# The motive of moduli spaces of rank two vector bundles over a curve.

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September 19, 1996

**Abstract**

We study the motive of the moduli spaces of semistable rank two vector bundles over an algebraic curve. When the degree is odd the moduli space is a smooth projective variety, we obtain the absolute Hodge motive of this, and in particular the Poincaré-Hodge polynomial. When the degree is even the moduli space is a singular projective variety, we compute the pure motivic Poincaré polynomial and show that only two weights can occur in each cohomology group. As corollaries we obtain the isogeny type of some intermediate jacobians of the smooth moduli space and the motive and Hodge numbers of Seshadri's smooth model for the singular moduli space.

## 1 Introduction.

The moduli space of stable vector bundles over an algebraic curve is a relatively well-known object, it has received great attention for the last twenty years, in particular when the rank and degree are coprime its cohomology has been shown to be torsion free and its Betti numbers are known. However the methods used in studying its cohomology are topological ([20]), number theoretical ([11], [9]) or infinite-dimensional ([1]), and these, at least in principle, do not yield information on the algebraic structure of the cohomology of the moduli space, say its Hodge numbers or Hodge structure.

In this paper we use a recent construction by M. Thaddeus ([25]) to give a description of the motivic Poincaré polynomial of the moduli space of rank two semistable vector bundles of fixed determinant on an algebraic curve. It is an idea of Grothendieck (see [23]) that one should work in the Grothendieck group $K_0$ of the category of motives, this is where the motivic Poincaré polynomial lives. We believe that the theory of motives is an effective language to express clearly and precisely how the algebro-geometric properties of the curve influence those of the moduli space of stable vector bundles, as a manifestation of this belief we show how to prove a result by I. Biswas [6] in this framework. However at the present moment we do not have at our disposal the true category of motives $\mathcal{M}_k$ of Grothendieck, since the standard conjectures remain unproven, so we use the definition by Deligne of absolute Hodge motives $\mathcal{M}_k^{AH}$.

We start by giving a quick review of the theory of absolute Hodge motives, the natural language in which motives are expressed is that of tannakian categories so we recall the

[0]1991 *Mathematics Subject Classification.* Primary 14C15, 14A20; Secondary 14C35.

*Partially supported by DGCYT grant PB93-0790.

basic facts of these, we also define the motivic Poincaré polynomial, this is done in section 2.

In order to carry out the calculations we need a motivic version of MacDonald's formula for the Betti numbers of a symmetric power, we do this in section 3. Note that in fact we get an expression for the motive of $X^{(n)}$ and not only for $P^{mot}X^{(n)} \in K_0\mathcal{M}_k^{AH}$.

Then in section 4 we give a short account of Thaddeus' construction of the moduli spaces of pairs.

In section 5 is where with the aid of Thaddeus' construction we manage to calculate the motivic Poincaré polynomial of the moduli space $N_0(2,1)$ of stable rank 2 vector bundles with fixed odd determinant.

In section 6 we study the singular moduli space $N_0(2,0)$ of rank 2 semistable vector bundles with fixed even determinant. We use the geometric Hecke correspondence as defined by Narasimhan an Ramanan in [18]. Our results differ from those of Kirwan ([13]) in that we use pure Poincaré polynomials whereas she finds the intersection cohomology Poincaré polynomial. As a corollary we obtain the motive of the smooth model of Seshadri ([24]) generalising the results of Balaji and Seshadri ([4], [5]).

Finally in section 7 we extract information concerning the intermediate jacobians of the moduli spaces from the motivic Poincaré polynomial.

*Acknowledgement:* I would like to express my gratitude to Vicente Navarro Aznar and Pere Pascual for the invaluable help and encouragement they have shown me during the preparation of this paper and P. A. Vishwanath for some illuminating discussions on the Hecke correspondence. I would also like to thank my colleagues at the Barcelona Cigrons Geometry Seminar.

Throught the paper $k$ will denote a field of characteristic zero.

## 2 Absolute Hodge motives.

In this section we give a brief review of the theory of absolute Hodge motives and related topics. For proofs of theorems and more precise statements refer to [8].

### 2.1 Tannakian categories.

By a tannakian category we shall mean a $k$-linear abelian neutral rigid tensor category with $End(\mathbb{1}) = k$. Let $\mathcal{C}$ be a tannakian category, the fact that $\mathcal{C}$ is neutral means there exists a faithful exact functor from $\mathcal{C}$ to the category of finite dimensional vector spaces over $k$

$$\omega : \mathcal{C} \longrightarrow \mathbf{Vec_k}$$

called a fibre functor.

In a rigid tensor category there is a concept of rank. However in the case of a tannakian category this definition of rank gets simplified by the use of a fibre functor, $\text{rank}M = \dim_k \omega(M)$.

An example of tannakian category is the category of finite dimensional $k$-representations of an affine group scheme $\mathcal{G}$ over $k$, the fibre functor being the obvious one. In fact a fundamental theorem ([8], 2.11) states that all tannakian categories arise in this way, so that if $\mathcal{C}$ is a tannakian category with fibre functor $\omega$ then there exists an affine group scheme $\mathcal{G}_\omega$ over $k$ and an equivalence of categories $\mathcal{C} \longrightarrow \textbf{Rep-}\mathcal{G}_\omega$ compatible with the fibre functors, $\mathcal{G}_\omega$ is then called the Galois group or fundamental group of the tannakian category $\mathcal{C}$ respect to $\omega$, $Gal(\mathcal{C},\omega)$.

**2.1** *Examples:*

1. Consider the category of finite dimensional graded vector spaces over $k$, it is a tannakian category which is easily seen to be equivalent to **Rep-**$\mathbb{G}_{\mathbf{m}}$.

2. The category of local systems of finite dimensional complex vector spaces over a topological space $X$ is a tannakian category. A fibre functor is obtained by assigning to each local system $\mathcal{L}$ the complex vector space $\mathcal{L}_x$, where $x$ is a point of $X$. The fundamental group of this tannakian category is naturally isomorphic to $\pi_1(X,x)$, the fundamental group of homotopy classes of loops based at $x$.

3. The category of rational pure Hodge structures, $\mathcal{HS}_{\mathbb{Q}}$ is a tannakian category.

One can also introduce the notion of a graded (by $\mathbb{Z}$) tannakian category ([8], §5). It is one where every object has a finite direct sum decomposition compatible with a Künneth formula. A pure object is defined to be one with trivial decomposition, its degree will also be refered to as its *weight*. This is better expressed by using the Galois group, a graded tannakian category is a tannakian category $\mathcal{C}$ together with a central morphism $\mathbb{G}_m \longrightarrow Gal(\mathcal{C})$.

However a richer structure appears quite naturally in the theory of motives: $(\mathcal{C}, w, T)$ is called a Tate triple if $\mathcal{C}$ is a tannakian category graded by $w : \mathbb{G}_m \longrightarrow Gal(\mathcal{C})$, and $T$ is a weight $-2$ invertible object called the Tate object. The result of tensoring an object by the Tate object is usually refered to as a Tate twist. A standard way to abreviate $A \otimes T^{\otimes i}$ is $A(i)$.

The following definition will be useful.

**Definition 2.2** *If $\mathcal{C}$ is a tensor category then $\mathcal{C}[[T]]$ is the tensor category whose objects are*

$$Ob(\mathcal{C}[[T]]) = \{(A_i)_{i\in\mathbb{N}} | A_i \in Ob(\mathcal{C})\}$$

*(also written as $\sum A_i T^i$). The morphisms are defined by*

$$Hom(\sum A_i T^i, \sum B_i T^i) = \prod Hom(A_i, B_i).$$

*The tensor product of $\sum A_i T^i$ and $\sum B_i T^i$ is defined as*

$$\sum_n \left( \bigoplus_{i+j=n} A_i B_j \right) T^n.$$

*This tensor product inherits associativity and commutativity constraints from $\mathcal{C}$. Define natural functors*

$$\mathrm{Coef}_{T^n} : \mathcal{C}[[T]] \longrightarrow \mathcal{C} \tag{1}$$

*sending $\sum A_i T^i$ to $A_n$ (this is not however a tensor functor).*

Recall that in a tensor category there are commutation constraints, that is for every pair of objects $M, N \in Ob(\mathcal{C})$ isomorphisms

$$M \otimes N \xrightarrow{\varphi} N \otimes M$$

**2.3** Let $\mathcal{C}$ be a graded tensor category. Consider the new commutation constraints, given on pure objects $M_i$, $N_j$ of weights $i$ and $j$ by $(-1)^{ij}\varphi$ where $\varphi$ are the old commutation constraints. Call $\dot{\mathcal{C}}$ the resulting tensor category.

In the case when $\mathcal{C}$ is a tannakian category $\dot{\mathcal{C}}$ need not be tannakian. For instance, if $\mathcal{C}$ is $\mathcal{M}_k^{AH}$ (see 2.2) then $\dot{\mathcal{C}}$ is called the false category of motives $\dot{\mathcal{M}}_k^{AH}$ ([8], §6).

## 2.2 Absolute Hodge motives.

We shall work with the category of smooth projective varieties over $k$, $\mathcal{V}_k$.

The main problem in the theory of motives is to find a tannakian category that factors all possible cohomology functors. Grothendieck gave a construction of such a category $\mathcal{M}_k$ (see [15]) but in order to prove it has the required properties one needs the standard conjectures which remain unproven.

