Exponential decay in one-dimensional porous-thermo-elasticity

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Acknowledgments

This work is supported by the project "Stability aspects in thermomechanics" (BFM2003-00309) of the Spanish Ministry of Science and Technology.

Keywords: porous-thermo-elasticity, exponential decay.

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Abstract

This paper concerns the one dimensional problem of the porous-thermo-elasticity. Two kinds of dissipation process are considered: the viscosity type in the porous structure and the thermal dissipation. It is known that when only thermal damping is considered or when only porous damping is considered we have the slow decay of the solutions. Here we prove that when both kinds of dissipation terms are taken into account in the evolution equations the solutions are exponentially stable.

1 INTRODUCTION

An increasing interest has been developed in recent years to determine the decay behavior of the solutions of several elasticity problems. It is known that combining the equations of elasticity with thermal effects provokes exponential stability of solutions in the one-dimensional case [1]. Several results of this kind were obtained by Muñoz-Rivera [2], Jiang [3] and Lebeau and Zuazua [4]. The classical theory has been studied in the book of Jiang and Racke [5]. The Lord and Shulmann theory of thermoelasticity has been studied by Racke [6] and the thermoelasticity of type III has been considered by Quintanilla and Racke [7] and Zhang and Zuazua [8]. It is worth recalling the book of Liu and Zheng [9]. There, the authors use in a systematic way a technique to prove the exponential stability of C^0- semigroups in order to apply them to several thermomechanical problems. In the present paper we consider the theory of thermoelastic solids with voids. In a recent paper the authors proved the slow decay of the solutions of the one-dimensional elasticity with porosity dissipation [10] and for the thermoelasticity without porous dissipation [11]. A natural question is: Does the combination of thermal dissipation and the porous viscosity damp exponentially the deformations of porous elastic solids?

In what follows we consider a theory for the behavior of porous solids such that the matrix material is elastic and the interstices are void of material; it is a generalization of the classical theory of elasticity. The theory of porous elastic material has been established by Cowin and Nunziato [12, 13, 14]. In this theory the bulk density is the product of two scalar fields, the matrix material density and the volume fraction field; it is studied in the book of Ciarletta and Iesan [15]. Thermal effects were included in the work of Iesan [16]. Results on linear and nonlinear problems have been obtained recently [17, 18, 19, 20]. In the one-dimensional case the evolution equations are:

$$\rho\ddot{u} = t_x, \quad \rho\kappa\ddot{\phi} = h_x + g, \quad d\dot{\eta} = q_x. \tag{1.1}$$

Here t is the stress, h is the equilibrated stress, g is the equilibrated body force and q is the heat flux. The variables u, ϕ and η are the displacement of the solid elastic material, the volume fraction and the entropy respectively. We assume that ρ, κ and d are positive constants whose physical meaning is well known. The constitutive equations are:

$$t = \mu u_x + b\phi - \beta\theta, \;\; h = \alpha\phi_x, \;\; \eta = \beta u_x + m\phi + c\theta, \;\; g = -bu_x - \xi\phi + m\theta - \tau\dot{\phi}, \;\; q = k\theta_x. \tag{1.2}$$

We assume that the internal energy density is a positive definite form. Thus, the constitutive coefficients satisfy the conditions:

$$\mu > 0, \;\; \alpha > 0, \;\; c > 0, \;\; \xi > 0, \mu\xi > b^2. \tag{1.3}$$

We assume as well the positivity of the thermal heat conduction constant k and that the parameter β is different from zero.

It is worth noting that the constitutive equations (1.2) and the assumptions (1.3) are compatible with the usual theories of thermoelastic solids with voids in case of viscosity with respect the variable ϕ [14].

If we introduce the constitutive equations in the evolution equations, we obtain the field equations:

$$\rho\ddot{u} = \mu u_{xx} + b\phi_x - \beta\theta_x, \tag{1.4}$$

$$J\ddot{\phi}=\alpha\phi_{xx}-bu_x-\xi\phi+m\theta-\tau\dot{\phi}, \quad (1.5)$$

$$c\dot{\theta}=k^*\theta_{xx}-\beta v_x-m\dot{\phi}. \quad (1.6)$$

Here $J=\rho\kappa$ and $k^*=k/d$, but from now on we omit the star.

The aim of this article is to prove that the solutions of the porous-thermo-elasticity system defined by (1.4)-(1.6) decay exponentially. So it is worth noting that, in this case, the sum of two slow decay processes determine a process that decay exponentially. It is worth noting that the proof we propose here is based in the methods developed by Liu and Zheng in his book [9].

