

# MONOIDAL FUNCTORS, ACYCLIC MODELS AND CHAIN OPERADS

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ABSTRACT. We prove that for a topological operad  $P$  the operad of oriented cubical chains,  $C_*^{ord}(P)$ , and the operad of singular chains,  $S_*(P)$ , are weakly equivalent. As a consequence,  $C_*^{ord}(P; \mathbb{Q})$  is formal if and only if  $S_*(P; \mathbb{Q})$  is formal, thus linking together some formality results spread in the literature. The proof is based on an acyclic models theorem for monoidal functors. We give different variants of the acyclic models theorem and apply the contravariant case to study the cohomology theories for simplicial sets defined by  $R$ -simplicial differential graded algebras.

## 1. INTRODUCTION

Since its introduction by S. Eilenberg and S. MacLane in [EM], the acyclic models theorem has been a powerful technique in algebraic topology in order to establish the existence of chain transformations, uniquely determined up to homotopy and consequently determining well-defined morphisms in homology.

The homology functor  $H_*(X; \mathbb{Z})$  from the category of topological spaces **Top** to the category of graded abelian groups is, by the acyclic models theorem of Eilenberg-MacLane cited above, a well-established functor (up to unique isomorphism), independent of the (singular or cubical) chain complex we use to compute it. This functor comes equipped with a Künneth morphism  $H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \times Y)$  which, for the same reasons, is also well defined. In categorical terms, the Künneth morphism defines a monoidal structure on the functor  $H_*(; \mathbb{Z})$ , so we can say that the monoidal functor of homology is well defined up to isomorphism.

In some situations (for example when one considers the dg operad defined by a topological operad) it is appropriate to work with complexes. If  $S_*(X)$  and  $C_*(X)$  denote the singular and cubical chain complexes of  $X$ , respectively, the classic acyclic models theorem proves that there is a morphism of complexes  $S_*(X) \longrightarrow C_*(X)$  which is a quasi-isomorphism (that is, a homology isomorphism). But this morphism is not necessarily a monoidal transformation and so, given a topological operad  $P$ , we do not necessarily obtain a morphism between  $S_*(P)$  and  $C_*(P)$ .

In this paper we prove an acyclic models theorem for monoidal functors from a monoidal category  $\mathcal{C}$  to the category of chain complexes of abelian groups  $\mathbf{C}_*(\mathbb{Z})$ , see corollary 4.3.3. We likewise establish several variations of the main result, which cover the symmetric monoidal and the contravariant monoidal settings. As a consequence of our results we will prove that  $S_*(X)$  and  $C_*(X)$  are weakly equivalent as monoidal functors.

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The acyclic models theorem for monoidal functors, 4.3.1, is stated and proved by adapting to the monoidal setting the categorical framework proposed by Barr for these kind of results, [B96]. Thus, if  $\mathcal{C}$  is a monoidal category,  $K_*, L_*$  are monoidal functors from  $\mathcal{C}$  to  $\mathbf{C}_*(\mathbb{Z})$ , and  $H_0(K_*) \rightarrow H_0(L_*)$  is a monoidal morphism, theorem 4.3.1 establishes sufficient conditions on  $K_*, L_*$  to extend  $f$  to a morphism  $f : K_* \rightarrow L_*$  defined up to quasi-isomorphism, that is, in the localized category obtained by inverting quasi-isomorphisms.

The main tools in the statement and proof of the acyclic models theorem are the models of  $\mathcal{C}$ , given by a suitable monoidal cotriple  $\mathbf{G}$  defined in  $\mathcal{C}$ , and a class of morphisms  $\Sigma$  in the category of monoidal functors from  $\mathcal{C}$  to  $\mathbf{C}_*(\mathbb{Z})$ , denoted by  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathbb{Z}))$ , which is stable under the total complex functor applied to double complexes. In the classical situation the cotriple  $\mathbf{G}$  is a model induced cotriple. For instance, in  $\mathbf{Top}$  one can take  $\mathbf{G}(X) = \sqcup \Delta^n$ , where the disjoint union is taken over all  $n \geq 0$  and all continuous maps  $\Delta^n \rightarrow X$ . In our topological applications we use a monoidal version of this cotriple and we use the cotriple introduced by Kleisli in [K] for the symmetric monoidal situation. As for the class of morphisms  $\Sigma$ , in the applications it is a class in between the class of objectwise homotopy equivalences and that of quasi-isomorphisms.

We apply the monoidal acyclic models theorem in two different contexts. First, taking  $\mathcal{C} = \mathbf{Top}$ , we prove that the singular and the oriented cubical chain functors are weakly homotopy equivalent (see 4.4.1 for a precise definition) as symmetric monoidal functors, so their extension to topological operads gives quasi-isomorphic dg operads, (see theorem 7.2.2). As a consequence, the formality of a given topological operad does not depend on the chosen chain functor. This enables us to relate two different versions of Kontsevich formality theorem for the little cube operad, the one proved by Kontsevich in [K] for oriented cubical chains and the one by Tamarkin, see [T], for singular chains, and to give a more precise statement of Deligne's conjecture.

The second application refers to a uniqueness result for cohomology theories defined by a simplicial differential graded algebra, (see [C]). Our uniqueness result includes the additive and the multiplicative structure of these theories, thus proving the two main theorems in Cartan's paper. In particular, for any simplicial set  $X$  it follows that the  $\mathbb{Q}$ -differential graded algebras of singular cochains  $S^*(X; \mathbb{Q})$  and of polynomial forms  $Su^*(X)$  are weakly equivalent, recovering the well-known result of rational homotopy theory, (see [S], [FHT]).

We now give an overview of the contents of the different sections. In section 2, we recall some basic facts about monoidal functors. We begin section §3 by reviewing the Eilenberg-MacLane shuffle product formulas and the Alexander-Whitney formulas, which we use to state the existence of several monoidal and comonoidal simple functors. Section §4 is devoted to the statement and proof of the main theorem 4.3.1.

The comparison of the monoidal singular and cubical chain functors defined on  $\mathbf{Top}$  is established in §5. The scheme for proving theorem 4.3.3 is general enough that it can be reproduced with minor variations to obtain a theorem for symmetric monoidal functors (§6) from which we obtain a comparison result for the singular and the oriented cubical chain functors and, as an application, a similar result for topological operads, §7. Finally, sections §8 and §9 are devoted to the statement of the contravariant version of the acyclic models theorem and its application to cohomology theories cited above.

We should point out that in this paper we do not work out the contravariant symmetric case. It will be dealt with elsewhere as an example of a more systematic study of acyclic models in the context of descent categories (see [GN]).

## 2. THE MONOIDAL BACKGROUND

In this section we recall the basic principles of monoidal categories and functors, (see [EK] or [ML] for details), and introduce notations that are to be used further in the paper. These remarks allow us to define in 3.2 the simple functors which are associated with simplicial and cosimplicial functors with values in a category of complexes.

### 2.1. Monoidal categories.

2.1.1. A *monoidal category* consists of a category  $\mathcal{C}$ , a functor  $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ , which we shall call the product functor, and an object  $\mathbf{1}$  of  $\mathcal{C}$ , in addition to natural isomorphisms of associativity for  $\otimes$  and unit for  $\mathbf{1}$ , which are subject to coherence constraints.

2.1.2. A monoidal category  $(\mathcal{C}, \otimes, \mathbf{1})$  is *symmetric* if, for any objects  $X, Y$  of  $\mathcal{C}$ , it comes equipped with a natural commutative isomorphism  $\tau_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$  that satisfies the commutativity constraints.

2.1.3. By an *additive monoidal category*, also called an additive tensor category, we understand an additive category  $\mathcal{A}$  which is monoidal in such a way that the product functor  $\otimes : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$  is biadditive.

**Examples 2.1.1.** In these examples we set out the monoidal structure of several categories that will appear later on in this paper.

1. Let  $\mathcal{A}$  be an additive category. We denote by  $\mathbf{C}_*(\mathcal{A})$  the category of uniformly bounded below chain complexes on  $\mathcal{A}$ , that is, there is an  $n \in \mathbb{Z}$  such that the objects of  $\mathbf{C}_*(\mathcal{A})$  are differential graded objects  $(C_*, d)$ , with differential  $d$  of degree  $-1$ , with  $C_i = 0$  for  $i < n$ .

If  $\mathcal{A}$  is an additive monoidal category, then the category  $\mathbf{C}_*(\mathcal{A})$  is an additive monoidal category with the product given by  $(C_* \otimes D_*)_n = \bigoplus_{p+q=n} C_p \otimes D_q$ , the differential on  $C_p \otimes D_q$  being  $d = d_C \otimes \text{id} + (-1)^p \text{id} \otimes d_D$ , and the unit given by the complex consisting of the unit of  $\mathcal{A}$  concentrated in degree zero.

If  $\mathcal{A}$  is a symmetric monoidal category, then  $\mathbf{C}_*(\mathcal{A})$  is also symmetric, with the natural commutativity isomorphism

$$\tau_{C_*, D_*} : C_* \otimes D_* \longrightarrow D_* \otimes C_*$$

that includes the signs, i.e., it is defined by

$$\tau_{C_*, D_*} = (-1)^{pq} \tau_{C_p, D_q}, \quad \text{on } C_p \otimes D_q.$$

If  $\mathcal{A}$  is the category of  $R$ -modules for some ring  $R$ , we will denote  $\mathbf{C}_*(\mathcal{A})$  by  $\mathbf{C}_*(R)$ .

2. Let  $\mathcal{A}$  be an additive monoidal category. By a double complex of  $\mathcal{A}$  we understand a bigraded object of  $\mathcal{A}$ ,  $C_{**}$ , with two differentials,  $d', d''$ , which commute; with the obvious morphisms,

they form a category. Thus, the category of uniformly bounded below double chain complexes  $\mathbf{C}_{**}(\mathcal{A})$  is isomorphic to  $\mathbf{C}_*(\mathbf{C}_*(\mathcal{A}))$ .

3. Let  $\mathcal{X}$  be a category and  $\mathcal{D}$  be a monoidal category. In the functor category  $\mathbf{Cat}(\mathcal{X}, \mathcal{D})$  (also denoted by  $\mathbf{Fun}(\mathcal{X}, \mathcal{C})$ , see [B96]) we define the product  $\boxtimes$  componentwise by

$$(F \boxtimes G)(X) = F(X) \otimes G(X),$$

and the unit  $\mathbf{1}$  as the constant functor defined by the unit of  $\mathcal{D}$ . With this structure  $\mathbf{Cat}(\mathcal{X}, \mathcal{D})$  becomes a monoidal category, which is symmetric if  $\mathcal{D}$  is symmetric.

In particular, if  $\Delta$  denotes the simplicial category, the category of simplicial objects of  $\mathcal{D}$ , which we denote by  $\Delta^{\circ}\mathcal{D}$ , is a monoidal category. Its objects are denoted by  $X_{\bullet}$ .

Analogously, the category of cosimplicial objects, denoted by  $\Delta\mathcal{D}$ , is a monoidal category, which is symmetric if  $\mathcal{D}$  is symmetric. Its objects will be denoted by  $X^{\bullet}$ .

4. Observe that if  $\mathcal{C}$  is a monoidal category, then the dual category  $\mathcal{C}^{\circ}$  is also monoidal. In particular, if  $\mathcal{A}$  is an additive monoidal category, the category of uniformly bounded below cochain complexes  $\mathbf{C}^*(\mathcal{A})$ , which is isomorphic to the dual category of  $\mathbf{C}_*(\mathcal{A}^{\circ})$ , is a monoidal category, which is symmetric if  $\mathcal{A}$  is symmetric. Likewise the category of uniformly bounded below double chain complexes  $\mathbf{C}^{**}(\mathcal{A})$  is a monoidal category.

## 2.2. Monoidal functors.

2.2.1. If  $(\mathcal{C}, \otimes, \mathbf{1})$ ,  $(\mathcal{D}, \otimes, \mathbf{1}')$  are monoidal categories, a *monoidal functor* (sometimes called a lax monoidal functor) is a triple

$$(F, \kappa, \eta) : (\mathcal{C}, \otimes, \mathbf{1}) \longrightarrow (\mathcal{D}, \otimes, \mathbf{1}'),$$

where  $F : \mathcal{C} \longrightarrow \mathcal{D}$  is a covariant functor, together with a natural morphism of  $\mathcal{D}$ ,

$$\kappa_{X,Y} : FX \otimes FY \longrightarrow F(X \otimes Y),$$

for all objects  $X, Y$  of  $\mathcal{C}$ , and a morphism of  $\mathcal{D}$ ,  $\eta : \mathbf{1}' \longrightarrow F\mathbf{1}$ , compatible with the constraints of associativity and unit. We will refer to  $\kappa$  as the *Künneth morphism* of  $F$ .