Deligne ([8]) has given a temporary working definition for motives, these are the absolute Hodge motives which we shall use in what follows. The category $\mathcal{M}_k^{AH}$ is constructed in exactly the same way as $\mathcal{M}_k$ but using absolute Hodge cycles instead of algebraic cycles. We recall that an absolute Hodge cycle of $X$ of codimension $p$ is an element of

$$F^0 H^{2p}_{DR}(X)(p) \times \prod_l H^{2p}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(p) \times \prod_{\sigma: k \hookrightarrow \mathbb{C}} H^{2p}_{sing}(X_\sigma, \mathbb{Q})(p)$$

such that it is compatible with the comparison isomorphisms between de Rham, singular and tale cohomology. We denote the group of such cycles by $Z^p_{AH}(X)$.

In the same manner as with Grothendieck motives we get a functor, $h$, from the category of smooth projective varieties over $k$ to the category of AH-motives.

One advantage of working with AH-motives is that the Künneth components of the diagonal in $H^{2d}(X \times X)$ are again AH-cycles, so we get a decomposition $hX = h^0X \oplus h^1X \oplus \cdots \oplus h^{2d}X$. This makes $\mathcal{M}_k^{AH}$ into a graded tannakian category, it is customary to refer to this grading as the weight grading. A motive that is zero in all degrees except in one is called a pure weight motive. As $\mathcal{M}_k^{AH}$ is a graded tannakian category one has a graded fibre functor

$$\begin{aligned} \mathcal{M}_k^{AH} &\longrightarrow \textbf{Grad-Vec}_k \\ M = \oplus M_i &\longrightarrow \oplus H^i_{DR}(M). \end{aligned}$$

**2.4** *Remarks:*

1. The cycle maps produce an absolute Hodge cycle for each p-codimensional algebraic cycle so one gets a morphism $Z^p(X) \longrightarrow Z^p_{AH}(X)$. In this way we get a functor $\mathcal{M}_k \longrightarrow \mathcal{M}_k^{AH}$.

2. An important thing to know about the category of AH-motives is that it is a full subcategory of the category of realization systems defined in [12]. The motive $h^i(X)$ can thus be seen as a triple $\left(H^i_{DR}(X,k), H^i_\sigma(X,\mathbb{Q}), H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)\right)$, where $H^i_{DR}(X,k)$

is a finite dimensional $k$-vector space with a filtration $F^{\cdot}$ (the Hodge filtration), for each embedding $k \overset{\sigma}{\hookrightarrow} \mathbb{C}$ $H^i_\sigma(X,\mathbb{Q})$ is a rational pure Hodge structure of weight $i$ and for each prime $\ell$ $H^i_{\text{ét}}(X,\mathbb{Q}_\ell)$ is a $Gal(\overline{k},k)$-module, together with comparison isomorphisms.

## 2.3 $\mathbf{K_0\mathcal{M}_k^{AH}}$ and the motivic Poincaré polynomial

Recall that to every abelian category $\mathcal{C}$ one can attach the Grothendieck group $K_0\mathcal{C}$. Moreover if $\mathcal{C}$ is a (graded) tensor category then $K_0\mathcal{C}$ is a (graded) unitary commutative ring. Given $A$ an object $\mathcal{C}$ we shall use the notation $[A]$ for its class in $K_0\mathcal{C}$. If $\mathcal{C}$ is a tensor category and $\mathcal{C}[[T]]$ is the tensor category defined in 2.2 then $K_0(\mathcal{C}[[T]]) = K_0\mathcal{C}[[T]]$ and the additive functor $\mathrm{Coef}_{T^n}$ induces the standard morphism $\mathrm{Coef}_{T^n} : K_0\mathcal{C}[[T]] \longrightarrow K_0\mathcal{C}$.

In particular if we put $\mathcal{C} = \mathcal{M}_k^{AH}$ we get a graded ring $K_0\mathcal{M}_k^{AH}$. In the category $\mathcal{M}_k^{AH}$ one has the Tate twist

$$\begin{array}{ccc} \mathcal{M}_k^{AH} & \overset{\cdot\otimes\mathbb{1}(n)}{\longrightarrow} & \mathcal{M}_k^{AH} \\ A & \longrightarrow & A(n) \end{array}$$

and the dualising functor

$$\begin{array}{ccc} \mathcal{M}_k^{AH} & \overset{\cdot^\vee}{\longrightarrow} & \mathcal{M}_k^{AH} \\ A & \longrightarrow & A^\vee \end{array}$$

both of which are exact functors and so descend to additive morphisms of the graded ring $K_0\mathcal{M}_k^{AH}$.

**Definition 2.5** *Let $M$ be an AH-motive then its motivic Poincaré polynomial is defined to be its class in the graded ring $K_0\mathcal{M}_k^{AH}$*

$$P^{mot}M = [M] \in K_0\mathcal{M}_k^{AH}$$

**2.6** *Remarks:*

1. Note that this is not really a polynomial, it is an element of a graded ring.
2. If $X$ is a smooth projective variety over $k$ then we shall write $P^{mot}X = P^{mot}hX$.
3. This is a generalisation of the usual Poincaré polynomial as can be seen by following $hX = \oplus h^iX$ through the commutative diagram

$$\begin{array}{ccc} Ob(\mathcal{M}_k^{AH}) & \overset{[\cdot]}{\longrightarrow} & K_0\mathcal{M}_k^{AH} \\ {\scriptstyle Ob(H^*_{DR})}\downarrow & & {\scriptstyle K_0(H^*_{DR})}\downarrow \\ Ob(\mathbf{Grad\text{-}Vec_k}) & \overset{[\cdot]}{\longrightarrow} & K_0\mathbf{Grad\text{-}Vec_k} = \mathbb{Z}[t,t^{-1}] \end{array}$$

4. If $\widetilde{X}$ is the blow up of a smooth variety $X$ along a smooth closed subvariety $Y$ then $P^{mot}\widetilde{X} = P^{mot}X - P^{mot}Y + P^{mot}Y(\mathbb{1} + \cdots + \mathbb{1}(-\mathrm{codim}_X Y))$. If $\mathcal{E}$ is a vector bundle over $X$ then $P^{mot}\mathbb{P}\mathcal{E} = P^{mot}X\,(\mathbb{1} + \cdots + \mathbb{1}(-\mathrm{rank}\mathcal{E})$ ([8], §6).

In general the map from isomorphism classes of objects of an abelian category $\mathcal{C}$ to $K_0\mathcal{C}$ is not injective, but as an application of the fact that $\mathcal{M}_k^{AH}$ is semisimple ([8], theorem 6.5) we show now that this is the case for $\mathcal{M}_k^{AH}$.

The following proposition is easily proven.

**Proposition 2.7** *Let $\mathcal{C}$ be an artinian abelian semisimple category (for example $\mathcal{M}_k^{AH}$) and $A, B, C \in Ob(\mathcal{C})$. If $A \oplus C \simeq B \oplus C$ then $A \simeq B$.*

**Corollary 2.8** *Let $M, N$ be AH-motives. If $P^{mot}M = P^{mot}N$ then $M \simeq N$.*

*Proof.* $P^{mot}M = P^{mot}N$ means that there exists $P \in Ob(\mathcal{M}_k^{AH})$ with $M \oplus P \simeq N \oplus P$, now use the previous proposition. □

Therefore whenever we need to prove an equality of motives it will be enough to prove it in $K_0$, and this is normally easier to write.

## 2.4 Mixed absolute Hodge motives.

The geometric methods in the definition of AH-motives do not extend at the present moment to the case of open or singular varieties. As already mentioned $\mathcal{M}_k^{AH}$ is a full subcategory of the category of realization systems, $\mathcal{R}_k$, this is very useful to construct a category of mixed absolute Hodge motives as there is a reasonable candidate for category of mixed systems of realizations, $\mathcal{MR}_k$, together with natural functors $h^i : \mathcal{W}_k \longrightarrow \mathcal{MR}_k$, where $\mathcal{W}_k$ denotes the category of varieties over $k$ (not necessarily smooth or proper) ([12], I.§2).

Let $\mathcal{V}_k^0$ denote the category of smooth varieties over $k$ (not necessarily proper). Jannsen ([12], I.§4) defines $\mathcal{MM}_k^{AH}$ to be the full tannakian subcategory of $\mathcal{MR}_k$ generated by the image of the $h^i : \mathcal{V}_k^0 \longrightarrow \mathcal{MR}_k$.

There is a functor

$$h : \mathcal{W}_k \longrightarrow \textbf{Grad-}\mathcal{MM}_{\mathbf{k}}^{\mathbf{AH}}$$

which assigns $\oplus h^i(X)$ to the variety $X$.

There is a natural fully faithful functor $\mathcal{M}_k^{AH} \longrightarrow \mathcal{MM}_k^{AH}$, $\mathcal{M}_k^{AH}$ can thus be seen as a full subcategory of $\mathcal{MM}_k^{AH}$. An object $M$ of $\mathcal{MM}_k^{AH}$ is provided with an increasing filtration $W_\cdot$ called the weight filtration and the associated graded object is a pure motive. This implies that the previous functor induces an isomorphism of rings $K_0\mathcal{M}_k^{AH} \xrightarrow{\simeq} K_0\mathcal{MM}_k^{AH}$. We now define a polynomial which via this isomorphism extends the motivic Poincaré polynomial.

**Definition 2.9** *([19]) Let $M = \oplus M_i \in Ob(\textbf{Grad-}\mathcal{MM}_k^{AH})$ be a graded mixed motive, then the pure motivic Poincaré polynomial is*

$$P^{mot}M = \sum_m \left( \sum_i (-1)^{m+i} [Gr_m^{W_\cdot} M_i] \right) \in K_0\mathcal{M}_k^{AH}.$$

**2.10** *Remarks:*

1. If $M = \oplus M_i$ with $M_i$ a pure motive of weight $i$, $Gr_m^{W_\cdot} M_i$ is equal to $M_i$ if $i = m$ and zero if $i \neq m$ so that this polynomial coincides with the one already defined.

