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# Polynomial differential systems having a given Darbouxian first integral <sup>☆</sup>

Jaume Llibre <sup>\*</sup>, Chara Pantazi

*Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*

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## Abstract

The Darbouxian theory of integrability allows to determine when a polynomial differential system in  $\mathbb{C}^2$  has a first integral of the kind  $f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/h)$  where  $f_i$ ,  $g$  and  $h$  are polynomials in  $\mathbb{C}[x, y]$ , and  $\lambda_i \in \mathbb{C}$  for  $i = 1, \dots, p$ . The functions of this form are called Darbouxian functions. Here, we solve the inverse problem, i.e. we characterize the polynomial vector fields in  $\mathbb{C}^2$  having a given Darbouxian function as a first integral.

On the other hand, using information about the degree of the invariant algebraic curves of a polynomial vector field, we improve the conditions for the existence of an integrating factor in the Darbouxian theory of integrability.

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## 1. Introduction and statement of the main results

By definition a *planar polynomial differential system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \quad \frac{dy}{dt} = \dot{y} = Q(x, y), \quad (1)$$

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<sup>\*</sup> Corresponding author.

*E-mail address:* jllibre@mat.uab.es (J. Llibre).

where  $P$  and  $Q$  are polynomials in the variables  $x$  and  $y$ . Moreover, the dependent variables  $x$  and  $y$ , the independent variable  $t$  (called the *time*), and the coefficients of the polynomials  $P$  and  $Q$  are complex.

Associated to the polynomial differential system (1) in  $\mathbb{C}^2$  there is the *polynomial vector field*

$$X = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \quad (2)$$

in  $\mathbb{C}^2$ . Sometimes, the polynomial vector field  $X$  will be denoted simply by  $(P, Q)$ .

The *degree*  $m$  of the polynomial differential system (1) or of the polynomial vector field  $X$  is the maximum of the degrees of the polynomials  $P$  and  $Q$ . The degree of a polynomial  $P$  is denoted by  $\delta P$ . The degree of a rational function  $P/Q$  is defined as  $\delta(P/Q) = \max\{\delta P, \delta Q\}$ .

If the polynomials  $P$  and  $Q$  are not coprime, let  $R$  be the greatest common divisor of  $P$  and  $Q$ . Then, the change in the independent variable  $t$  given by  $ds = R dt$  transforms the polynomial vector field (2) into the polynomial vector field  $(P/R, Q/R)$  with  $P/R$  and  $Q/R$  coprime. Since if  $(P/R, Q/R)$  has a first integral, we also have a first integral for  $(P, Q)$ , in what follows we shall work with polynomial vector fields  $(P, Q)$  with  $P$  and  $Q$  coprime.

A Darbouxian function can be written into the form

$$H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(\frac{g}{f_1^{n_1} \cdots f_p^{n_p}}\right), \quad (3)$$

where  $f_1, \dots, f_p$  are irreducible polynomials in  $\mathbb{C}[x, y]$ ,  $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ ,  $n_1, \dots, n_p \in \mathbb{N} \cup \{0\}$  (i.e. the  $n_i$  are non-negative integers) and the polynomial  $g$  of  $\mathbb{C}[x, y]$  is coprime with  $f_i$  if  $n_i \neq 0$ .

First we want to characterize when a polynomial vector field  $X$  in  $\mathbb{C}^2$  has the Darbouxian function  $H(x, y)$  as a first integral; i.e. when  $H$  is constant on the trajectories of  $X$  contained in the domain of definition  $U$  of  $H$ , or equivalently when  $dH/dt = XH = P\partial H/\partial x + Q\partial H/\partial y = 0$ , on  $U$ .

Given a polynomial vector field  $X$  the Darbouxian theory of integrability provides sufficient conditions in order that  $X$  has a Darbouxian first integral, see for more details Section 2. This theory started with Darboux [10] in 1878. For more details and results on the Darbouxian theory of integrability for planar polynomial vector fields, see [1,3,4,6,12, 14–19]. Here, we study the inverse problem. Our main results on the inverse problem are summarized in what follows.

**Theorem 1.** *Let  $H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$  be a Darbouxian function with  $f_1, \dots, f_p$  irreducible polynomials in  $\mathbb{C}[x, y]$ ,  $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ ,  $n_1, \dots, n_p \in \mathbb{N} \cup \{0\}$  and the polynomial  $g$  of  $\mathbb{C}[x, y]$  is coprime with  $f_i$  if  $n_i \neq 0$ . We denote by  $l$  the degree of the rational function  $g/(f_1^{n_1} \cdots f_p^{n_p})$ . Then,  $H$  is a first integral for the polynomial vector field  $X = (P, Q)$  of degree  $m$  with  $P$  and  $Q$  coprimes if and only if*

$$(a) \quad l + \sum_{i=1}^p \delta f_i = m + 1 \text{ and}$$

$$\begin{aligned}
 X = & \left( \prod_{l=1}^p f_l^{n_l} \right) \sum_{i=1}^p \lambda_i \left( \prod_{j=1, j \neq i}^p f_j \right) X_f \\
 & - g \sum_{i=1}^p n_i \left( \prod_{j=1, j \neq i}^p f_j \right) X_f + \left( \prod_{j=1}^p f_j \right) X_g,
 \end{aligned} \tag{4}$$

where  $X_{f_i}$  is the Hamiltonian vector field  $(-f_{iy}, f_{ix})$ .

