Central configurations of the planar coorbital satellite problem

Josep M. Cors[1] Jaume Llibre[2] Mercè Ollé[3]

Abstract

We study the planar central configurations of the $1 + n$ body problem where one mass is large and the other n masses are infinitesimal and equal. We find analytically all these central configurations when $2 \leq n \leq 4$. Numerically, first we provide evidence that when $n \geq 9$ the only central configuration is the regular n–gon with the large mass in its barycenter, and second we provide also evidence of the existence of an axis of symmetry for every central configuration.

1. Introduction

A very old problem in Celestial Mechanics is the study of central configurations of the N–body problem. *Central configurations* are the configurations such that the total Newtonian acceleration on every body is equal to a constant multiplied by the position vector of this body with respect to the center of mass of the configuration.

There is an extensive literature concerning these solutions. For a classical background, see the sections on central configurations in [40] and [13]. For a modern background one can see [36], [37] and [30]. More recent work can be found in [2]–[12], [20]–[34] and [41]. One of the reasons why central configurations are interesting is that they allow to obtain explicit homographic solutions of the N–body problem. This was already pointed out by Laplace and, historically, the problem of central configurations was first formulated in this context. Moulton [26] in 1910 characterized the number of collinear central configurations by showing that there exist exactly $N!/2$ classes of central configurations of the N–body problem for a given set of positive masses. The number of classes of planar central configurations of the N–body problem for an arbitrary given set of positive masses has been only solved for $N = 3$, see Wintner [40] and Smale [38].

Central configurations also appear as a key point when we study the topology of the set of points of the phase space having energy h and angular momentum c, see [36, 37]. Every motion starting and ending in a total collision is asymptotic to a central configuration, and every parabolic motion of the N bodies (i.e. the N bodies tend to infinity as $t \to \infty$ with zero radial velocity) is asymptotic to a central configuration, see [11, 31, 39].

In this paper we consider a restricted version of the problem of planar central configurations, i.e., N is equal to $1 + n$ and we study the limit case of one large mass and n small equal masses as the small mass tends to zero. We mention that this problem may be interesting from the practical point of view, in the sense that it can model (in a first approximation) the motion of several coorbital satellites located in the same circular orbit. In fact, this problem was first consider by Maxwell [19] trying to construct a model for Saturn's rings. The unpublished paper of Hall [14] shows that if $n \geq e^{27000}$, then there is a unique class of central configuration, the regular polygon. In [9] the same result is proved under the assumption that $n \geq e^{73}$.

In Section 2 we give the equations for the central configurations of the planar $1+n$ body problem as well as some definitions. Section 3 is devoted to state a summary of our numerical results. Thus, first we have checked the numerical results of Salo and Yoder [32] for $n = 2, \ldots, 9$ and after we have explored bigger values of n up to 15. Second we give numerical evidence that all the configurations are symmetric with respect to a straight line.

After that, we study analytically the central configurations of the $1+n$ body problem for n small. In Section 4 we study the cases $n = 2$ and $n = 3$ that were completely solved in [14]. Since [14] is an unpublished paper and the proofs of $n = 2$ and $n = 3$ are shorter we provide them here and give some hints for n large. Finally in Section 5 and 6 we prove that the number of classes of central configurations for the $1 + 4$ body problem is three.

2. Definitions and equations

We start by defining the central configurations for N particles in the plane: consider N particles of masses $m_1, \ldots, m_n$ in $\mathbb{R}^2$ subject to their mutual Newtonian gravitational interaction. In an inertial reference frame with origin at the center of mass of these N bodies and choosing suitable units, the equations of motion are

$$Mq'' = V_q,$$

where M is the mass matrix $M = \text{diag}(m_1, m_1, \ldots, m_N, m_N)$, $q = (q_1, \ldots, q_N)$ is the position vector with $q_i \in \mathbb{R}^2$, V the potential function

$$V(q_1, \ldots, q_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{\|q_i - q_j\|},$$

and $V_q = (\partial V/\partial q_1, \ldots, \partial V/\partial q_N)$. Excluding the singularities of the equations, the configuration space of the planar N body problem associated with the mass matrix M is

$$\mathcal{M}(m_1, ..., m_N) = \left\{ q \in \mathbb{R}^{2N} : \sum_{i=1}^{N} m_i q_i = 0, q_i \neq q_j, \text{ for } i \neq j \right\}.$$

Given a matrix M, we say that $q \in \mathcal{M}$ represents a central configuration of the associated planar N body problem if there exists a positive constant λ^2 such that

$$M^{-1} V_q = \lambda^2 q,$$

i.e., if the acceleration vector of every particle is directed towards the center of mass and its modulus is proportional to the distance from the particle to the center of mass.