2. Note that in the mixed case $P^{mot}M$ does not coincide with the class of $M$ in $K_0\mathcal{M}_k^{AH}$.

3. Let $X$ be a variety over $k$, $\oplus h^i X$ its mixed motive and $P^{mot}X$ its motivic Poincaré polynomial. Composition with the ring morphism

$$K_0\mathcal{M}_k^{AH} \longrightarrow K_0\textbf{Grad-Vec}_k = \mathbb{Z}[t, t^{-1}]$$

does not yield the classical Poincaré polynomial, $P_tX := \sum \dim H^i(X,\mathbb{Q})t^i$, but rather the pure Poincaré polynomial defined by

$$P_t^{pur}(X) = \sum_m \chi_m^{pur}(X)t^m, \text{ where } \chi_m^{pur}(X) = \sum_i (-1)^{i+m}\dim Gr_m^{W_\cdot} H^i(X,\mathbb{Q})$$

(c.f. [10], 185-191, [23] and [19]), which is better suited for computations than the ordinary Poincaré polynomial. For example if $Y$ is a closed subvariety of $X$ of codimension $d$ and both $X$ and $Y$ are smooth one has the Gysin exact sequence,

$$\cdots \longrightarrow h^{i-2d}Y(-d) \longrightarrow h^iX \longrightarrow h^i(X-Y) \longrightarrow \cdots$$

and as the functor $Gr_m^{W_\cdot}$ is exact one gets an equality in $K_0\mathcal{M}_k^{AH}$

$$\sum_i (-1)^i[Gr_m^{W_\cdot} h^iX] = \sum (-1)^i[Gr_m^{W_\cdot} h^i(X-Y)] + \sum (-1)^i[Gr_{m-2d}^{W_\cdot} h^{i-2d}Y](-d)$$

so that $P^{mot}X = P^{mot}(X-Y) + P^{mot}Y(-d)$.

4. One can define in the same fashion the mixed motive of a variety with compact supports, $h_cX = \oplus h_c^iX$, and its pure motivic Poincaré polynomial $P_c^{mot}X$ . If $Y$ is a closed subvariety of a variety $X$ then $P_c^{mot}X = P_c^{mot}(X-Y) + P_c^{mot}Y$. If $X$ is proper then $P_c^{mot}X = P^{mot}X$ and if $X$ is smooth then $P_c^{mot}X = (P^{mot}X)^\vee(-\dim X)$.

# 3 A motivic MacDonald formula.

Let $X$ be a compact polyhedron and consider $X^{(n)}$ the symmetric power of $X$, this is the quotient of $X^n$ by the natural action of the symmetric group $\mathfrak{S}_n$. MacDonald gave a formula ([14]) that computes Betti numbers of $X^{(n)}$ in terms of those of $X$, explicitly

$$P_tX^{(n)} = \mathrm{Coef}_{T^n} \frac{(1+tT)^{b_1(X)} \cdot (1+t^3T)^{b_3(X)} \cdots}{(1-T)^{b_0(X)} \cdot (1-t^2T)^{b_2(X)} \cdots}.$$

In this paragraph we give a motivic version of MacDonald's formula valid in any neutral $k$-linear graded tannakian category, in particular that of Absolute Hodge Motives or conjecturally Grothendieck's category of pure motives.

Let $\mathcal{C}$ be a tannakian category, in [8] (proposition 1.5) it is shown that the commutation constraints can be extended to cover the case of more than two factors so that for every $\sigma \in \mathfrak{S}_n$, we get isomorphisms

$$\varphi_\sigma : M_1 \otimes \cdots \otimes M_n \longrightarrow M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(n)}.$$

In particular if $M \in Ob(\mathcal{C})$ this defines an action of $\mathfrak{S}_n$ on $M^{\otimes n}$

$$\mathfrak{S}_n \xrightarrow{\varphi} Aut(M^{\otimes n}).$$

Let $\varepsilon : \mathfrak{S}_n \longrightarrow \{+1,-1\}$ denote the signature.

**Definition 3.1** *Given $M \in Ob(\mathcal{C})$ define $S^iM$ (resp. $\wedge^iM$) to be the image of the morphism $\frac{1}{i!}\sum_{\sigma\in\mathfrak{S}_i}\varphi_\sigma : M^{\otimes i} \longrightarrow M^{\otimes i}$ (resp. $\frac{1}{i!}\sum_{\sigma\in\mathfrak{S}_i}\varepsilon(\sigma)\cdot\varphi_\sigma$). Extend this definition to the case $i=0$ by putting $S^0M = \wedge^0M = \mathbb{1}$.*

**3.2** *Remarks:*

1. As the fibre functor $\omega$ is a tensor functor it sends $\varphi_\sigma$ to the canonical commutation constraints in $\mathbf{Vec}_k$. Combining this with the fact that $\omega$ is exact gives immediately that $\omega\left(\wedge^i M\right) = \wedge^i\omega(M)$, and using the faithfulness of $\omega$ we see that $\wedge^i M = 0$ for $i > \mathrm{rank} M$.

2. If $M$ is a rank one object then using again the fibre functor one immediately sees that $S^i M = M^{\otimes i}$.

**Definition 3.3** *If $M \in Ob(\mathcal{C})$ define*

$$\begin{aligned}(1+T)^M &= \sum \wedge^i M \cdot T^i \in Ob(\mathcal{C}[T])\\ (1-T)^{-M} &= \sum S^i M \cdot T^i \in Ob(\mathcal{C}[[T]]).\end{aligned}$$

If the rank of $M$ is one then $(1-T)^{-M} = \sum M^{\otimes i} T^i$ so we shall also use the notation $\frac{1}{1-MT}$ in this case. If $M$ and $N$ are rank one objects then $\frac{1}{M-NT}$ will stand for

$$M^{-1} \otimes \left(\frac{1}{1-M^{-1}NT}\right) \in Ob(\mathcal{C}[[T]]).$$

For the rest of the section $\mathcal{C}$ will denote a graded tannakian category over $k$. Recall that for a graded tensor category $\mathcal{C}$ we defined in 2.3 a tensor category $\dot{\mathcal{C}}$ by changing certain signs in the commutation constraints.

**Definition 3.4** *Define the symmetric power of $M$, $M^{(i)}$, in the same way as $S^i M$ but using the commutation constraints from $\dot{\mathcal{C}}$.*

**Proposition 3.5** *Let $M$ be a pure degree object of weight $n$ then $M^{(i)}$ is $S^i M$ if $n$ is even and $\wedge^i M$ if $n$ is odd.*

*Proof.* If $M$ is pure of even weight then the commutation constraints $M \otimes M \to M \otimes M$ are the same in $\dot{\mathcal{C}}$ and in $\mathcal{C}$ so $M^{(n)} = S^n M$.

In the odd weight case the commutation constraints change sign and when more than one factor appears then the sign is given by the signature $\varepsilon$ so we get $M^{(n)} = \wedge^n M$. □

The next theorem gives an expression for the symmetric power of an object in terms of symmetric powers of its pure components, it is our motivic version of the MacDonald formula.

**Theorem 3.6** *Let $M = \oplus M_i$ be an object in a graded neutral $k$-linear tannakian category, then*

$$M^{(n)} \simeq \mathrm{Coef}_{T^n} \frac{\cdots \otimes (1+T)^{M_{-1}} \otimes (1+T)^{M_1} \otimes (1+T)^{M_3} \otimes \cdots}{\cdots \otimes (1-T)^{M_{-2}} \otimes (1-T)^{M_0} \otimes (1-T)^{M_2} \otimes \cdots}. \tag{2}$$

*Proof.* We need to see that $M^{(n)}$ is isomorphic to

$$\begin{aligned}\mathrm{Coef}_{T^n} \quad &(\cdots \otimes \textstyle\sum_i \wedge^i M_{-1} T^i \otimes \sum_i \wedge^i M_1 T^i \otimes \sum_i \wedge^i M_3 T^i \otimes \cdots\\ &\otimes \textstyle\sum_i S^i M_{-2} T^i \otimes \sum_i S^i M_0 T^i \otimes \sum_i S^i M_2 T^i \otimes \cdots)\\ &= \textstyle\sum_{\lambda_1+\cdots+\lambda_k=n} M_{r_1}^{(\lambda_1)} \otimes \cdots \otimes M_{r_k}^{(\lambda_k)}.\end{aligned}$$

On the other hand, by Künneth

$$M^{\otimes n} = \bigoplus_{r_1,\ldots,r_n\in\mathbb{Z}} M_{r_1}\otimes\cdots\otimes M_{r_n}$$

Applying the exactness and faithfulness of a fibre functor one can check that the following morphism is a monomorphism

$$\left(M^{\otimes n}\right)^{\mathfrak{S}_n} \hookrightarrow M^{\otimes n} \longrightarrow \bigoplus_{\substack{s_1<\cdots<s_k\\ \sum\lambda_i=n}} M_{s_1}\otimes\overset{\lambda_1}{\cdots}\otimes M_{s_1}\otimes\cdots\otimes M_{s_k}\otimes\overset{\lambda_k}{\cdots}\otimes M_{s_k}.$$