Moreover, the vector field given by (4) has the integrating factor

$$R_1 = (f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p})^{-1}.$$

- (b)  $l + \sum_{i=1}^p \delta f_i > m + 1$  and  $X$  is as in (4) dividing its components by their greatest common divisor  $D$ . Moreover,  $DR_1$  is a rational integrating factor of  $X$ .

Theorem 1 will be proved in Section 3. Also in that section we shall show that the second part of statement (a) cannot be extended to the integrating factors of the form (3) with  $g \neq 0$ . In Section 4 we provide examples of all statements of Theorem 1.

**Corollary 2.** *Under the assumptions of Theorem 1 if (3) is a first integral for the polynomial vector field  $X = (P, Q)$  of degree  $m$  with  $P$  and  $Q$  coprimes, then  $l + \sum_{i=1}^p \delta f_i \geq m + 1$ .*

Corollary 2 follows directly from Theorem 1. Note that Corollary 2 says that the degree of a polynomial vector field having the first integral (3) is not independent of the degrees of the polynomials appearing in (3).

Prelle and Singer in [15] proved the following result.

**Theorem 3.** *If a polynomial vector field  $X$  has a first integral of the form  $H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$  where  $f_1, \dots, f_p$  are irreducible polynomials in  $\mathbb{C}[x, y]$ ,  $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ ,  $n_1, \dots, n_p \in \mathbb{N} \cup \{0\}$  and the polynomial  $g$  of  $\mathbb{C}[x, y]$  is coprime with  $f_i$  if  $n_i \neq 0$ , then the vector field has an integrating factor of the form*

$$\left( \frac{a(x, y)}{b(x, y)} \right)^{\frac{1}{N}}$$

with  $a, b \in \mathbb{C}[x, y]$  and  $N$  an integer.

We improve Theorem 3 as follows.

**Corollary 4.** *We assume that the polynomial vector field  $X$  has a first integral of the form  $H(x, y) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$  where  $f_1, \dots, f_p$  are irreducible polynomials in  $\mathbb{C}[x, y]$ ,  $\lambda_1, \dots, \lambda_p \in \mathbb{C}$ ,  $n_1, \dots, n_p \in \mathbb{N} \cup \{0\}$  and the polynomial  $g$  of  $\mathbb{C}[x, y]$  is coprime with  $f_i$  if  $n_i \neq 0$ . We denote by  $l = \delta(g/(f_1^{n_1} \cdots f_p^{n_p}))$ .*

- (a) *If  $l + \sum_{i=1}^p \delta f_i = m + 1$  then the inverse of the polynomial  $f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p}$  is an integrating factor.*

(b) *Otherwise, a function of the form  $a(x, y)/(f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p})$  with  $a \in \mathbb{C}[x, y]$  is an integrating factor.*

The results of Corollary 4 are strongly related with Proposition 3.2 and Corollary 3.3 of Walcher [20].

Other aspects of the inverse problem of the Darbouxian theory of integrability have been studied, see for more details Theorem 10 (due to Christopher [5], Żołądek [21] and Christopher, Llibre, Pantazi and Ziang [8]) and Proposition 12 (due to Christopher and Kooij [5]) in Section 2. In fact, the next result improves statement (b) of Theorem 10 and Proposition 12.

**Theorem 5.** *Let  $X = (P, Q)$  be a polynomial vector field with  $P$  and  $Q$  coprime having  $f_1 = 0, \dots, f_p = 0$  as irreducible invariant algebraic curves satisfying the generic conditions:*

- (i) *There are no points at which  $f_i$  and its first derivatives are all vanish.*
- (ii) *The highest order terms of  $f_i$  have no repeated factors.*
- (iii) *If two curves intersect at a point in the finite plane, they are transversal at this point.*
- (iv) *There are no more than two curves  $f_i = 0$  meeting at any point in the finite plane.*
- (v) *There are no two curves having a common factor in the highest order terms.*

*Then,  $X$  has the first integral  $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$  with  $\lambda_i \in \mathbb{C}$  if and only if  $\sum_{i=1}^p \delta f_i = m + 1$ . Moreover,*

$$X = \sum_{i=1}^p \lambda_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) X_{f_i}. \quad (5)$$

Theorem 5 will be proved in Section 3. An example of a polynomial vector field satisfying Theorem 5 will be given in Section 4.

A function  $R(x, y)$  is an *integrating factor* of the vector field  $X = (P, Q)$  on the domain of definition  $U$  of  $R$  if  $\operatorname{div}(RP, RQ) = 0$  on  $U$ . As usual the *divergence* of the vector field  $X$  is defined by

$$\operatorname{div}(X) = \operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

For the next theorem see the definitions of irreducible invariant algebraic curve, exponential factor, their cofactors and (weak) independent singular in Section 2. This theorem improves the conditions for the existence of an integrating factor in the Darbouxian theory of integrability using information about the degree of the invariant algebraic curves, specifically it improves statement (e) of Theorem 9. As far as we know, this is the first time that information about the degree of the invariant algebraic curves, instead of the number of these curves, is used for studying the integrability of a polynomial vector field.