A monoidal functor  $F$  is said to be a *strong monoidal* functor if the Künneth morphisms are isomorphisms.

2.2.2. If  $\mathcal{C}$  and  $\mathcal{D}$  are symmetric monoidal categories, a monoidal functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  is said to be *symmetric* if  $\kappa$  is compatible with the commutativity isomorphisms  $\tau$ .

**Examples 2.2.1.** Let  $\mathcal{A}$  be an additive (symmetric) monoidal category.

1. Given a double complex  $C_{**}$  of  $\mathcal{A}$  with finite anti-diagonals, the associated total complex is the complex  $\text{Tot}(C_{**})$  which in degree  $n$  is given by  $\text{Tot}_n(C_{**}) = \bigoplus_{p+q=n} C_{pq}$  with differential defined by  $d = d' + (-1)^p d''$  on  $C_{pq}$ . For bounded below double complexes the total complex defines a functor  $\text{Tot} : \mathbf{C}_{**}(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A})$ . It is a strong (symmetric) monoidal functor.

Similarly, the total complex functor defines a strong (symmetric) monoidal functor for cochain complexes  $\text{Tot} : \mathbf{C}^{**}(\mathcal{A}) \longrightarrow \mathbf{C}^*(\mathcal{A})$ .

2. The *homology functor*  $H_* : \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A})$  is a (symmetric) monoidal functor, taking as  $\kappa$  the usual Künneth morphism

$$\kappa_{X,Y} : H_*(X) \otimes H_*(Y) \longrightarrow H_*(X \otimes Y).$$

2.3. Monoidal functors are stable under composition, so the category of monoidal categories and monoidal functors,  $\mathbf{Mon}$ , is a subcategory of  $\mathbf{Cat}$ .

2.3.1. Let  $F, G : \mathcal{C} \longrightarrow \mathcal{D}$  be two monoidal functors. A natural transformation  $\phi : F \Rightarrow G$  is said to be *monoidal* if it is compatible with  $\kappa$  and  $\eta$ .

Monoidal functors between two monoidal categories  $\mathcal{C}, \mathcal{D}$ , together with monoidal natural transformations, define a subcategory of the functor category  $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$  which will be denoted by  $\mathbf{Mon}(\mathcal{C}, \mathcal{D})$ .

Moreover, monoidal functors and monoidal natural transformations are compatible, so they equip the class of monoidal categories,  $\mathbf{Mon}$ , with a 2-category structure.

2.3.2. Similarly, one can extend the notions above to the symmetric setting. Hence, the symmetric monoidal functors between two symmetric monoidal categories  $\mathcal{C}, \mathcal{D}$ , together with symmetric monoidal natural transformations define a subcategory of the functor category  $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$  which is denoted by  $\mathbf{Symon}(\mathcal{C}, \mathcal{D})$ .

2.4. **Comonoidal functors.** After the definition of monoidal functor we obtain, by duality, the notion of comonoidal functor between monoidal categories (called op-monoidal functor in [KS]).

2.4.1. By definition, a *comonoidal functor* between  $\mathcal{C}$  and  $\mathcal{D}$  is a covariant monoidal functor between the dual categories  $\mathcal{C}^o, \mathcal{D}^o$ .

That is to say, a comonoidal functor  $F : \mathcal{C} \longrightarrow \mathcal{D}$  is a covariant functor together with natural morphisms

$$\kappa^o : F(X \otimes Y) \longrightarrow F(X) \otimes F(Y),$$

for objects  $X, Y$  of  $\mathcal{C}$ , and a morphism  $\eta^o : F\mathbf{1} \longrightarrow \mathbf{1}'$  of  $\mathcal{D}$ , satisfying constraints of associativity and unit. If  $\mathcal{C}, \mathcal{D}$ , are symmetric monoidal categories, a comonoidal functor  $F$  is said to be symmetric if  $\kappa^o$  is compatible with the commutativity constraint.

2.4.2. The definition of comonoidal natural transformations is clear. We denote by  $\mathbf{CoMon}(\mathcal{C}, \mathcal{D})$  the category of comonoidal functors and comonoidal natural transformations. The category  $\mathbf{CoMon}(\mathcal{C}, \mathcal{D})$  is the dual category of  $\mathbf{Mon}(\mathcal{C}^o, \mathcal{D}^o)$ , so we will focus our attention on monoidal functors.

2.4.3. We have defined the notions of monoidal and comonoidal functors for covariant functors between two monoidal categories  $\mathcal{C}, \mathcal{D}$ . It is straightforward to define the corresponding notions for contravariant functors from  $\mathcal{C}$  to  $\mathcal{D}$ . Nevertheless, we should point out that a contravariant monoidal functor  $\mathcal{C} \rightarrow \mathcal{D}$  (resp. a contravariant comonoidal functor) is equivalent to a comonoidal functor (resp. monoidal functor) defined in the dual category,  $\mathcal{C}^o \rightarrow \mathcal{D}$ , so nothing new is gained.

### 3. SHUFFLE AND ALEXANDER-WHITNEY MAPS

3.1. **Simplicial chain complexes.** Let  $\mathcal{A}$  be an additive monoidal category. We use  $\sim: \Delta^o \mathcal{A} \rightarrow \mathbf{C}_*(\mathcal{A})$  to denote the functor which takes a simplicial object  $C_\bullet$  of  $\mathcal{A}$  to the chain complex  $\widetilde{C}_\bullet$  given by  $(\widetilde{C}_\bullet)_n = C_n$  with differential  $d = \partial_0 - \partial_1 + \cdots + (-1)^n \partial_n$ .

3.1.1. *Shuffle map.* The functor  $\sim$  with the shuffle map is monoidal (see [EM2]): recall that if  $C_\bullet, D_\bullet$  are objects of  $\Delta^o \mathcal{A}$ , the shuffle product  $sh: C_* \otimes D_* \rightarrow \widetilde{C_\bullet \boxtimes D_\bullet}$  is defined in degree  $n = p + q$ ,  $C_p \otimes D_q \rightarrow C_{p+q} \otimes D_{p+q}$ , by the formula

$$sh_{pq} = \sum_{(\mu, \nu)} \varepsilon(\mu, \nu) (s_{\nu_q} \cdots s_{\nu_1}) \otimes (s_{\mu_p} \cdots s_{\mu_1}),$$

where the sum is taken over all  $(p, q)$ -shuffles  $(\mu, \nu)$  and  $\varepsilon(\mu, \nu)$  is the signature of the associated permutation.

3.1.2. *Alexander-Whitney map.* The functor  $\sim$  with the Alexander-Whitney map is also comonoidal, (loc. cit.). Recall that if  $C_\bullet, D_\bullet$  are simplicial objects of  $\mathcal{A}$ , the Alexander-Whitney morphism  $AW: \widetilde{C_\bullet \boxtimes D_\bullet} \rightarrow C_* \otimes D_*$  is given by morphisms  $AW: C_n \otimes D_n \rightarrow \bigoplus_{i=0}^n C_i \otimes D_{n-i}$  which are defined by

$$AW = \sum_{i=0}^n \widetilde{\partial}^{n-i} \otimes \partial_0^i,$$

where  $\widetilde{\partial}^{p-i}$  is the last face operator  $\partial_{i+1} \cdots \partial_p$ .

3.1.3. *Cosimplicial cochain complexes.* Let  $\mathcal{A}$  be an additive monoidal category. The category of cosimplicial objects of  $\mathcal{A}$ ,  $\Delta \mathcal{A}$ , is a monoidal category. Let  $\sim: \Delta \mathcal{A} \rightarrow \mathbf{C}^*(\mathcal{A})$  represent the functor which takes a cosimplicial object  $K^\bullet$  to the cochain complex  $K^*$  with differential  $d = \partial^0 + \cdots + (-1)^n \partial^n$ . By dualizing 3.1.1 and 3.1.2, one obtains comonoidal and monoidal structures, respectively, for this functor  $\sim$ . We use the monoidal structure issuing from the Alexander-Whitney formulas to study contravariant functors in §8.

3.2. **The simple functors.** When  $\mathcal{A}$  is an additive monoidal category we can use the shuffle and Alexander-Whitney structures above to put monoidal and comonoidal structures in the classic simple complex associated to a simplicial complex of  $\mathcal{A}$ .

3.2.1. As  $\mathbf{C}_*(\mathcal{A})$  is an additive monoidal category, the functor  $\sim$  defines a functor

$$\Delta^\circ \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_{**}(\mathcal{A}), \quad (1)$$

which, with the shuffle product, is monoidal. The composition of this functor with the total functor (which, as pointed out in example 2.2.1, is monoidal) is a monoidal functor

$$s_{EM} : \Delta^\circ \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A}),$$

which will be called the *Eilenberg-MacLane simple functor* (or the simple functor, for short).

3.2.2. If  $\mathcal{C}, \mathcal{D}$  are monoidal categories, there is an equivalence of categories  $\Delta^\circ \mathbf{Mon}(\mathcal{C}, \mathcal{D}) \cong \mathbf{Mon}(\mathcal{C}, \Delta^\circ \mathcal{D})$ , so if  $\mathcal{A}$  is an additive monoidal category, the composition of this isomorphism with the Eilenberg-MacLane simple functor is a functor, also denoted by  $s_{EM}$ ,

$$s_{EM} : \Delta^\circ \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \longrightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})),$$

which associates a monoidal functor to a simplicial monoidal functor between  $\mathcal{C}$  and  $\mathbf{C}_*(\mathcal{A})$ . We will also refer to this composition as the *Eilenberg-MacLane simple functor*.

3.2.3. If in 3.2.1 we use the comonoidal structure on  $\sim$  that comes from the Alexander-Whitney map, we obtain a comonoidal functor

$$s_{AW} : \Delta^\circ \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A}),$$

which we call the *Alexander-Whitney simple functor*. Thus, in a way completely analogous to 3.2.2, we obtain a simple functor for comonoidal functors

$$s_{AW} : \Delta^\circ \mathbf{CoMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \longrightarrow \mathbf{CoMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})).$$

3.2.4. By duality, there are similarly defined simple functors for the categories of monoidal and comonoidal functors with values in the cochain category  $\mathbf{C}^*(\mathcal{A})$ . In §8 we will use, in the contravariant setting, the Alexander-Whitney simple functor

$$s_{AW} : \Delta \mathbf{Mon}(\mathcal{C}^\circ, \mathbf{C}^*(\mathcal{A})) \longrightarrow \mathbf{Mon}(\mathcal{C}^\circ, \mathbf{C}^*(\mathcal{A})).$$

Although  $s_{AW}$  denotes the two functors introduced above, it will be clear from the context which we are referring to.

**3.3. Symmetric simple functor.** Let  $\mathcal{A}$  be an additive symmetric monoidal category. As the shuffle product is symmetric, ([EM2]), the Eilenberg-MacLane simple functor  $s_{EM} : \Delta^\circ \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A})$  is a symmetric monoidal functor. Thus, if  $\mathcal{C}$  is a symmetric monoidal category, we can follow the reasoning in 3.2.2 to deduce the existence of an *Eilenberg-MacLane symmetric simple functor*

$$s_{EM} : \Delta^\circ \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \longrightarrow \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})).$$

## 4. AN ACYCLIC MODELS THEOREM FOR MONOIDAL FUNCTORS

In this section we prove the main technical tool, the acyclic models theorem for monoidal functors. Our presentation is a variation of the scheme devised by Barr in [B96] for these kinds of results. According to Barr, three main ingredients are required to state an acyclic models theorem: the total complex associated to a double complex, the acyclic classes of complexes that define the associated classes of weak equivalences, and the cotriples constructed from the models. The total complex functor and its monoidal counterparts were the object of the previous section, we begin now by introducing acyclic classes.

**4.1. Acyclic classes and weak equivalences.** Let  $\mathcal{A}$  be an abelian category. If  $C_*$  is a chain complex on  $\mathcal{A}$ , we denote by  $C_*[-1]$  the chain complex given by  $(C_*[-1])_n = C_{n-1}$  with differential defined by  $-d_{n-1}$ .

4.1.1. Recall the following definition from [B96], (see also [B02]),

**Definition 4.1.1.** A class  $\Gamma$  of complexes in  $\mathbf{C}_*(\mathcal{A})$  is called an *acyclic class* if the following conditions are satisfied:

- (AC1) The complex 0 is in  $\Gamma$ .
- (AC2) Stability:  $C_*$  is in  $\Gamma$  if and only if  $C_*[-1]$  is in  $\Gamma$ .
- (AC3) Let  $C_*, D_*$  be two chain homotopic complexes. Then  $C_*$  is in  $\Gamma$  if and only if  $D_*$  is in  $\Gamma$ .
- (AC4) Every complex in  $\Gamma$  is acyclic.
- (AC5) If  $C_{**}$  is a double complex in  $\mathbf{C}_{**}(\mathcal{A})$  all of whose rows are in  $\Gamma$ , then the total complex  $\text{Tot } C_{**}$  is in  $\Gamma$ .

