\mathcal{U}}$. This is clear from the definition of both sets, except for what concerns the lower bound on $|h_1(\Lambda)|$. But using lemma 5.2 we have,

$$
|h_1(\Lambda')| \geq |h_1(\Lambda)| - |h_1(\Lambda') - h_1(\Lambda)| \geq 2(M^{(0)})^{\alpha/2} - \frac{2c_2 R}{a}|\mu' - \mu| - \frac{8c_1 R}{a}|\Omega_2' - \Omega_2| \geq (M^{(0)})^{\alpha/2},
$$

provided that R is small enough.

Next to that we proceed by induction with respect to n. Concretely, we want to prove that, for any $n \geq 0$, the following properties hold,

$$|\partial_\mu \Omega_1^{(n)}|_{\hat{\mathcal{E}}^{(n-1)}+2\nu^{(n-1)}} \leq 2c_3 R, \qquad |\partial_{\Omega_2} \Omega_1^{(n)}|_{\hat{\mathcal{E}}^{(n-1)}+2\nu^{(n-1)}} \leq 2c_3 R, \qquad (96)$$

$$\hat{\mathcal{E}}^{(n-1)} + 3\nu^{(n-1)} \subset \bar{\mathcal{E}}^{(n-1)}, \qquad \mathrm{Lip}_{\hat{\mathcal{E}}^{(n-1)}+\nu^{(n-1)}}(\Omega_1^{(n)}) \leq 4c_3 R. \qquad (97)$$

From the discussions above and lemma 5.3 it is clear that (96) and (97) hold when $n = 0$ (recall that $\hat{\mathcal{U}} \subset \overline{\mathcal{U}} \subset \mathcal{U}$). Let us suppose them true for a given $n \geq 0$ and we verify them for the next case.

We first prove that $\hat{\mathcal{E}}^{(n)} + 3\nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$. Let $\Lambda \in \hat{\mathcal{E}}^{(n)}$ be fixed and take Λ' such that $|\Lambda' - \Lambda| \leq 3\nu^{(n)}$. As the set $\hat{\mathcal{E}}^{(n)}$ is defined from $\hat{\mathcal{E}}^{(n-1)}$ and $3\nu^{(n)} \leq \nu^{(n-1)}$, it is clear that both $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n-1)} + \nu^{(n-1)}$, so that we can use the Lipschitz estimate (97) on $\Omega_1^{(n)}$. To check that $\Lambda' \in \bar{\mathcal{E}}^{(n)}$ we compute, for any $k \in \mathbb{Z}^2$ with $0 < |k|_1 < 2\bar{N}^{(n)}$ and $\ell \in \{0, 1, 2\}$ (recall that $\Omega^{(n)}(\Lambda) = (\Omega_1^{(n)}(\Lambda), \Omega_2)$),

$$
\begin{aligned}
|\langle k, \Omega^{(n)}(\Lambda')\rangle + \ell\mu'| &\geq |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| - |k_1||\Omega_1^{(n)}(\Lambda') - \Omega_1^{(n)}(\Lambda)| - |k_2||\Omega_2' - \Omega_2| - |\ell||\mu' - \mu| \\
&\geq b_n (M^{(0)})^{\alpha/2}|k|_1^{-\tau} - 4c_3 R|k_1||\Lambda' - \Lambda| - |k_2||\Omega_2' - \Omega_2| - 2|\mu' - \mu| \\
&\geq \left(b_n (M^{(0)})^{\alpha/2} - 4\nu^{(n)}(2\bar{N}^{(n)})^{\tau+1}\right)|k|_1^{-\tau} \\
&\geq a_n (M^{(0)})^{\alpha/2}|k|_1^{-\tau},
\end{aligned}
$$

if R is small enough (condition depending only on c_3). Here, we use definition (94) and $b_n - a_n = 2^{-n}$.

The following step is to control the partial derivatives of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$. From the iterative application of lemma 5.4, we can define the analytic function $\Omega_1^{(j+1)}$ in the complex set $\bar{\mathcal{E}}^{(j)}$, with

$$|\Omega_1^{(j+1)} - \Omega_1^{(j)}|_{\bar{\mathcal{E}}^{(j)}} \leq \frac{\bar{\kappa}\bar{M}^{(j)}}{(\bar{\delta}^{(j)})^\tau (M^{(0)})^{13\alpha/2}}, \qquad j \geq 0. \qquad (98)$$

Using standard Cauchy estimates and the inductive inclusion $\hat{\mathcal{E}}^{(j)} + 3\nu^{(j)} \subset \bar{\mathcal{E}}^{(j)}$, for $j = 0, \ldots, n$, we obtain (in order to bound the sum below compare (87) and recall (84)),

$$
\begin{aligned}
|\partial_\mu(\Omega_1^{(n+1)} - \Omega_1^{(0)})|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} &\leq \sum_{j=0}^{n} \frac{|\Omega_1^{(j+1)} - \Omega_1^{(j)}|_{\bar{\mathcal{E}}^{(j)}}}{\nu^{(j)}} \\
&\leq \sum_{j=0}^{\infty} 2^{-(j+2)(\tau+3)+3}(\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{12\alpha}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}(\bar{\kappa}^{(0)})^{2^j} \\
&\leq 2^{2\tau+5}\bar{\kappa}\frac{(M^{(0)})^{1-7\alpha}}{(\bar{\delta}^{(0)})^{2\tau+1}}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}. \qquad (99)
\end{aligned}
$$

If we require R small enough so that (99) is bounded by $c_3 R$, then we have (see lemma 5.3)

$$|\partial_\mu \Omega_1^{(n+1)}|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} \leq |\partial_\mu \Omega_1^{(0)}|_{\mathcal{U}} + |\partial_\mu(\Omega_1^{(n+1)} - \Omega_1^{(0)})|_{\hat{\mathcal{E}}^{(n)}+2\nu^{(n)}} \leq 2c_3 R. \qquad (100)$$

The same works for the other partial derivative, $\partial_{\Omega_2}(\Omega_1^{(n+1)})$, so that (96) holds for any $n \geq 0$.

Finally, we discuss the Lipschitz constant of $\Omega_1^{(n+1)}$ in $\hat{\mathcal{E}}^{(n)} + \nu^{(n)}$. Clearly, this Lipschitz constant can be locally bounded in terms of the partial derivatives by $4c_3 R$. But it only holds for points such that their union segment is contained inside the domain $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$. However, the sequence of Diophantine conditions we have imposed to the original (convex) domain of basic frequencies $\mathcal{U}$ (see (62)), have created multiple holes in the set $\hat{\mathcal{E}}^{(n)}$. Thus, we can only guarantee linear connectivity inside $\hat{\mathcal{E}}^{(n)} + 2\nu^{(n)}$ for points $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n)} + \nu^{(n)}$ so that $|\Lambda' - \Lambda| \leq \nu^{(n)}$. But if we pick up points so

that $|\Lambda' - \Lambda| \geq \nu^{(n)}$, we can alternatively control their "Lipschitz constant" using the norm of the function and the lower bound on their separation. Concretely,

$$|(\Omega_1^{(n+1)}(\Lambda') - \Omega_1^{(n)}(\Lambda')) - (\Omega_1^{(n+1)}(\Lambda) - \Omega_1^{(n)}(\Lambda))| \leq \frac{2|\Omega_1^{(n+1)} - \Omega_1^{(n)}|_{\bar{\mathcal{E}}^{(n)}}}{\nu^{(n)}}|\Lambda' - \Lambda|,$$

which is the same bound that we obtain in the "local case" using Cauchy estimates. Then, taking into account this methodology for controlling the "Lipschitz constant" for "separated points", we can adapt the procedure used in (99) and (100) in order to control $|\Omega_1^{(n+1)}(\Lambda') - \Omega_1^{(n+1)}(\Lambda)|$ for $\Lambda, \Lambda' \in \hat{\mathcal{E}}^{(n)} + \nu^{(n)}$, independently of their distance. We leave the details to the reader.

5.12   Measure estimates

Now, we have at hand all the ingredients needed to discuss the Lebesgue measure of the set of basic frequencies giving an invariant torus linked to the Hopf bifurcation. But, as we are only interested in *real* basic frequencies, we first introduce the following sets:

$$\hat{\mathcal{E}}^{(\infty)}(R) := \bigcap_{n \geq 0} \hat{\mathcal{E}}^{(n)}(R), \qquad \mathcal{E}^{(\infty)}(R) := \hat{\mathcal{E}}^{(\infty)} \cap \mathbb{R}^2, \qquad \mathcal{E}^{(n)}(R) := \hat{\mathcal{E}}^{(n)}(R) \cap \mathbb{R}^2, \quad n \geq -1, \quad (101)$$

and (see (62))

$$\mathcal{V}(R) := \mathcal{U}(R) \cap (\mathbb{R}^+ \times \mathbb{R}) = \{\Lambda = (\mu, \Omega_2) \in \mathbb{R}^2 : 0 < \mu \leq c_2 R, \ |\Omega_2 - \omega_2| \leq c_2 R\}. \quad (102)$$

In few words, $\mathcal{V} = \mathcal{V}(R)$ is the initial set of real basic frequencies in which we look for invariant tori and $\mathcal{E}^{(\infty)} = \mathcal{E}^{(\infty)}(R)$ is the corresponding subset in which we have proved convergence of the KAM process. To size up the holes between invariant tori, we have to control the Lebesgue measure meas $(\mathcal{V} \setminus \mathcal{E}^{(\infty)})$. For this purpose we write:

$$\mathcal{V} \setminus \mathcal{E}^{(\infty)} = \left(\mathcal{V} \setminus \mathcal{E}^{(-1)}\right) \bigcup \left(\bigcup_{n \geq 0} (\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})\right). \quad (103)$$