**Theorem 6.** *Suppose that a polynomial vector field  $X = (P, Q)$  of degree  $m$ , with  $P$  and  $Q$  coprime, admits  $p$  irreducible invariant algebraic curves  $f_i = 0$  with cofactors  $K_i$  for  $i = 1, \dots, p$ ;  $q$  exponential factors  $\exp(g_j/h_j)$  with cofactors  $L_j$  for  $j = 1, \dots, q$ ; and  $r$  independent singular points  $(x_k, y_k)$  such that  $f_i(x_k, y_k) \neq 0$  for  $i = 1, \dots, p$  and for  $k = 1, \dots, r$ . Then, the irreducible factors of the polynomials  $h_j$  are some  $f_i$ 's and we can write*

$$\begin{aligned} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} &= \exp\left(\frac{\mu_1 g_1}{h_1} + \cdots + \frac{\mu_q g_q}{h_q}\right) \\ &= \exp\left(\frac{g}{f_1^{n_1} \cdots f_p^{n_p}}\right), \end{aligned}$$

where  $\mu_1, \dots, \mu_q \in \mathbb{C}$ ,  $n_1, \dots, n_p \in \mathbb{N} \cup \{0\}$  and the polynomial  $g$  of  $\mathbb{C}[x, y]$  is coprime with  $f_i$  if  $n_i \neq 0$ . We denote by  $l = \max\{\sum_{i=1}^p n_i \delta f_i, \delta g\}$ .

If  $p + q + r = m(m + 1)/2$ ,  $l + \sum_{i=1}^p \delta f_i < m + 1$ , and the  $r$  independent singular points are weak, then the (multi-valued) function

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{6}$$

for convenient  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero is an integrating factor of  $X$ .

Theorem 6 is also proved in Section 3. An example of a polynomial vector field satisfying Theorem 6 will be given in Section 4.

As far as we know, this theorem uses by first time information about the degree of the invariant algebraic curves for studying the integrability of a polynomial vector field, because until now the Darbouxian theory of integrability only used of the invariant algebraic curves of a polynomial vector field its number for studying its integrability looking for, either a first integral, or an integrating factor, see Theorem 9.

## 2. Darbouxian theory of integrability

The Darbouxian theory of integrability for planar polynomial vector fields can be summarized in the next theorem. As far as we know, the problem of integrating a polynomial vector fields by using its invariant algebraic curves was started to be considered by Darboux in [10]. The version that we present improves Darboux's one essentially because here we also take into account the exponential factors (see [4,9]), and the independent singular points (see [3]). Some more complete versions can also consider the Darbouxian invariants (see [1,2]), but since these more complete versions will not play any role in this paper here we omit them.

First we introduce the main three notions in the Darbouxian theory of integrability. Let  $f \in \mathbb{C}[x, y]$ . The algebraic curve  $f(x, y) = 0$  is an *invariant algebraic curve* of the polynomial vector field  $X$  if for some polynomial  $K \in \mathbb{C}[x, y]$  we have

$$Xf = P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = Kf.$$

The polynomial  $K$  is called the *cofactor* of the invariant algebraic curve  $f = 0$ . Of course, the curve  $f = 0$  is formed by trajectories of the polynomial vector field  $X$ . We note that since the polynomial vector field has degree  $m$ , then any cofactor has at most degree  $m - 1$ . The following result is well known, see for instance [7].

**Proposition 7.** *We suppose that  $f \in \mathbb{C}[x, y]$  and let  $f = f_1^{n_1} \cdots f_r^{n_r}$  be its factorization in irreducible factors over  $\mathbb{C}[x, y]$ . Then, for the polynomial system (1),  $f = 0$  is an invariant algebraic curve with cofactor  $K_f$  if and only if  $f_i = 0$  is an invariant algebraic curve for each  $i = 1, \dots, r$  with cofactor  $K_{f_i}$ . Moreover  $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$ .*

By Proposition 7, in what follows we can restrict our attention to the irreducible invariant algebraic curves.

Let  $h, g \in \mathbb{C}[x, y]$  and assume that  $h$  and  $g$  are relatively prime in the ring  $\mathbb{C}[x, y]$ . Then the function  $\exp(g/h)$  is called an *exponential factor* of the polynomial vector field  $X$  if for some polynomial  $L \in \mathbb{C}[x, y]$  of degree at most  $m - 1$  it satisfies

$$X\left(\exp\left(\frac{g}{h}\right)\right) = L \exp\left(\frac{g}{h}\right).$$

As before we say that  $L$  is the *cofactor* of the exponential factor  $\exp(g/h)$ .

**Proposition 8.** *If  $\exp(g/h)$  is an exponential factor for the polynomial vector field  $X$ , then  $h = 0$  is an invariant algebraic curve of  $X$ .*

**Proof.** See [4].  $\square$

In fact, in Proposition 8  $h = 0$  is an invariant algebraic curve with multiplicity larger than 1 as solution of  $X$ , for more details see [9].