We shall denote by $\mathcal{C}$ the set of planar central configurations associated with a given matrix M. Notice that $\mathcal{C}$ is invariant with respect to homothetic transformations and rotations in $\mathbb{R}^2$. We shall denote by $\tilde{\mathcal{C}}$ the set of planar central configurations modulus the group $SO(2)$ of plane rotations.

Now we deal with the central configurations of the planar $1+n$ body problem with infinitesimal equal masses. That is, we consider $N = 1+n$, and let $q(\epsilon) = (q_0(\epsilon), q_1(\epsilon), \ldots, q_n(\epsilon)) \in \tilde{\mathcal{C}}$ be a central configuration of the planar $1+n$ body problem with $m_0 = 1$, $m_i = \epsilon$, $i = 1, ..., n$.

We say that $q = (q_0, q_1, ..., q_N)$ is a *central configuration* of the planar $1+n$ body problem if there exists $\lim_{\epsilon \to 0} q(\epsilon)$ and this limit is equal to q. We have then the following two results (for a proof see [9]).

Proposition 1. *All central configurations of the planar $1+n$ body problem lie on a circle centered at $q_0 = 0$.*

Proposition 2. *Let $q = (q_0, ..., q_n)$ be a non–collision central configuration of the planar $1+n$ body problem. Denoting by α_i the angle defined by the position of the i–th infinitesimal mass on a circle centered at $q_0 = 0$, we have*

$$\sum_{j=1, j\neq i}^{n} \sin(\alpha_j - \alpha_i) \left(1 - \frac{1}{2\sqrt{2}\sqrt{(1-\cos(\alpha_j-\alpha_i))^3}} \right) = 0, \qquad i = 1, ..., n. \tag{1}$$

Since we are interested in central configurations modulus rotations and homothetic transformations, we can assume that the circle has radius 1 and that $\alpha_1 = 0$. Since we exclude collisions in the definition of central configurations, that is, $\alpha_i \neq \alpha_j$ for $i \neq j$, we shall take as coordinates the angles between two consecutive particles, i.e.,

$$\theta_i = \alpha_{i+1} - \alpha_i, \qquad i = 1, ..., n-1.$$

and it is convenient to work with a n–th redundant coordinate angle

$$\theta_n = 2\pi - \sum_{i=1}^{n-1} \theta_i$$

which measures the angular distance between the n–th particle and the first one. In this way the configuration space for the central configurations is the simplex

$$S = \left\{ (\theta_1, ..., \theta_n) : \theta_i > 0\,, \sum_{i=1}^{n} \theta_i = 2\pi \right\}.$$

Let

$$f(\theta) = \sin\theta \left(1 - \frac{1}{2\sqrt{2}\sqrt{(1-\cos\theta)^3}} \right).$$

We note that equations (1) for the central configurations can be expressed in terms of the function f, as

$$f(\theta_1) + f(\theta_1 + \theta_2) + \cdots + f(\theta_1 + ... + \theta_{n-1}) \;=\; 0,$$

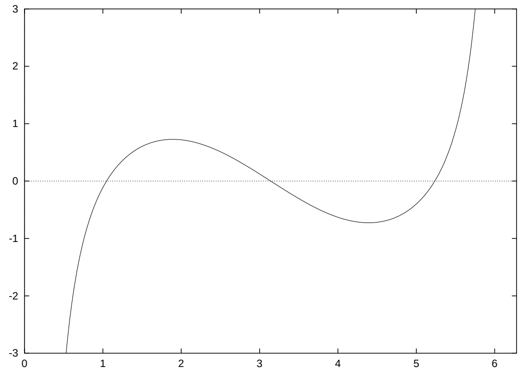

Figure 1: The graph of $f(\theta)$.