We start by controlling meas $(\mathcal{V} \setminus \mathcal{E}^{(-1)})$. From the definition of $\hat{\mathcal{E}}^{(-1)} = \hat{\mathcal{U}}$ (see (92)), we have

$$\mathcal{E}^{(-1)} = \Big\{\Lambda \in \mathbb{R}^2 \quad : \quad 2(M^{(0)})^{\alpha/2} \leq \mu \leq c_2 R - (M^{(0)})^{\alpha/2}, \ |\Omega_2 - \omega_2| \leq c_2 R - (M^{(0)})^{\alpha/2},$$
$$|h_1(\Lambda)| \geq 2(M^{(0)})^{\alpha/2}\Big\}.$$

So, if we get rid off the lower bound $|h_1(\Lambda)| \geq 2(M^{(0)})^{\alpha/2}$, then we clearly obtain an estimate of $\mathcal{O}((M^{(0)})^{\alpha/2})$ for this measure. Unfortunately, this estimate is worsened when adding the condition on h_1. We recall that the vector-function $h^{(R)} = (h_1, h_2)$, depending on R, and so on the selected normal form order, has been introduced in lemma 5.2. Concretely, h denotes the inverse of the transformation $\zeta = (\xi, \eta) \mapsto \Lambda = (\mu, \Omega_2)$, defined by the parametrization in terms of ζ of the bifurcated invariant tori of the normal form (see theorem 4.2), i. e., $\xi = h_1(\Lambda)$. For further uses, and to prevent from possible confusions with the basic frequencies itself, we denote by $\Upsilon = \Upsilon^{(R)}(\zeta)$ the vector-function having as components $\Upsilon_1 = \mu(\zeta)$ and $\Upsilon_2 = \Omega_2(\zeta)$, defined by the R-depending parametrizations (24), (27) and (28). Therefore, due to the square root of the definition of μ in (28), we have to be very careful to select the domain for the vector-function Υ.

According to lemma 5.1, the function $\mathbb{I} = \mathbb{I}^{(R)}$ is analytic in the (complex) domain $\Gamma = \Gamma(R)$ (see (60)) and so is $\Omega_2(\zeta)$ (see (24)). Thus, it is natural to consider the following (real) domain for Υ (see remark 4.3 for more details):

$$\Gamma^* = \Gamma^*(R) := \{\zeta \in \Gamma \cap \mathbb{R}^2 : 4\eta^2 + 2a\xi + 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) > 0\}.$$

Moreover, we consider the auxiliar sets $A = A(R)$ and $B = B(R)$ given by

$$
\begin{aligned}
A &:= \{\Lambda \in \mathbb{R}^2 : 0 < \mu \leq c_2 R, \ |\Omega_2 - \omega_2| \leq c_2 R, \ |h_1(\Lambda)| \leq 2(M^{(0)})^{\alpha/2}\}, \\
B &:= \{\zeta \in \Gamma^* : |\xi| \leq 2(M^{(0)})^{\alpha/2}, \ |\eta| \leq c_1 R\},
\end{aligned}
$$

where c_1 has been introduced in lemma 5.1. We stress that the restriction $\zeta \in \Gamma^*$ in the definition of B also implies that $\Upsilon_1(\zeta) > 0$. It is clear that by bounding $\mathrm{meas}(A)$ we control the effect of the lower bound $|h_1(\Lambda)| \geq 2(M^{(0)})^{\alpha/2}$ on $\mathrm{meas}(\mathcal{V} \setminus \mathcal{E}^{(-1)})$. Then, the important thing is that $A \subset \Upsilon(B)$, so that bounding $\mathrm{meas}(A)$ can be done by bounding the Jacobian of Υ in B. From the expressions (24), (27), (28), the bounds in (59) on the partial derivatives of $Z = Z^{(R)}$ and of lemma 5.1 on $\mathbb{I} = \mathbb{I}^{(R)}$, we have that, for any $\zeta \in \Gamma^*$ and R small enough,

$$
|\partial_\xi \Upsilon_1(\zeta)| = \frac{\partial_\xi(\Upsilon_1^2(\zeta))}{2\Upsilon_1(\zeta)}, \qquad |\partial_\eta \Upsilon_1(\zeta)| \leq \frac{8c_1 R}{\Upsilon_1(\zeta)}, \qquad |\partial_\xi \Upsilon_2(\zeta)| \leq 2|e| + 4\frac{a|f|}{|d|}, \qquad |\partial_\eta \Upsilon_2(\zeta)| \leq 2,
$$

where we take special care on making explicit the effect on the derivatives of the square root defining $\Upsilon_1 = \mu$. From here, we can bound the Jacobian of Υ by

$$
|\det(\partial_\zeta \Upsilon)(\zeta)| \leq \frac{3}{2} \frac{\partial_\xi(\Upsilon_1^2(\zeta))}{\Upsilon_1(\zeta)},
$$

where the key point is that $\partial_\xi(\Upsilon_1^2(\zeta)) \geq a > 0$, if R is small enough. Then, we have

$$
\mathrm{meas}(A) = \int\int_A d\mu \, d\Omega_2 \leq \int\int_{\Upsilon(B)} d\mu \, d\Omega_2 \leq \int\int_B \frac{3}{2} \frac{\partial_\xi(\Upsilon_1^2(\zeta))}{\Upsilon_1(\zeta)} d\xi \, d\eta.
$$

The integral with respect to ξ on the right hand side can be computed explicitly, giving an expression of the form $3(\Upsilon_1(\xi', \eta) - \Upsilon_1(\xi, \eta))$, for certain $\xi = \xi(\eta)$ and $\xi' = \xi'(\eta)$. On its turn, this expression has to be integrated with respect to η. In order to avoid the square root defining Υ_1 we recall the Hölder bound

$$
|\sqrt{x} - \sqrt{y}| \leq |x - y|^{1/2}, \qquad x, y > 0.
$$

Hence we have, for small R and $(\xi, \eta), (\xi', \eta) \in B$,

$$
\begin{aligned}
|\Upsilon_1(\xi', \eta) - \Upsilon_1(\xi, \eta)| &\leq |\Upsilon_1^2(\xi', \eta) - \Upsilon_1^2(\xi, \eta)|^{1/2} \\
&= |2a(\xi' - \xi) + 2\xi' \partial_{1,1}^2 Z(\xi', \mathbb{I}(\xi', \eta), \xi'\eta) - 2\xi \partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta)|^{1/2} \\
&\leq 4\sqrt{a}(M^{(0)})^{\alpha/4},
\end{aligned}
$$

which finally gives

$$
\mathrm{meas}(A) \leq 24\sqrt{a} c_1 R (M^{(0)})^{\alpha/4}.
$$

To obtain this estimate, apart from the explicit expression of $\Upsilon_1 = \mu$, we use the bounds on the partial derivatives of Z and the definition of the set B.

As a conclusion, we have established the following bound

$$
\mathrm{meas}(\mathcal{V} \setminus \mathcal{E}^{(-1)}) \leq c_4 (M^{(0)})^{\alpha/4}, \tag{104}
$$

for certain $c_4 > 0$ independent of R.

Next step is to bound $\mathrm{meas}(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})$ for any $n \geq 0$. This is a standard process (compare for instance [25]) which only requires of suitable transversality conditions in order to deal with the Diophantine conditions defining the sets at hand. In the present context, these transversality conditions are immediate from the bifurcation scenario we are discussing and our "adequate" choice of the basic frequencies (see (31)). We consider the following decomposition,

$$
\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)} = \bigcup_{\ell \in \{0,1,2\}} \bigcup_{0 < |k|_1 < 2\bar{N}^{(n)}} \mathcal{R}_{\ell,k}^{(n)},
$$

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where $\mathcal{R}_{\ell,k}^{(n)}$ contains the basic frequencies for which one of the Diophantine conditions defining $\mathcal{E}^{(n)}$ fails (see (93) and (101)). Concretely,

$$\mathcal{R}_{\ell,k}^{(n)} = \{\Lambda \in \mathcal{E}^{(n-1)} : |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| < b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}\}.$$

To control $\text{meas}\,(\mathcal{R}_{\ell,k}^{(n)})$, we suppose first that $(\ell, k_2) \neq (0,0)$ and take a couple $\Lambda, \Lambda' \in \mathcal{R}_{\ell,k}^{(n)}$ such that $\Lambda - \Lambda'$ is parallel to the vector (ℓ, k_2). Then we have

$$
\begin{aligned}
|\Lambda - \Lambda'|_2 &= \frac{1}{|(\ell, k_2)|_2}|\langle \Lambda - \Lambda', (\ell, k_2)\rangle| = \frac{1}{|(\ell, k_2)|_2}|(k_2\Omega_2 + \ell\mu) - (k_2\Omega_2' + \ell\mu')| \\
&\leq \frac{1}{|(\ell, k_2)|_2}\left(2b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + |k_1||\Omega_1^{(n)}(\Lambda) - \Omega_1^{(n)}(\Lambda')|\right) \\
&\leq \frac{1}{|(\ell, k_2)|_2}\left(6(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 4c_3R|k_1||\Lambda - \Lambda'|\right),
\end{aligned}
$$

where we recall that $b_n \leq 3$ and that $\text{Lip}_{\hat{\mathcal{E}}^{(n-1)}}(\Omega_1^{(n)}) \leq 4c_3R$ (see (97)). So, we obtain

$$\left(1 - 4c_3R\frac{|k_1|}{|(\ell, k_2)|_2}\right)|\Lambda - \Lambda'| \leq 6(M^{(0)})^{\alpha/2}\frac{|k|_1^{-\tau}}{|(\ell, k_2)|_2}.$$