If

$$S(x, y) = \sum_{i+j=0}^{m-1} a_{ij} x^i y^j$$

is a polynomial of degree  $m - 1$  with  $m(m + 1)/2$  coefficients in  $\mathbb{C}$ , then we write  $S \in \mathbb{C}_{m-1}[x, y]$ . We identify the linear vector space  $\mathbb{C}_{m-1}[x, y]$  with  $\mathbb{C}^{m(m+1)/2}$  through the isomorphism  $S \rightarrow (a_{00}, a_{10}, a_{01}, \dots, a_{m-1,0}, a_{m-2,1}, \dots, a_{0,m-1})$ .

We say that  $r$  points  $(x_k, y_k) \in \mathbb{C}^2, k = 1, \dots, r$ , are *independent* with respect to  $\mathbb{C}_{m-1}[x, y]$  if the intersection of the  $r$  hyperplanes

$$\left\{ (a_{ij}) \in \mathbb{C}^{m(m+1)/2}: \sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, k = 1, \dots, r \right\},$$

is a linear subspace of  $\mathbb{C}^{m(m+1)/2}$  of dimension  $m(m + 1)/2 - r > 0$ .

We recall that  $(x_0, y_0)$  is a singular point of system (1) if  $P(x_0, y_0) = Q(x_0, y_0) = 0$ .

We remark that the maximum number of isolated singular points of the polynomial system (1) is  $m^2$  (by Bezout theorem), that the maximum number of independent isolated singular points of the system is  $m(m + 1)/2$ , and that  $m(m + 1)/2 < m^2$  for  $m \geq 2$ .

A singular point  $(x_0, y_0)$  of system (1) is called *weak* if the divergence,  $\text{div } X$ , of system (1) at  $(x_0, y_0)$  is zero.

**Theorem 9.** *Suppose that a polynomial vector field  $X$  of degree  $m$  admits  $p$  irreducible invariant algebraic curves  $f_i = 0$  with cofactors  $K_i$  for  $i = 1, \dots, p$ ;  $q$  exponential factors  $\exp(g_j/h_j)$  with cofactors  $L_j$  for  $j = 1, \dots, q$ ; and  $r$  independent singular points  $(x_k, y_k)$  such that  $f_i(x_k, y_k) \neq 0$  for  $i = 1, \dots, p$  and for  $k = 1, \dots, r$ . Moreover, the irreducible factors of the polynomials  $h_j$  are some  $f_i$ 's.*

(a) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0,$$

*if and only if the (multi-valued) function*

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} \left( \exp\left(\frac{g_1}{h_1}\right) \right)^{\mu_1} \dots \left( \exp\left(\frac{g_q}{h_q}\right) \right)^{\mu_q} \tag{7}$$

*is a first integral of  $X$ .*

(b) *If  $p + q + r = [m(m + 1)/2] + 1$ , then there exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0.$$

(c) *If  $p + q + r \geq [m(m + 1)/2] + 2$ , then  $X$  has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.*

(d) *There exist  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero such that*

$$\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = -\text{div } X,$$

*if and only if the function (7) is an integrating factor of  $X$ .*

(e) *If  $p + q + r = m(m + 1)/2$  and the  $r$  independent singular points are weak, then function (7) for convenient  $\lambda_i, \mu_j \in \mathbb{C}$  not all zero is a first integral or an integrating factor.*

Note that in Theorem 9 the fact that the irreducible factors of the polynomials  $h_j$  are some  $f_i$ 's is due to Proposition 8.

Statements (a), (b), (d) and (e) restricted only to invariant algebraic curves are due essentially to Darboux [10]. These statements taking into account the exponential factors and the independent singular points can be found in [4,6,7]. Statement (c) is due to Jouanolou [12], for an easy proof see [7].

The next theorem is another kind of inverse problem of the Darbouxian theory of integrability, in it the invariant algebraic curves are given and we want to obtain all the polynomial vector fields having these invariant algebraic curves. This theorem was stated by Christopher without proof in [5], and used in other papers as [1,11,13]. Żołądek in [21]

(see also Theorem 3 of [22]) stated a similar result using an analytic approach, but as far as we know the paper [21] has not been published. A first complete proof of it, using mainly algebraic tools, has been given in [8].

**Theorem 10.** *Let  $f_i = 0$ , for  $i = 1, \dots, p$ , be irreducible algebraic curves in  $\mathbb{C}^2$ . We assume that all  $f_i$  satisfy the generic conditions of Theorem 5. Then any polynomial vector field  $X$  of degree  $m$  having all  $f_i = 0$  as invariant algebraic curves satisfies one of the following statements.*

(a) *If  $\sum_{i=1}^p \delta f_i < m + 1$ , then*

$$X = \left( \prod_{i=1}^p f_i \right) Y + \sum_{i=1}^p h_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) X_{f_i},$$

*where the  $h_i$  are polynomials such that  $\delta h_i \leq m + 1 - \sum_{i=1}^p \delta f_i$ , and  $Y$  is a polynomial vector field with degree  $\leq m - \sum_{i=1}^p \delta f_i$ .*

(b) *If  $\sum_{i=1}^p \delta f_i = m + 1$ , then  $X$  is of the form (5).*

(c) *If  $\sum_{i=1}^p \delta f_i > m + 1$ , then  $X = 0$ .*

In [8] we show that all the assumptions of Theorem 10 are necessary in order that the result hold. More specifically, we proved the next result.