To the system (1.4)-(1.6) we adjoin the initial conditions

$$u(x,0)=u^0(x),\quad \phi(x,0)=\phi^0(x),\quad \theta(x,0)=\theta^0(x),\quad x\in[0,\pi], \quad (1.7)$$

$$\dot{u}(x,0)=v^0(x),\quad \dot{\phi}(x,0)=\varphi^0(x),\quad x\in[0,\pi], \quad (1.8)$$

and the homogeneous boundary conditions:

$$u(x,t)=\phi_x(x,t)=\theta_x(x,t)=0,\quad x=0,\pi,\ \ t\in(0,\infty). \quad (1.9)$$

It is worth noting that the existence of solutions of the problem determined by system (1.4)-(1.6) can be obtained by means of the semigroup theory [18]. In fact, we obtain contractive a semigroup of contractions.

The systems of equations considered here introduce new mathematical difficulties in order to determine the asymptotic behavior. As far as the authors know, there are no contributions made in this sense.

Whenever boundary conditions (1.9) are taken into account we impose that

$$\int_0^\pi \varphi^0(x)dx=\int_0^\pi \phi^0(x)dx=\int_0^\pi \theta^0(x)dx=0. \quad (1.10)$$

It is worth noting that conditions (1.10) are imposed to guarantee that the solutions decay. It is known that for the problem determined by (1.4)-(1.9) we can always take solutions where ϕ and θ are constants. Thus, if we want to avoid this behavior we need to impose conditions (1.10).

It will be seen (in the proofs) that alternative boundary conditions on ϕ and θ could be studied in a similar way.

2 EXPONENTIAL DECAY

In this section we study the behavior of solutions of the porous-thermo-viscoelasticity problem determined by the system of equations (1.4)-(1.6) and the conditions (1.7)-(1.9). It is worth noting that if we do not consider thermal effects, the solutions of the porous-viscoelasticity decay slowly. In a recent article [11] we have studied the system of equations of thermo-microstretch elastic solids. We have also proved slow decay of solutions. In the one-dimensional problem the system of equations (without porous viscous effects) agrees with the system of equations of thermo-microstretch elastic solids. Here, we prove that when we consider both dissipation effects we obtain exponential decay. Thus, the main goal of this note is to establish the following result:

Theorem 2.1 *Let (u, ϕ, θ) be a solution of the problem determined by system (1.4)-(1.6), initial conditions (1.7), (1.8) and boundary conditions (1.9). If the initial data satisfy condition (1.10), then, (u, ϕ, θ) decays exponentially.*

To prove this theorem, we consider the Hilbert space

$$\mathcal{H} = \{(u, v, \phi, \varphi, \theta) \in H_0^1 \times L^2 \times H^1 \times L^2 \times L^2, \int_0^\pi \phi dx = \int_0^\pi \varphi dx = \int_0^\pi \theta dx = 0\}.$$

The general solutions of our problem are given by the semigroup of contractions generated by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & Id & 0 & 0 & 0 \\ \rho^{-1}\mu D^2 & 0 & \rho^{-1} b D & 0 & -\rho^{-1}\beta D \\ 0 & 0 & 0 & Id & 0 \\ -J^{-1} b D & 0 & J^{-1}(\alpha D^2 - \xi) & -J^{-1}\tau & J^{-1} m \\ 0 & -c^{-1}\beta D & 0 & -c^{-1} m & c^{-1} k D^2 \end{pmatrix}, \quad D = \frac{d}{dx}. \tag{2.1}$$

In fact, if we consider the scalar product

$$< (u, v, \phi, \varphi, \theta), (u^*, v^*, \phi^*, \varphi^*, \theta^*) >_{\mathcal{H}} \tag{2.2}$$

$$= \int_0^\pi (\rho v \overline{v}^* + J \varphi \overline{\varphi}^* + c \theta \overline{\theta}^* + \mu u_x \overline{u}_x^* + \alpha \phi_x \overline{\phi}_x^* + \xi \phi \overline{\phi}^* + b(u_x \overline{\phi}^* + \overline{u}_x^* \phi)) dx,$$

we have that

$$\text{Re} < \mathcal{A}(u, v, \phi, \varphi, \theta), (u, v, \phi, \varphi, \theta) >= - \int_0^\pi (\tau |\varphi|^2 + k |\theta_x|^2) dx. \tag{2.3}$$

We shall prove first some lemmas.