Its image obviously lies in

$$\bigoplus_{\substack{s_1<\cdots<s_k\\ \sum\lambda_i=n}} \left(M_{s_1}\otimes\overset{\lambda_1}{\cdots}\otimes M_{s_1}\otimes\cdots\otimes M_{s_k}\otimes\overset{\lambda_k}{\cdots}\otimes M_{s_k}\right)^{\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_k}}.$$

This way we get an injection of the LHS into the RHS of (2), to complete the proof its enough to see that they both have the same rank but this is just the assertion of MacDonald's main theorem ([14]). □

Let $X$ be a smooth projective variety over $k$. Write $hX = \oplus h^iX \in Ob(\mathcal{M}_k^{AH})$, then by Proposition 6.8 in [8] $hX^{(n)} = (hX^{\otimes n})^{\mathfrak{S}_n}$, where the action of $\mathfrak{S}_n$ is the one arising from the geometric commutations $X\times\cdots\times X \longrightarrow X\times\cdots\times X$, so that $(hX^{\otimes n})^{\mathfrak{S}_n} = hX^{(n)}$ and this is computed using the formula in the Theorem, so we have

**Corollary 3.7** *The motive of $X^{(n)}$ is*

$$hX^{(n)} = \mathrm{Coef}_{T^n}\frac{(1+T)^{h^1X}\otimes(1+T)^{h^3X}\otimes\cdots}{(1-T)^{h^0X}\otimes(1-T)^{h^2X}\otimes\cdots}.$$

In particular if $C$ is a smooth projective curve

$$hC^{(n)} = \mathrm{Coef}_{T^n}\frac{(1+T)^{h^1C}}{(1-\mathbb{1}T)(1-\mathbb{1}(-1)T)}.$$

**3.8** *Remark:* If we apply the graded fibre functor

$$H^*_{DR}:\mathcal{M}_k^{AH}\longrightarrow \textbf{Grad-Vec}_\mathbf{k}$$

followed by $Ob(\textbf{Grad-Vec}_\mathbf{k}) \xrightarrow{[\cdot]} \mathbb{Z}[t,t^{-1}]$ we get the classical MacDonald formula, whereas if we do the same with

$$(H^*_{DR},F^{\cdot}):\mathcal{M}_k^{AH}\longrightarrow \textbf{Grad-Fil-Vec}_\mathbf{k}$$

and $Ob(\textbf{Grad-Fil-Vec}_\mathbf{k}) \xrightarrow{[\cdot]} \mathbb{Z}[x,y,x^{-1},y^{-1}]$ we get the Hodge numbers as in [7].

# 4 Thaddeus' construction.

In this section we review the basic construction of Thaddeus we shall use, for a more complete exposition see [25].

Let $C$ be a fixed smooth projective algebraic curve of genus $g \geq 2$ over $k$ and $\mathcal{L}$ a line bundle over C of *large* degree $d$. The moduli spaces we are primarily interested in are $N_0(2,d)(C)$ the moduli space of rank 2 semistable vector bundles with fixed determinant over $C$. They depend on the curve $C$ however we shall simply write $N_0(2,d)$.

Thaddeus considers the problem of giving a moduli space for pairs $(E,s)$, where $E$ is a rank 2 vector bundle over the curve $C$ with fixed determinant $\mathcal{L}$ and $s$ is a non-zero section of $E$. It appears that there are many possible definitions for stability of a pair depending on a parameter $\sigma \in [0, \frac{d}{2}]$. For $\sigma$ varying in certain open disjoint intervals there are no strictly semistable pairs and one obtains a finite list of fine moduli spaces of pairs $M_0$, ..., $M_\omega$ ($\omega = \left[\frac{d-1}{2}\right]$).

These different moduli spaces are all birational and are related by a special kind of birational maps called flips. In this context a flip between two varieties $X$ and $Y$ means that $X$ and $Y$ have a common blow-up, $\tilde{X} \simeq \tilde{Y}$, with the same exceptional locus. The centers of these blow-ups are a couple of subvarieties of $M_j$ called $\mathbb{P}W_i^+$ and $\mathbb{P}W_{i+1}^-$ isomorphic to certain projective bundles over symmetric products of the curve: $\mathbb{P}W_i^+$ is a $\mathbb{P}^{d-2i+g-2}$-bundle over $C^{(i)}$ and $\mathbb{P}W_{i+1}^-$ is a $\mathbb{P}^i$-bundle over $C^{(i+1)}$. To summarize, the blow-up of $M_i$ along $\mathbb{P}W_i^-$ is isomorphic to the blow-up of $M_{i-1}$ along $\mathbb{P}W_i^+$. We can picture this chain of flips:

$$\begin{array}{ccccccccccccc}
 & & \tilde{M}_2 & & & & \tilde{M}_3 & & & & \tilde{M}_\omega & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \cdots & \swarrow & & \searrow & \\
M_1 & & & & M_2 & & & & & & & & M_\omega \\
\downarrow & & & & & & & & & & & & \downarrow \\
M_0 & & & & & & & & & & & & N_0(2,d).
\end{array}$$

Moreover, it is easy to see that $M_0$ is a projective space of dimension $d+g-2$. In the other extreme we have $M_\omega$, in the case when deg$\mathcal{L}$ is odd $M_\omega$ is a projective bundle of relative dimension $d-2g+1$ over $N_0(2,1)$, the moduli space of rank two stable vector bundles over $C$ with fixed odd determinant, whereas if deg$\mathcal{L}$ is even we have a map from $M_\omega$ to the analogous moduli space which is only a projective fibration over the stable locus.

# 5 The motive of $N_0(2,1)$.

The purpose of this section is to give an expression for $hN_0(2,1)$ in terms of $h^1C$ and $\mathbb{1}(1)$. An immediate consequence is that $hN_0(2,1)$ is in the tannakian subcategory of $\mathcal{M}_k^{AH}$ generated by $hC$ and $\mathbb{1}(1)$.

The calculation of the Poincaré polynomial of the moduli space involves some infinite sums of motives thus falling outside of the ring $K_0\mathcal{M}_k^{AH}$, to formalise this we need to construct a greater ring $\widehat{K_0\mathcal{M}_k^{AH}}$, the ring of Laurent series of motives, this is done as follows: first consider the subring

$$K_0\mathcal{M}_k^{AH^+} = \{x \in K_0\mathcal{M}_k^{AH} | \text{deg} x \geq 0\} \subset K_0\mathcal{M}_k^{AH},$$

complete it with respect to the ideal $I$ formed by the strictly positive degree elements, tensor the result by $K_0\mathcal{M}_k^{AH}$ over $K_0\mathcal{M}_k^{AH+}$, then the result is the ring we were looking for.

**Definition 5.1**

$$\widehat{K_0\mathcal{M}_k^{AH}} = K_0\mathcal{M}_k^{AH} \otimes_{K_0\mathcal{M}_k^{AH+}} \widehat{K_0\mathcal{M}_k^{AH}}_I^{+}.$$

An easy but crucial result is the following.

**Lemma 5.2** *The natural graded ring morphism* $K_0\mathcal{M}_k^{AH} \longrightarrow \widehat{K_0\mathcal{M}_k^{AH}}$ *is a monomorphism.*

Note that if $A, B$ are invertible motives with $\deg B > \deg A$ then $A - B \in K_0\mathcal{M}_k^{AH}$ is invertible in $\widehat{K_0\mathcal{M}_k^{AH}}$ and its inverse is given by

$$\frac{1}{A-B} = A^{-1} + A^{-2}\cdot B + A^{-3}\cdot B^2 + \cdots.$$

**Proposition 5.3** *The motive of the moduli space of pairs* $M_i$ *is given by*

$$hM_i = \sum_{j=0}^{i} hC^{(j)} \otimes (\mathbb{1}(-j) \oplus \cdots \oplus \mathbb{1}(-d+2j-g+2))$$

*and its motivic Poincaré polynomial is*

$$\frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-1)} \cdot \mathrm{Coef}_{T^i}\left(\frac{\mathbb{1}(-d+2i-g+1)}{\mathbb{1}(-2)T-\mathbb{1}} - \frac{\mathbb{1}(-i-1)}{T-\mathbb{1}(-1)}\right) \cdot \frac{(1+T)^{h^1C}}{(\mathbb{1}-T)(\mathbb{1}-\mathbb{1}(-1)T)}.$$

*Proof.* From Thaddeus' construction and 2.6.4 we get

$$\begin{aligned} P^{mot}\widetilde{M}_j &= P^{mot}M_{j-1} + P^{mot}E_j - P^{mot}\mathbb{P}W_j^- \\ P^{mot}\widetilde{M}_j &= P^{mot}M_j + P^{mot}E_j - P^{mot}\mathbb{P}W_j^+ \end{aligned}$$

combining both equalities

$$P^{mot}M_j = P^{mot}M_{j-1} + P^{mot}\mathbb{P}W_j^+ - P^{mot}\mathbb{P}W_j^-. \tag{3}$$

Projective bundles are rationally cohomologically trivial (2.6.4) so

$$\begin{aligned} P^{mot}M_j = \quad & P^{mot}M_{j-1} + P^{mot}C^{(j)}(\mathbb{1} + \cdots + \mathbb{1}(-d+2j-g+2)) \\ & -P^{mot}C^{(j)}(\mathbb{1} + \cdots + \mathbb{1}(-j+1)) \end{aligned}$$

and putting this in (3)

$$P^{mot}M_j = P^{mot}M_{j-1} + P^{mot}C^{(j)}(\mathbb{1}(-j) + \cdots + \mathbb{1}(-d+2j-g+2)).$$