**Proposition 11.** *If one of the conditions (i)–(v) of Theorem 10 is not satisfied, then its statements do not hold.*

An interesting complement to Theorem 10(b) due to Christopher and Kooij [5] is the following.

**Proposition 12.** *Under the assumptions of Theorem 10(b) a polynomial system (5) has an integrating factor of the form  $(f_1 \cdots f_p)^{-1}$  and a first integral of the form  $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ .*

The second part of statement (a) of Theorem 1 is in some sense the equivalent to Proposition 12 for our inverse problem.

### 3. Proof of our main results

In this section we shall prove Theorems 1, 5 and 6.

**Proof of Theorem 1.** By a direct calculation we prove that system (4) in statements (a) and (b) of Theorem 1 has (3) as a first integral. So, the “only if” part of Theorem 1 is proved. Now, we shall prove the “if” part.



We assume that  $H = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F$  with  $F = \exp(g/(f_1^{n_1} \cdots f_p^{n_p}))$  is a first integral of the polynomial vector field  $X = (P, Q)$  of degree  $m$ . So, we have

$$\begin{aligned}
 0 &= PH_x + QH_y \\
 &= PF \left( \sum_{i=1}^p \lambda_i f_i^{\lambda_i-1} f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^p f_j^{\lambda_j} + g_x \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j^{\lambda_j} \right) \right. \\
 &\quad \left. - g \left( \sum_{i=1}^p n_i f_i^{n_i-1} f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^p f_j^{n_j} \right) \left( \prod_{r=1}^p f_r^{-2n_r} \right) \left( \prod_{j=1}^p f_j^{\lambda_j} \right) \right) \\
 &\quad + QF \left( \sum_{i=1}^p \lambda_i f_i^{\lambda_i-1} f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^p f_j^{\lambda_j} + g_y \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j^{\lambda_j} \right) \right. \\
 &\quad \left. - g \left( \sum_{i=1}^p n_i f_i^{n_i-1} f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^p f_j^{n_j} \right) \left( \prod_{r=1}^p f_r^{-2n_r} \right) \left( \prod_{j=1}^p f_j^{\lambda_j} \right) \right) \\
 &= \left[ P \left( \sum_{i=1}^p \lambda_i f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^p f_j + g_x \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j \right) \right. \right. \\
 &\quad \left. \left. - g \sum_{i=1}^p n_i f_{ix} \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j \right) \right) \right. \\
 &\quad \left. + Q \left( \sum_{i=1}^p \lambda_i f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^p f_j + g_y \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j \right) \right. \right. \\
 &\quad \left. \left. - g \sum_{i=1}^p n_i f_{iy} \left( \prod_{r=1}^p f_r^{-n_r} \right) \left( \prod_{j=1}^p f_j \right) \right) \right] F \prod_{j=1}^p f_j^{\lambda_j-1}.
 \end{aligned}$$

Since the last expression is equal to zero, we can cancel the non-zero product  $F \prod_{j=1}^p f_j^{\lambda_j-1}$  and we can replace it with the non-zero product  $\prod_{r=1}^p f_r^{n_r}$ . So we get

$$0 = PG_1 + QG_2, \tag{8}$$

with

$$\begin{aligned}
 G_1 &= \left( \sum_{i=1}^p \lambda_i f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) \prod_{r=1}^p f_r^{n_r} + g_x \prod_{j=1}^p f_j - g \sum_{i=1}^p n_i f_{ix} \prod_{\substack{j=1 \\ j \neq i}}^p f_j, \\
 G_2 &= \left( \sum_{i=1}^p \lambda_i f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) \prod_{r=1}^p f_r^{n_r} + g_y \prod_{j=1}^p f_j - g \sum_{i=1}^p n_i f_{iy} \prod_{\substack{j=1 \\ j \neq i}}^p f_j.
 \end{aligned}$$

We remark that, since  $P$  and  $Q$  are coprime, from  $PH_x + QH_y = 0$  it follows that  $H_x$  and  $H_y$  cannot be zero. Consequently,  $G_1$  and  $G_2$  are not zero.

Since  $P$  and  $Q$  are coprime, from (8) we have that  $P$  must divide the polynomial  $G_2$ , and  $Q$  must divide the polynomial  $G_1$ , which is impossible if  $\delta G_i < m = \max\{\delta P, \delta Q\}$  for  $i = 1, 2$ . Due to the fact that  $\delta G_i = l - 1 + \sum_{i=1}^p \delta f_i$ , we get that  $l + \sum_{i=1}^p \delta f_i \geq m + 1$ .

Since  $P$  and  $Q$  are coprime, if  $\sum_{i=1}^p \delta f_i + l = m + 1$  we have that there is a constant  $\lambda \in \mathbb{C} \setminus \{0\}$  such that  $P = -\lambda G_2$  and  $Q = \lambda G_1$ . Doing the change of time  $t \rightarrow (1/\lambda)t$  the first part of statement (a) is proved. Now we shall show the second part of statement (a).