If we assume, for the moment, that $4c_3R|k_1|/|(\ell, k_2)|_2 \leq 1/2$, then we finally end with the estimate

$$|\Lambda - \Lambda'| \leq 12(M^{(0)})^{\alpha/2}\frac{|k|_1^{-\tau}}{|(\ell, k_2)|_2}. \tag{105}$$

To ensure this assumption, we study for which values of k_1 it could be $\mathcal{R}_{\ell,k}^{(n)} \neq \emptyset$. Thus, let us suppose that $\Lambda = (\mu, \Omega_2)$ belongs to $\mathcal{R}_{\ell,k}^{(n)}$. If we set R small enough so that $\max\{|\Omega_2|, \mu\} \leq 2|\omega_2|$, we have

$$|k_1||\Omega_1^{(n)}(\Lambda)| \leq |\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| + |k_2||\Omega_2| + |\ell||\mu| \leq 3(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 4|\omega_2||(\ell, k_2)|_2.$$

If we also assume R small enough such that $|\Omega_1^{(n)}(\Lambda)| \geq |\omega_1|/2$, then

$$|k_1| \leq \frac{6}{|\omega_1|}(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + 8\frac{|\omega_2|}{|\omega_1|}|(\ell, k_2)|_2,$$

which clearly implies $4c_3R|k_1|/|(\ell, k_2)|_2 \leq 1/2$, for R small. Moreover, if $(\ell, k_2) \neq (0,0)$, we also deduce, for small R,

$$|k|_1 = |k_1| + |k_2| \leq \frac{6}{|\omega_1|}(M^{(0)})^{\alpha/2}|k|_1^{-\tau} + \left(1 + 8\frac{|\omega_2|}{|\omega_1|}\right)|(\ell, k_2)|_2 \leq 2\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)|(\ell, k_2)|_2.$$

From here, we can rewrite (105) as

$$|\Lambda - \Lambda'| \leq 24\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)(M^{(0)})^{\alpha/2}\frac{1}{|k|_1^{\tau+1}}.$$

Once we have bounded the width of $\mathcal{R}_{\ell,k}^{(n)}$, in the direction given by the vector (ℓ, k_2), then, by taking into account the diameter of the set $\mathcal{V}$ (see (102)), we obtain the following estimate for its measure,

$$\text{meas}\,(\mathcal{R}_{\ell,k}^{(n)}) \leq 24\sqrt{5}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2R(M^{(0)})^{\alpha/2}\frac{1}{|k|_1^{\tau+1}}. \tag{106}$$

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It remains to control this measure when $(\ell, k_2) = (0,0)$. However, these cases can be omitted because $\mathcal{R}^{(n)}_{0,(k_1,0)} = \emptyset$ if $k_1 \neq 0$. Indeed,

$$|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| = |k_1||\Omega_1^{(n)}(\Lambda)| \geq |\omega_1|/2.$$

Then, using decomposition (103), we have

$$\text{meas}\left(\bigcup_{n\geq 0}(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})\right) = \sum_{n=0}^{\infty}\text{meas}\left(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)}\right) \leq \sum_{n=0}^{\infty}\sum_{\ell\in\{0,1,2\}}\sum_{0<|k|_1<2\bar{N}^{(n)}}\text{meas}\left(\mathcal{R}^{(n)}_{\ell,k}\right).$$

Unfortunately, this expression diverges if we just use the estimate (106), because it does not depend on the index n. Nevertheless, we will show below that, for any $(\ell, k) \in \{0, 1, 2\} \times (\mathbb{Z}^2 \setminus \{0\})$, there is at most one $n = n^*(|k|_1)$ so that $\mathcal{R}^{(n)}_{\ell,k}$ is non-empty. Assuming this assertion true, we have

$$
\begin{aligned}
\text{meas}\left(\bigcup_{n\geq 0}(\mathcal{E}^{(n-1)} \setminus \mathcal{E}^{(n)})\right) &\leq \sum_{\ell\in\{0,1,2\}}\sum_{0<|k|_1<2\bar{N}^{(n)}}\text{meas}\left(\mathcal{R}^{(n^*(|k|_1))}_{\ell,k}\right) \\
&\leq 288\sqrt{5}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2 R(M^{(0)})^{\alpha/2}\sum_{j=1}^{\infty}\frac{1}{j^\tau} \\
&\leq 288\sqrt{5}\frac{\tau}{\tau-1}\left(1 + 4\frac{|\omega_2|}{|\omega_1|}\right)c_2 R(M^{(0)})^{\alpha/2}, \quad (107)
\end{aligned}
$$

where we use that $\#\{k \in \mathbb{Z}^2 : |k|_1 = j\} = 4j$ and $\sum_{j=1}^{+\infty}j^{-\tau} \leq 1 + \int_1^\infty x^{-\tau}dx$ (recall $\tau > 1$).

Let us prove the assertion above. To be precise, given a fixed $k \in \mathbb{Z}^2 \setminus \{0\}$, we denote by $n^* = n^*(|k|_1) \geq 0$ the first index so that $|k|_1 < 2\bar{N}^{(n^*)}$. Then, we are going to show that $\mathcal{R}^{(n)}_{\ell,k} = \emptyset$ for any $n \neq n^*(|k|_1)$ and $\ell \in \{0, 1, 2\}$. In few words, this means that being n^* the first index n for which the small divisor of order (ℓ, k) is taken into account in the definition of the set of valid basic frequencies $\mathcal{E}^{(n)}$ (see (93) and (101)), then the "resonant zone" $\mathcal{R}^{(n^*)}_{\ell,k}$ determines completely the values of Λ for which the Diophantine condition of order (ℓ, k) can fail at any step. Thus, if the required Diophantine condition of order (ℓ, k) is fulfilled for certain Λ at the step $n^*(|k|_1)$, then this basic frequency cannot fall into a resonant zone $\mathcal{R}^{(n)}_{\ell,k}$ for any $n > n^*$. We prove this property as follows.

Let $n \geq 1$ and $\Lambda \in \mathcal{E}^{(n-1)}$. This means that $\Omega_1^{(n-1)}(\Lambda)$ verifies

$$|\langle k, \Omega^{(n-1)}(\Lambda)\rangle + \ell\mu| \geq b_{n-1}(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \qquad 0 < |k|_1 < 2\bar{N}^{(n-1)}, \ell \in \{0, 1, 2\}.$$

Then, we want to show that the following Diophantine conditions on $\Omega_1^{(n)}(\Lambda)$ are verified automatically,

$$|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| \geq b_n(M^{(0)})^{\alpha/2}|k|_1^{-\tau}, \qquad 0 < |k|_1 < 2\bar{N}^{(n-1)}, \ell \in \{0, 1, 2\}. \quad (108)$$

Thus, bounds (108) imply that to define $\mathcal{E}^{(n)}$ we only have to worry about the Diophantine conditions on $\Omega_1^{(n)}(\Lambda)$ of order $2\bar{N}^{(n-1)} \leq |k|_1 < 2\bar{N}^{(n)}$. Indeed,

$$
\begin{aligned}
|\langle k, \Omega^{(n)}(\Lambda)\rangle + \ell\mu| &\geq |\langle k, \Omega^{(n-1)}(\Lambda)\rangle + \ell\mu| - |k_1||\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}} \\
&\geq \left(b_{n-1} - \frac{(2\bar{N}^{(n-1)})^{\tau+1}}{(M^{(0)})^{\alpha/2}}|\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}}\right)(M^{(0)})^{\alpha/2}|k|_1^{-\tau}.
\end{aligned}
$$

On the other hand, using (83), (86), (98) and that $\bar{\kappa}^{(0)} \leq 1$, we have, for any $n \geq 1$,

$$
\begin{aligned}
\frac{(2\bar{N}^{(n-1)})^{\tau+1}}{(M^{(0)})^{\alpha/2}}|\Omega_1^{(n)} - \Omega_1^{(n-1)}|_{\bar{\mathcal{E}}^{(n-1)}} &\leq 2^{-n(4+\tau)-\tau-1}(\bar{\delta}^{(0)})^{2\tau+5}(M^{(0)})^{12\alpha}\left(\log\left(\frac{1}{\bar{\kappa}^{(0)}}\right)\right)^{\tau+1}(\bar{\kappa}^{(0)})^{2^{n-1}} \\
&\leq 2^{-n+1} = b_{n-1} - b_n,
\end{aligned}
$$

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provided that $M^{(0)}$ is small enough (i. e., R small enough). Hence, we conclude that (108) holds.

Finally, if we put estimate (104) together with (107), we obtain the measure estimates of (13)

$$\text{meas}\,(\mathcal{V}(R) \setminus \mathcal{E}^{(\infty)}(R)) \leq c_5 (M^{(0)}(R))^{\alpha/4},$$

for certain $c_5 > 0$ independent of R.

5.13 Real invariant tori

Now, it is time to return to the original system of coordinates of the problem, and to discuss which of the tori we have obtained are *real* tori when expressed in such coordinates. We recall that we have settled $\mathcal{H}$ in (10) to be our initial Hamiltonian. This system is written in a canonical set of (real) coordinates, namely $(\theta, x_1, x_2, I, y_1, y_2)$, that have been introduced with the requirement that the normal variational equations of the critical periodic orbit are of constants coefficients. Later on, we have modified those original coordinates through the paper, according to the different steps of the proof of theorem 3.1. Let us summarize here this sequence of changes.