$$\begin{array}{rcl} f(\theta_2)+f(\theta_2+\theta_3)+\cdots+f(\theta_2+...+\theta_n) & = & 0, \\ f(\theta_3)+f(\theta_3+\theta_4)+\cdots+f(\theta_3+...+\theta_n+\theta_1) & = & 0, \\ \cdots & & \\ f(\theta_{n-1})+f(\theta_{n-1}+\theta_n)+\cdots+f(\theta_{n-1}+\theta_n+\theta_1+\cdots+\theta_{n-3}) & = & 0, \\ \theta_1+\cdots+\theta_n & = & 2\pi. \end{array} \tag{2}$$

Due the important role that the function $f(\theta)$ plays in this problem we state some properties which will be used throughout this work.

Proposition 3. *The function $f(\theta)$ satisfies:*

1. $f(2\pi-\theta)=-f(\theta)$,
2. $f(\pi-\theta)=-f(\pi+\theta)$,
3. $f'''(\theta)>0$ *for all* $\theta\in(0,2\pi)$.

Proof. Statements (1) and (2) follow directly from the analytical expression of f. By straightforward computations one finds

$$\begin{array}{rcl} f'''(\theta) & = & -\left(1-\dfrac{1}{2\sqrt{2}\,(1-\cos\theta)^{\frac{3}{2}}}\right)\cos\theta+\dfrac{9\cos^2\theta}{4\sqrt{2}\,(1-\cos\theta)^{\frac{5}{2}}} \\ & - & \dfrac{3\sin^2\theta}{\sqrt{2}\,(1-\cos\theta)^{\frac{5}{2}}}-\dfrac{45\,\cos\theta\sin^2\theta}{4\sqrt{2}\,(1-\cos\theta)^{\frac{7}{2}}}+\dfrac{105\sin^4\theta}{16\sqrt{2}\,(1-\cos\theta)^{\frac{9}{2}}}. \end{array}$$

We claim that $f'''(\theta)$ has no real zeros in $(0,2\pi)$. Then, since $f'''(\pi)>0$, it follows that $f'''(\theta)>0$ for all $\theta\in(0,2\pi)$.

Now we shall prove the claim. We consider the equation $(1-\cos\theta)^{9/2}f'''(\theta)=0$. In the variable $x=\cos\theta$ this equation becomes

$$(1-x)^2\left[\frac{1}{16\sqrt{2}}(57+38x+x^2)-x(1-x)^{5/2}\right]=0\,. \tag{3}$$

Then, eliminating the term $\sqrt{1-x}$ taking squares, we obtain the polynomial equation

$$(x-1)^4(3249+4332x+1046x^2+2636x^3-5119x^4+5120x^5-2560x^6+512x^7)=0\,.$$

The unique real solutions of this equation are $-0.54498679137..$ and 1. Since both values are not solutions of (3) with $x\in(-1,1)$, the claim follows. ☐

Proposition 4. *The regular n–gon is always a central configuration of the planar $1+n$ body problem*

Proof. We need to proof that $\theta_1=\theta_2=\ldots=\theta_n=\frac{2\pi}{n}$ is a solution of equations (2).

Clearly, equations (2) become

$$f(\frac{2\pi}{n})+f(2\frac{2\pi}{n})+f(3\frac{2\pi}{n})+\ldots+f((n-2)\frac{2\pi}{n})+f((n-1)\frac{2\pi}{n})=0. \tag{4}$$

Note that $f((n-i)\frac{2\pi}{n})=f(2\pi-i\frac{2\pi}{n})$ for $i=1,2,\ldots,n-1$, and using the symmetries of f we have that $f(2\pi-i\frac{2\pi}{n})=-f(i\frac{2\pi}{n})$. So, if n is odd equation (4) holds because all the terms cancel and if n is even all the term cancel except $f(\left(\frac{n}{2}\right)\frac{2\pi}{n})=f(\pi)$, that is zero since $\sin\pi=0$. ☐