The algebraic curve  $f_k = 0$  is invariant for the vector field (4) with cofactor

$$K_k = \left( \prod_{l=1}^p f_l^{n_l} \right) \sum_{i=1}^p \lambda_i (f_{ix} f_{ky} - f_{iy} f_{kx}) \left( \prod_{\substack{j=1 \\ j \neq i, k}}^p f_j \right) + (g_x f_{ky} - g_y f_{kx}) \left( \prod_{\substack{j=1 \\ j \neq k}}^p f_j \right) + g \sum_{i=1}^p n_i (f_{iy} f_{kx} - f_{ix} f_{ky}) \left( \prod_{\substack{j=1 \\ j \neq i, k}}^p f_j \right).$$

The vector field (4) has divergence

$$\begin{aligned} \operatorname{div} X = & - \left( \prod_{l=1}^p f_l^{n_l} \right)_x \sum_{i=1}^p \lambda_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) f_{iy} + \left( \prod_{l=1}^p f_l^{n_l} \right)_y \sum_{i=1}^p \lambda_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) f_{ix} \\ & + \left( \prod_{l=1}^p f_l^{n_l} \right) \sum_{i=1}^p \lambda_i \left( \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right)_y f_{ix} - \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right)_x f_{iy} \right) \\ & + g_x \sum_{i=1}^p n_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) f_{iy} - g_y \sum_{i=1}^p n_i \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) f_{ix} \\ & + g \sum_{i=1}^p n_i \left( \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right)_x f_{iy} - \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right)_y f_{ix} \right) \\ & + \left( \prod_{j=1}^p f_j \right)_y g_x - \left( \prod_{j=1}^p f_j \right)_x g_y, \end{aligned}$$

or equivalently,

$$\operatorname{div} X = \sum_{i, k=1}^p n_k \lambda_i (f_{ky} f_{ix} - f_{kx} f_{iy}) \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) \left( \prod_{\substack{l=1 \\ l \neq k}}^p f_l^{n_l} \right) f_k^{n_k - 1}$$

$$\begin{aligned}
 &+ \left( \prod_{l=1}^p f_l^{n_l} \right) \sum_{j,k=1}^p \lambda_i (f_{jy} f_{ix} - f_{jx} f_{iy}) \left( \prod_{\substack{k=1 \\ k \neq i, j}}^p f_k \right) \\
 &+ \sum_{i=1}^p n_i (g_x f_{iy} - g_y f_{ix}) \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right) \\
 &+ g \sum_{i=1}^p n_i \sum_{\substack{j=1 \\ j \neq i}}^p (f_{jx} f_{iy} - f_{jy} f_{ix}) \left( \prod_{\substack{k=1 \\ k \neq i, j}}^p f_k \right) \\
 &+ \sum_{i=1}^p (g_x f_{iy} - g_y f_{ix}) \left( \prod_{\substack{j=1 \\ j \neq i}}^p f_j \right),
 \end{aligned}$$

and it is easy to check that

$$\sum_{r=1}^p K_r + \sum_{r=1}^p n_r K_r = \operatorname{div} X.$$

Therefore, by Theorem 9(b),  $R_1 = (f_1 \cdots f_p f_1^{n_1} \cdots f_p^{n_p})^{-1}$  is an integrating factor of the vector field (4).

Suppose that  $l + \sum_{i=1}^p \delta f_i > m + 1$ . Since  $P$  and  $Q$  are coprime, from (8) we have that there is a polynomial  $F$  such that  $G_1 = FQ$  and  $G_2 = -FP$ . So, dividing  $G_1$  and  $G_2$  by  $F$  we obtain the polynomial vector field  $(P, Q)$  of degree  $m$ . This completes the proof of statement (b), and consequently of Theorem 1.  $\square$

**Proof of Theorem 5.** Assume that the assumptions of Theorem 5 hold. Suppose that  $\sum_{i=1}^p \delta f_i = m + 1$ . Then, by Theorem 10(b) it follows that the polynomial vector field satisfying the assumptions of Theorem 5 is of the form (5), and by Proposition 12 it has the first integral  $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ .

Now we shall prove the converse statement. Suppose that the polynomial vector field satisfying the assumptions of Theorem 5 has the first integral  $f_1^{\lambda_1} \cdots f_p^{\lambda_p}$ . So, for this first integral  $l = 0$ , using the notation of Theorem 1. Then, by Corollary 2 we have that  $\sum_{i=1}^p \delta f_i \geq m + 1$ . Since all the invariant algebraic curves  $f_i = 0$  are generic, by Theorem 10, it follows that  $\sum_{i=1}^p \delta f_i \leq m + 1$ . Hence,  $\sum_{i=1}^p \delta f_i = m + 1$ , and the proof of the theorem is completed.  $\square$

Now we shall show that the second part of statement (a) of Theorem 1 cannot be extended to integrating factors of the form (3) with  $g \neq 0$ . The system

$$\begin{aligned}
 \dot{x} &= x(x + y + 1), \\
 \dot{y} &= y(x + y),
 \end{aligned} \tag{9}$$

has the two invariant algebraic curves  $f_1 = x = 0$  and  $f_2 = y = 0$ , and the exponential factor  $F = \exp(-(1 + x)/y)$  with cofactors  $K_1 = x + y + 1$ ,  $K_2 = x + y$  and  $L = 1$ ,

respectively. Since  $-K_1 + K_2 + L = 0$ , by Theorem 9(a) system (9) has the first integral  $H = f_1^{-1} f_2 F$ . Doing simple computations we observe that system (9) can be written into the form (4) with  $\lambda_1 = -1$ ,  $\lambda_2 = 1$ ,  $n_1 = 0$  and  $n_2 = 1$ . We also note that the polynomials  $P$  and  $Q$  are coprime.