By (AC1) and (AC3) every contractible complex is in  $\Gamma$  and by (AC4) all complexes in  $\Gamma$  are acyclic, thus  $\Gamma$  is a class between the class of all contractible complexes and the class of all acyclic complexes. These two extreme cases are examples of acyclic classes, (see [B96], §4).

4.1.2. Given an acyclic class  $\Gamma$  in  $\mathbf{C}_*(\mathcal{A})$ , we denote by  $\Sigma$  the class of morphisms between chain complexes of  $\mathcal{A}$  whose mapping cone is in  $\Gamma$ . Morphisms in  $\Sigma$  are referred to as *weak equivalences* (with respect to  $\Gamma$ ).

By (AC1) and (AC3) every homotopy equivalence is in  $\Sigma$  and by (AC4) the mapping cone of a morphism  $f$  of  $\Sigma$  is acyclic, thus it follows from the exact sequence of the mapping cone that any such  $f$  is a quasi-isomorphism.

**Remark 4.1.2.** Let  $\mathbf{K}_*(\mathcal{A})$  be the category of chain complexes up to homotopy, that is, its objects are complexes of  $\mathcal{A}$  and its morphisms are homotopy classes of morphisms of complexes. It is a triangulated category. An acyclic class  $\Gamma$  determines a triangulated subcategory of  $\mathbf{K}_*(\mathcal{A})$  and, as a consequence, the class of morphisms  $\Sigma$  associated with  $\Gamma$  inherits some properties from the general setup of triangulated categories, (i.e. it admits a calculus of fractions). For the sake of simplicity we will follow Barr's treatment and refer to [B95] for the properties of  $\Sigma$  that will be used.



4.1.3. We use  $\mathbf{C}_*(\mathcal{A})[\Sigma^{-1}]$  to denote the localized category of  $\mathbf{C}_*(\mathcal{A})$  with respect to  $\Sigma$ , which exists in a suitable universe, (see [GZ]), and is uniquely determined up to isomorphism.

The class of morphisms  $\Sigma$  is stable under composition and satisfies the 2 out of 3 property (see the proof of proposition 3.3 in [B96]), that is, for every pair of morphisms  $f, g$  of  $\mathbf{C}_*(\mathcal{A})$  so that  $gf$  exists, if *two* of  $f, g$  and  $gf$  are in  $\Sigma$ , then so is the third. Moreover,  $\Sigma$  has a homotopy calculus of fractions (see loc. cit.). In many cases, such as when  $\Sigma$  is the class of quasi-isomorphisms or the class of homotopy equivalences,  $\Sigma$  is a saturated class of morphisms, in other words, it is precisely the class of morphisms of  $\mathbf{C}_*(\mathcal{A})$  which become isomorphisms in  $\mathbf{C}_*(\mathcal{A})[\Sigma^{-1}]$ .

Two objects  $C_*, D_*$  of  $\mathbf{C}_*(\mathcal{A})$  are said to be *weakly equivalent* (with respect to  $\Gamma$ ) if there exists a sequence of morphisms of  $\mathbf{C}_*(\mathcal{A})$ ,

$$C_* \longleftarrow C_*^1 \longrightarrow \dots \longleftarrow C_*^m \longrightarrow D_*,$$

which are weak equivalences. Hence, weakly equivalent objects are isomorphic in  $\mathbf{C}_*(\mathcal{A})[\Sigma^{-1}]$ .

4.1.4. Let  $\mathcal{C}$  be a category and  $\mathcal{A}$  an abelian category. The functor category  $\mathbf{Cat}(\mathcal{C}, \mathcal{A})$  is an abelian category. We fix a class  $\Sigma$  of weak equivalences in  $\mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  associated to an acyclic class  $\Gamma$  in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{C}, \mathcal{A})) = \mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ .

Let us now assume that  $\mathcal{C}$  is a monoidal category and  $\mathcal{A}$  is an abelian monoidal category. As  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  is a subcategory of  $\mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ ,  $\Sigma$  determines a class of morphisms in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , which will be represented by the same symbol. Thus, a morphism of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  is a weak equivalence if it is in  $\Sigma$  as a morphism of  $\mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ .

However, note that for two monoidal functors from  $\mathcal{C}$  to  $\mathbf{C}_*(\mathcal{A})$ , the weak equivalence relation in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  is not the same as it is in  $\mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , since in the first case the intermediate functors in the chain 4.1.3 have to be monoidal.

We use  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$  to denote the category obtained by inverting the weak equivalences in  $\Sigma$ , so the natural functor  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \longrightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$  transforms weak equivalences to isomorphisms.

4.1.5. We note that the class of weak equivalences in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  contains the homotopy equivalences and is compatible with the functor  $s_{EM}$ , so it defines a functor

$$s_{EM} : (\Delta^o \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})))[\widetilde{\Sigma}^{-1}] \longrightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}],$$

where  $\widetilde{\Sigma}$  is the class of morphisms in  $\Delta^o \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  which in each simplicial degree are in  $\Sigma$ , as follows from (AC5).

This compatibility is one of the basic properties of acyclic classes that is needed to prove the acyclic models theorem below. Instead of Barr's acyclic classes we could work in other settings where there is such a compatibility (see also 4.5).

**4.2. G-presentable objects and G-weak equivalences.** Let  $\mathcal{C}$  be a monoidal category,  $\mathcal{A}$  an abelian monoidal category,  $\Sigma$  a class of weak equivalences in  $\mathbf{Cat}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  which contains the homotopy equivalences and is compatible with  $s_{EM}$ .

4.2.1. *The standard construction.* Recall that if  $\mathcal{X}$  is a category, a cotriple (also called a *comonad*, see [ML])  $\mathbf{G} = (G, \varepsilon, \delta)$  in  $\mathcal{X}$  is given by a functor  $G : \mathcal{X} \rightarrow \mathcal{X}$  and natural transformations  $\varepsilon : G \Rightarrow \text{id}$  and  $\delta : G \Rightarrow G^2$ , which satisfies

$$\delta G \cdot \delta = G\delta \cdot \delta : G \Rightarrow G^3, \quad \varepsilon G \cdot \delta = \mathbf{1}_G = G\varepsilon \cdot \delta : G \Rightarrow G,$$

where the dot denotes the composition of natural transformations, (see [ML]). Given a cotriple  $\mathbf{G}$  in  $\mathcal{X}$ , every object  $X$  of  $\mathcal{X}$  has a functorial augmented simplicial object associated with it,  $B_\bullet(X)$ , which will be called the *standard construction* of  $\mathbf{G}$  applied to  $X$ , ([ML], chapter VII). In degree  $n$ ,  $n \geq 0$ , the simplicial object  $B_\bullet(X)$  is given by  $G^{n+1}(X)$  with face and degeneracy transformations given by

$$\begin{aligned} \partial_i &= G^i \varepsilon G^{n-i} : G^{n+1}(X) \rightarrow G^n(X), & 0 \leq i \leq n, \\ s_i &= G^i \delta G^{n-i} : G^{n+1}(X) \rightarrow G^{n+2}(X), & 0 \leq i \leq n. \end{aligned}$$

In this way we obtain a simplicial object with an augmentation defined by  $\varepsilon : B_0(X) = G(X) \rightarrow X$ .

By the functoriality of the standard construction we obtain an augmented simplicial functor  $B_\bullet$  in  $\mathbf{Cat}(\mathcal{X}, \mathcal{X})$ .

**Proposition 4.2.1.** (1) *The augmented simplicial functor  $G(\varepsilon) : G \circ B_\bullet \Rightarrow G$  is contractible.*  
(2) *The augmented simplicial functor  $G(\varepsilon) : B_\bullet \circ G \Rightarrow G$  is contractible.*

*Proof.* (1) The face and degeneracy morphisms of  $G \circ B_\bullet$  are given by

$$\begin{aligned} G(\partial_i^n) &= G^{i+1} \varepsilon G^{n-i} = \partial_{i+1}^{n+1}, & 0 \leq i \leq n, \\ G(s_i^n) &= G^{i+1} \delta G^{n-i} = s_{i+1}^{n+1}, & 0 \leq i \leq n, \end{aligned}$$

where the latter are the faces and degeneracies of  $B_\bullet$ . Now, the extra degeneracy from  $B_\bullet$ ,  $s = s_0^{n+1}$ , gives a contraction for  $G \circ B_\bullet$ .

For (2) use the last extra degeneracy  $s_{n+1}^{n+1}$  in  $B_n \circ G$ .  $\square$

4.2.2. *Monoidal structure in the standard construction.* Let  $\mathbf{G}$  be a cotriple in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ . We say that  $\mathbf{G}$  is *compatible* with  $\Sigma$  if  $G(\Sigma) \subseteq \Sigma$ . We will associate to  $\mathbf{G}$  a monoidal functor  $B_* : \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \rightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ .

The standard construction defines a functor

$$B_\bullet : \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \rightarrow \Delta^o \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})).$$

By composing  $B_\bullet$  with the Eilenberg-MacLane simple functor defined in 3.2.2,  $s_{EM}$ , we obtain a functor

$$B_* = s_{EM} B_\bullet : \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \rightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})).$$

The natural transformation  $\varepsilon$  gives a natural transformation  $\varepsilon : B_* \Rightarrow \text{id}$ .

4.2.3. **G-presentable objects.** Barr introduces in [B96] the  $\varepsilon$ -presentable objects. We will refer to these objects as **G**-presentable objects. Recall its definition:

**Definition 4.2.2.** We say that  $K_*$  of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  is **G-presentable** (with respect to  $\Gamma$ ) if the augmentation morphism  $\varepsilon : B_*(K_*) \longrightarrow K_*$  is a weak equivalence.

As remarked in 4.1.4,  $\Sigma$  contains the homotopy equivalences, so if  $K_*$  is an object of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  such that  $\varepsilon_{K_n}$  splits for all  $n$ , then  $\varepsilon_{K_n} \in \Sigma$ , and since  $\Sigma$  is compatible with  $s_{EM}$ , it follows that  $\varepsilon_K \in \Sigma$ . Hence we obtain the following result,

**Proposition 4.2.3.** *Let  $K_*$  be an object of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  such that  $\varepsilon_{K_n}$  splits for all  $n \in \mathbb{Z}$ , that is to say, for each  $n$  there is a natural transformation  $\theta_n : K_n \longrightarrow K_n G$  such that  $\varepsilon_{K_n} \theta_n = \text{id}$ . Then,  $K_*$  is **G-presentable**.*

**Example 4.2.4.** If  $K_*$  is an object of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , then  $G(K_*)$  is **G-presentable**, since for each  $n$  we can split  $G(K_n)$  by  $\delta$ .

4.2.4. **G-Weak equivalences.** Barr introduces in [B96] the **G**-acyclic objects. More generally we can speak of objects that are **G**-equivalent in the following sense,

**Definition 4.2.5.** Let  $f : K_* \longrightarrow L_*$  be a morphism of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ . We say that  $f$  is a **BG-weak equivalence**, (respectively, a **G-weak equivalence**), with respect to  $\Gamma$ , if  $B_*(f) \in \Sigma$ , (respectively, if  $G(f) \in \Sigma$ ).

**Proposition 4.2.6.** *If  $G(\Sigma) \subseteq \Sigma$ , then a morphism  $f : K_* \longrightarrow L_*$  of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  is a **G-weak equivalence** if, and only if, it is a **BG-weak equivalence**.*

*Proof.* If  $f$  is a **G-weak equivalence** then, for all  $n \geq 0$ ,  $G^{n+1}(f)$  is a weak equivalence, by hypothesis. Therefore, by (AC5),  $f$  is a **BG-weak equivalence**.

Reciprocally, if we assume that  $f$  is a **BG-weak equivalence**, then  $G(B(f)) : G(B_*(K_*)) \longrightarrow G(B_*(L_*))$  is a weak equivalence, by hypothesis. Consider the commutative diagram

$$\begin{array}{ccc} G(B_*(K_*)) & \xrightarrow{G(B(f))} & G(B_*(L_*)) \\ \downarrow G(\varepsilon) & & \downarrow G(\varepsilon) \\ G(K_*) & \xrightarrow{G(f)} & G(L_*) \end{array}$$

By proposition 4.2.1 (2), the two vertical morphisms are isomorphisms, so the result follows from the 3 out of 2 property of  $\Sigma$ .  $\square$

Now we recover Barr's definition:

**Definition 4.2.7.** An object  $K_*$  of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , in non-negative degrees, (that is, with  $K_p = 0$  if  $p < 0$ ), is said to be **G-acyclic** (with respect to  $\Gamma$ ) if the augmentation  $K_* \longrightarrow H_0(K_*)$  is a **G-weak equivalence**.