When $j = 0$ this is still valid taking $M_{-1} = \emptyset$ since $M_0$ is just $\mathbb{P}^{d+g-2}$. Now we add all these expressions from $j = 0$ to $j = i$ to get

$$P^{mot}M_i = \sum_{j=0}^{i} P^{mot}C^{(j)}(\mathbb{1}(-j) + \cdots + \mathbb{1}(-d+2j-g+2)), \tag{4}$$

which by 2.8 proves the first part of the proposition. For the rest rewrite (4)

$$P^{mot}M_i = \sum_{j=0}^{i} P^{mot}C^{(j)} \frac{\mathbb{1}(-j) - \mathbb{1}(-d+2j-g+1)}{\mathbb{1} - \mathbb{1}(-1)}$$

and apply corollary 3.7

$$\begin{aligned}
P^{mot}M_i &= \sum_{j=0}^{i} \mathrm{Coef}_{T^j} \frac{(1+T)^{h^1C}}{(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)} \frac{\mathbb{1}(-j) - \mathbb{1}(-d+2j-g+1)}{\mathbb{1}-\mathbb{1}(-1)} \\
&= \mathrm{Coef}_{T^i} \sum_{j=0}^{i} \frac{(1+T)^{h^1C}T^{i-j}}{(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)} \frac{\mathbb{1}(-j) - \mathbb{1}(-d+2j-g+1)}{\mathbb{1}-\mathbb{1}(-1)} \\
&= \mathrm{Coef}_{T^i} \sum_{j=0}^{i} T^{i-j}(-j) - T^{i-j}(-d+2j-g+1) \\
&\quad \cdot \frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-1)} \frac{(1+T)^{h^1C}}{(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)} \\
&= \mathrm{Coef}_{T^i} \left( \frac{T^{i+1} - \mathbb{1}(-i-1)}{T - \mathbb{1}(-1)} + \frac{(\mathbb{1} - T^{i+1}(-2i+2))\mathbb{1}(-d-g+1+2i)}{\mathbb{1}(-2)T - \mathbb{1}} \right) \\
&\quad \cdot \frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-1)} \frac{(1+T)^{h^1C}}{(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)}
\end{aligned}$$

and this completes the proof of the proposition. □

In the odd degree case, if $d > 4g-4$ $M_\omega$ is a $\mathbb{P}^{d-2g+1}$-fibration over $N_0(2,d)$. As $N_0(2,d) \simeq N_0(2,1)$ we can choose any convenient value of $d$, if we use $d = 4g-3$, then $\omega = 2g-2$. Then by (2.6.4)

$$P^{mot}M_\omega = \frac{\mathbb{1} - \mathbb{1}(-2g+1)}{\mathbb{1} - \mathbb{1}(-1)} P^{mot}N_0. \tag{5}$$

If we put the formula for $P^{mot}M_i$ in proposition 5.3 into (5) we obtain the following expression for $P^{mot}N_0(2,1)$

$$\begin{aligned}
\frac{-\mathbb{1}(-g)}{\mathbb{1}-\mathbb{1}(-2g+1)} \quad & \mathrm{Coef}_{T^{2g-2}} \frac{(1+T)^{h^1C}}{(\mathbb{1}-\mathbb{1}(-2)T)(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)} \\
& + \frac{\mathbb{1}(-2g+2)}{\mathbb{1}-\mathbb{1}(-2g+1)} \mathrm{Coef}_{T^{2g-2}} \frac{(1+T)^{h^1C}}{(\mathbb{1}-\mathbb{1}(2)T)(\mathbb{1}-\mathbb{1}T)(\mathbb{1}-\mathbb{1}(-1)T)}
\end{aligned}$$

Our aim now is to simplify this in $K_0\mathcal{M}_k^{AH}$. We shall need a definition.

**Definition 5.4** *Let $A, B$ and $M$ be objects in $\mathcal{M}_k^{AH}$ with $r = \mathrm{rank}M$, define $(A+B)^M$ to be the Newton binomial*

$$(A+B)^M = \sum \wedge^i M \cdot A^{r-i} \cdot B^i \in K_0\mathcal{M}_k^{AH}.$$

**Lemma 5.5** *If $M = h^1C$, with $C$ a curve of genus $g$, we have*

$$(A+B)^M = (B(-1)+A)^M(g).$$

*Proof.* Poincaré duality on the Jacobian of $C$ says $\wedge^i M \simeq (\wedge^{2g-i} M(g))^\vee$, and by Poincaré duality on $C$, $M^\vee \simeq M(1)$ so that

$$\begin{aligned} \wedge^i M &\simeq (\wedge^{2g-i} M(g))^\vee \simeq (\wedge^{2g-i} M^\vee)(-g) \\ &\simeq \wedge^{2g-i}(M(1))(-g) \simeq \wedge^{2g-i} M(g-i). \end{aligned}$$

Apply this to the definition of $(A+B)^M$,

$$\begin{aligned} (A+B)^M &= \wedge^0 M A^{2g} + \wedge^1 M A^{2g-1} B + \wedge^2 M A^{2g-2} B^2 + \cdots + \wedge^{2g} M B^{2g} \\ &= \wedge^{2g} M(g) A^{2g} + \wedge^{2g-1} M(g-1) A^{2g-1} B + \cdots + \wedge^0 M(-g) B^{2g} \\ &= (\wedge^{2g} M A^{2g} + \wedge^{2g-1} M(-1) A^{2g-1} B + \cdots + \wedge^0 M(-2g) B^{2g})(g) \\ &= (B(-1)+A)^M(g). \end{aligned}$$

□

**Theorem 5.6** *If $N_0(2,1)$ denotes the moduli space of rank two vector bundles with fixed odd degree on a curve $C$ then its motivic Poincaré polynomial in $K_0\mathcal{M}_k^{AH}$ is*

$$P^{mot} N_0(2,1) = \frac{(\mathbb{1}+\mathbb{1}(-1))^{h^1C} - (\mathbb{1}+\mathbb{1})^{h^1C}(-g)}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}.$$

*Proof.* We have seen that

$$P^{mot} N_0 = \tfrac{-\mathbb{1}(-g)}{\mathbb{1}-\mathbb{1}(-2g+1)} F(\mathbb{1}, \mathbb{1}(-1), \mathbb{1}(-2)) + \tfrac{\mathbb{1}(-2g+2)}{\mathbb{1}-\mathbb{1}(-2g+1)} F(\mathbb{1}, \mathbb{1}(-1), \mathbb{1}(1))$$

where in analogy with [25], if $a$, $b$ and $c$ are rank one motives, $F(a,b,c)$ means

$$F(a,b,c) = \mathrm{Coef}_{T^{2g-2}} \frac{(\mathbb{1}+T)^{h^1C}}{(\mathbb{1}-aT)(\mathbb{1}-bT)(\mathbb{1}-cT)}.$$

By direct calculation one can prove the same identity as in [25],

$$F(a,b,c) = \frac{(a+\mathbb{1})^{h^1C}}{(a-b)(a-c)} + \frac{(b+\mathbb{1})^{h^1C}}{(b-c)(b-a)} + \frac{(c+\mathbb{1})^{h^1C}}{(c-a)(c-b)}.$$

Then $P^{mot} N_0$ equals

$$\begin{aligned} &\tfrac{\mathbb{1}(-2g+2)}{\mathbb{1}-\mathbb{1}(-2g+1)} \left( \tfrac{(\mathbb{1}+\mathbb{1})^{h^1C}}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(1))} + \tfrac{(\mathbb{1}(-1)+\mathbb{1})^{h^1C}}{(\mathbb{1}(-1)-\mathbb{1})(\mathbb{1}(-1)-\mathbb{1}(1))} + \tfrac{(\mathbb{1}(1)+\mathbb{1})^{h^1C}}{(\mathbb{1}(1)-\mathbb{1})(\mathbb{1}(1)-\mathbb{1}(-1))} \right) \\ +&\tfrac{-\mathbb{1}(-g)}{\mathbb{1}-\mathbb{1}(-2g+1)} \left( \tfrac{(\mathbb{1}+\mathbb{1})^{h^1C}}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))} + \tfrac{(\mathbb{1}(-1)+\mathbb{1})^{h^1C}}{(\mathbb{1}(-1)-\mathbb{1})(\mathbb{1}(-1)-\mathbb{1}(-2))} + \tfrac{(\mathbb{1}(-2)+\mathbb{1})^{h^1C}}{(\mathbb{1}(-2)-\mathbb{1})(\mathbb{1}(-2)-\mathbb{1}(-1))} \right) \end{aligned}$$

Call $S_1$ the result of adding the third summand in both sums and $S_2$ the rest, we shall first calculate $S_1$,

$$\frac{\mathbb{1}(-2g+2)(\mathbb{1}(1)+\mathbb{1})^{h^1C}}{(\mathbb{1}(1)-\mathbb{1})(\mathbb{1}(1)-\mathbb{1}(-1))} = \frac{(\mathbb{1}+\mathbb{1}(-1))^{h^1C}}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}$$

and

$$\frac{-\mathbb{1}(-g)(\mathbb{1}(-2)+\mathbb{1})^{h^1C}}{(\mathbb{1}(-2)-\mathbb{1})(\mathbb{1}(-2)-\mathbb{1}(-1))} = \frac{-\mathbb{1}(-2g)(\mathbb{1}+\mathbb{1}(-1))^{h^1C}\mathbb{1}(1)}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}.$$

Adding and dividing by $(\mathbb{1}-\mathbb{1}(-2g+1))$

$$S_1 = \frac{(\mathbb{1}+\mathbb{1}(-1))^{h^1C}}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))},$$

similarly we calculate $S_2$

$$S_2 = -\frac{(\mathbb{1}+\mathbb{1})^{h^1C}\mathbb{1}(-g)}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}.$$

Sum $S_1$ and $S_2$ to get the desired expression for $P^{mot}N_0$. This proves the theorem. □

By applying the ring morphism $K_0(H^*_{DR}) : K_0\mathcal{M}_k^{AH} \to \mathbb{Z}[t,t^{-1}]$ we obtain the formula of Desale and Ramanan ([9]) for the Poincaré polynomial of $N_0(2,1)$

$$\frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

In the following corollary we obtain the Hodge numbers of $N_0(2,1)$.