(i) We have applied to $\mathcal{H}$ the R-depending normal form transformation $\widehat{\Psi}^{(R)}$ given by theorem 4.1.

(ii) We have introduced action–angle–like coordinates to this (partially) normalized system through the (R-independent) change (19). This transformation is not properly a complexification, but we need $q > 0$ in order to have real tori.

(iii) We have considered the Λ-depending coordinate change (32), that moves to the "origin" the unperturbed bifurcated torus having vector of basic frequencies Λ and arranges its variational equations. This transformation involves the complex "diagonalizing" change (33) but, as we have discussed in section 5.2, all the invariant tori we compute are real when written in the action–angle coordinates (19). Then, we only have to worry about the condition $q > 0$.

(iv) Finally, we have performed the KAM process. Then, we have to compose all these coordinate changes with the limit KAM transformation $\widetilde{\Psi}^{(\infty)} = \widetilde{\Psi}_\Lambda^{(\infty)}(\theta_1, \theta_2, x, I_1, I_2, y)$ (see section 5.10), that is well-defined for any $\Lambda \in \mathcal{E}^{(\infty)}$ (see (101)).

The most important property of the KAM transformation $\widetilde{\Psi}^{(\infty)} = \text{Id} + (\widetilde{\Theta}^{(\infty)}, \widetilde{\mathcal{X}}^{(\infty)}, \widetilde{\mathcal{I}}^{(\infty)}, \widetilde{\mathcal{Y}}^{(\infty)})$ is that if we set $x = y = I_1 = I_2 = 0$, then we obtain the parametrization, as function of $\theta = (\theta_1, \theta_2)$, of the corresponding Λ-invariant torus of the "full" system $H_\Lambda^{(0)}$ in (34). After composition of this parametrization with the transformations described above, we obtain the invariant tori of the initial system (written in the original variables). If we want to detect which of these tori are real, we have to study the sign of the variable q evaluated on any of them. Thus, abusing notation, we denote by $q^{(\infty)}(\theta, \Lambda)$ this coordinate-function, which is obtained by replacing in (32) the variables (x, y, I_1, I_2) by the parametrization of the tori. Indeed,

$$q^{(\infty)}(\theta, \Lambda) := \xi + \widetilde{\mathcal{X}}^{(\infty)} - \frac{\xi}{\lambda_+}\widetilde{\mathcal{Y}}^{(\infty)} - \frac{2\xi}{\mu^2}(\partial_{J_1, q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\widetilde{\mathcal{I}}_1^{(\infty)} - \frac{2\xi}{\mu^2}(\partial_{J_2, q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\widetilde{\mathcal{I}}_2^{(\infty)}, \qquad (109)$$

where the components of the KAM transformation $\widetilde{\Psi}^{(\infty)}$ above are evaluated at $x = y = I_1 = I_2 = 0$. To help in the understanding of this expression, we recall that, given a vector of basic frequencies $\Lambda = (\mu, \Omega_2)$, the values of $\zeta = (\xi, \eta)$ in (109) are related with Λ through the R-depending vector-function $h = h^{(R)}$ introduced in lemma 5.2, i. e., $\zeta = h(\Lambda)$. Then, using the estimates provided by this lemma, the lower bounds $|\mu| = |\lambda_+| \geq (M^{(0)})^{\alpha/2}$ and $|\xi| = |h_1(\Lambda)| \geq (M^{(0)})^{\alpha/2}$, the explicit expression of $\breve{\mathcal{Z}}$ in (21), the bounds (59) on the partial derivatives of Z and those of (89), (90) and (91) on the components of $\widetilde{\Psi}^{(\infty)}$, we easily obtain an estimate of the form

$$|q^{(\infty)} - \xi|_{\bar{\mathcal{E}}^{(\infty)}, \rho^{(\infty)}, 0} \leq c_6 (M^{(0)})^{1-9\alpha}, \qquad (110)$$

This motivates to introduce the real set $\breve{\mathcal{E}}^{(\infty)} = \breve{\mathcal{E}}^{(\infty)}(R)$ defined as (see (101))

$$\breve{\mathcal{E}}^{(\infty)} := \{ \Lambda \in \mathcal{E}^{(\infty)} : \xi = h_1(\Lambda) \geq (M^{(0)})^{\alpha/2} \}.$$

It is clear that if $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$ then we have a *real* analytic invariant torus of the initial system (10). Additionally, we introduce the R-depending function $\Phi^{(\infty)}(\theta, \Lambda)$,

$$\Phi^{(\infty)} : \mathbb{T}^2 \times \breve{\mathcal{E}}^{(\infty)} \to \mathbb{T} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2, \tag{111}$$

giving the parametrization (in the original phase space) of the 4D-Cantor manifold of 2D-bifurcated elliptic tori. This parametrization is defined through the composition of the changes $(i) - (iv)$ above evaluated at $x = y = I_1 = I_2 = 0$.

Remark 5.7. *We recall that the curves in the Λ-space defined by the condition $\xi = 0$, giving he (stable) periodic orbits of the family, do not change with the selected order of the normal form. This implies that the "boundary" $\xi = 0$ of the set $\breve{\mathcal{E}}^{(\infty)}(R)$ does not change with R. See remark 4.2.*

5.14 Whitney-smoothness of the surviving tori

After showing the persistence of a Cantor family of 2D-real bifurcated invariant tori of $\mathcal{H}$, labelled by $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, in this section we are going to prove the Whitney-C^∞ regularity of this construction. More precisely, we show that the function $\Omega_1^{(\infty)}(\Lambda)$, giving the first component of the vector of intrinsic frequencies of these tori, and the vector-function $\Phi^{(\infty)}$, giving their parametrization, admit a Whitney-C^∞ extension. Albeit Whitney-smoothness is a very classical subject, in section A.3 we include a brief summary with the main definitions and results we require.

In order to achieve these results, we apply the Inverse Approximation Lemma A.9 to $\Omega_1^{(\infty)}$ as a limit of $\{\Omega_1^{(n)}\}_{n \geq 0}$ (see section 5.11) and $\Phi^{(\infty)}$ as a limit of $\{\Phi^{(n)}\}_{n \geq 0}$ (see below). In what follows, we discuss the application of the lemma to $\Phi^{(\infty)}$ (this is the most involved case), but leave the details for $\Omega_1^{(\infty)}$ to the reader.

The sequence of (analytic) "approximate" parametrizations $\{\Phi^{(n)}\}_{n \geq 0}$ is constructed in terms of the sequence of canonical transformation $\{\widetilde{\Psi}^{(n)}\}_{n \geq 0}$ (see (88)) provided by the KAM iterative procedure. Thus, to define $\Phi^{(n)}$ we proceed analogously as we did for $\Phi^{(\infty)}$ in (111). Concretely, we have to compose $\widetilde{\Psi}^{(n)}$, evaluated at $x = y = I_1 = I_2 = 0$, with the changes $(i) - (iv)$ summarized in section 5.13.

We first consider the coordinate change (32) and, performing the same abuse of notation as in (109), we define, for each $n \geq 0$,

$$
\begin{aligned}
\phi_j^{(n)}(\theta, \Lambda) &:= \theta_j + \widetilde{\Theta}_j^{(n)} - \frac{2\xi}{\mu^2}(\partial_{J_j,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}) \left(\frac{\lambda_+}{2\xi} \widetilde{\mathcal{X}}^{(n)} + \frac{1}{2}\widetilde{\mathcal{Y}}^{(n)} \right), \qquad j = 1, 2, \\
q^{(n)}(\theta, \Lambda) &:= \xi + \widetilde{\mathcal{X}}^{(n)} - \frac{\xi}{\lambda_+}\widetilde{\mathcal{Y}}^{(n)} - \frac{2\xi}{\mu^2}(\partial_{J_1,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\widetilde{\mathcal{I}}_1^{(n)} - \frac{2\xi}{\mu^2}(\partial_{J_2,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}})\widetilde{\mathcal{I}}_2^{(n)}, \\
J_1^{(n)}(\theta, \Lambda) &:= \mathbb{I}(\zeta) + \widetilde{\mathcal{I}}_1^{(n)}, \qquad J_2^{(n)}(\theta, \Lambda) := 2\xi\eta + \widetilde{\mathcal{I}}_2^{(n)}, \qquad p^{(n)}(\theta, \Lambda) := \frac{\lambda_+}{2\xi}\widetilde{\mathcal{X}}^{(n)} + \frac{1}{2}\widetilde{\mathcal{Y}}^{(n)},
\end{aligned}
\tag{112}
$$

with all the components of $\widetilde{\Psi}^{(n)}$ evaluated at $x = y = I_1 = I_2 = 0$ (see comments following (109) for a better understanding of these expressions). Moreover, for convenience, we also extend these definitions to the case $n = -1$ by setting $\widetilde{\Psi}^{(-1)} := 0$. By using the bounds of section 5.10 on the transformations

$\widetilde{\Psi}^{(n)}$ (see also the comments linked to (110)) it is not difficult to obtain the following estimates,

$$|\phi_j^{(n)} - \phi_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_7 (M^{(0)})^{19\alpha/2} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{113}$$

$$|q^{(n)} - q^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_7 (M^{(0)})^{10\alpha} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n},$$

$$|J_j^{(n)} - J_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_7 (M^{(0)})^{23\alpha/2} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{114}$$

$$|p^{(n)} - p^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_7 (M^{(0)})^{10\alpha} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n},$$

for $j = 1, 2$, $n \geq 0$ and R small enough, being $c_7 > 0$ independent of R.