**4.3. The acyclic models theorem for monoidal functors.** The following result and its corollaries are a variation of the classical acyclic models theorem in the context of monoidal functors (see [EM] and also [BB], [B96]).

Let  $\mathcal{C}$  be a monoidal category,  $\mathcal{A}$  an abelian monoidal category, and  $\Sigma$  a class of weak equivalences in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  which contains the homotopy equivalences and is compatible with  $s_{EM}$ . Let  $\mathbf{G}$  be a cotriple in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ .

**Theorem 4.3.1.** *Let  $K_*$  be a  $\mathbf{G}$ -presentable object of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  and  $s : L_* \rightarrow M_*$  a  $\mathbf{G}$ -weak equivalence. Suppose that  $\mathbf{G}$  is compatible with  $\Sigma$ , (that is,  $G(\Sigma) \subseteq \Sigma$ ). Then, for any  $\alpha : K_* \rightarrow M_*$  there exists a unique morphism  $\tilde{\alpha} : K_* \rightarrow L_*$  in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$  such that  $\alpha = s\tilde{\alpha}$  in this localized category.*

*Proof.* By the naturality of  $\varepsilon$ , we have a commutative diagram

$$\begin{array}{ccccc} B_*(K_*) & \xrightarrow{B(\alpha)} & B_*(M_*) & \xleftarrow{B(s)} & B_*(L_*) \\ \downarrow \varepsilon_K & & \downarrow \varepsilon_M & & \downarrow \varepsilon_L \\ K_* & \xrightarrow{\alpha} & M_* & \xleftarrow{s} & L_* \end{array}$$

As  $K_*$  is  $\mathbf{G}$ -presentable,  $\varepsilon_K$  is a weak equivalence. Moreover, by proposition 4.2.6,  $B(s)$  is also a weak equivalence. So they are isomorphisms in the localized category  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$ .

In this category we define the morphism

$$\tilde{\alpha} = \varepsilon_L B(s)^{-1} B(\alpha) (\varepsilon_K)^{-1}.$$

We have  $s\tilde{\alpha} = \alpha$ , since by the commutativity of the diagram above we have  $\alpha = \varepsilon_M B(\alpha) (\varepsilon_K)^{-1}$  and  $\varepsilon_M = s\varepsilon_L B(s)^{-1}$ , so it follows that

$$\alpha = \varepsilon_M B(\alpha) (\varepsilon_K)^{-1} = s\varepsilon_L B(s)^{-1} B(\alpha) (\varepsilon_K)^{-1} = s\tilde{\alpha}.$$

With respect to uniqueness, assume that  $\gamma : K_* \rightarrow L_*$  is another lifting of  $\alpha$  in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$ , so that  $\alpha = s\gamma$ . Since the standard construction is functorial and compatible with weak equivalences because  $G(\Sigma) \subseteq \Sigma$ , we have  $B(s)B(\tilde{\alpha}) = B(\alpha) = B(s)B(\gamma)$ , but  $B(s) \in \Sigma$  so  $B(\gamma) = B(\tilde{\alpha})$ . Moreover, as  $\varepsilon$  is a natural transformation, we have  $\varepsilon_L B(\gamma) = \gamma\varepsilon_K$  and  $\varepsilon_L B(\tilde{\alpha}) = \tilde{\alpha}\varepsilon_K$ , and since  $\varepsilon_K \in \Sigma$  we deduce that  $\gamma = \tilde{\alpha}$ . This ends the proof of the theorem.  $\square$

**Corollary 4.3.2.** *Let  $K_*, L_*$  be objects of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  in non-negative degrees. Suppose that  $K_*$  is  $\mathbf{G}$ -presentable and  $L_*$  is  $\mathbf{G}$ -acyclic. Then, any monoidal natural transformation  $H_0(K_*) \rightarrow H_0(L_*)$  has a unique extension to a morphism  $K_* \rightarrow L_*$  in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$ .*

*Proof.* It follows from the previous theorem applied to the diagram

$$\begin{array}{ccccc} & & & & L_* \\ & & & \nearrow \text{dotted} & \downarrow \\ K_* & \xrightarrow{\quad} & H_0(K_*) & \xrightarrow{\quad} & H_0(L_*) \end{array}$$

□

**Corollary 4.3.3.** *Let  $K_*, L_*$  be objects of  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  in non-negative degrees. Suppose that  $K_*, L_*$  are  $\mathbf{G}$ -acyclic and  $\mathbf{G}$ -presentable. Then any monoidal natural isomorphism  $H_0(K_*) \longrightarrow H_0(L_*)$  lifts to a unique isomorphism  $f_* : K_* \longrightarrow L_*$  in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$ .*

**4.4. Weak homotopy type.** In the following sections we will apply the general results above to some chain-valued functors defined in the category of topological spaces or in the category of simplicial sets. In all cases, the acyclic classes in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{C}, \mathcal{A}))$  will come from acyclic classes in  $\mathbf{C}_*(\mathcal{A})$ , and the cotriples in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{C}, \mathcal{A}))$  will come from monoidal cotriples in  $\mathcal{C}$ . Let us describe the main features of this situation.

4.4.1. Let  $\Gamma$  be an acyclic class in  $\mathbf{C}_*(\mathcal{A})$  and  $\Sigma$  the associated class of weak equivalences. We extend  $\Sigma$  to a class of weak equivalences  $\tilde{\Sigma}$  in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{X}, \mathcal{A}))$  componentwise: a morphism  $f : K_* \longrightarrow L_*$  is in  $\tilde{\Sigma}$  if, and only if, for all objects  $X$  of  $\mathcal{C}$ ,  $f_X : K_*(X) \longrightarrow L_*(X)$  is in  $\Sigma$ .

If we take the contractible complexes in  $\mathbf{C}_*(\mathcal{A})$  as the acyclic class  $\Gamma$ , we will say that  $\tilde{\Gamma}$  is the acyclic class of *weakly contractible* functors in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{X}, \mathcal{A}))$ . The morphisms in  $\tilde{\Sigma}$  will be called *weak homotopy equivalences*. If  $\Gamma$  is the class of acyclic complexes, we will say that  $\Sigma$  is the class of *weak quasi-isomorphisms*.

4.4.2. *Monoidal cotriples.* Given a monoidal category  $\mathcal{C}$ , a *monoidal cotriple* in  $\mathcal{C}$  is a cotriple  $\mathbf{G} = (G, \varepsilon, \delta)$ , such that  $G : \mathcal{C} \longrightarrow \mathcal{C}$  is a monoidal functor and  $\varepsilon : G \Rightarrow \text{id}_{\mathcal{C}}$  and  $\delta : G \Rightarrow G^2$  are monoidal natural transformations.

If  $\mathbf{G} = (G, \varepsilon, \delta)$  is a monoidal cotriple in  $\mathcal{C}$ , define a functor

$$\tilde{G} : \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \longrightarrow \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})),$$

by composition  $\tilde{G}(F_*) = F_*G$ , and let  $\tilde{\varepsilon} : \tilde{G} \Rightarrow \text{id}$  and  $\tilde{\delta} : \tilde{G} \Rightarrow \tilde{G}^2$  be the natural transformations induced by  $\varepsilon, \delta$ . It follows easily from the definitions that  $\tilde{\mathbf{G}} = (\tilde{G}, \tilde{\varepsilon}, \tilde{\delta})$  is a cotriple in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ .

**Proposition 4.4.1.** *Let  $\Gamma$  be an acyclic class in  $\mathbf{C}_*(\mathcal{A})$  and  $\mathbf{G}$  a monoidal cotriple in  $\mathcal{C}$ . Consider the class of morphisms  $\tilde{\Sigma}$  and the cotriple  $\tilde{\mathbf{G}}$  induced in  $\mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ . Then,  $\tilde{\mathbf{G}}$  is compatible with  $\tilde{\Sigma}$ , that is,  $\tilde{G}(\tilde{\Sigma}) \subseteq \tilde{\Sigma}$ .*

*Proof.* If  $f : K_* \longrightarrow L_*$  is in  $\tilde{\Sigma}$ , then  $f_X : K_*(X) \longrightarrow L_*(X)$ , for all objects  $X$  of  $\mathcal{C}$ . In particular,  $f_{G(X)} \in \Sigma$ , thus  $\tilde{G}(f) \in \tilde{\Sigma}$ . □

4.5. A detailed analysis of the proof of the acyclic models theorem shows that, in fact, the main ingredients of our proof are a monoidal functor  $s_{EM} : \Delta^o \mathbf{C}_*(\mathcal{A}) \longrightarrow \mathbf{C}_*(\mathcal{A})$  and a saturated class of morphisms  $\Sigma$  of  $\mathbf{C}_*(\mathcal{A})$ , such that

- (1)  $\Sigma$  contains the homotopy equivalences.
- (2) If  $f_{**} : C_{**} \longrightarrow D_{**}$  is a morphism of double complexes such that  $f_{n*}$  is in  $\Sigma$ , then  $s_{EM}(f_{**})$  is also in  $\Sigma$ .

We will analyze this more general situation elsewhere.

## 5. APPLICATION: COMPARISON OF SINGULAR AND CUBICAL CHAINS

In this section we compare the singular and cubical chains of a topological space in the monoidal setting by applying the results above, extending the well-known classic comparison theorem, [EM].

5.1. Let  $\mathbf{Top}$  denote the category of topological spaces, which is a monoidal category under cartesian product, and let  $\mathbf{C}_*(\mathbb{Z})$  be the category of chain complexes of abelian groups. We fix the class  $\Sigma$  of weak homotopy equivalences in the functor category  $\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_*(\mathbb{Z}))$ , 4.4.1.

It is well known, [EM], that the functor of *singular chains*

$$S_* : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z}),$$

together with the shuffle product as the Künneth morphism is a monoidal functor.

We can also consider the functor of *cubical chains*,

$$C_* : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z}).$$

For a topological space  $X$ ,  $C_p(X)$  is generated by the continuous maps  $I^p \longrightarrow X$ , where  $I$  is the unit interval of the real line, modulo the degenerated ones, (see e.g. [Mas]). Together with the cross product

$$\times : C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \times Y),$$

which for singular cubes  $c : I^p \longrightarrow X$  and  $d : I^q \longrightarrow Y$  is defined as the cartesian product

$$c \times d : I^{p+q} = I^p \times I^q \longrightarrow X \times Y,$$

it is a monoidal functor.

5.2. **A monoidal cotriple in  $\mathbf{Top}$ .** In order to compare  $S_*$  and  $C_*$  as monoidal functors following corollary 4.3.3, we introduce a monoidal cotriple in  $\mathbf{Top}$ . It is a model induced cotriple, (see [B02], §4.2).

In the classical version of the acyclic models theorem the models are the standard simplexes  $\Delta^m$ ,  $m \geq 0$ . To obtain a monoidal version we take products of these spaces.

Therefore, for any sequence  $\underline{n} = (n_1, \dots, n_r)$ ,  $n_i \geq 0$ , take  $\Delta^{\underline{n}} = \Delta^{n_1} \times \dots \times \Delta^{n_r}$ . We define  $\mathbf{G} = (G, \varepsilon, \delta)$  as the model induced cotriple with models  $\Delta^{\underline{n}}$ . That is, the functor  $G : \mathbf{Top} \longrightarrow \mathbf{Top}$  takes a topological space  $X$  to

$$G(X) = \bigsqcup_{\alpha: \Delta^{\underline{n}} \rightarrow X} (\Delta^{\underline{n}}, \alpha),$$

where  $(\Delta^{\underline{n}}, \alpha)$  is a copy of  $\Delta^{\underline{n}}$  indexed by the continuous map  $\alpha : \Delta^{\underline{n}} \rightarrow X$  and where the disjoint union is over all sequences  $\underline{n}$  and maps  $\alpha$ .  $G$  takes a continuous map  $f : X \longrightarrow Y$ , to the map

$$G(f) : \bigsqcup_{\alpha: \Delta^{\underline{n}} \rightarrow X} (\Delta^{\underline{n}}, \alpha) \longrightarrow \bigsqcup_{\beta: \Delta^{\underline{n}} \rightarrow Y} (\Delta^{\underline{n}}, \beta),$$

which is the identity from  $(\Delta^n, \alpha)$  to  $(\Delta^n, f \circ \alpha)$ .