Definition 5. *A central configuration $(\theta_1,\theta_2,\ldots,\theta_n)$ of the planar $1+n$ body problem is symmetric with respect to a straight line L containing the large mass, m_0, if modulus a cyclic permutation of the angles we have*

- *when n is even, either*

  $$\theta_1=\theta_n,\quad \theta_2=\theta_{n-1},\quad \ldots,\quad \theta_{\frac{n}{2}}=\theta_{\frac{n+2}{2}},$$

  (in this case the symmetry axis L contains two infinitesimal masses) (see Figure 2), or

  $$\theta_1=\theta_{n-1},\quad \theta_2=\theta_{n-2},\quad \ldots,\quad \theta_{\frac{n-2}{2}}=\theta_{\frac{n+2}{2}},$$

  (in this case the symmetry axis L does not contain any infinitesimal masses) (see Figure 3);

- *and when n is odd*

  $$\theta_1=\theta_n,\quad \theta_2=\theta_{n-1},\quad \ldots,\quad \theta_{\frac{n-1}{2}}=\theta_{\frac{n+3}{2}},$$

  (in this case the symmetry axis L contains one infinitesimal mass) (see Figure 4).

3. Numerical results

Concernig the numerical results, we only know the ones of Salo and Yoder (1988) who gave the number of central configurations for $n\leq 9$. This section is devoted to check those results and to explore other bigger values of n up to 15.

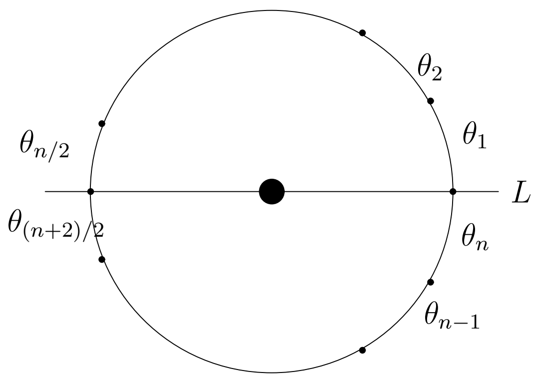

Figure 2: n is even, and L contains two infinitesimal masses.

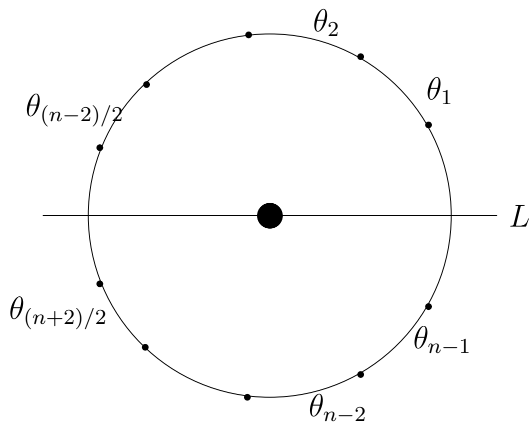

Figure 3: n is even, and L does not contain two infinitesimal masses.

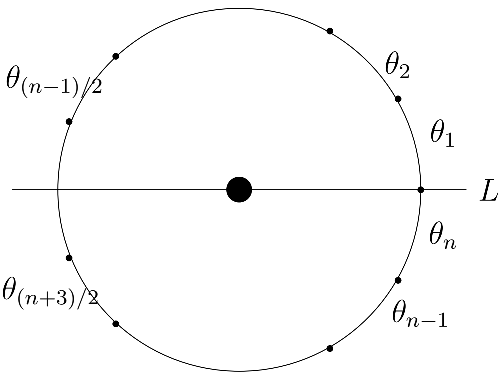

Figure 4: n is odd, and L contains one infinitesimal mass.