Next to that, we apply change (19) to the parametrizations (112). Thus, the coordinates ϕ_1 and J_1 remain unchanged and for the other ones we have, for $n \geq -1$,

$$x_1^{(n)}(\theta, \Lambda) := \sqrt{2q^{(n)}} \cos \phi_2^{(n)}, \qquad y_1^{(n)}(\theta, \Lambda) := -\frac{J_2^{(n)}}{\sqrt{2q^{(n)}}} \sin \phi_2^{(n)} + p^{(n)} \sqrt{2q^{(n)}} \cos \phi_2^{(n)},$$

$$x_2^{(n)}(\theta, \Lambda) := -\sqrt{2q^{(n)}} \sin \phi_2^{(n)}, \qquad y_2^{(n)}(\theta, \Lambda) := -\frac{J_2^{(n)}}{\sqrt{2q^{(n)}}} \cos \phi_2^{(n)} - p^{(n)} \sqrt{2q^{(n)}} \sin \phi_2^{(n)}.$$

Then, using bounds above on the parametrizations (112) we obtain, for $j = 1, 2$ and $n \geq 0$,

$$|x_j^{(n)} - x_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_8 (M^{(0)})^{19\alpha/2} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{115}$$

$$|y_j^{(n)} - y_j^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_8 (M^{(0)})^{37\alpha/4} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{116}$$

for certain $c_8 > 0$ independent of R. Among the technical results on the weighted norm we have used here, we stress the mean value theorem of lemma A.2, combined with the lower bound $|q^{(n)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \geq (M^{(0)})^{\alpha/2}/2$, which follows in a completely analogous way as done for $q^{(\infty)}$ in (109).

Finally, we apply the (partial) normal form transformation $\widehat{\Psi} = \widehat{\Psi}^{(R)}$ to the components of the parametrizations obtained after change (19) and we end with the desired (R-depending) sequence $\Phi^{(n)}$. In particular, we point out that

$$\Phi^{(-1)}(\theta, \Lambda) = \widehat{\Psi}(\theta_1, \sqrt{2\xi} \cos \theta_2, -\sqrt{2\xi} \sin \theta_2, \mathbb{I}(\zeta), -\eta\sqrt{2\xi} \sin \theta_2, -\eta\sqrt{2\xi} \cos \theta_2),$$

where we recall that $\zeta = (\xi, \eta) = h(\Lambda)$. To bound $\Phi^{(n)} - \Phi^{(n-1)}$ we rely on the mean value theorem of lemma A.2. Hence, we use Cauchy estimates on the bounds of point (ii) of theorem 4.1 in order to control the size the partial derivatives of $\widehat{\Psi}$. According to the bounds of section 5.7 on the adapted system of coordinates, we observe that the Cauchy estimates on these partial derivatives can be done in such a way that the worst of them involves, at most, a denominator of order R^2 —but not any power of $M^{(0)}(R)$ at all—. Then, if we combine them with (113), (114), (115) and (116), we can easily establish the following bound,

$$|\Phi^{(n)} - \Phi^{(n-1)}|_{\bar{\mathcal{E}}^{(n)}, \bar{\rho}_-^{(n+1)}, 0} \leq c_9 R^{-2} (M^{(0)})^{37\alpha/4} 2^{-n(2\tau+4)} (\bar{\kappa}^{(0)})^{2^n}, \tag{117}$$

for some $c_9 > 0$ independent of R.

Once we have bounded the "convergence speed" of $\Phi^{(n)}$, now we have to control, in geometric form, the width of the complex widening of the set $\check{\mathcal{E}}^{(\infty)}$ to which we can apply the n-step of the KAM process. Thus, we introduce the sequence of complex sets $\{W^{(n)}\}_{n \geq 0}$, $W^{(n)} \subset \mathbb{C}^2$, given by $W^{(n)} := \check{\mathcal{E}}^{(\infty)} + r^{(n)}$, where the R-depending quantities $r^{(n)} = r^{(n)}(R)$ are defined as

$$r^{(n)} := r^{(0)} \chi^n, \qquad r^{(0)} := \frac{(\bar{\delta}^{(0)})^{\tau+1} (M^{(0)})^{\alpha/2}}{2^{2\tau+4}} \left(\log \left(\frac{1}{\bar{\kappa}^{(0)}} \right) \right)^{-\tau-1}, \qquad \chi := 2^{-3-2\tau}.$$

We notice that (95) implies that $\nu^{(n)} \geq r^{(n)}$, so that $W^{(n)} \subset \hat{\mathcal{E}}^{(n)} + \nu^{(n)} \subset \bar{\mathcal{E}}^{(n)}$.

At this point, we start verifying the conditions of the Inverse Approximation Lemma A.9 for $\Phi^{(\infty)}$ as a limit of $\Phi^{(n)}$. We remark that besides the Λ-dependence, that is the only one taken into account for the Whitney part, the sequence $\Phi^{(n)}$ also depends in analytic and periodic way in the variables $\theta \in \Delta_2(\rho^{(\infty)})$ (see (6) and (77)). Then, according to remark A.1, both, the analytic and periodic dependence, are preserved by the Whitney-extension by simply dealing with θ as a parameter. Thus, we define $U^{(n)}(\theta, \Lambda) := \Phi^{(n-1)} - \Phi^{(-1)}$, for $n \geq 0$. By definition we have that $U^{(0)} = 0$ and, using (117),

$$|U^{(n)} - U^{(n-1)}|_{\Delta_2(\rho^{(\infty)}) \times W^{(n-1)}} \leq c_9 R^{-2} (M^{(0)})^{37\alpha/4} 2^{-(n-1)(2\tau+4)} (\bar{\kappa}^{(0)})^{2^{n-1}}, \qquad n \geq 0. \qquad (118)$$

Our purpose is to show that, for any $\beta > 0$, there is $S = S(\beta) > 0$ (also depending on R) so that (118) is bounded by $S(r^{(n-1)})^\beta$. Indeed, we have the following conditions on S:

$$S \geq 2^{2\tau+4+\beta} c_9 (\bar{\delta}^{(0)})^{-\beta(\tau+1)} R^{-2} (M^{(0)})^{37\alpha/4 - \alpha\beta/2} \left(\log \left(\frac{1}{\bar{\kappa}^{(0)}} \right) \right)^{\beta(\tau+1)} 2^{n(\beta-1)(2\tau+4)} (\bar{\kappa}^{(0)})^{2^{n-1}}, \quad n \geq 1.$$

Due to the super-exponential term $(\bar{\kappa}^{(0)})^{2^{n-1}}$ —compared with the geometric growth of $r^{(n-1)}$— it is easy to realize of the existence of such S. Hence, lemma A.9 ensures that the limit vector-function $U^{(\infty)} = \Phi^{(\infty)} - \Phi^{(-1)}$ is of class Whitney-C^β with respect to $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, for any $\beta > 0$, and so is $\Phi^{(\infty)}$ (observe that $\Phi^{(-1)}$ is analytic in Λ). Consequently, using the Whitney Extension Theorem A.10, the function $\Phi^{(\infty)}$ can be extended to a C^∞-function of Λ in the whole $\mathbb{R}^2$. Abusing notation, we keep the name $\Phi^{(\infty)}$ for this extension. As pointed before, it keeps the analytic and periodic dependence with respect to $\theta \in \Delta_2(\rho^{(\infty)})$.

Acknowledgements

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A Appendix

In this final section we have compiled those contents that, in our opinion, are necessary for the self-containment of the article, but that we have preferred not to include in the body of the paper in order to facilitate its readability. Concretely, in section A.1 we present the technical results on weighted norms we use to prove theorem 3.1. In section A.2 we prove a technical bound concerning the statement of theorem 4.1. Finally, in section A.3 we present a brief introduction to Whitney-smoothness.

A.1 Basic properties of the weighted norm

The following lemmas review some properties of the weighted norm $|\cdot|_{\rho,R}$ introduced in (3). These properties are completely analogous to those for the usual supremum norm.

In lemmas from A.1 to A.4 we discuss the bounds in terms of this weighted norm for the product of functions, partial derivatives (Cauchy estimates), composition of functions, the mean value theorem, estimates on Hamiltonian flows and on small divisors. In lemma A.5 we discuss the convergence of an infinite composition of canonical transformations. In lemma A.6 we give a technical result on the norm of the square root and, finally, in lemma A.7 we give another technical result referring to the norm $|\cdot|_R$ introduced at the end of section 2.

For most of these results we omit the proof, because it can be done simply by expanding the functions in Taylor-Fourier series (2) and then bounding the resulting expressions. For full details we refer to [28]. Along this section we use the notations introduced in section 2, sometimes without explicit mention.

Lemma A.1. *Let $f = f(\theta, x, I, y)$ and $g = g(\theta, x, I, y)$ be analytic functions defined in $\mathcal{D}_{r,s}(\rho, R)$ with 2π-periodic dependence in θ. Then we have:*

(i) $|f \cdot g|_{\rho,R} \le |f|_{\rho,R} \cdot |g|_{\rho,R}.$

(ii) For any $0 < \delta \le R$, $0 \le \chi < 1$, $i = 1, \ldots, r$ and $j = 1, \ldots, 2r$ we have:

$$|\partial_{\theta_i} f|_{\rho-\delta,R} \le \frac{|f|_{\rho,R}}{\delta \exp(1)}, \qquad |\partial_{I_i} f|_{\rho,R\chi} \le \frac{|f|_{\rho,R}}{(1-\chi^2)R^2}, \qquad |\partial_{z_j} f|_{\rho,R\chi} \le \frac{|f|_{\rho,R}}{(1-\chi)R},$$

being $z = (x, y)$. All these bounds can be extended to the case in which f and g take values in $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$ (assuming the matrix product of (i) defined).