For a topological space  $X$  define the map

$$\varepsilon_X : G(X) = \bigsqcup_{\alpha: \Delta^n \rightarrow X} (\Delta^n, \alpha) \longrightarrow X,$$

which over  $(\Delta^n, \alpha)$  is  $\alpha : \Delta^n \longrightarrow X$ , so we have a natural transformation  $\varepsilon : G \Rightarrow \text{id}$ . Finally we define a natural transformation  $\delta : G \Rightarrow G^2$ : the iteration of  $G$  gives the functor

$$G^2(X) = \bigsqcup_{\Delta^m \xrightarrow{\beta} \Delta^n \xrightarrow{\alpha} X} ((\Delta^m, \alpha), \beta),$$

so we can define  $\delta : G(X) \longrightarrow G^2(X)$  as the identity from  $(\Delta^n, \alpha)$  to  $((\Delta^n, \alpha), \text{id})$ .

Moreover, this cotriple has a monoidal structure. To define the Künneth morphisms of  $G$  consider topological spaces  $X, Y$  and take the map

$$\kappa_{X,Y} : G(X) \times G(Y) = \left( \bigsqcup_{\alpha: \Delta^n \rightarrow X} (\Delta^n, \alpha) \right) \times \left( \bigsqcup_{\beta: \Delta^m \rightarrow Y} (\Delta^m, \beta) \right) \longrightarrow G(X \times Y) = \bigsqcup_{\gamma: \Delta^r \rightarrow X \times Y} (\Delta^r, \gamma),$$

given by

$$(\Delta^n, \alpha) \times (\Delta^m, \beta) \xrightarrow{\text{id}} (\Delta^n \times \Delta^m, \alpha \times \beta).$$

It is straightforward to prove that  $\mathbf{G}$  is a monoidal cotriple in  $\mathbf{Top}$ .

5.3. As in previous sections, we denote also by  $\mathbf{G} = (G, \varepsilon, \delta)$  the cotriple induced in  $\mathbf{Mon}(\mathbf{Top}, \mathbf{C}_*(\mathbb{Z}))$  by the monoidal cotriple in  $\mathbf{Top}$  defined above.

**Theorem 5.3.1.** *The singular and cubical chain monoidal functors  $S_*, C_* : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z})$  are weakly homotopy equivalent monoidal functors, that is, they are weakly equivalent in  $\mathbf{Mon}(\mathbf{Top}, \mathbf{C}_*(\mathbb{Z}))$ .*

*Proof.* Both  $H_0 S_*$  and  $H_0 C_*$  are the functors that associate to a topological space  $X$  the free group generated by the points of  $X$ , so the result will follow from corollary 4.3.3 after we prove that  $S_*, C_*$  are  $\mathbf{G}$ -presentable and  $\mathbf{G}$ -acyclic as objects of  $\mathbf{Mon}(\mathbf{Top}, \mathbf{C}_*(\mathbb{Z}))$ .

With respect to  $\mathbf{G}$ -acyclicity for  $S_*$  we have to prove that  $G(S_*) \longrightarrow G(H_0)$  is a weak equivalence. That is to say, that the morphisms

$$\bigoplus_{\alpha: \Delta^n \rightarrow X} S_*(\Delta^n) \longrightarrow \bigoplus_{\alpha: \Delta^n \rightarrow X} \mathbb{Z},$$

where in each summand the morphism  $S_*(\Delta^n) \longrightarrow \mathbb{Z}$  is the natural augmentation, are homotopy equivalences, for all  $X$ . As  $\Delta^n$  are contractible spaces, the result is clear. Similarly, we can prove the  $\mathbf{G}$ -acyclicity of  $C_*$ .

To prove  $\mathbf{G}$ -presentability it is sufficient, by proposition 4.2.3, to prove that  $\varepsilon_{C_n}$  and  $\varepsilon_{S_n}$  split, for all  $n \geq 0$ . We can define a natural transformation

$$\theta_S : S_n \longrightarrow G(S_n),$$

by sending a singular simplex  $\sigma : \Delta^n \longrightarrow X$  on the topological space  $X$  to the simplex of  $G(X)$  given by  $\text{id} : \Delta^n \longrightarrow (\Delta^n, \sigma)$ . It follows from the definition that  $\varepsilon \circ \theta_S = \text{id}$ , so this morphism splits the standard resolution  $B_*(S_*)$ .

For the cubical chains, observe that  $I^n$  is homeomorphic to  $\Delta^1 \times \dots \times \Delta^1$ , so that a  $n$ -cube on a space  $X$  is given by a map  $\Delta^1 \times \dots \times \Delta^1 \rightarrow X$ . We can now follow the definition of  $\theta_S$  to define a natural transformation

$$\theta_C : C_n \rightarrow G(C_n),$$

by sending a cubical simplex  $c : \Delta^1 \times \dots \times \Delta^1 \rightarrow X$  to the cubical simplex of  $G(X)$  given by  $\text{id} : \Delta^1 \times \dots \times \Delta^1 \rightarrow (\Delta^1 \times \dots \times \Delta^1 \rightarrow X, c)$ , which splits  $\varepsilon_{C_n}$ .  $\square$

## 6. SYMMETRIC MONOIDAL FUNCTORS

In this section we indicate how to extend the results of the previous sections to symmetric monoidal functors. We will focus our attention on the symmetric version of corollary 4.3.3, which we will apply to obtain a comparison result in the symmetric setting for the singular and ordered cubical functors acting on topological spaces.

**6.1. Acyclic models for symmetric monoidal functors.** Let  $\mathcal{C}$  be a symmetric monoidal category,  $\mathcal{A}$  a symmetric abelian monoidal category,  $\Gamma$  an acyclic class in  $\mathbf{C}_*(\mathbf{Cat}(\mathcal{C}, \mathcal{A}))$  and  $\Sigma$  the class of weak equivalences.  $\Sigma$  determines a class of morphisms in  $\mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , which will be represented by the same symbol. Let  $\mathbf{G}$  be a cotriple in  $\mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  compatible with  $\Sigma$ , i.e.,  $G(\Sigma) \subseteq \Sigma$ .

Following the general procedure in 4.2.1, given  $\mathbf{G}$  in  $\mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$ , we can associate a functor

$$B_\bullet : \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \rightarrow \Delta^o \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})),$$

in such a way that the simplicial object  $B_\bullet(K_*)$  is augmented to  $K_*$  by  $\varepsilon$ . By composing this functor with the symmetric simple functor, 3.3, we obtain a functor

$$B_* : \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})) \rightarrow \mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A})).$$

The natural transformation  $\varepsilon$  gives a natural transformation  $B_* \Rightarrow \text{id}$ .

We can now reproduce the definitions and results from section §4 in the symmetric setting. In particular, we have the following result, which is analogous to corollary 4.3.3.

**Theorem 6.1.1.** *Let  $K_*, L_*$  be objects of  $\mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))$  in non-negative degrees. Suppose that  $K_*, L_*$  are  $\mathbf{G}$ -acyclic and  $\mathbf{G}$ -presentable. Then any symmetric monoidal natural isomorphism  $H_0(K_*) \rightarrow H_0(L_*)$  lifts to a unique isomorphism  $f_* : K_* \rightarrow L_*$  in  $\mathbf{SyMon}(\mathcal{C}, \mathbf{C}_*(\mathcal{A}))[\Sigma^{-1}]$ .*

A symmetric monoidal cotriple  $\mathbf{G} = (G, \varepsilon, \delta)$  on a symmetric monoidal category  $\mathcal{C}$  is a monoidal cotriple such that  $G$  is a symmetric monoidal functor and  $\varepsilon, \delta$  are monoidal transformations. As in 4.4.2, if  $\mathcal{D}$  is another symmetric monoidal category, a symmetric monoidal cotriple on  $\mathcal{C}$  induces a cotriple on  $\mathbf{SyMon}(\mathcal{C}, \mathcal{D})$ , that will also be denoted by  $\mathbf{G}$ , which satisfies  $G(\Sigma) \subseteq \Sigma$  (see proposition 4.4.1).



**6.2. The Kleisli cotriple.** The cotriple in **Top** defined in 5.2 is not symmetric. Therefore in order to apply theorem 6.1.1 with  $\mathcal{C} = \mathbf{Top}$  we must first define a suitable symmetric monoidal cotriple on **Top**. The following cotriple, introduced by Kleisli in [K], will turn out to be symmetric and monoidal: for a topological space  $X$  and an element  $x \in X$ , let  $P(X, x)$  denote the space of pointed continuous paths  $\alpha : (I, 0) \rightarrow (X, x)$ , which is topologized by the compact-open topology. Then, define  $G$  on  $X$  by

$$G(X) = \bigsqcup_{x \in X} P(X, x),$$

so  $G(X)$  is the set of paths in  $X$ , but with a topology that is not the path space topology. Nevertheless, note that if  $Z$  is a connected space, a map from  $Z$  to  $G(X)$  is equivalent to a map  $\beta : Z \times I \rightarrow X$  such that  $\beta(z, 0) = x$ , for some  $x \in X$  and all  $z \in Z$ . The action of  $G$  on a continuous map  $f : X \rightarrow Y$  is given by composition, that is, if  $\alpha \in P(X, x)$ , then  $G(f)(\alpha) = f \circ \alpha$ .

Define a natural transformation  $\varepsilon : G \Rightarrow \text{id}$  by evaluating paths at 1. Finally, note that the iteration  $G^2$  over a space  $X$  is given by

$$G^2(X) = \bigsqcup_{x \in X} \bigsqcup_{\alpha \in P(X, x)} P(P(X, x), \alpha).$$

An element of  $P(P(X, x), \alpha)$  is determined by a map  $\tilde{\alpha} : I^2 \rightarrow X$  with  $\tilde{\alpha}(t, 0) = \alpha(t)$ , for all  $t \in I$ , and  $\tilde{\alpha}(0, s) = x$ , for all  $s \in I$ . We define the natural transformation  $\delta : G \Rightarrow G^2$  by maps  $P(X, x) \rightarrow P(P(X, x), x)$  that send  $\alpha$  to the map  $\tilde{\alpha}(t, s) = \alpha(ts)$ .

The cotriple  $\mathbf{G} = (G, \varepsilon, \delta)$  will be called the Kleisli cotriple. It is a symmetric monoidal: for topological spaces  $X, Y$  define the Künneth morphism

$$\kappa_{X, Y} : G(X) \times G(Y) = \bigsqcup_{x \in X} P(X, x) \times \bigsqcup_{y \in Y} P(Y, y) \rightarrow G(X \times Y) = \bigsqcup_{(x, y) \in X \times Y} P(X \times Y, (x, y)),$$

by sending the paths  $\alpha \in P(X, x)$  and  $\beta \in P(Y, y)$  to the path  $\alpha \times \beta \in P(X \times Y, (x, y))$ . It is evident that these morphisms are compatible with the symmetric structure of **Top**.

**6.3. Ordered cubical chains.** The cubical chain functor is not symmetric. However, the ordered cubical chains define a symmetric monoidal functor that may be compared directly with  $S_*$ .

6.3.1. The ordered cubical chains of a topological space  $X$  are defined as follows (see [K]). If  $c : I^n \rightarrow X$  is a singular  $n$ -cube and  $\pi \in \Sigma_n$  define the chain  $\pi c$  as

$$(\pi c)(t_1, \dots, t_n) = \varepsilon(\pi) c(t_{\pi(0)}, \dots, t_{\pi(n)}),$$

and extend this action to  $C_n(X)$  linearly. Let  $D_n(X)$  be the subgroup of  $C_n(X)$  generated by chains of the form  $c - \pi c$ . Then  $D_*(X)$  is a subcomplex of  $C_*(X)$ , so we can define the *ordered cubical chains* of  $X$  as the quotient complex

$$C_*^{ord}(X) = C_*(X)/D_*(X).$$

There is a natural transformation  $C_* \Rightarrow C_*^{ord}$ .

6.3.2. The monoidal structure of the cubical chain functor  $C_*$  with the usual cross product is carried over the quotient by  $D_*$ . In fact, if  $X, Y$  are topological spaces, the cross product maps  $C_*(X) \otimes D_*(Y)$  to  $D_*(X \times Y)$  since if  $c \in C_p(X), d \in C_q(Y)$  are singular cubes and  $\pi \in \Sigma_p$ , then

$$(c - \pi c) \times d = (c \times d) - \tilde{\pi}(c \times d),$$

where  $\tilde{\pi}$  is the element of  $\Sigma_{p+q}$  that acts as  $\pi$  on the first  $p$  elements and fixes the rest. Analogously,  $\times$  maps  $D_*(X) \otimes C_*(Y)$  to  $D_*(X \times Y)$ .