Therefore, we consider the nonlinear system of equations (2). Of course we can always assume that

$$\theta_1, \cdots, \theta_{n-1} < \pi \tag{5}$$

and since $\theta_n \neq 0$, we can also assume that

$$\theta_1 + \cdots + \theta_{n-1} < 2\pi. \tag{6}$$

For a fixed value of n, system (2) may be regarded as $F(\theta) = 0$ with $\theta = (\theta_1, ..., \theta_n)$. In order to find the roots of this system, we have implemented an algorithm that combines the rapid local convergence of Newton's method with a globally convergent strategy that will guarantee some progress towards the solution at each iteration. We outline the method described in [27] and we refer the interested reader there for additional details. We know that the Newton step for the set of equations $F(\theta) = 0$ is $\theta_{new} = \theta_{old} + \delta\theta_{old}$, where $\delta\theta_{old} = -J^{-1}(\theta_{old}) \cdot F(\theta_{old})$, $J(\theta)$ being the Jacobian matrix of $F(\theta)$. Then a reasonable strategy to decide whether to accept the Newton step is to require that at every step the function $\frac{1}{2}|F|^2 = \frac{1}{2}F \cdot F$ decreases. This is the same requirement we would impose if we tried to minimize $g = \frac{1}{2}F \cdot F$. Of course, every solution of $F(\theta) = 0$ minimizes g but there may be a local minimum of g that does not vanish F; furthermore, it can be seen that the Newton step is a descent direction for g (that is, $\bigtriangledown g \cdot \delta\theta < 0$).

The method is as follows: we first try the Newton step and check if it decreases g (because once we are close enough to the solution we will get quadratic convergence). If not, we backtrack along the Newton direction until we have an acceptable step; that is, we consider a suitable $t \in (0, 1]$ such that the point

$$\theta_{new} = \theta_{old} + t\delta\theta_{old}$$

decreases g. We can see that this method essentially minimizes g by taking Newton steps designed to bring F to zero.

We also remark that as initial approximation we have taken a set of points chosen in a random way on every interval $(0, \pi)$. We also point out that any point of the net $(\theta_1, ..., \theta_{n-1})$ which does not satisfy condition (6) is eliminated as an initial point for the Newton method.

Our results for $n = 2, ..., 15$, and with all the computations in double precision, are the following: for $N = 10$ in all the different values of $n = 2, ..., 9$, we have obtained the same number of central configurations and the same values (with more precision) than the ones given by Yoder and Salo (1985). For $n = 10, ..., 15$, the numerical exploration gives only the *trivial solution* corresponding to equally spaced angles $\theta_i = 2\pi/n$, for $i = 1, ..., n$. Recently, Carles Simó has obtained numerical evidence that for $n = 9, 10, \ldots, 100$ the unique central configurations is the trivial one.

The actual values (in degrees) of the angles θ_i, $i = 1, ..., n$, for the central configurations of the planar $1 + n$ problem, when varying n are the following (we do not write the trivial solution of equally spaced angles):

θ_1	60.
θ_2	300.

Table 1. Non–trivial central configurations for $n = 2$

θ_1	47.3608595705	82.4690381116
θ_2	47.3608595705	138.765480944
θ_3	265.278280859	138.765480944

Table 2. Non–trivial central configurations for $n = 3$

θ_1	60.	239.648650392
θ_2	60.	41.4977207411
θ_3	120.	37.3559081255
θ_4	120.	41.4977207411

Table 3. Non–trivial central configurations for $n = 4$

θ_1	46.0925284527	32.6600023394
θ_2	51.3270261777	32.6600023394
θ_3	105.626709595	38.2019365548
θ_4	105.626709595	218.276122211
θ_5	51.3270261777	38.2019365548

Table 4. Non–trivial central configurations for $n = 5$

θ_1	47.5240066355	30.0127857084
θ_2	40.5198693768	28.5367336774
θ_3	40.5198693768	30.0127857084
θ_4	47.5240066354	36.3094838501
θ_5	91.9561239876	198.818727205
θ_6	91.9561239876	36.3094838501

Table 5. Non–trivial central configurations for $n = 6$

θ_1	49.2823954323	28.5355057842	48.7784534573	51.6428684752
θ_2	41.5947333658	26.2776693930	45.3999709091	45.8974643028
θ_3	39.5812025241	26.2776693930	45.3999709101	44.1356981194
θ_4	41.5947333658	28.5355057842	48.7784534605	45.8974643028
θ_5	49.2823954323	35.4632374907	55.5475074031	51.6428684754
θ_6	69.3322699397	179.447174663	60.5481364614	60.3918181621
θ_7	69.3322699397	35.4632374907	55.5475074031	60.3918181621