**Corollary 5.7** *The Poincaré-Hodge polynomial of $N_0(2,1)$ is*

$$P_{xy}N_0(2,1) = \frac{(1+x^2y)^g(1+xy^2)^g - x^gy^g(1+x)^g(1+y)^g}{(1-xy)(1-x^2y^2)},$$

*Proof.* Let **Bi-Grad-Vec**$_k$ denote the category of finite dimensional vector spaces over $k$ with a double graduation $V=\oplus_{p,q}V^{p,q}$. Sometimes we shall write $V^{p,q}x^py^q$ instead of $V^{p,q}$ to remind us of the graduation. Note that $K_0$**Bi-Grad-Vec**$_k = \mathbb{Z}[x,y,x^{-1},y^{-1}]$ and $[V]=\sum \dim V^{p,q}x^py^q$.

If $M$ is a AH-motive and $M=\oplus M_i$ is its weight grading define $H^{p,q}_{DR}M = Gr^p_{F^\cdot}M_{p+q}$, this defines an exact functor $H^{*,*}_{DR}$

$$\begin{array}{rcl} H^{*,*}_{DR} : \mathcal{M}_k^{AH} & \longrightarrow & \textbf{Bi-Grad-Vec}_k \\ M & \longmapsto & \oplus_{p,q}Gr^p_{F^\cdot}M_{p+q}. \end{array}$$

We have to calculate the image of

$$\frac{(\mathbb{1}+\mathbb{1}(-1))^{h^1C} - \mathbb{1}(-g)(\mathbb{1}+\mathbb{1})^{h^1C}}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}$$

by the morphism

$$K_0(H^{*,*}_{DR}) : K_0\mathcal{M}_k^{AH} \longrightarrow K_0\textbf{Bi-Grad-Vec}_k = \mathbb{Z}[x,y,x^{-1},y^{-1}],$$

this is a morphism of rings and it is enough to calculate the image of $\frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-1)}$, $\frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-2)}$, $\mathbb{1}(-g)$, $(\mathbb{1}+\mathbb{1})^{h^1C}$ and $(\mathbb{1}+\mathbb{1}(-1))^{h^1C}$.

As $H^{*,*}_{DR}(\mathbb{1}(-i))\,\mathbb{C}x^iy^i$ the image of $\frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-1)}$, $\frac{\mathbb{1}}{\mathbb{1}-\mathbb{1}(-2)}$ and $\mathbb{1}(-g)$ by $K_0(H^{*,*}_{DR})$ is just $\frac{1}{1-xy}$, $\frac{1}{1-x^2y^2}$ and $x^gy^g$.

The functor $H^{*,*}_{DR}$ sends $(\mathbb{1}+\mathbb{1}(-k))^{h^1C} = \oplus_n \wedge^n h^1C(-nk)$ to $\oplus_n \wedge^n (\mathbb{C}^g x \oplus \mathbb{C}^g y)\otimes \mathbb{C}x^{nk}y^{nk} \simeq \oplus_n \oplus_{i+j=n} \wedge^i(\mathbb{C}^g x^{k+1}y^k)\otimes \wedge^j(\mathbb{C}^g x^ky^{k+1})$ and taking the class in $K_0$ we obtain $(1+x^{k+1}y^k)^g(1+x^ky^{k+1})^g$.

Putting all this together we obtain the Poincaré-Hodge polynomial of the moduli space. □

Recall that by [21] $h^{0,p}N_0(2,1) = 0$ for all $p > 0$, that is, the border of the Hodge diamond contains zeroes, in fact the Hodge diamond is quite thin, for this recall the definition of the level of a Hodge structure: $\mathrm{Max}_{h^{p,q}\neq 0}|p-q|$. Then one can prove that the level of the Hodge structure $H^iN_0(2,1)$ is lower o equal to $\left[\frac{i}{3}\right]$. This can be proven by using the Newstead generators of the cohomology ring ([22]) or by working out the Poincaré-Hodge polynomial in the following way: Put $A = (1+xy^2)(1+x^2y)$ and $B = xy(1+x)(1+y)$ then

$$\begin{aligned} P_{xy}N_0(2,1) &= \frac{A^g - B^g}{A - B} \\ &= A^{g-1} + A^{g-2}B + \cdots + B^{g-1}, \end{aligned}$$

and as the only monomials in $A$ and $B$ are $x^iy^j$ with $i = j$, $i = 2j$ or $2i = j$ one can now see that the level of $H^i$ is less than or equal to $\left[\frac{i}{3}\right]$.

As a byproduct of our result and theorem 6.25 in [8] we can give another proof of the following result of I. Biswas

**Corollary 5.8** *([6]) Let $\sigma$ be an embedding of the field $k$ in $\mathbb{C}$, then $\sigma$-Hodge cycle on the variety $N_0(2,1)$ is an absolute Hodge cycle.*

Let $C$ be a curve defined over a finite field, $\mathbb{F}_q$, and note by $\zeta_C(s)$ its zeta function. The moduli space is also defined over $\mathbb{F}_q$ and we can deduce the following result concerning its number of points (see [11], in fact this corollary is equivalent to the Siegel formula for $SL_2$ over the function field of $C$).

**Corollary 5.9** *The number of $\mathbb{F}_q$-points of $N_0(2,1)$ is*

$$\#N_0(2,1)(\mathbb{F}_q) = \zeta_C(2) - \frac{q^g}{(1-q)(1-q^2)}\#Jac(C).$$

## 6 The mixed motive of $N_0(2,0)$.

If $g = 2$ then $N_0(2,0)$ is isomorphic to $\mathbb{P}^3$ and its motive is well known. However if $g > 2$ moduli space $N_0(2,d)$ is a singular projective variety with singular locus the Kummer variety associated to the Jacobian of the curve. In this section we assume $g > 2$ and study the mixed motive $hN_0(2,0)$, in particular we find an expression for its motivic Poincaré polynomial and show that only two weights can appear. We do this by relating $N_0(2,1)$ and $N_0(2,0)$ via the Hecke correspondence introduced by Narasimhan and Ramanan in [18]. This consists of a variety $\widetilde{M}$ and morphisms $p_1$ and $p_2$

$$\begin{array}{ccc} & \widetilde{M} & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ N_0(2,1) & & N_0(2,0). \end{array}$$

The basic properties of $\widetilde{M}$ are:

1. The morphism $p_1$ makes $\widetilde{M}$ into a $\mathbb{P}^1$-bundle. This projective bundle arises from a vector bundle over $N_0(2,1)$.

2. The morphism $p_2$ has fibres as follows:

   (a) Over the stable locus,

   $$p_2^{-1}N_0(2,0)^s \longrightarrow N_0(2,0)^s$$

   is a $\mathbb{P}^1$-bundle. Unlike the odd degree case this projective bundle does not arise from a vector bundle.

   (b) The strictly semistable locus is the Kummer variety, $K$, associated to the Jacobian of the curve. Let $K_0$ be its $2^{2g}$ singular points. Then the fibre of $p_2$ over a point of $K - K_0$ is isomorphic to $\mathbb{P}^{g-1} \vee \mathbb{P}^{g-1}$. It is an interesting fact that the double cover associated to this over $K - K_0$ is not trivial.

   (c) Over $K_0$ the reduced fibre is isomorphic to $\mathbb{P}^{g-1}$.

3. One can define $\widetilde{M}$ as a moduli space of semistable parabolic rank two vector bundles with trivial determinant over the curve with small enough parabolic weights ([16]).

We set some notations. Let $J$ be the Jacobian of the curve and $J_0$ its 2-torsion points. The strictly semistable locus of the moduli space $N_0(2,0)$ is isomorphic to the Kummer variety associated to the Jacobian of $C$, we shall note this by $K$ and $K_0$ its $2^{2g}$ singular points. The projection $J \longrightarrow K$ restricts to a double cover $J - J_0 \longrightarrow K - K_0$. Fix a point $x \in C$ and choose a Poincaré line bundle, $\mathcal{L}$, over $C \times J$ normalised so that $\mathcal{L}_x = \mathcal{L}_{|\{x\}\times J}$ is trivial. The action of $\mathbb{Z}/2\mathbb{Z}$ on $(J - J_0) \times C$ lifts to the vector bundle $\mathcal{L} \oplus \mathcal{L}^{-1}$ and by descent we get a vector bundle on $(K - K_0) \times C$ which we still note $\mathcal{L} \oplus \mathcal{L}^{-1}$ (of course $\mathcal{L}$ is not defined over $(K - K_0) \times C$). The projections of a product, $X \times Y$, over its factors will be written $p_X$ and $p_Y$.