**Proposition 6.3.1.** *The functor  $C_*^{ord} : \mathbf{Top} \rightarrow \mathbf{C}_*(\mathbb{Z})$  is a symmetric monoidal functor and the natural transformation  $C_* \Rightarrow C_*^{ord}$  is a monoidal natural transformation.*

*Proof.* The proof follows immediately from the definitions. Note that the cross product in  $C_*^{ord}$  is symmetric, since if  $X, Y$  are spaces and  $c \otimes d \in C_p^{ord}(X) \otimes C_q^{ord}(X)$ , then

$$\begin{aligned} \tau(c \otimes d)(t_1, \dots, t_{p+q}) &= (-1)^{pq}(d \otimes c)(t_1, \dots, t_{p+q}) \\ &= (d \otimes c)(t_q, \dots, t_{p+q}, t_1, \dots, t_p) \\ &= \tau_{X,Y}(c \times d)(t_1, \dots, t_{p+q}), \end{aligned}$$

where in the second equality we used the invariance of the oriented cubical chains by the action of the symmetric group.  $\square$

6.3.3. We fix in  $\mathbf{Symon}(\mathbf{Top}, \mathbf{C}_*(\mathbb{Z}))$  the class  $\Sigma$  of weak homotopy equivalences, that is, the symmetric monoidal functors  $f$  such that  $f_X$  is a homotopy equivalence for each space  $X$ . Let  $\mathbf{G}$  be the cotriple induced by the Kleisli cotriple, which satisfies  $G(\Sigma) \subseteq \Sigma$ , see 4.4.1.

**Theorem 6.3.2.** *The singular and ordered cubical chain functors  $S_*, C_*^{ord} : \mathbf{Top} \rightarrow \mathbf{C}_*(\mathbb{Z})$  are weakly homotopy equivalent symmetric monoidal functors.*

*Proof.* For a topological space  $X$ , both groups  $H_0(S_*(X))$  and  $H_0(C_*^{ord}(X))$  are isomorphic to the free group generated by the points of  $X$ , so the result will follow from theorem 6.1.1 after we prove that  $S_*$  and  $C_*^{ord}$  are  $\mathbf{G}$ -presentable and  $\mathbf{G}$ -acyclic with respect to the Kleisli cotriple. This has been proved by Barr (see [B02]) for the singular chains functor. Let us prove it for the ordered cubical chains.

Firstly we prove the  $\mathbf{G}$ -acyclicity of  $C_*^{ord}$ . We will prove that there is a chain contraction  $s$  for the complex  $C_*G \rightarrow H_0G$  such that for any singular  $n$ -cube  $c$  and any  $\sigma \in \Sigma_n$ ,  $s(\sigma c) = \sigma s(c)$ .

Let  $X$  be any topological space and  $c : I^n \rightarrow G(X)$  a singular  $n$ -cube. By the connectedness of the standard cube, there is a point  $x \in X$  such that the map  $c$  factors through a map  $c : I^n \rightarrow P(X, x)$ . By adjunction,  $c$  is equivalent to a map  $\tilde{c} : I^{n+1} \rightarrow X$ , which satisfies  $\tilde{c}(t_1, \dots, t_n, 0) = x$ , for  $(t_1, \dots, t_n) \in I^n$ . Taking into account the product decomposition  $I^{n+1} = I^n \times I$ , we will write the value of  $\tilde{c}$  at the point  $(t_1, \dots, t_n) \in I^n$  and  $u \in I$  by  $\tilde{c}(t_1, \dots, t_n; u)$ .

Note that  $H_0(G(X))$  is the free group generated by the elements of  $X$ , since the spaces  $P(X, x)$  are contractible, and define  $s : H_0(G(X)) \rightarrow C_0(G(X))$  by  $s(x) = p_x$ , where  $p_x$  denotes the constant path at  $x$ .

For  $n \geq 0$  define  $\mu : I^{n+1} \times I \longrightarrow I^n \times I$  by

$$\mu(t_1, \dots, t_{n+1}; u) = (t_1, \dots, t_n; t_{n+1}u)$$

and define  $s : C_n(G(X)) \longrightarrow C_{n+1}(G(X))$  by

$$s(\tilde{c}) = (-1)^{n+1} \tilde{c} \circ \mu.$$

Next, we recall the definition of the differential  $d : C_n(X) \longrightarrow C_{n-1}(X)$  of the cubical chain complex: for  $1 \leq i \leq n$ ,  $\epsilon \in \{0, 1\}$ , let  $\delta_i^\epsilon : I^{n-1} \longrightarrow I^n$  denote the face defined by  $\delta_i^\epsilon(t_1, \dots, t_{n-1}) = (t_1, \dots, \epsilon, \dots, t_{n-1})$ , where  $\epsilon$  is in the  $i$ -th place; if  $c \in C_n(X)$ ,  $d(c)$  is defined by

$$d(c) = \sum_{i, \epsilon} (-1)^{i+\epsilon} c \circ \delta_i^\epsilon.$$

It is clear that on  $C_{n+1}$  we have  $\delta_i^\epsilon \circ \mu = \mu \circ \delta_i^\epsilon$ , for  $1 \leq i \leq n$ , and that

$$\begin{aligned} (\mu \circ \delta_{n+1}^0)(t_1, \dots, t_n; u) &= (t_1, \dots, t_n, 0), \\ (\mu \circ \delta_{n+1}^1)(t_1, \dots, t_n; u) &= (t_1, \dots, t_n; u), \end{aligned}$$

thus, an easy calculation proves that

$$(ds + sd)(\tilde{c}) = \tilde{c} - x,$$

but, as  $x$  represents the constant path  $p_x$ , which is degenerated, it is zero on  $C_n(X)$ , so we have proved the equality  $ds + sd = \text{id}$  in  $C_*(X)$ .

The contraction  $s$  is trivially compatible with the action of the symmetric group, so it defines a contraction for  $C_*^{\text{ord}}$ , and as a consequence  $C_*^{\text{ord}}$  is  $\mathbf{G}$ -acyclic.

To prove  $\mathbf{G}$ -presentability, notice that the natural transformation  $\theta_n : C_n \longrightarrow C_n G$  given by

$$\theta_n(\sigma)(t_1, \dots, t_n)(s) = c(st_1, \dots, st_n),$$

is a section of  $\varepsilon$  and that it is compatible with the action of the symmetric group  $\Sigma_n$ , so we can apply proposition 4.2.3.  $\square$

**Remark 6.3.3.** Note that if  $S_*^N : \mathbf{Top} \longrightarrow \mathbf{C}_*(\mathbb{Z})$  denotes the functor of normalized singular chains, which is also a symmetric monoidal functor since the degenerated singular chains are invariant by shuffle product, (see [EM2]), then the projection  $S_* \Rightarrow S_*^N$  is a monoidal natural transformation of symmetric monoidal functors which is a weak equivalence. This is a classical result and follows directly from the fact that the degenerated singular chains are a direct factor of singular chains.

## 7. APPLICATION TO OPERADS

**7.1. Operads.** Let us recall some definitions and notations about operads (see [MSS]).

**7.1.1.** Let  $\Sigma$  be the *symmetric groupoid*, that is, the category whose objects are the sets  $\{1, \dots, n\}$ ,  $n \geq 1$ , and the only morphisms are those of the symmetric groups  $\Sigma_n$ .

7.1.2. Let  $\mathcal{C}$  be a monoidal category. The category of contravariant functors from  $\Sigma$  to  $\mathcal{C}$  is called the category of  $\Sigma$ -modules and is represented by  $\Sigma\mathbf{Mod}_{\mathcal{C}}$ . We identify its objects with sequences of objects in  $\mathcal{C}$ ,  $E = ((E(l))_{l \geq 1})$ , with a right  $\Sigma_l$ -action on each  $E(l)$ . If  $E$  and  $F$  are  $\Sigma$ -modules, a *morphism of  $\Sigma$ -modules*  $f : E \rightarrow F$  is a sequence of  $\Sigma_l$ -equivariant morphisms  $f(l) : E(l) \rightarrow F(l)$ ,  $l \geq 1$ .

7.1.3. A *unital  $\Sigma$ -operad* (an *operad* for short) in  $\mathcal{C}$  is a  $\Sigma$ -module  $P$  together with a family of *composition morphisms*

$$\gamma_{l; m_1, \dots, m_l} : P(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l) \rightarrow P(m_1 + \cdots + m_l),$$

and a *unit morphism*

$$\eta : \mathbf{1} \rightarrow P(1),$$

which satisfies the axioms of equivariance, associativity and unit. A *morphism of operads* is a morphism of  $\Sigma$ -modules that is compatible with structure morphisms. We use  $\mathbf{Op}_{\mathcal{C}}$  to denote the category of operads in  $\mathcal{C}$  and its morphisms.

An operad in  $\mathbf{Top}$  is called a *topological operad*. If  $R$  is a ring, an operad in  $\mathbf{C}_*(R)$  is called a *dg operad*. We are especially interested in the cases  $R = \mathbb{Z}, \mathbb{Q}$ .

7.2. We can now extend the comparison result between  $S_*$  and  $C_*^{ord}$  to topological operads.

7.2.1. If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a symmetric monoidal functor between monoidal categories, it is easy to prove that, applied componentwise,  $F$  induces a functor between the corresponding categories of operads

$$\mathbf{Op}_F : \mathbf{Op}_{\mathcal{C}} \rightarrow \mathbf{Op}_{\mathcal{D}},$$

also denoted by  $F$ . Therefore, singular and cubical ordered chains induce functors

$$S_*, C_*^{ord} : \mathbf{Op}_{\mathbf{Top}} \rightarrow \mathbf{Op}_{\mathbf{C}_*(\mathbb{Z})}.$$

Moreover, if  $\mathcal{D}$  has a notion of weak equivalence, it extends to  $\mathbf{Op}_{\mathcal{D}}$  componentwise. For instance, we can consider the weak equivalence relation in the category of chain complexes  $\mathbf{C}_*(R)$  induced by quasi-isomorphisms. Now it easily follows from the definitions that

**Proposition 7.2.1.** *Let  $\mathcal{C}$  be a monoidal category,  $R$  a ring and  $F, G \in \mathbf{Mon}(\mathcal{C}, \mathbf{C}_*(R))$ . If  $F, G$  are weakly equivalent (with respect to quasi-isomorphism in  $\mathbf{C}_*(R)$ ), then the functors*

$$\mathbf{Op}_F, \mathbf{Op}_G : \mathbf{Op}_{\mathcal{C}} \rightarrow \mathbf{Op}_{\mathbf{C}_*(R)}$$

*are weakly equivalent.*

This proposition together with theorem 6.3.2 applied to  $S_*, C_*^{ord}$  gives

**Theorem 7.2.2.** *The functors*

$$S_*, C_*^{ord} : \mathbf{Op}_{\mathbf{Top}} \rightarrow \mathbf{Op}_{\mathbf{C}_*(\mathbb{Z})}$$

*are weakly equivalent (with respect to quasi-isomorphism). In particular, for a topological operad  $P$  the dg operads  $S_*(P), C_*^{ord}(P)$  are weakly equivalent.*

If  $\mathrm{HoOp}_{\mathbf{C}_*(\mathbb{Z})}$  denotes the localization of  $\mathbf{Op}_{\mathbf{C}_*(\mathbb{Z})}$  with respect to quasi-isomorphisms, we obtain

**Corollary 7.2.3.** *There is an isomorphism of functors*

$$S_* \cong C_*^{ord} : \mathbf{Op}_{\mathbf{Top}} \longrightarrow \mathrm{HoOp}_{\mathbf{C}_*(\mathbb{Z})}.$$

7.2.2. Theorem 7.2.2 may be applied to compare the categories of  $S_*(P)$  and  $C_*^{ord}(P)$  algebras up to homotopy.

Recall that, given a dg operad  $P$ , a  $P$ -algebra is a chain complex  $V$  of finite type together with a morphism  $P \rightarrow \mathcal{E}[V]$ , where  $\mathcal{E}[V]$  denotes the operad of endomorphisms of  $V$ . Following [GNPR], definition 7.3.1, define a  $P$ -algebra up to homotopy as a finite type complex  $V$  together with a morphism  $P \rightarrow \mathcal{E}[V]$  in  $\mathrm{HoOp}_{\mathbf{C}_*(\mathbb{Z})}$ . From theorem 7.2.2, it immediately follows that

**Corollary 7.2.4.** *Let  $P$  be a topological operad. The categories of  $S_*(P)$  and  $C_*^{ord}(P)$  algebras up to homotopy are equivalent.*

This remark applies to Deligne's conjecture, which is commonly expressed as follows: let  $A$  be an associative algebra over a ring  $R$ , and let  $C^*(A; A)$  be the associated Hochschild complex.