Table 6. Non–trivial central configurations for $n = 7$

θ_1	28.1137392778	49.4914946678
θ_2	25.2481328157	36.6843730729
θ_3	24.4600505522	32.1858180623
θ_4	25.2481328157	30.9746420140
θ_5	28.1137392778	32.1858180623
θ_6	35.9024722662	36.6843730729
θ_7	157.011260727	49.4914946678
θ_8	35.9024722662	92.3019863797

Table 7. Non–trivial central configurations for $n = 8$

These numerical results provide us evidence for the following two conjectures:

Conjecture 6. *For $n \geq 9$ there is only one central configuration, the trivial one.*

Conjecture 7. *All central configurations of the $1+n$ body problem are symmetric with respect to a straight line according with the Definition 5.*

4. The cases $n = 2$ and $n = 3$

In this section we prove analytically that the central configurations computed numerically for $n = 2, 3$ are the unique ones.

Proposition 8. *The $1 + 2$ body problem has two and only two central configurations.*

Proof. For $n = 2$ system (2) becomes

$$f(\theta_1) = 0, \quad \theta_1 + \theta_2 = 2\pi.$$

So we wish to solve $\sin\theta_1 = 0$ or $\cos\theta_1 = 1/2$, These are $\theta_1 = \pi$, that corresponds to the collinear configuration with one small particle on either side of the origin, and $\theta_1 = \pi/3$, $\theta_1 = 5\pi/3$, that give us the same configuration and correspond to the equilateral triangle solution. □

Proposition 9. *The* 1+3 *body problem has three and only three central configurations.*

Proof. For $n = 3$ system (2) becomes

$$\begin{aligned} f(\theta_1) + f(\theta_1 + \theta_2) &= 0, \\ f(\theta_2) + f(\theta_2 + \theta_3) &= 0, \\ \theta_1 + \theta_2 + \theta_3 &= 2\pi. \end{aligned} \tag{7}$$

Using the property of f that $f(2\pi-\theta_1) = -f(\theta_1)$ the second equation of (7) becomes $f(\theta_2) - f(\theta_1) = 0$, that is, $f(\theta_1) = f(\theta_2)$. In a similar way the first equation of (7) implies $f(\theta_1) = f(\theta_3)$. So we have that $f(\theta_1) = f(\theta_2) = f(\theta_3)$.

Lemma 10. *At least two of the angles $\theta_1, \theta_2, \theta_3$ satisfying (7) are equal.*

Proof. Without loss of generality we can assume that θ_1 is the smallest of these three angles and no pair of these angles are equal.

If $\theta_1 < \pi/3$, then from the fact that $f(\theta_1) = f(\theta_2) = f(\theta_3)$ we can see from the graph of f that θ_2 and θ_3 must be bigger that π, that is a contradiction with the fact that $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

If $\theta_1 \geq \pi/3$, then one of the angles θ_2 or θ_3 must be bigger than or equal to $5\pi/3$. But again we have contradicted $\theta_1 + \theta_2 + \theta_3 = 2\pi$. □

Doing a rotation (if necessary) we can consider that $\theta_1 = \theta_2 = \theta \in [0, \pi)$. Hence, we must solve

$$f(\theta) + f(2\theta) = 0, \text{ for } \theta \in (0, \pi).$$

From Proposition 3 we get that

$$\frac{d^3}{d\theta^3}(f(\theta) + f(2\theta)) > 0.$$

Clearly, $f(\theta)+f(2\theta)$ tends to $-\infty$ when θ goes to zero, and it tends to $+\infty$ when θ goes to π. So, $f(\theta) + f(2\theta)$ have no more than three zeros in $(0, \pi)$. It is easily computed that

$$f(2\pi/3) + f(4\pi/3) = 0,$$

$$f'(2\pi/3) + 2f'(4\pi/3) < 0.$$

So, $f(\theta) + f(2\theta)$ must have exactly two more zeros, one in $(0, 2\pi/3)$ and the other in $(2\pi/3, \pi)$, both with positive derivative. Finally since that $f(\pi/2) + f(\pi) > 0$, the smaller of these zeros is in $(0, \pi/2)$. □

These three configurations correspond to an equilateral triangle and to an isosceles triangle with the large mass at their barycenter of mass and to a convex configuration with the large mass at one of its vertices.