**Theorem 6.1** *The motivic Poincaré polynomial of $N_0(2,0)$ and $N_0(2,0)^s$ are*

$$P^{mot}N_0(2,0) = P^{mot}N_0(2,1) + \frac{(\mathbb{1} - \mathbb{1}(-1))\,(-1)P^{mot}K - (\mathbb{1}(-1) - \mathbb{1}(-g))\,P^{mot}J}{\mathbb{1} + \mathbb{1}(-2)},$$

$$P^{mot}N_0(2,0)^s = P^{mot}N_0(2,1) - \frac{(\mathbb{1} - \mathbb{1}(-1))\,P^{mot}K + (\mathbb{1}(-1) - \mathbb{1}(-g))\,P^{mot}J}{\mathbb{1} + \mathbb{1}(-2)}.$$

The proof of the theorem relies on the following lemma

**Lemma 6.2** *The morphism $p_2^{-1}(K - K_0) \longrightarrow K - K_0$ has a section $\sigma$ such that if $P$ is the projective bundle over $J - J_0$ defined by the vector bundle $R^1p_{(J-J_0)*}\mathcal{L}^2 \oplus \mathcal{O}_{J-J_0}$ and $\sigma'$ is the section defined by the natural morphism $R^1p_{(J-J_0)*}\mathcal{L}^2 \oplus \mathcal{O}_{J-J_0} \longrightarrow \mathcal{O}_{J-J_0}$ then there exists an isomorphism $\varphi$ yielding a commutative diagram*

$$\begin{array}{ccc} P - \sigma'(J - J_0) & \xrightarrow{\varphi} & p_2^{-1}(K - K_0) - \sigma(K - K_0) \\ \downarrow & & \downarrow \\ J - J_0 & \longrightarrow & K - K_0. \end{array}$$

*Proof.* We first define $\sigma$. Let $\mathcal{F} = \mathcal{L} \oplus \mathcal{L}^{-1} \longrightarrow (K - K_0) \times C$ be the family of rank two vector bundles defined above. If $i_x : \{x\} \hookrightarrow C$ is the natural inclusion, put $\mathcal{F}_x = (id \times i_x)^*\mathcal{F}$. The projective bundle $\pi : \mathbb{P}\mathcal{F}_x^\vee \longrightarrow K - K_0$ parametrizes a family of parabolic rank two vector bundles: $(id \times \pi)^*\mathcal{F}$, $\mathcal{O}(1) \hookrightarrow \pi^*\mathcal{F}_x$. If we see a point of $\mathbb{P}\mathcal{F}_x^\vee$ as a line, $\ell$, in $L_x \oplus L_x^{-1}$ with $L \in J - J_0$ one can see that the parabolic bundle is stable iff

$\ell \neq L_x$ and $\ell \neq L_x^{-1}$. As we are assuming $\mathcal{L}_x = \mathcal{O}$, $\mathbb{P}\mathcal{F}_x^\vee \simeq \mathbb{P}^1 \times (K - K_0)$ and we see that the stable locus in the previous family is $(\mathbb{P}^1 - \{0, \infty\}) \times (K - K_0)$. From the definition of $\widetilde{M}$ we obtain a modular morphism, $j$,

$$\begin{array}{ccc} (\mathbb{P}^1 - \{0,\infty\}) \times (K - K_0) & \stackrel{j}{\longrightarrow} & \widetilde{M} \\ \downarrow & & \downarrow \\ K - K_0 & \hookrightarrow & N_0(2,0). \end{array}$$

And it is easy to prove that $j(\ell, L \oplus L^{-1})$ is independent of $\ell$ thus yielding a section of $\widetilde{M} \longrightarrow N_0(2,0)$ over $K - K_0$.

Note next that the sheaf $R^1 p_{(J-J_0)*}\mathcal{L}^2$ is locally free of rank $g-1$, let $R$ be the projective bundle over $J - J_0$ associated to it and let $\pi$ be the projection. Over $R \times C$ we have an extension of line bundles

$$0 \longrightarrow \mathcal{O}_R(1) \otimes \pi^*\mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \pi^*\mathcal{L}^{-1} \longrightarrow 0.$$

Taking pullback by $i_x$ we obtain the following exact sequence of vector bundles over $R$

$$0 \longrightarrow \mathcal{O}_R(1) \longrightarrow \mathcal{E}_x \longrightarrow \mathcal{O}_R \longrightarrow 0.$$

Let $Q$ be the $\mathbb{P}^1$ bundle over $R$ associated to the vector bundle $\mathcal{E}_x$, $Q$ parametrizes a family of parabolic vector bundles. If we see a point of $Q$ as an extension of line bundles $0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \longrightarrow 0$ together with a line $\ell \subset E_x$ then this defines an unstable parabolic bundle iff $\ell \subset L_x$. This way we see that the stable locus, $Q^s$, of $Q$ is is the complementary of the section of $Q = \mathbb{P}\mathcal{E}_x \simeq \mathbb{P}(\mathcal{O}_R(1) \oplus \mathcal{O}_R)$ given by $\mathcal{O}_R(1) \oplus \mathcal{O}_R \longrightarrow \mathcal{O}_R$, that is $Q^s$ is isomorphic to the total space of the line bundle $\mathcal{O}_R(1)$.

From the definition of $\widetilde{M}$ we get a modular morphism $\varphi : Q^s \longrightarrow \widetilde{M}$ which by obvious considerations makes the following diagram commutative

$$\begin{array}{ccc} Q^s & \stackrel{\varphi}{\longrightarrow} & \widetilde{M} \\ \downarrow & & \downarrow \\ J - J_0 & \longrightarrow & N_0(2,0). \end{array}$$

We see that $\varphi$ gives an isomorphism $Q^s \longrightarrow p_2^{-1}(K - K_0)$ because it induces a bijection of $S$-points for every $S \in Ob(\mathbf{Sch})$.

Moreover $Q^s$ is the complementary of the stated section of the projective bundle $\mathbb{P}(R^1 p_{(J-J_0)*}\mathcal{L}^2 \oplus \mathcal{O})$ to see this just note that if $V \longrightarrow X$ is a vector bundle then $\mathcal{O}_{\mathbb{P}V}(1)$ is isomorphic to the complementary of the trivial section of $\mathbb{P}(V \oplus \mathcal{O})$. $\square$

*Proof of the Theorem.* From the Hecke correspondence we see that

$$P_c^{mot}\widetilde{M} = (\mathbb{1} + \mathbb{1}(1))\, P_c^{mot} N_0(2,1),$$

and

$$P_c^{mot}\widetilde{M} = (\mathbb{1} + \mathbb{1}(1))\, P_c^{mot} N_0(2,0)^s + P_c^{mot} p_2^{-1}(K - K_0) + P_c^{mot} p_2^{-1} K_0.$$

Now $p_2^{-1}K_0$ consists of $2^{2g}$ copies of $\mathbb{P}^{g-1}$ and so

$$P_c^{mot} p_2^{-1} K_0 = 2^{2g} \frac{\mathbb{1} - \mathbb{1}(-g)}{\mathbb{1} - \mathbb{1}(-1)}.$$

On the other hand by lemma 6.2 $P_c^{mot} p_2^{-1}(K - K_0)$ equals

$$\left(\frac{\mathbb{1} - \mathbb{1}(-g)}{\mathbb{1} - \mathbb{1}(-1)} - \mathbb{1}\right)\left(P^{mot} J - 2^{2g}\right) - P^{mot} K + 2^{2g}.$$

Therefore

$$P_c^{mot} N_0(2,0)^s = P^{mot} N_0(2,1) - \frac{P^{mot} K + \left(\frac{\mathbb{1}-\mathbb{1}(-g)}{\mathbb{1}-\mathbb{1}(-1)} - \mathbb{1}\right) P^{mot} J}{\mathbb{1} + \mathbb{1}(1)},$$

and $P^{mot} N_0(2,0) = P_c^{mot} N_0(2,0)^s + P^{mot} K$. □

Seshadri constructs in [24] a desingularization of the moduli space $N_0(2,0)$,

$$p : M \longrightarrow N_0(2,0)$$

the fibres of this morphism are given by

1. Over the stable locus $N_0(2,0)^s$ it is an isomorphism.
2. Over $K - K_0$ it is a $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$-bundle.
3. The fibre over a point of $K_0$ is the disjoint union of $Grass_3 V$ and a rank $g-2$ vector bundle over $Grass_2 V$ where $V = H^0(C, \mathcal{O}_C)$.

From theorem 6.1 one can easily derive the following corollary.

**Corollary 6.3** *The motivic Poincaré polynomial of $M$ is*

$$\begin{aligned} P^{mot} N_0(2,1) + \quad & P^{mot} K \left(\frac{(\mathbb{1}-\mathbb{1}(3-g))(2\mathbb{1}-\mathbb{1}(-1)-\mathbb{1}(2-g))(-1)}{(\mathbb{1}-\mathbb{1}(-1))(\mathbb{1}-\mathbb{1}(-2))}\right) - P^{mot} J \left(\frac{(\mathbb{1}-\mathbb{1}(1-g))}{\mathbb{1}-\mathbb{1}(-2)}\right) + \\ & + 2^{2g}\left(P^{mot} Grass_3 V + P^{mot} Grass_2 V(2-g) - \left(\frac{\mathbb{1}-\mathbb{1}(2-g)}{\mathbb{1}-\mathbb{1}(-1)}\right)^2\right) \end{aligned}$$

*Proof.* From the description of $M$ we see that

$$P^{mot} M = P_c^{mot} N_0(2,0)^s + P_c^{mot} p^{-1}(K - K_0) + P_c^{mot} p^{-1}(K_0)$$

We have computed $P_c^{mot} N_0(2,0)^s$ in theorem 6.1, on the other hand

$$\begin{aligned} P_c^{mot} p^{-1}(K - K_0) &= \left(\frac{\mathbb{1} - \mathbb{1}(-g+2)}{\mathbb{1} - \mathbb{1}(-1)}\right)^2 \left(P^{mot} K - 2^{2g}\right) \\ P_c^{mot} p^{-1} K_0 &= 2^{2g}\left(P^{mot} Grass_3 V + P^{mot} Grass_2 V(g-2)\right). \end{aligned}$$