**Deligne's conjecture.** *For any associative  $R$ -algebra  $A$ , the complex  $C^*(A; A)$  is naturally an algebra over the singular chains of the little discs operad  $\mathcal{D}_2$  or a suitable version of it.*

There is some ambiguity in this statement with reference to the chain model of  $\mathcal{D}_2$  to be used and whether obtaining a solution for one model means that a solution is obtained for any other model. From our results it follows that there is only one chain model of a topological operad up to homotopy, therefore, this ambiguity disappears when we work with structures up to homotopy, linking in this way the solutions of Deligne's conjecture given by Kontsevich, who used  $C_*^{ord}(\mathcal{D}_2)$ , (see [K]), and MacClure-Smith, Tamarkin and Voronov, who used singular chains (see, for example, [T]).

7.3. **Formality.** Recall that a dg operad  $P$  is *formal* if it is quasi-isomorphic to its homology operad  $H(P)$ , (see [MSS]). From theorem 7.2.2 we can deduce the following:

**Corollary 7.3.1.** *Let  $P$  be a topological operad. Then  $S_*(P; \mathbb{Q})$  is a formal operad if, and only if,  $C_*^{ord}(P; \mathbb{Q})$  is a formal operad.*

M. Kontsevich proved the formality of the ordered cubical chains of the little  $k$ -discs operad after tensoring by the real numbers,  $\mathbb{R}$ , (see [K], theorem 2). The independent nature of formality on the base field proved in [GNPR] implies that  $C_*^{ord}(\mathcal{D}_k; \mathbb{Q})$  is also formal over the rational field, so by corollary 7.3.1,  $S_*(\mathcal{D}_k; \mathbb{Q})$  is formal.

7.4. **Modular operads.** The results above may be extended to chain models of modular operads. We refer to [GK] and [MSS] for the definitions concerning modular operads. Given a symmetric monoidal category we denote by  $\mathbf{MOp}_{\mathcal{C}}$  the category of modular operads of  $\mathcal{C}$ .

As for operads, it follows from the definitions that every symmetric monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  applied componentwise induces a functor

$$\mathbf{MOp}_F : \mathbf{MOp}_{\mathcal{C}} \longrightarrow \mathbf{MOp}_{\mathcal{D}}.$$

In particular, the singular and cubical chain functors of topological spaces extend to functors defined in the category of topological modular operads. Now, as in theorem 7.2.2, one has

**Theorem 7.4.1.** *The functors*

$$S_*, C_*^{ord} : \mathbf{MOp}_{\mathbf{Top}} \longrightarrow \mathbf{MOp}_{\mathbf{C}_*(\mathbb{Z})}$$

*are weakly equivalent. In particular, for a topological modular operad  $P$ , the dg modular operads  $S_*(P), C_*^{ord}(P)$  are weakly equivalent.*

From theorem 7.4.1 it immediately follows that

**Proposition 7.4.2.** *Let  $P$  be a topological modular operad. Then  $S_*(P; \mathbb{Q})$  is a formal modular operad if and only if  $C_*^{ord}(P; \mathbb{Q})$  is a formal modular operad.*

We can apply this result to the modular operad  $\mathcal{M}$ : the family  $\overline{\mathcal{M}}((g, l)) = \overline{\mathcal{M}}_{g,l}$  of Deligne-Knudsen-Mumford moduli spaces of stable genus  $g$  algebraic curves with  $l$  marked points, with the maps that identify marked points, is a modular operad in the category of projective smooth Deligne-Mumford stacks, [GK]. In [GNPR] we proved that  $S_*(\overline{\mathcal{M}}; \mathbb{Q})$  is a formal modular operad, so any other chain model is also formal. For instance, for cubical chains we can state that

**Corollary 7.4.3.** *The dg modular operad  $C_*^{ord}(\overline{\mathcal{M}}; \mathbb{Q})$  is a formal modular operad.*

## 8. CONTRAVARIANT FUNCTORS

If we work in the category of contravariant monoidal functors between monoidal categories,  $\mathbf{Mon}(\mathcal{C}^o, \mathcal{D})$ , a cotriple on  $\mathcal{C}$  induces a triple on  $\mathbf{Mon}(\mathcal{C}^o, \mathcal{D})$ , so that the standard construction produces cosimplicial objects instead of simplicial objects. As remarked in 2.4.3 this can be avoided by identifying such functors with covariant comonoidal functors between  $\mathcal{C}$  and  $\mathcal{D}^o$ . However, for functors with values in  $\mathbf{C}^*(\mathbb{Z})$ , which are the functors that appear in the study of the cohomology of topological spaces, the dual category  $\mathbf{C}^*(\mathbb{Z})^o$  is not the same as  $\mathbf{C}^*(\mathbb{Z})$ , so the acyclic models theorem 4.3.3 has to be appropriately dualized to cover this case.

In this section we present the minor modifications of the constructions and results of §4 that are necessary to cover this situation, and we apply them to compare the singular and cubical cochain functors.

**8.1. The standard construction.** Let  $\mathbf{T} = (T, \eta, \mu)$  be a triple (or *monad*, see [ML]) in a category  $\mathcal{X}$ . There is a functor associated to  $\mathbf{T}$ , which we call the *standard construction*,

$$B^\bullet : \mathcal{X} \longrightarrow \Delta\mathcal{X},$$

defined by  $B^n(X) = T^{n+1}(X)$  and with faces and degeneracies defined analogously as in 4.2.1.

8.1.1. Let  $\mathcal{C}$  be a monoidal category and  $\mathcal{A}$  an additive monoidal category, and  $\Sigma$  a class of weak equivalences of  $\mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))$  which contains the homotopy equivalences and is compatible with the Alexander-Whitney functor  $s_{AW}$ . A monoidal cotriple  $\mathbf{G} = (G, \varepsilon, \delta)$  in  $\mathcal{C}$  induces a triple  $\mathbf{T}$  on  $\mathcal{X} = \mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))$ , whose endofunctor is given by  $T(K^*) = K^*G$ , and with  $\eta, \mu$  induced by  $\varepsilon, \delta$ . By the dual of proposition 4.4.1,  $\mathbf{T}$  is compatible with  $\Sigma$ , that is,  $T(\Sigma) \subseteq \Sigma$ .

We can compose the standard cosimplicial construction with the Alexander-Whitney simple functor defined in 3.2.4 to obtain a functor

$$B^* : \mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A})) \longrightarrow \mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A})).$$

Moreover, the natural transformation  $\varepsilon$  of  $\mathbf{G}$  defines a natural transformation  $\varepsilon : \text{id} \Rightarrow B^*$ .

8.2. **Acyclic models for contravariant monoidal functors.** By analogy to section §4, it is clear how to define the notions of  $\mathbf{T}$ -presentability and  $\mathbf{T}$ -equivalence in  $\mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))$ , so that we have all the ingredients to transpose the proof of 4.3.1 and its corollaries to the category of contravariant monoidal functors between  $\mathcal{C}$  and  $\mathbf{C}^*(\mathcal{A})$ . In particular, we can state the contravariant version of corollary 4.3.3 referred to the assumptions made in the previous paragraph:

**Theorem 8.2.1.** *Let  $K^*, L^*$  be objects of  $\mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))$  concentrated in non-negative degrees, and  $\mathbf{T}$  a triple in  $\mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))$  induced by a monoidal cotriple  $\mathbf{G}$  on  $\mathcal{C}$ . Suppose that  $K^*, L^*$  are  $\mathbf{T}$ -acyclic and  $\mathbf{T}$ -presentable. Then any monoidal transformation  $H^0(K^*) \longrightarrow H^0(L^*)$  lifts to a unique morphism  $K^* \longrightarrow L^*$  in  $\mathbf{Mon}(\mathcal{C}^o, \mathbf{C}^*(\mathcal{A}))[\Sigma^{-1}]$ . In particular, if  $H^0(K^*)$  and  $H^0(L^*)$  are isomorphic, then  $K^*$  and  $L^*$  are weakly homotopy equivalent.*

8.3. **Application to singular and cubical cochains.** For a topological space  $X$ , let  $S^*(X), C^*(X)$  denote, respectively, the complexes of singular and of cubical cochains defined on  $X$ :

$$\begin{aligned} S^*(X) &= \text{Hom}(S_*(X), \mathbb{Z}), \\ C^*(X) &= \text{Hom}(C_*(X), \mathbb{Z}). \end{aligned}$$

They define contravariant functors

$$S^*, C^* : \mathbf{Top}^o \longrightarrow \mathbf{C}^*(\mathbb{Z}).$$

By dualizing the Alexander-Whitney morphism for  $S_*$ ,  $S^*$  becomes a contravariant monoidal functor. There is also an explicit Alexander-Whitney associative morphism for cubical chains  $C_*(X \times Y) \longrightarrow C_*(X) \otimes C_*(Y)$  (see [Mas], XI§5 and exercise XIII.5.1), and as a consequence  $C^*$  is also a contravariant monoidal functor.

The monoidal cotriple  $\mathbf{G}$  on  $\mathbf{Top}$  defined in 5.2 induces a triple in  $\mathbf{Mon}(\mathbf{Top}^o, \mathbf{C}^*(\mathbb{Z}))$ , denoted by  $\mathbf{T}$ , and it is easily seen (compare 5.3.1) that  $S^*$  and  $C^*$  are  $\mathbf{T}$ -presentable and  $\mathbf{T}$ -acyclic. Thus we are able to apply theorem 8.2.1 and deduce:

**Theorem 8.3.1.** *The singular and cubical cochain functors  $S^*, C^* : \mathbf{Top}^o \longrightarrow \mathbf{C}^*(\mathbb{Z})$  are weakly homotopy equivalent contravariant monoidal functors, that is, they are weakly homotopy equivalent in  $\mathbf{Mon}(\mathbf{Top}^o, \mathbf{C}^*(\mathbb{Z}))$ .*

## 9. APPLICATION TO COHOMOLOGY THEORIES

In this section we apply the acyclic models theorem 8.2.1 to compare cohomology theories on simplicial sets arising from simplicial differential graded algebras over a ring  $R$ .

**9.1. Cohomology theories.** Denote by  $\Delta^{\circ}\mathbf{Sets}$  the category of simplicial sets. If  $A_{\bullet}^*$  is a simplicial differential graded  $\mathbb{Z}$ -algebra, then to any simplicial set  $X$  we can associate a differential graded algebra  $A^*(X)$  by taking morphisms, in  $\Delta^{\circ}\mathbf{Sets}$ , from  $X$  to  $A_{\bullet}^*$ . In order to obtain a good cohomology theory we impose some conditions on  $A_{\bullet}^*$ , following Cartan (see [C]).

9.1.1. Let  $R$  be a ring and  $A_{\bullet}^*$  a simplicial differential graded  $R$ -algebra. We will assume that  $A_{\bullet}^*$  satisfies the following axioms:

- (a) *Homology axiom.* For each  $p \geq 0$ , the natural morphism  $R \longrightarrow A_p^*$  is a homotopy equivalence of cochain complexes. In particular, it is a quasi-isomorphism.
- (b) *Homotopy axiom.* For each  $q \geq 0$ , the simplicial set  $A_{\bullet}^q$  is (simplicially) contractible, i.e. the homotopy groups of  $A_{\bullet}^q$  are zero.
- (c) *Freeness axiom.*  $R$  is a principal ideal domain (PID) and, for all  $p, q$ , the  $R$ -module  $A_p^q$  is free.

9.1.2. Associated to  $A_{\bullet}^*$  there is a *cohomology theory* defined by

$$A^*(X) = \mathrm{Hom}_{\Delta}(X_{\bullet}, A_{\bullet}^*),$$

where  $\mathrm{Hom}_{\Delta}$  stands for the homomorphism set in the simplicial category.  $A^*(X)$  is a cochain complex, which in degree  $q$  is equal to  $\Delta^{\circ}\mathbf{Sets}(X_{\bullet}, A_{\bullet}^q)$ .

**Example 9.1.1.** Given a ring  $R$ , the singular cochain complex  $S^*(\ ; R)$  is an example of cohomology theory in the sense above. To see this it suffices to take the simplicial  $R$ -algebra  $S_{\bullet}^*(R)$  which in simplicial degree  $p$  is the cochain  $R$ -algebra of the simplicial set represented by  $p$ ,  $\Delta[p]$ , with evident face and degeneracies.

Cartan proves in [C], theorem 1 (see also [M]), that the cohomology of cohomology theories associated with simplicial differential graded algebras  $A_{\bullet}^*$ , which satisfies the homology and homotopy axioms, are isomorphic, and that this isomorphism comes from a true morphism of complexes if  $A_{\bullet}^*$  satisfies the normalization  $A_p^q = 0$  if  $q > p$ , (loc. cit. theorem 2). Moreover, it is compatible with products if the cohomology theories satisfy some flatness conditions (loc. cit. theorem 3). We will apply 8.2.1 to obtain a comparison result at the chain level, which is stronger than Cartan's theorems.