5. The case $n = 4$. Numerical approach.

In Section 3 we have shown that the number of configurations of the $1 + 4$ body body problem is at least three. We are going to prove now (numerically) that there are exactly three central configurations.

To do so, we consider the tetrahedron T limited by the vertices $A = (2\pi, 0, 0, 0)$, $B = (0, 2\pi, 0, 0)$ $C = (0, 0, 2\pi, 0)$ and $D = (0, 0, 0, 2\pi)$. Each point in the segment $\bar{AB}$ can be represented by a coordinate λ given by the map $(\theta_1, \theta_2, 0, 0) \in \bar{AB} \to \lambda \in [0, 1]$, such that $(\theta_1, \theta_2) = (2\pi(1 - \lambda), 2\pi\lambda)$. In the same way, we can use $(\lambda, \mu) \in [0, 1]^2$ to represent a point in the triangle limited by the vertices A, B and C, or any point in the tetrahedron by three coordinates $(\lambda, \mu, \delta) \in [0, 1]^3$–called from now on *normalized* coordinates– such that for any $(\theta_1, \theta_2, \theta_3, \theta_4) \in T$, we have

$$\begin{aligned} \theta_1 &= 2\pi(1-\delta)(1-\mu)(1-\lambda) \\ \theta_2 &= 2\pi(1-\delta)(1-\mu)\lambda \\ \theta_3 &= 2\pi(1-\delta)\mu \\ \theta_4 &= 2\pi\delta \end{aligned}$$

and the nonlinear system (2) in the *four* coordinates θ_i, $i = 1, ..., 4$ inside the tetrahedron $T \in R^4$ becomes a nonlinear system in the *three* coordinates (λ, μ, δ) defined in the open cube $C = (0, 1)^3 \in R^3$ (we remark that a point in the boundary of C corresponds to $\theta_i = 0$, for some i), which in principle should be easier to visualize. However each one of the equations of system (2) reduces to a surface in the cube C, and the intersections of the three surfaces (that is the central configurations) obtained from the first three equations of (2) are still hard to distinguish. Therefore we consider slices of such surfaces, that is, for any fixed $\lambda = \lambda_0$, we consider the curves obtained from the intersection between the surfaces and the plane $\lambda = \lambda_0$. Of course, a central configuration corresponds to a point belonging to the intersection of the three surfaces, or in the slice context, to the intersection of the three curves (we remark that each curve may have more than one component). Now the method to determine the exact number of central configurations consists of varying $\lambda_0 \in (0, 1)$, and following the different shapes of the curves in order to guarantee the number of possible intersections. We note that, besides the number of central configurations, we can also compute their values but this task will be detailed in an analytical way in the next section.

We remark that for any central configuration we can always assume that either $\theta_1 < \theta_2$ (that is $\lambda = 1/2$) or $\theta_1 < \theta_2$ ($\lambda > 1/2$). On one hand, we plot the curves of the slice for $\lambda = 1/2$ in figure 5 (left); this figure shows three intersection points $(\lambda, \mu, \delta) = (1/2, 1/5, 1/6)$, $(\lambda, \mu, \delta) = (1/2, 1/2, 1/3)$, both corresponding to the central configuration (in degrees) $(\theta_1, \theta_2, \theta_3, \theta_4) = (60, 60, 120, 120)$, and $(\lambda, \mu, \delta) = (1/2, 1/3, 1/4)$ corresponding to the central configuration (in degrees) $(\theta_1, \theta_2, \theta_3, \theta_4) = (90, 90, 90, 90)$. On the other hand, we show the evolution of the curves in each slice for the different values of $\lambda > 1/2$: $\lambda = 0.505$ (figure 5), $\lambda = 0.51$, $\lambda = 0.52$ (figure 6), $\lambda = 0.5267$, $\lambda = 0.53$ (figure 7), $\lambda = 0.54$, $\lambda = 0.55$ (figure 8), $\lambda = 0.6$, $\lambda = 0.6657$ (figure 9), $\lambda = 0.7$, $\lambda = 0.8$ (figure 10), $\lambda = 0.8524$, $\lambda = 0.9$ (figure 11). So for $\lambda > 1/2$, there are only three intersection points for $\lambda = 0.5267, 0.6657, 0.8524$ corresponding to