Lemma A.2. *Let us take $0 < \rho_0 < \rho$ and $0 < R_0 < R$, and consider analytic vector-functions $\Theta^{(i)}$, $\mathcal{I}^{(i)}$, $\mathcal{X}^{(i)}$ and $\mathcal{Y}^{(i)}$ defined for $(\theta, x, I, y) \in \mathcal{D}_{r,m}(\rho_0, R_0)$, 2π-periodic in θ and taking values in $\mathbb{C}^{r'}$, $\mathbb{C}^{r'}$, $\mathbb{C}^{m'}$ and $\mathbb{C}^{m'}$, respectively, for $i = 0, 1$. We assume that $|\Theta^{(i)}|_{\rho_0, R_0} \leq \rho - \rho_0$, $|\mathcal{I}^{(i)}|_{\rho_0, R_0} \leq R^2$ and that $|\mathcal{Z}^{(i)}|_{\rho_0, R_0} \leq R$, for $i = 0, 1$, where $\mathcal{Z}^{(i)} = (\mathcal{X}^{(i)}, \mathcal{Y}^{(i)})$. Let $f(\theta', x', I', y')$ be a given analytic function defined in $\mathcal{D}_{r',m'}(\rho, R)$ and 2π-periodic in θ'. We introduce:*

$$F^{(i)}(\theta, x, I, y) = f(\theta + \Theta^{(i)}, \mathcal{X}^{(i)}, \mathcal{I}^{(i)}, \mathcal{Y}^{(i)}), \qquad G(\theta, x, I, y) = F^{(1)} - F^{(0)}.$$

Then, we have:

(i) $|F^{(i)}|_{\rho_0, R_0} \leq |f|_{\rho, R}, \quad i = 0, 1.$

(ii) $|G|_{\rho_0, R_0} \leq r' |\partial_\theta f|_{\rho, R} |\Theta^{(1)} - \Theta^{(0)}|_{\rho_0, R_0} + r' |\partial_I f|_{\rho, R} |\mathcal{I}^{(1)} - \mathcal{I}^{(0)}|_{\rho_0, R_0} + 2m' |\partial_z f|_{\rho, R} |\mathcal{Z}^{(1)} - \mathcal{Z}^{(0)}|_{\rho_0, R_0}.$

Lemma A.3. *Let $S = S(\theta, x, I, y)$ be a function such that ∇S is analytic in $\mathcal{D}_{r,m}(\rho, R)$ and 2π-periodic in θ. We also assume that*

$$|\nabla_\theta S|_{\rho, R} \leq R^2 (1 - \chi^2), \qquad |\nabla_I S|_{\rho, R} \leq \delta, \qquad |\nabla_z S|_{\rho, R} \leq R(1 - \chi),$$

for certain $0 < \chi < 1$ and $0 < \delta < \rho$, being $z = (x, y)$. If we denote by Ψ_t^S the flow time t of the Hamiltonian system S, then it is defined as $\Psi_t^S : \mathcal{D}_{r,m}(\rho - \delta, R\chi) \rightarrow \mathcal{D}_{r,m}(\rho, R)$, for every $-1 \leq t \leq 1$. Moreover, if we write $\Psi_t^S - \mathrm{Id} = (\Theta_t^S, \mathcal{X}_t^S, \mathcal{I}_t^S, \mathcal{Y}_t^S)$ and $\mathcal{Z}_t^S = (\mathcal{X}_t^S, \mathcal{Y}_t^S)$, then all these components are 2π-periodic in θ and the following bounds hold for any $-1 \leq t \leq 1$,

$$|\Theta_t^S|_{\rho - \delta, R\chi} \leq |t| |\nabla_I S|_{\rho, R}, \qquad |\mathcal{I}_t^S|_{\rho - \delta, R\chi} \leq |t| |\nabla_\theta S|_{\rho, R}, \qquad |\mathcal{Z}_t^S|_{\rho - \delta, R\chi} \leq |t| |\nabla_z S|_{\rho, R}.$$

Lemma A.4. *Let $f = f(\theta)$ be an analytic and 2π-periodic function in the r-dimensional complex strip $\Delta_r(\rho)$, for some $\rho > 0$, and $\{d_k\}_{k \in \mathbb{Z}^r \setminus \{0\}} \subset \mathbb{C}^*$ with $|d_k| \geq \gamma / |k|_1^\tau$, for some $\gamma > 0$ and $\tau > 0$. We expand f in Fourier series, $f = \sum_{k \in \mathbb{Z}^r} f_k \exp(\mathrm{i} \langle k, \theta \rangle)$, and assume that the average of f is zero, i. e., $\langle f \rangle_\theta = f_0 = 0$. Then, for any $0 < \delta \leq \rho$, we have that the function g defined as*

$$g(\theta) = \sum_{k \in \mathbb{Z}^r \setminus \{0\}} \frac{f_k}{d_k} \exp(\mathrm{i} \langle k, \theta \rangle),$$

satisfies the bound

$$|g|_{\rho - \delta, 0} \leq \left(\frac{\tau}{\delta \exp(1)} \right)^\tau \frac{|f|_{\rho, 0}}{\gamma}.$$

Lemma A.5. *We consider strictly decreasing sequences of positive numbers $\rho^{(n)}$, $R^{(n)}$, a_n, b_n and c_n, defined for $n \geq 0$, and such that the series $A = \sum_{n \geq 0} a_n$, $B = \sum_{n \geq 0} b_n$ and $C = \sum_{n \geq 0} c_n$ are convergent. Additionally, for a given $0 < \delta \leq 1/2$, we define $\rho_-^{(n)} = \rho^{(n)} - \delta$, $R_-^{(n)} = R^{(n)} \exp(-\delta)$ and suppose that $\lim_{n \to +\infty} \rho_-^{(n)} = \rho^{(\infty)}$ and $\lim_{n \to +\infty} R_-^{(n)} = R^{(\infty)}$ are both positive and that*

$$a_n \leq \rho_-^{(n)} - \rho_-^{(n+1)}, \qquad b_n \leq (R_-^{(n)})^2 - (R_-^{(n+1)})^2, \qquad c_n \leq R_-^{(n)} - R_-^{(n+1)}. \tag{119}$$

Let $\Psi^{(n)} : \mathcal{D}_{r,s}(\rho^{(n+1)}, R^{(n+1)}) \rightarrow \mathcal{D}_{r,s}(\rho^{(n)}, R^{(n)})$ be a sequence of analytic canonical transformations with the following bounds for the components of $\Psi^{(n)} - \mathrm{Id}$:

$$|\Theta^{(n)}|_{\rho^{(n+1)}, R^{(n+1)}} \leq a_n, \qquad |\mathcal{I}^{(n)}|_{\rho^{(n+1)}, R^{(n+1)}} \leq b_n, \qquad |\mathcal{Z}^{(n)}|_{\rho^{(n+1)}, R^{(n+1)}} \leq c_n.$$

If we define the composition $\widetilde{\Psi}^{(n)} = \Psi^{(0)} \circ \cdots \circ \Psi^{(n)}$, for any $n \geq 0$, then we have that $\widetilde{\Psi}^{(\infty)} = \lim_{n \to +\infty} \widetilde{\Psi}^{(n)}$ defines an analytic canonical transformation verifying

(i) $\widetilde{\Psi}^{(\infty)} : \mathcal{D}_{r,s}(\rho^{(\infty)}, R^{(\infty)}) \rightarrow \mathcal{D}_{r,s}(\rho^{(0)}, R^{(0)}).$

(ii) *The components of* $\widetilde{\Psi}^{(\infty)} - \mathrm{Id}$ *verify*

$$|\widetilde{\Theta}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leq A, \qquad |\widetilde{\mathcal{I}}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leq B, \qquad |\widetilde{\mathcal{Z}}^{(\infty)}|_{\rho^{(\infty)},R^{(\infty)}} \leq C.$$

(iii) *If we define, for each* $n \geq 0$,

$$\Pi_n = \frac{1}{\delta}\left(\frac{ra_n}{\exp(1)} + \frac{rb_n}{(R^{(\infty)})^2} + \frac{4sc_n}{R^{(\infty)}} \right),$$

then the components of $\widetilde{\Psi}^{(n)} - \mathrm{Id}$ *satisfy:*

$$|\widetilde{\Theta}^{(n)} - \widetilde{\Theta}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leq a_n + A\Pi_n, \qquad |\widetilde{\mathcal{I}}^{(n)} - \widetilde{\mathcal{I}}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leq b_n + B\Pi_n,$$
$$|\widetilde{\mathcal{Z}}^{(n)} - \widetilde{\mathcal{Z}}^{(n-1)}|_{\rho_-^{(n+1)},R_-^{(n+1)}} \leq c_n + C\Pi_n.$$