□

Upon application of the ring morphism

$$K_0(H^*_{DR}) : K_0 \mathcal{M}_k \longrightarrow K_0 \textbf{Grad-Vec}_k = \mathbb{Z}[t, t^{-1}]$$

we obtain the formula for the Poincaré polynomial of $M$ in [5]. If instead we apply the ring morphism

$$K_0\left(H^*_{DR}, F^{\cdot}\right) : K_0 \mathcal{M}_k \longrightarrow K_0 \textbf{Grad-Fil-Vec}_k = \mathbb{Z}[x, y, x^{-1}, y^{-1}]$$

we obtain the following formula for the Poincaré-Hodge polynomial of $M$

$$\frac{(1+xy^2)^g(1+x^2y)^g-(1+x)^g(1+y)^g x^g y^g}{(1-xy)(1-x^2y^2)}+\frac{(1+x)^g(1+y)^g+(1-x)^g(1-y)^g}{2}\left(\frac{\left(1-(xy)^{g-3}\right)\left(2-xy-(xy)^{g-2}\right)xy}{(1-xy)(1-(xy)^2)}\right)-$$
$$-(1+x)^g(1+y)^g\left(\frac{\left(1-(xy)^{g-1}\right)}{1-x^2y^2}\right)+2^{2g}\left(P_{xy}Grass_3V+P_{xy}Grass_2V(xy)^{g-2}-\left(\frac{1-(xy)^{g-2}}{1-xy}\right)^2\right).$$

The following proposition shows *how near* is $P^{mot}N_0(2,0)$ from the true motive of $N_0(2,0)$.

**Proposition 6.4** *The mixed AH-motive $h^iN_0(2,0)$ has only weights $i$ and $i-1$.*

*Proof.* First note that $p^{-1}K_0$ admits a cell decomposition and therefore $h^i_cp^{-1}K_0$ is a pure motive of weight $i$. Now write the Gysin exact sequence for $p^{-1}K_0\subset p^{-1}K$,

$$\cdots\longrightarrow h^i_cp^{-1}(K-K_0)\longrightarrow h^i_cp^{-1}K\longrightarrow h^i_cp^{-1}K_0\longrightarrow\cdots.$$

If one writes the Gysin exact sequence for $K_0\subset K$ one sees that $W_{i-1}h^i_c(K-K_0)$ is the image of $h^{i-1}_cK_0$ by the connecting morphism, from this and the description of the fibres of $p$ it can be seen that the image of $W_{i-1}h^i_cp^{-1}(K-K_0)$ in $h^i_cp^{-1}K$ is zero, it follows that $h^i_cp^{-1}K$ is a pure motive of weight $i$.

Now write the Gysin sequence for $p^{-1}K\subset M$,

$$\cdots\longrightarrow h^{i-1}_cp^{-1}K\longrightarrow h^i_cN_0(2,0)^s\longrightarrow h^i_cM\longrightarrow h^i_cp^{-1}K\longrightarrow\cdots.$$

Being $h^{i-1}_cp^{-1}K$ and $h^i_cM$ pure motives it follows that $h^i_cN(2,0)^s$ is a mixed motive with weights $i$ and $i-1$, by writing one more Gysin sequence one proves the same fact about $h^iN_0(2,0)$. □

# 7 Intermediate Jacobians.

If $X$ is a smooth projective variety over the complex numbers then Griffiths associates a complex torus to the integer pure Hodge structure $H^{2i-1}(X,\mathbb{Z})$,

$$J^i(X)=\frac{H^{2i-1}(X,\mathbb{C})}{F^iH^{2i-1}(X,\mathbb{C})+H^{2i-1}(X,\mathbb{Z})},$$

called the $i$-th intermediate Jacobian. If $X$ is defined over a field $k$ then for each embedding $\sigma:k\hookrightarrow\mathbb{C}$ we have an associated intermediate jacobian $J^i(X\otimes_\sigma\mathbb{C})$.

Note that if we know $h^{2i-1}(X)$ and are interested in $J^i(X_\sigma)$ there is only one piece of data missing: the entire structure on the singular cohomology group $H^{2i-1}(X_\sigma,\mathbb{Q})$, so we can recover $J^i(X_\sigma)$ up to isogeny from $h^{2i-1}(X)$

The intermediate jacobian $J^iX$ is isomorphic to the group $Ext^1_{\mathcal{MHS}}\left(\mathbb{Z},H^{2i-1}(X,\mathbb{Z})(j)\right)$, where $\mathcal{MHS}$ is the category of mixed Hodge, (see Lemma 9.2 in [12]) this motivates the definition of $\ell$-adic intermediate jacobian of a variety, $X$, defined over an arbitrary field $k$, as the group of extensions $Ext^1_{Rep\text{-}G_k}\left(\mathbb{Z}_\ell,H^{2i-1}(X,\mathbb{Z}_\ell)(j)\right)$ where *Rep*-$G_k$ is the category of continuous $\ell$-adic representations of the Galois group $G_k=Gal(\overline{k}|k)$. Define the $\ell$-adic intermediate jacobians *up to isogeny* as the same extension groups but replacing $\mathbb{Z}_\ell$ by $\mathbb{Q}_\ell$.

In the following corollary, for notational purposes, assume that either an embedding of the field $k$ in $\mathbb{C}$ or a prime $\ell$ have been chosen and use the corresponding definition of intermediate jacobian.

**Corollary 7.1** *The i-th intermediate jacobian of the moduli space $N_0(2,1)$ is isogenous to*

$$\prod_{k=1}^{[\frac{g+1}{2}]} \left(J^k Jac(C)\right)^{c_{i,k,g}},$$

*where* $c_{i,k,g} = \mathrm{Coef}_{t^{i-3k+1}} \left(1 + t + t^2 + \cdots + t^{g-2k}\right)\left(1 + t^2 + t^4 + \cdots + t^{2g-4k}\right)$.

*Proof.* Theorem 5.6 says

$$P^{mot} N_0(2,1) = \frac{(\mathbb{1} + \mathbb{1}(-1))^{h^1C} - (\mathbb{1} + \mathbb{1})^{h^1C}(-g)}{(\mathbb{1} - \mathbb{1}(-1))(\mathbb{1} - \mathbb{1}(-2))},$$

this is equal to

$$\bigoplus_{k=0}^{2g} \wedge^k h^1 C \frac{\mathbb{1}(-k) - \mathbb{1}(-g)}{(\mathbb{1} - \mathbb{1}(-1))\,(\mathbb{1} - \mathbb{1}(-2))},$$

use Poincaré duality on the Jacobian of $C$ as in the proof of lemma 5.5 to get

$$\bigoplus_{k=0}^{g} \wedge^k h^1 C \frac{\mathbb{1}(-k) - \mathbb{1}(-g)}{(\mathbb{1} - \mathbb{1}(-1))\,(\mathbb{1} - \mathbb{1}(-2))} \oplus \bigoplus_{k=0}^{g} \wedge^k h^1 C(-g+k) \frac{\mathbb{1}(k-2g) - \mathbb{1}(-g)}{(\mathbb{1} - \mathbb{1}(-1))\,(\mathbb{1} - \mathbb{1}(-2))},$$

adding this we obtain the following expression for $P^{mot} N_0(2,1)$

$$\bigoplus_{k=0}^{g} \wedge^k h^1 C \frac{\mathbb{1} - \mathbb{1}(-g+k)}{\mathbb{1} - \mathbb{1}(-1)} \frac{\mathbb{1} - \mathbb{1}(-2g+2k)}{\mathbb{1} - \mathbb{1}(-2)}(-k). \tag{6}$$

Note that this is the class in $K_0\mathcal{M}_k^{AH}$ of an object of $\mathcal{M}_k^{AH}$ so that in fact, by corollary 2.8, we have obtained $hN_0(2,1)$.

In order to compute $J^i N_0(2,1)$ we have to find $h^{2i-1} N_0(2,1)$, by formula (6) this is

$$\bigoplus_{k=1}^{[\frac{g+1}{2}]} \wedge^{2k-1} h^1 C \otimes \mathbb{1}(k-i)^{\oplus c_{i,k,g}}$$

Now the result follows. □

**7.2** *Examples:*

1. *Putting $i = 1$ we obtain $J^1 N_0(2,1) = 0$ which is reasonable since $H^1 N_0(2,1) = 0$. The value $i = 2$ gives an isogeny $J^2 N_0(2,1) \sim Jac(C)$ which is a result of Mumford and Newstead ([17]) modulo isogeny. One can easily check that if $g > 2$ $J^3 N_0(2,1)$ is also isogenous to $Jac(C)$. Of course, if $g = 2$, $J^3 N_0(2,1) = 0$.*

2. *For $g > 3$ the value $i = 4$ gives an isogeny $J^4 N_0(2,1) \sim Jac(C) \times Jac(C)$. If $g = 2$ $J^4 N_0(2,1)$ is clearly zero for dimensional reasons, if $g = 3$ then $J^4 N_0(2,1) = Jac(C)$ by duality.*

3. *If $g > 3$ then there are non abelian intermediate jacobians, the first is $J^5 N_0(2,1)$ which for $g > 4$ is isogenous to $J^2 Jac(C) \times Jac(C) \times Jac(C)$ whereas for $g = 4$ is isogenous to $J^2 Jac(C) \times Jac(C)$.*

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