Since the divergence of system (9) is  $\text{div} = 1 + 3x + 3y$  and we have that  $K_1 + K_2 \neq \text{div}$  and  $K_1 + K_2 + L \neq \text{div}$ , by Theorem 9(d) there is no integrating factors of the form  $(f_1 f_2)^{-1}$  or  $(f_1 f_2 \exp F)^{-1}$ . So, although system (9) can be written into the form (4), the second part of statements (a) of Theorem 1 cannot be extended to integrating factors of the form (3) with  $g \neq 0$ . However, since  $K_1 + 2K_2 = \text{div}$ , this system has the integrating factor  $R_1 = f_1^{-1} f_2^{-2}$ .

**Proof of Theorem 6.** Assume that the assumptions of Theorem 6 hold. By Theorem 9(e) function (6) is either a first integral, or an integrating factor of  $X$ . But, from Corollary 2 function (6) cannot be a first integral of  $X$  because  $l + \sum_{i=1}^p \delta f_i < m + 1$ . Hence, the proof is completed.  $\square$

#### 4. The examples

First, we provide three examples of a first integral satisfying statement (a) of Theorem 3.

The Darbouxian function  $H = y^{-3} \exp(3x^3/y)$  is of the form (3) with  $f_1 = y$ ,  $\lambda_1 = -3$ ,  $n_1 = 1$  and  $g = 3x^3$ . Then, the  $l$  defined in Theorem 3 satisfies  $l = 3$ . Therefore, since  $l + \sum_{i=1}^p \delta f_i = 4$ , and the polynomial vector field given by (4) is  $X = 3(y + x^3, 3x^2y)$  with  $m = 3$ , it follows that  $H$  and  $X$  satisfy statement (a) of Theorem 3.

The next first integral and its corresponding polynomial vector field provide examples satisfying Theorem 3(a) and Theorem 5. The Darbouxian function  $H = xy(x - 1 + y/3)$  is of the form (3) with  $f_1 = x$ ,  $f_2 = y$ ,  $f_3 = x - 1 + y/3$ ,  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ ,  $n_1 = n_2 = n_3 = 0$  and  $g = 0$ . Then, the  $l = 0$ . Therefore, since  $l + \sum_{i=1}^p \delta f_i = 3$ , and the polynomial vector field given by (4) is  $X = (x(1 - x - 2y/3), y(-1 + 2x + y/3))$  with  $m = 2$ , we get that  $H$  and  $X$  satisfy statement (a) of Theorem 3, because  $X$  has the first integral  $H$  and the integrating factor  $1/H$ . Additionally, this is an example satisfying Theorem 5.

Now the third example satisfying Theorem 3(a). The Darbouxian function  $H = (x^2 + y^2) \exp(2y)$  is of the form (3) with  $f_1 = x + iy$ ,  $f_2 = x - iy$ ,  $\lambda_1 = \lambda_2 = 1$ ,  $n_1 = n_2 = 0$  and  $g = 2y$ . Then, the  $l = 1$ . Therefore, since  $l + \sum_{i=1}^p \delta f_i = 3$ , and the polynomial vector field given by (4) is  $X = 2(-y - x^2 - y^2, x)$  with  $m = 2$ , we have that  $H$  and  $X$  satisfy statement (a) of Theorem 3, because  $X$  has the first integral  $H$  and the integrating factor  $1/(x^2 + y^2)$ .

Now we shall provide two examples satisfying statement (b) of Theorem 3. The Darbouxian function  $H = y^{-4}(x^3 + x^4 + y^4)$  is of the form (3) with  $f_1 = y$ ,  $f_2 = x^3 + x^4 + y^4$ ,  $\lambda_1 = -4$ ,  $\lambda_2 = 1$ ,  $n_1 = n_2 = 0$  and  $g = 0$ . Then, the  $l = 0$ . Therefore,  $l + \sum_{i=1}^p \delta f_i = 5$ , and the polynomial vector field given by (4) is  $(P, Q) = (4x^3(1 + x), x^2(3 + 4x)y)$  with  $P$  and  $Q$  non-coprime. So,  $X = (4x(1 + x), y(3 + 4x))$  with  $m = 2$  is the polynomial vector field satisfying statement (b) of Theorem 3.