Lemma 2.1 *Let $\mathcal{A}$ defined in (2.1). Then 0 is in the resolvent of $\mathcal{A}$.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we want to find $\omega \in \mathcal{H}$ such that

$$\mathcal{A}\omega = \mathcal{F}, \tag{2.4}$$

i. e.

$$v = f_1, \tag{2.5}$$

$$\rho^{-1}\mu D^2 u + \rho^{-1} b D\phi - \rho^{-1}\beta D\theta = f_2, \tag{2.6}$$

$$\varphi = f_3, \tag{2.7}$$

$$-J^{-1} b D u + J^{-1}(\alpha D^2 - \xi)\phi - J^{-1}\tau\varphi + J^{-1} m\theta = f_4, \tag{2.8}$$

$$-c^{-1}\beta D v - c^{-1} m \varphi + c^{-1} k D^2 \theta = f_5. \tag{2.9}$$

By (2.5), (2.7) and (2.9) we can write

$$c^{-1} k D^2 \theta = f_5 + c^{-1}\beta D f_1 + c^{-1} m f_3 \in L^2. \tag{2.10}$$

From the usual elliptic arguments we conclude that there exists a unique function $\theta \in H^2$ satisfying (2.10). Then, we plug θ just obtained to get

$$\rho^{-1}\mu D^2 u + \rho^{-1} b D\phi = f_2 + \rho^{-1}\beta D\theta, \tag{2.11}$$

$$-J^{-1} b Du + J^{-1}(\alpha D^2 - \xi)\phi = f_4 + J^{-1}\tau f_3 - J^{-1} m\theta. \tag{2.12}$$

In view of the last condition in (1.3) and the usual arguments on elliptic systems the unique solvability follows. It is clear from the regularity theory of linear elliptic equations that

$$||\omega||_{\mathcal{H}} \leq K||\mathcal{F}||_{\mathcal{H}}, \tag{2.13}$$

where K is a constant independent of ω.

Lemma 2.2 *Let $\mathcal{A}$ defined in (2.1). Then*

$$\{i\lambda, \lambda \in (-\infty, \infty)\} \textit{ is contained in the resolvent of } \mathcal{A}. \tag{2.14}$$

Following the arguments in [9], the proof consists of the following steps:

(i) Since 0 is in the resolvent of $\mathcal{A}$, using the contraction mapping theorem, we have that for any real λ such that $|\lambda| < ||\mathcal{A}^{-1}||^{-1}$, the operator $i\lambda\mathcal{I}\mathcal{D} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I}\mathcal{D})$ is invertible. Moreover, $||(i\lambda\mathcal{I}\mathcal{D} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1}, ||\mathcal{A}^{-1}||^{-1})$.

(ii) If $\sup\{||(i\lambda\mathcal{I}\mathcal{D} - \mathcal{A})^{-1}||, |\lambda| < ||\mathcal{A}^{-1}||^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I}\mathcal{D} - \mathcal{A} = (i\lambda_0\mathcal{I}\mathcal{D} - \mathcal{A})\Big(\mathcal{I}\mathcal{D} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I}\mathcal{D} - \mathcal{A})^{-1}\Big),$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $||\mathcal{A}^{-1}||^{-1}$ as we can, the set $\{\lambda, |\lambda| < ||\mathcal{A}^{-1}||^{-1} + M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $||(i\lambda\mathcal{I}\mathcal{D} - \mathcal{A})^{-1}||$ is a continuous function of λ in the interval $(-||\mathcal{A}^{-1}||^{-1} - M^{-1}, ||\mathcal{A}^{-1}||^{-1} + M^{-1})$.

(iii) If (2.14) is not true, then there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I}\mathcal{D} - \mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore there exists a sequence of real numbers λ_n with $\lambda_n \to \varpi,\ |\lambda_n| < |\varpi|$ and a sequence of vectors $\omega_n = (u_n, v_n, \phi_n, \varphi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$||(i\lambda_n\mathcal{I}\mathcal{D} - \mathcal{A})\omega_n|| \to 0. \tag{2.15}$$

This is

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{2.16}$$

$$i\lambda_n v_n - \rho^{-1}\mu D^2 u_n - \rho^{-1} b D\phi_n + \rho^{-1}\beta D\theta_n \to 0 \text{ in } L^2, \tag{2.17}$$

$$i\lambda_n \phi_n - \varphi_n \to 0 \text{ in } H^1, \tag{2.18}$$

$$i\lambda_n \varphi_n - J^{-1}(\alpha D^2\phi_n - \xi\phi_n) + J^{-1} b Du_n + J^{-1}\tau\varphi_n - J^{-1} m\theta_n \to 0 \text{ in } L^2, \tag{2.19}$$

$$i\lambda_n \theta_n - c^{-1} k D^2\theta_n + c^{-1}\beta Dv_n + c^{-1} m\varphi_n \to 0 \text{ in } L^2. \tag{2.20}$$