Note that in [Ma], M.Mandell obtains uniqueness results for cochain theories that satisfy a different set of axioms related to the classical Eilenberg-Steenrod axioms.

9.1.3. The category of simplicial sets  $\Delta^{\circ}\mathbf{Sets}$  is a monoidal category under the cartesian product. Using the algebra structure of  $A_{\bullet}^*$  it follows that the cohomology theory  $A^*$  defines a contravariant monoidal functor

$$A^* : (\Delta^{\circ}\mathbf{Sets})^{\circ} \longrightarrow \mathbf{C}^*(R).$$



**Theorem 9.1.2.** *Let  $A_\bullet^*, B_\bullet^*$  be simplicial differential graded  $R$ -algebras satisfying axioms (a)-(c). Then the contravariant monoidal functors*

$$A^*, B^* : (\Delta^{\circ}\mathbf{Sets})^{\circ} \longrightarrow \mathbf{C}^*(R),$$

*are weakly quasi-isomorphic in the category  $\mathbf{Mon}((\Delta^{\circ}\mathbf{Sets})^{\circ}, \mathbf{C}^*(R))$ . In particular, any such cohomology theory is weakly quasi-isomorphic to the singular cochain complex functor  $S^*( ; R)$ .*

*Proof.* Let  $\mathbf{G}$  be the cotriple in  $\Delta^{\circ}\mathbf{Sets}$  which is defined, as in the topological case, by

$$G(X) = \sqcup_{\alpha}(\Delta[\underline{n}], \alpha),$$

where  $\Delta[\underline{n}] = \Delta[n_1] \times \cdots \times \Delta[n_r]$ .  $\mathbf{G}$  induces a triple  $\mathbf{T} = (T, \eta, \mu)$  in  $\mathbf{Mon}(\Delta^{\circ}\mathbf{Sets}^{\circ}, \mathbf{C}^*(R))$ .

By hypothesis,  $H^0(A^*(X))$  and  $H^0(B^*(X))$  are isomorphic to  $H^0(X)$ , so in order to apply the acyclic models theorem 8.2.1 we have to prove that any cohomology theory  $A^*$  that satisfies axioms (a)-(c) is  $\mathbf{T}$ -acyclic and  $\mathbf{T}$ -presentable, the case of  $S^*( ; R)$  being well known. The  $\mathbf{T}$ -acyclicity will follow from the acyclicity of  $A_\bullet^*$  with respect to the differential degree, while the  $\mathbf{T}$ -presentability will follow from the contractibility with respect to the simplicial degree.

Let us first prove that  $A^*$  is  $\mathbf{T}$ -acyclic. We have to prove that for any sequence  $\underline{n} = (n_1, \dots, n_r)$  the complex

$$\dots \longrightarrow A^q(\Delta^{\underline{n}}) \longrightarrow A^{q-1}(\Delta^{\underline{n}}) \longrightarrow \dots \longrightarrow A^0(\Delta^{\underline{n}}) \longrightarrow H^0(A^*(\Delta^{\underline{n}})) \longrightarrow 0,$$

is acyclic. We prove this statement for any contractible simplicial set  $X$ ,  $X = \Delta^{\underline{n}}$  being a special case.

Note that the  $q$ -cocycles of  $A^*(X)$  are equal to  $\text{Hom}(X, Z^q A)$ , where  $Z^q A$  denotes the simplicial group of  $q$ -cocycles of  $A_\bullet^*$ . By the homology axiom  $d : A_\bullet^{q-1} \longrightarrow Z^q A$  is surjective, hence it is a Kan fibration with fiber  $Z^{q-1} A$ , and it follows also that  $Z^q A$  is connected.

Take a  $q$ -cocycle  $f : X \longrightarrow Z^q A$ . As  $X$  is contractible,  $f$  is homotopic to zero and  $Z^q A$  is connected,  $f$  admits an extension to a morphism  $X \longrightarrow A_\bullet^{q-1}$ , and consequently it is a boundary.

Let us now turn to  $\mathbf{T}$ -presentability. By the contravariant version of proposition 4.2.3, if each  $A^q$ ,  $q \geq 0$ , is  $\mathbf{T}$ -split then  $A^*$  will be  $\mathbf{T}$ -presentable. Therefore we want to define, for each  $q$ , a natural transformation  $\theta : T(A^q) = A^q G \Rightarrow A^q$  such that  $\theta \eta = \text{id}$ , where for a simplicial set  $X$ ,

$$\eta_X : A^q(X) \longrightarrow A^q G(X) = \prod_{\underline{n}} \prod_{\alpha : \Delta^{\underline{n}} \rightarrow X} A^q(\Delta^{\underline{n}}, \alpha)$$

is the morphism given by composition:  $\eta_X(w) = (w \circ \alpha, \alpha)$ .

We denote by  $X(p)$  the  $p$ -skeleton of the simplicial set  $X$ , so  $X = \varinjlim (X(p))$ . First of all, note that the functors  $A^q$  and  $A^q G$  are compatible with the skeleton decomposition, that is, we have

$$\begin{aligned} A^q(X) &= \varprojlim A^q(X(p)), \\ A^q G(X) &= \varprojlim A^q G(X(p)). \end{aligned}$$

The first isomorphism results from the compatibility of  $\text{Hom}$  functors with limits, while for the second we observe that  $G$  commutes with filtered colimits.

Thus, to define  $\theta$  it is sufficient to define morphisms  $\theta_p : A^q G(X(p)) \longrightarrow A^q(X(p))$ ,  $p \geq 0$ , which are sections of  $\eta_{X(p)}$ , and such that the diagrams

$$\begin{array}{ccc} A^q G(X(p)) & \xrightarrow{\theta_p} & A^q(X(p)) \\ \downarrow & & \downarrow \\ A^q G(X(p-1)) & \xrightarrow{\theta_{p-1}} & A^q(X(p-1)), \end{array}$$

where the vertical morphisms are induced by the inclusion  $X(p-1) \subseteq X(p)$ , are commutative.

We will define  $\theta_p$  inductively on  $p$ . The inductive step will be based on the following auxiliary result:

**Extension lemma.** *For any  $p$ , the inclusion  $\partial\Delta[p] \longrightarrow \Delta[p]$  induces a surjection  $f_p : A^q(\Delta[p]) \longrightarrow A^q(\partial\Delta[p])$  which has a  $R$ -linear section  $s : A^q(\partial\Delta[p]) \longrightarrow A^q(\Delta[p])$ .*

In fact, note that, as a consequence of the homotopy axiom, for any simplicial subset  $Y \subseteq X$  the induced morphism  $A^q(X) \longrightarrow A^q(Y)$  is surjective, since  $A_*^q$  is a contractible Kan complex and, as the inclusion  $Y \subseteq X$  is a cofibration, any map  $Y \longrightarrow A_*^q$  extends to a map  $X \longrightarrow A_*^q$ , (see [BG], and [FHT] for a more elementary proof).

In particular, the morphism  $f_p : A^q(\Delta[p]) \longrightarrow A^q(\partial\Delta[p])$  is surjective. But  $A^q(\partial\Delta[p])$  is  $R$ -projective, because  $R$  is a PID and  $A^q(\partial\Delta[p])$  is a submodule of the free  $R$ -module  $(A_{p-1}^q)^{p+1}$ . Thus,  $f_p$  has an  $R$ -linear section  $s : A^q(\partial\Delta[p]) \longrightarrow A^q(\Delta[p])$ .

Let us now return to the inductive definition of  $\theta_p$ . For  $p = -1$  there is nothing to prove, so let us assume that  $\theta_{p-1}$  has been constructed. The  $p$ -skeleton  $X(p)$  is obtained from the  $(p-1)$ -skeleton  $X(p-1)$  by the pushout diagram

$$\begin{array}{ccc} \sqcup \partial\Delta[p] & \longrightarrow & X(p-1) \\ \downarrow & & \downarrow \\ \sqcup \Delta[p] & \longrightarrow & X(p), \end{array}$$

where the disjoint union is over all nondegenerated maps  $\Delta[p] \longrightarrow X(p)$ . Since  $A^q$  transforms pushouts to pullbacks, we get the pullback diagram

$$\begin{array}{ccc} A^q(X(p)) & \longrightarrow & \prod A^q(\Delta[p]) \\ \downarrow & & \downarrow \\ A^q(X(p-1)) & \longrightarrow & \prod A^q(\partial\Delta[p]). \end{array}$$

Analogously, by applying  $A^q G$  we obtain the pullback diagram

$$\begin{array}{ccc} A^q G(X(p)) & \longrightarrow & \prod A^q G(\Delta[p]) \\ \downarrow & & \downarrow \\ A^q G(X(p-1)) & \longrightarrow & \prod A^q G(\partial\Delta[p]). \end{array}$$

By induction we have morphisms  $\theta_{p-1} = \theta_{X(p-1)}$  and  $\theta'_{p-1} = \theta_{\partial\Delta[p]}$ , so it suffices to define  $\theta'_p = \theta_{\Delta[p]}$  to make the diagram

$$\begin{array}{ccccc} A^q G(\Delta[p]) & \xrightarrow{g_p} & A^q G(\partial\Delta[p]) & \longleftarrow & A^q G(X(p-1)) \\ \theta'_p \downarrow & & \theta'_{p-1} \downarrow & & \theta_{p-1} \downarrow \\ A^q(\Delta[p]) & \xrightarrow{f_p} & A^q(\partial\Delta[p]) & \longleftarrow & A^q(X(p-1)) \end{array}$$

commutative and which is a section of  $\eta_{\Delta[p]}$ , (the square on the right commutes by induction).

Let  $\pi : A^q G(\Delta[p]) \longrightarrow A^q(\Delta[p])$  be the natural projection morphism, so that  $\pi\eta = \text{id}$ , and let  $s$  be a section of  $f_p$ ,  $f_p s = \text{id}$ , whose existence is guaranteed by the extension lemma above. We define the morphism

$$\theta'_p = \pi + s(\theta'_{p-1} g_p - f_p \pi).$$

Commutativity easily follows, since

$$f_p \theta'_p = f_p \pi + f_p s(\theta'_{p-1} g_p - f_p \pi) = f_p \pi + \theta'_{p-1} g_p - f_p \pi = \theta'_{p-1} g_p.$$

Moreover,  $\theta'_p$  is a section of  $\eta_{\Delta[p]}$ , since

$$\theta'_p \eta_{\Delta[p]} = \pi \eta_{\Delta[p]} + s \theta'_{p-1} g_p \eta_{\Delta[p]} - s f_p \pi \eta_{\Delta[p]} = \text{id} + s f_p - s f_p = \text{id},$$

where we have used the commutativity proved above.  $\square$

9.2. Any contravariant monoidal functor  $F^* : (\Delta^\circ \mathbf{Sets})^\circ \longrightarrow \mathbf{C}^*(R)$  gives rise to a functor on  $\Delta^\circ \mathbf{Sets}$  with values in the category of  $R$ -differential graded algebras by composing the Künneth morphism of  $F^*$  with the morphism induced by the diagonal of the space

$$F^*(X) \otimes F^*(X) \xrightarrow{\kappa} F^*(X \times X) \xrightarrow{\Delta^*} F^*(X).$$

Now, from theorem 9.1.2 we deduce

**Corollary 9.2.1.** *Let  $A_\bullet^*$  be a simplicial  $R$ -dg algebra that satisfies axioms (a)-(c). For any simplicial set  $X$ , the algebras  $S^*(X; R)$  and  $A^*(X)$  are weakly equivalent in the category of  $R$ -differential graded algebras.*

**Example 9.2.2.** As application we obtain a proof, for any simplicial set  $X$ , of the equivalence of the cochain algebra  $S^*(X; \mathbb{Q})$  and the polynomial De Rham algebra (compare with [S] and [BG]). For this, take  $L_\bullet^*$  the simplicial cochain complex defined by the regular differential forms on the  $\mathbb{Q}$ -cosimplicial scheme  $H^\bullet$ , whose  $p$  component is the hyperplane  $H^p$  of  $\mathbb{A}^{p+1}$  defined by the equation  $t_0 + \cdots + t_p = 1$ . It is a simplicial  $\mathbb{Q}$ -dga that satisfies the homology and homotopy axioms (see [BG]), so if we define the algebra of polynomial De Rham forms of  $X$  as

$$Su^*(X) = \text{Hom}_\Delta(X_\bullet, L_\bullet^*(\mathbb{Q})),$$

the last corollary ensures that  $S^*(X; \mathbb{Q})$  and  $Su^*(X)$  are weakly equivalent  $\mathbb{Q}$ -differential graded algebras. The rational field  $\mathbb{Q}$  may be replaced by any field  $\mathbf{k}$  of characteristic zero.

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