The second example is the following one. The Darbouxian function  $H = (x + 1)^{-2}(y - x^2) \exp(-1/(x + 1))$  is of the form (3) with  $f_1 = x + 1$ ,  $f_2 = y - x^2$ ,  $\lambda_1 = -2$ ,  $\lambda_2 = 1$ ,

$n_1 = 1$ ,  $n_2 = 0$  and  $g = -1$ . Then,  $l = 1$ . Therefore,  $l + \sum_{i=1}^p \delta f_i = 4$ , and the polynomial vector field given by (4) is  $X = (-(x+1)^2, -2x - y - 3x^2 - 2xy)$  with  $m = 2$  satisfying statement (b) of Theorem 3.

Finally we provide an example satisfying Theorem 6. The polynomial vector field  $X = (x(y+1), -y(x+1))$  with  $m = 2$  has the invariant algebraic curve  $f_1 = x$  with cofactor  $K_1 = y+1$ , the exponential factor  $\exp(x+y+1)$  with cofactor  $L = x - y = -\text{div}$ , and the weak independent singular point  $(-1, -1)$  which is not on  $f_1 = 0$ . Therefore,  $l = 1$ ,  $p = q = r = 1$ , and consequently it satisfies  $p + q + r = m(m+1)/2 = 3$  and  $l + \sum_{i=1}^p \delta f_i = 2 < m + 1 = 3$ , and it has  $f_1^0 \exp(x+y+1)$  as integrating factor. Hence,  $X$  is an example of a polynomial vector field satisfying Theorem 6. We note that, from Theorem 9(a), there does not exist a first integral given by a Darbouxian function of the form  $f_1^{\lambda_1} \exp(x+y+1)^{\mu_1}$ .

## References

- [1] L. Cairó, M.R. Feix, J. Llibre, Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems, *Resenhas de Universidade de São Paulo* 4 (2000) 127–161.
- [2] L. Cairó, J. Llibre, Darbouxian first integrals and invariants for real quadratic systems having an invariant conic, *J. Phys. A* 35 (2002) 589–608.
- [3] J. Chavarriga, J. Llibre, J. Sotomayor, Algebraic solutions for polynomial systems with emphasis in the quadratic case, *Expositiones Math.* 15 (1997) 161–173.
- [4] C.J. Christopher, Invariant algebraic curves and conditions for a center, *Proc. Roy. Soc. Edinburgh A* 124 (1994) 1209–1229.
- [5] C. Christopher, R. Kooij, Algebraic invariant curves and the integrability of polynomial systems, *Appl. Math. Lett.* 6 (1993) 51–53.
- [6] C. Christopher, J. Llibre, Algebraic aspects of integrability for polynomial systems, *Qualitative Theory of Planar Differential Equations* 1 (1999) 71–95.
- [7] C. Christopher, J. Llibre, Integrability via invariant algebraic curves for planar polynomial differential systems, *Ann. Differential Equations* 16 (2000) 5–19.
- [8] C.J. Christopher, J. Llibre, C. Pantazi, X. Ziang, Darboux integrability and invariant algebraic curves for planar polynomial systems, *J. Phys. A: Gen. Math.* 35 (2002) 2457–2476.
- [9] C. Christopher, J. Llibre, J.V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Preprint, 2003.
- [10] G. Darboux, Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré (Mélanges), *Bull. Sci. Math. 2ème série* 2 (1878) 60–96; 123–144; 151–200.
- [11] R.E. Kooij, Real polynomial systems of degree  $n$  with  $n + 1$  line invariants, *J. Differential Equations* 116 (1995) 249–264.
- [12] J.P. Jouanolou, Equations de Pfaff algébriques, in: *Lectures Notes in Mathematics*, vol. 708, Springer-Verlag, New York, 1979.
- [13] J. Llibre, J.S. Pérez del Río, J.A. Rodríguez, Phase portraits of a new class of integrable quadratic vector fields, *Dynamics of Continuous, Discrete and Impulsive Systems* 7 (2000) 595–616.
- [14] J.V. Pereira, Integrabilidade de equações diferenciais no plano complexo, in: *Monografias del IMCA*, vol. 25, Lima, Peru, 2002.
- [15] M.J. Prellé, M.F. Singer, Elementary first integrals of differential equations, *Trans. Amer. Math. Soc.* 279 (1983) 613–636.
- [16] D. Schlomiuk, Elementary first integrals and algebraic invariant curves of differential equations, *Expositiones Math.* 11 (1993) 433–454.
- [17] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, *Trans. Amer. Math. Soc.* 338 (1993) 799–841.

- [18] D. Schlomiuk, Algebraic and geometric aspects of the theory of polynomial vector fields, in: D. Schlomiuk (Ed.), *Bifurcations and Periodic Orbits of Vector Fields*, 1993, pp. 429–467.
- [19] M.F. Singer, Liouvillian first integrals of differential equations, *Trans. Amer. Math. Soc.* 333 (1992) 673–688.
- [20] S. Walcher, Plane polynomial vector fields with prescribed invariant curves, *Proc. Roy. Soc. Edinburgh A* 130 (2000) 633–649.
- [21] H. Żołądek, The solution of the center-focus problem, Preprint, 1992.
- [22] H. Żołądek, On algebraic solutions of algebraic Pfaff equations, *Studia Math.* 114 (1995) 117–126.