Proof. Along the proof we use the results on the weighted norm stated in lemmas A.1 and A.2. To prove convergence of $\widetilde{\Psi}^{(n)}$ we write

$$\widetilde{\Psi}^{(n)} - \mathrm{Id} = \sum_{j=1}^{n} (\widetilde{\Psi}^{(j)} - \widetilde{\Psi}^{(j-1)}) + (\widetilde{\Psi}^{(0)} - \mathrm{Id}),$$

and study the absolute convergence of this sum, when $n \to +\infty$, by using the norm $|\cdot|_{\rho^{(\infty)},R^{(\infty)}}$. To do that, first we control the components of $\widetilde{\Psi}^{(n)} - \mathrm{Id}$. We observe that

$$\begin{aligned}\widetilde{\Psi}^{(n)} - \mathrm{Id} &= \Psi^{(n)} - \mathrm{Id} + (\widetilde{\Psi}^{(n-1)} - \mathrm{Id}) \circ \Psi^{(n)} \\ &= \Psi^{(n)} - \mathrm{Id} + (\Psi^{(n-1)} - \mathrm{Id}) \circ \Psi^{(n)} + (\widetilde{\Psi}^{(n-2)} - \mathrm{Id}) \circ \Psi^{(n-1)} \circ \Psi^{(n)}.\end{aligned}$$

Hence, reading this expression by components and proceeding by induction, we obtain the estimate

$$|\widetilde{\Theta}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq \sum_{l=0}^{n} |\Theta^{(l)}|_{\rho^{(l+1)},R^{(l+1)}} \leq \sum_{l=0}^{n} a_l \leq A.$$

Similarly, we also have $|\widetilde{\mathcal{I}}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq B$ and $|\widetilde{\mathcal{Z}}^{(n)}|_{\rho^{(n+1)},R^{(n+1)}} \leq C$. At this point, if we assume *a priori* convergence of $\widetilde{\Psi}^{(n)}$, we clearly obtain the bounds in (ii) for $\widetilde{\Psi}^{(\infty)}$. Next to that, we write

$$\widetilde{\Psi}^{(j)} - \widetilde{\Psi}^{(j-1)} = \widetilde{\Psi}^{(j-1)} \circ \Psi^{(j)} - \widetilde{\Psi}^{(j-1)} = \Psi^{(j)} - \mathrm{Id} + (\widetilde{\Psi}^{(j-1)} - \mathrm{Id}) \circ \Psi^{(j)} - (\widetilde{\Psi}^{(j-1)} - \mathrm{Id}),$$

and consider this expression by components. For instance,

$$\widetilde{\Theta}^{(j)} - \widetilde{\Theta}^{(j-1)} = \Theta^{(j)} + \widetilde{\Theta}^{(j-1)} \circ \Psi^{(j)} - \widetilde{\Theta}^{(j-1)}.$$

Then, using the previous bound on $\widetilde{\Psi}^{(j-1)}$, Cauchy estimates and the mean value theorem, we obtain

$$\begin{aligned}|\widetilde{\Theta}^{(j)} - \widetilde{\Theta}^{(j-1)}|_{\rho_-^{(j+1)},R_-^{(j+1)}} &\leq |\Theta^{(j)}|_{\rho^{(j+1)},R^{(j+1)}} + |\widetilde{\Theta}^{(j-1)}|_{\rho^{(j)},R^{(j)}} \left(r\frac{|\Theta^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{\exp(1)(\rho^{(j)} - \rho_-^{(j)})} \right. \\ &\quad \left. + r\frac{|\mathcal{I}^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{(R^{(j)})^2 - (R_-^{(j)})^2} + 2s\frac{|\mathcal{Z}^{(j)}|_{\rho^{(j+1)},R^{(j+1)}}}{R^{(j)} - R_-^{(j)}} \right) \\ &\leq a_j + A\left(\frac{ra_j}{\delta \exp(1)} + \frac{rb_j}{(R^{(j)})^2(1 - \exp(-2\delta))} + \frac{2sc_j}{R^{(j)}(1 - \exp(-\delta))} \right) \\ &\leq a_j + A\Pi_j,\end{aligned}$$

for any $j \geq 1$. To be more precise, we have bounded the partial derivatives of $\widetilde{\Theta}^{(j-1)}$ in the domain $\mathcal{D}_{r,s}(\rho_-^{(j)}, R_-^{(j)})$, and then we have used hypothesis (119) to guarantee that $\Psi^{(j)}(\mathcal{D}_{r,s}(\rho_-^{(j+1)}, R_-^{(j+1)})) \subset \mathcal{D}_{r,s}(\rho_-^{(j)}, R_-^{(j)})$. Finally, we have also used that $R^{(j)} \geq R^{(\infty)}$ and that $(1 - \exp(-x))^{-1} \leq 2/x$, whenever $0 < x \leq 1$. Analogously, we can derive bounds in (iii) for the remaining components. Finally, the convergence of $\sum_{j \geq 1} |\widetilde{\Psi}^{(j)} - \widetilde{\Psi}^{(j-1)}|_{\rho^{(\infty)},R^{(\infty)}}$ follows immediately using these bounds. $\qquad\square$

Lemma A.6. *Let $f(\theta, x, I, y)$ be an analytic function defined in $\mathcal{D}_{r,m}(\rho, R)$, 2π-periodic in θ and such that $|f|_{\rho,R} \leq L < 1$. Let $g(\theta, x, I, y)$ and $h(\theta, x, I, y)$ be given by*

$$g(\theta, x, I, y) = \sqrt{1 + f(\theta, x, I, y)}, \qquad h(\theta, x, I, y) = (\sqrt{1 + f(\theta, x, I, y)})^{-1}.$$

Then, one has $|g|_{\rho,R} \leq 2 - \sqrt{1 - L}$ and $|h|_{\rho,R} \leq 1/\sqrt{1 - L}$.

Proof. To prove both inequalities, we simply develop the square roots using the binomial expansion,

$$|g|_{\rho,R} \leq \sum_{j \geq 0} \left| \binom{1/2}{j} \right| L^j = 2 - \sqrt{1 - L}, \qquad |h|_{\rho,R} \leq \sum_{j \geq 0} \left| \binom{-1/2}{j} \right| L^j = \frac{1}{\sqrt{1 - L}}.$$

$\square$

The next relation between norms is used to establish the bounds on $|Z|_{R^2}$ in (59) and in section A.2.

Lemma A.7. *Let $f(u, v)$ be an analytic function around the origin and $F(x, y)$ the same function written in terms of (x, y) through the changes $u = (x_1^2 + x_2^2)/2$ and $v = (y_1 x_2 - x_1 y_2)/2$, i. e., $F(x, y) = f(u, v)$. Then, for any $R > 0$ we have $|f|_{R^2} = |F|_R$.*

Proof. We consider the following expansion for $f(u, v)$,

$$f(u, v) = \sum_{\bar{k} \in \mathbb{Z}_+^2} a_{\bar{k}} 2^{|\bar{k}|_1} u^{\bar{k}_1} v^{\bar{k}_2},$$

for certain coefficients $a_{\bar{k}}$. By definition, $|f|_{R^2} = \sum_{\bar{k}} 2^{|\bar{k}|_1} |a_{\bar{k}}| R^{2|\bar{k}|_1}$. Then, we can write F as

$$F(x, y) = \sum_{k \in \mathbb{Z}_+^4} (-1)^{k_4} \binom{k_1 + k_2}{k_1} \binom{k_3 + k_4}{k_3} a_{(k_1 + k_2, k_3 + k_4)} x_1^{2k_1} x_2^{2k_2} (y_1 x_2)^{k_3} (y_2 x_1)^{k_4}.$$

We point out that all the monomials in the sum above are different for different k, so that

$$|F|_R = \sum_{k \in \mathbb{Z}_+^4} \binom{k_1 + k_2}{k_1} \binom{k_3 + k_4}{k_3} |a_{(k_1 + k_2, k_3 + k_4)}| R^{2|k|_1} = |f|_{R^2}.$$

$\square$

A.2   Bound on the term $\widetilde{Z}^{(R)}$ of the normal form

As we pointed out at the end of section 4.1, the estimate $|\widetilde{Z}^{(R)}|_{0,R} \leq \tilde{c} R^6$ on the statement of theorem 4.1 is not explicitly contained in [36]. In this section we show how this bound can be derived from the estimate $|\mathcal{Z}^{(R)}|_{0,R} \leq |\mathcal{H}|_{\rho_0, R_0}$ and the special structure of the normal form.

To establish this estimate on $\widetilde{Z}^{(R)}$ we take advantage on the fact that the normal form can be expanded in powers of $(q, I, L/2)$, where $q = (x_1^2 + x_2^2)/2$ and $L = y_1 x_2 - x_1 y_2$. Concretely, $\widetilde{Z}^{(R)}(x, I, y) = Z^{(R)}(q, I, L/2)$, with $Z^{(R)}(u)$ starting at degree three in $u = (u_1, u_2, u_3)$. We refer to point (iii) of theorem 4.1 for more details. Then, our purpose now is to bound the norm $|Z^{(R)}|_{R^2}$, which corresponds to the weighted norm for the expansion of $Z^{(R)}(u)$ in powers of u. Then, using lemma A.7 we can relate the norms $|\widetilde{Z}^{(R)}|_{0,R} = |Z^{(R)}|_{R^2}$.

Once we have fixed the value of $\varepsilon > 0$, we take any $0 < R \leq R^*$ and consider the following decomposition:

$$Z^{(R)} = \check{Z} + \widehat{Z}^{(R)},$$

where $\check{Z}$ is independent on R and contains the affine terms in u_1 and u_3 of the normal form $Z^{(R)}$ as described in point (iii) of theorem 4.1 (see also the comments following the statement of the theorem).