Taking the inner product of $(i\lambda_n\mathcal{I}\mathcal{D} - \mathcal{A})\omega_n$ times ω_n in $\mathcal{H}$ and then taking its real part yields

$$\tau||\varphi_n||^2 + k||D\theta_n||^2 \to 0. \tag{2.21}$$

From (2.20) it follows that

$$-kD^2\theta_n + \beta Dv_n \to 0 \text{ in } L^2. \tag{2.22}$$

Integrating (2.22) from 0 to x yields

$$-kD\theta_n + \beta v_n \to 0 \text{ in } L^2.$$

Then

$$v_n \to 0 \text{ in } L^2.$$

Taking the inner product of (2.17) times u_n and (2.19) times ϕ_n in L^2 and integrating by parts we obtain

$$||Du_n||,\ ||D\phi_n|| \to 0. \tag{2.23}$$

Thus we have shown that ω_n can not be of unit norm and the proof of (2.14) is complete.

Lemma 2.3 *Let $\mathcal{A}$ be the operator defined in (2.1). Then*

$$\overline{\lim}_{|\lambda|\to\infty}||(i\lambda\mathcal{I} - \mathcal{A})^{-1}|| < \infty. \tag{2.24}$$

Proof. We prove this by a contradiction argument. Suppose that (2.24) is not true. Then there exists a sequence λ_n with $|\lambda_n| \to \infty$ and a sequence of complex functions $\omega_n = (u_n, v_n, \phi_n, \varphi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ with unit norm such that (2.16)-(2.20) hold. Taking the real part of their inner product with ω_n in the Hilbert space yields conditions (2.21).

Dividing (2.20) by λ_n and using the Poincaré inequality, we get

$${\lambda_n}^{-1}(c^{-1}kD^2\theta_n - c^{-1}\beta Dv_n) \to 0 \text{ in } L^2. \tag{2.25}$$

Dividing (2.16) by λ_n and using (2.25), we obtain

$${\lambda_n}^{-1}c^{-1}kD^2\theta_n - ic^{-1}\beta Du_n \to 0 \text{ in } L^2. \tag{2.26}$$

Since $||Du_n||$ is bounded, we see that $||{\lambda_n}^{-1}D^2\theta_n||$ is bounded. Taking the inner product of (2.26) times Du_n yields

$$\Big({\lambda_n}^{-1}c^{-1}kD^2\theta_n, Du_n\Big) - ic^{-1}\beta||Du_n||^2 \to 0. \tag{2.27}$$

An integration by parts gives

$$\Big({\lambda_n}^{-1}c^{-1}kD^2\theta_n, Du_n\Big) = -\Big({\lambda_n}^{-1}c^{-1}kD\theta_n, D^2u_n\Big). \tag{2.28}$$

Dividing (2.17) by λ_n, we deduce that $||{\lambda_n}^{-1}D^2u_n||$ is bounded. In view of (2.21) we have

$$\Big(c^{-1}kD\theta_n, {\lambda_n}^{-1}D^2u_n\Big) \to 0. \tag{2.29}$$

From (2.27), we obtain that

$$||Du_n|| \to 0, \tag{2.30}$$

and

$${\lambda_n}^{-1}Dv_n \to 0. \tag{2.31}$$

Taking the inner product of (2.17) times v_n and dividing by λ_n, one obtains that

$$i||v_n||^2 + \rho^{-1}\mu\Big(Du_n, {\lambda_n}^{-1}Dv_n\Big) - \rho^{-1}b\Big(D\phi_n, {\lambda_n}^{-1}v_n\Big) \to 0. \tag{2.32}$$

Therefore, we obtain that

$$v_n \to 0 \text{ in } L^2. \tag{2.33}$$

Now, we multiply (2.19) by ϕ_n to obtain

$$\Big(i\lambda_n\varphi_n, \phi_n\Big) + J^{-1}\alpha||D\phi_n||^2 + J^{-1}\xi||\phi_n||^2 \to 0. \tag{2.34}$$

In view of (2.18) we can obtain that

$$-\Big(\varphi_n, \varphi_n\Big) + J^{-1}\alpha||D\phi_n||^2 + J^{-1}\xi||\phi_n||^2 \to 0, \tag{2.35}$$

which implies that $D\phi_n$ tends to zero in L^2. But these behaviors contradict that the sequence ω_n has norm unity.

Now, we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1 In view Lemmas 2.1 to 2.3, we can use the result due to Gearhart (see Wyler [21] and [9]) which states that a semigroup of contractions on a Hilbert space is exponentially stable if and only if conditions (2.14) and (2.24) hold. Thus the theorem is proved.

It is not difficult to extend our analysis to the case of other boundary conditions.

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