The term $\widehat{Z}^{(R)}$ is a polynomial on u, but $\breve{Z}$ allows generic (analytic) dependence on u_2. But due to the fact that $\breve{Z}$ is independent of R, we easily have that there is $\breve{A}$ (independent of R) such that $|\breve{Z}|_{R^2} \leq \breve{A}R^6$ for any R small enough. For the remaining terms we only have $|\widehat{Z}^{(R)}|_{R^2} \leq A$, with $A := |\mathcal{H}|_{\rho_0, R_0}$ also independent of R. In this case, we know that $\widehat{Z}^{(R)}$ is a polynomial of degree less than or equal to $\lfloor r_{opt}(R)/2 \rfloor$, where $r_{opt}(R)$ depends on ε (see (16)). Let us assume that R is small enough such that this degree is bigger than three and expand

$$\widehat{Z}^{(R)} = \sum_{p=3}^{\lfloor r_{opt}(R)/2 \rfloor} \widehat{Z}_p,$$

where $\widehat{Z}_p = \widehat{Z}_p(u)$ contains the terms of degree p in u of the normal form, except those included in $\breve{Z}$. We remark that the particular expression of the homogeneous polynomials $\widehat{Z}_p$ is independent of R. By using Cauchy estimates we have the following bound for these terms

$$|\widehat{Z}_p|_{R^2} \leq A \frac{R^{2p}}{R_p^{2p}},$$

where $R_p = R_p(\varepsilon)$ denotes the first value of R for which $\lfloor r_{opt}(R)/2 \rfloor \geq p$. Concretely, we observe that

$$p = \left\lfloor 1 + \frac{1}{2} \exp\left(W\left(\log\left(\frac{1}{R_p^\alpha} \right) \right) \right) \right\rfloor,$$

where $\alpha = 1/(\tau + 1 + \varepsilon)$. By skipping the integer part, we obtain the bounds

$$\frac{1}{2} \exp\left(W\left(\log\left(\frac{1}{R_p^\alpha} \right) \right) \right) \leq p \leq 1 + \frac{1}{2} \exp\left(W\left(\log\left(\frac{1}{R_p^\alpha} \right) \right) \right).$$

Then, simple computations show that $(2(p-1))^{2(p-1)/\alpha} \leq R_p^{-1} \leq (2p)^{2p/\alpha}$. As a consequence, we obtain the following ε-depending bound

$$|\widehat{Z}_p|_{R^2} \leq A(2p)^{4p^2/\alpha} R^{2p}.$$

If we let $\varepsilon \to 0^+$, then we obtain a ε-independent bound. Then, we have:

$$|\widehat{Z}|_{R^2} \leq \sum_{p=3}^{\lfloor r_{opt}(R)/2 \rfloor} A(2p)^{4(\tau+1)p^2} R^{2p} = AR^6 \sum_{p=3}^{\lfloor r_{opt}(R)/2 \rfloor} (2p)^{4(\tau+1)p^2} R^{2p-6}.$$

In order to bound this last sum by an expression independent of R, we remark that, once we have fixed the value of R, then for all the indexes p appearing in the sum above we have $R \leq R_p$. Consequently, by using the upper bound for R_p previously derived we obtain

$$|\widehat{Z}|_{R^2} \leq AR^6 \sum_{p \geq 3} \frac{(2p)^{4(\tau+1)p^2}}{(2(p-1))^{2(p-1)(2p-6)(\tau+1+\varepsilon)}} \leq AR^6 \sum_{p \geq 3} \frac{e^{6(\tau+1)p}}{(2(p-1))^{4\varepsilon p^2 + (12-16p)(\tau+1+\varepsilon)}} = \widehat{A}R^6,$$

where we have used that $(p/(p-1))^p \leq e^{3/2}$. Therefore the convergence of $\widehat{A} = \widehat{A}(\varepsilon)$ is clear, for any $\varepsilon > 0$, and then we define $\tilde{c} = \tilde{c}(\varepsilon) := \breve{A} + \widehat{A}$.

A.3 Whitney-smoothness

In this section we review the main definitions about Whitney-smoothness and the basic results on the topic we have used in section 5.14. See appendix 6 in [9] for a straightforward survey on the subject.

Definition A.8. *Let $A \subset \mathbb{R}^n$ be a closed set and $\beta > 0$ with $\beta \notin \mathbb{N}$. A Whitney-C^β function u on A —we shall write $u \in C^\beta_{Wh}(A)$—, consists of a collection $u = \{u_q\}_{0 \leq |q|_1 \leq k}$, with $k = \lfloor \beta \rfloor$ and $q \in \mathbb{Z}^n_+$, of functions defined on A satisfying the following property: there exists $\widehat{\gamma} > 0$ such that,*

$$|u_q(x)| \leq \widehat{\gamma}, \quad |u_q(x) - P_q(x,y)| \leq \widehat{\gamma}|x-y|^{\beta-|q|_1}, \qquad \forall x, y \in A, \ \forall q \in \mathbb{Z}^n_+, \ 0 \leq |q|_1 \leq k, \qquad (120)$$

where $P_q(x,y)$ is the analogous of the $(k-|q|_1)$-th order Taylor polynomial of u_q. More precisely,

$$P_q(x,y) = \sum_{j=0}^{k-|q|_1} \sum_{|l|_1=j} \frac{1}{l!} u_{q+l}(y)(x-y)^l, \qquad l \in \mathbb{Z}^n_+,$$

with the multi-index notation $l! = \prod_{i=1}^n l_i!$ and $(x-y)^l = \prod_{i=1}^n (x_i - y_i)^{l_i}$. The norm $\|u\|_{C^\beta_{Wh}(A)}$ is defined as the smallest $\widehat{\gamma}$ for which (120) holds. If $u \in C^\beta_{Wh}(A)$ for all $\beta \notin \mathbb{N}$, we will refer to u as a Whitney-C^∞ function —we shall write $u \in C^\infty_{Wh}(A)$—.

Of course, conditions (120) are not easy to fulfill for a given function defined on an arbitrary closed set A. However, in case it is constructed as a limit of analytic functions, next result provides a way to verify those properties (see [48] for a proof).

Lemma A.9 (Inverse Approximation Lemma). *Take a geometric sequence $r_j = r_0 \chi^j$, with $r_0 > 0$ and $0 < \chi < 1$. Let $A \subset \mathbb{R}^n$ be an open or closed set and define $W_j = A + r_j$, $j \in \mathbb{Z}_+$ (see (9)). Consider a sequence of real analytic functions $\{U^{(j)}\}_{j \in \mathbb{Z}_+}$, with $U^{(0)} = 0$, such that $U^{(j)}$ is defined in W_{j-1} and*

$$|U^{(j)} - U^{(j-1)}|_{W_{j-1}} \leq S r_{j-1}^\beta, \qquad j \in \mathbb{N},$$

for some constants $S \geq 0$ and $\beta > 0$, with $\beta \notin \mathbb{N}$. Then, there exists a unique function $U^{(\infty)}$, defined on A, which is of class Whitney-C^β and such that,

$$\|U^{(\infty)}\|_{C^\beta_{Wh}(A)} \leq c_{\chi,\beta,n} S, \qquad \lim_{j \to +\infty} \|U^{(\infty)} - U^{(j)}\|_{C^\alpha_{Wh}(A)} = 0,$$

for all $\alpha < \beta$, where the constant $c_{\chi,\beta,n} > 0$ does not depend on A.

The following result states the classical *Whitney Extension Theorem*, claiming that Whitney-C^β functions defined on closed subsets of $\mathbb{R}^n$ can be extended to C^β functions on the whole space $\mathbb{R}^n$. See [46, 47].

Theorem A.10 (Whitney Extension Theorem). *For any $\beta > 0$, $\beta \notin \mathbb{N}$, and any closed set $A \subset \mathbb{R}^n$ there exists a (non-unique) linear extension operator $\mathcal{F}_\beta : C^\beta_{Wh}(A) \to C^\beta(\mathbb{R}^n)$, such that for each $u = \{u_q\}_q \in C^\beta_{Wh}(A)$ and $U = \mathcal{F}_\beta(u)$, we have, for all $0 \leq |q|_1 \leq \lfloor \beta \rfloor$,*

$$D^q U|_A = u_q, \qquad \|U\|_{C^\beta(\mathbb{R}^n)} \leq c_{\beta,n} \|u\|_{C^\beta_{Wh}(A)},$$

where $c_{\beta,n}$ does not depend on A. The norm in $C^\beta(\mathbb{R}^n)$ is the usual Hölder one, i. e., if $k = \lfloor \beta \rfloor$ then

$$\|U\|_{C^\beta(\mathbb{R}^n)} = \sup_{\substack{q \in \mathbb{Z}^n_+, \ |q|_1 \leq k \\ x \in \mathbb{R}^n}} \{|D^q U(x)|\} + \sup_{\substack{q \in \mathbb{Z}^n_+, \ |q|_1 = k \\ x, y \in \mathbb{R}^n, \ x \neq y}} \left\{ \frac{|D^q U(x) - D^q U(y)|}{|x-y|^{\beta-k}} \right\}.$$

If $\beta = +\infty$, then there is a (non-unique) linear extension operator $\mathcal{F} : C^\infty_{Wh}(A) \to C^\infty(\mathbb{R}^n)$, such that if $U = \mathcal{F}(u)$ then for all the derivatives —in the sense of Whitney— of u, $D^q U|_A = u_q$.

Remark A.1. *It is worth remarking that the functions $U^{(j)}$, $j \in \mathbb{Z}_+$, and the limit function $U^{(\infty)}$ of lemma A.9 may depend in analytic, smooth or periodic way on other variables. In such case, these other variables must be thought of as parameters. Moreover, one can choose extension operators $\mathcal{F}_\beta$ and $\mathcal{F}$ preserving the analyticity (respectively smoothness), as well as periodicity with respect to these parameters. See [9].*