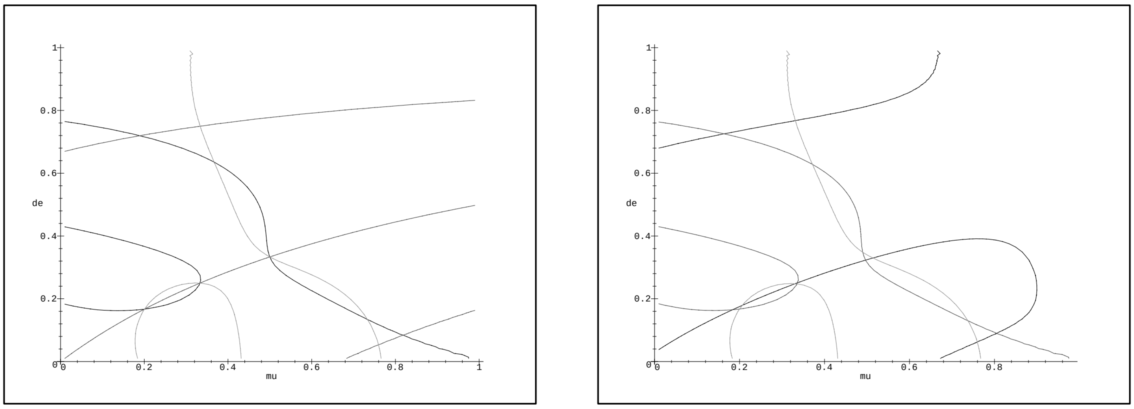

Figure 5: $1 + 4$ body problem in normalized coordinates. Slice in μ (horizontal axis) and δ (vertical one). Left: $\lambda = 0.5$, right: $\lambda = 0.505$

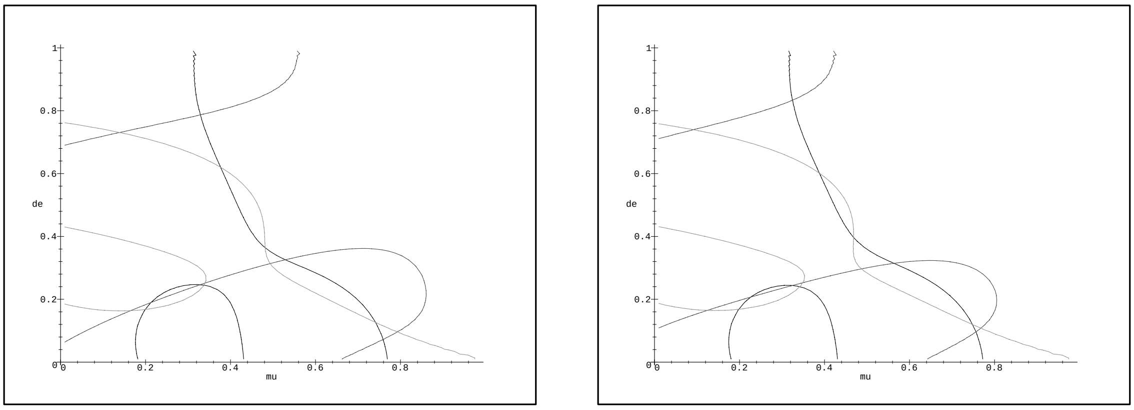

Figure 6: $\lambda = 0.51$, $\lambda = 0.52$

the same central configuration $(\theta_1, \theta_2, \theta_3, \theta_4) = (37.35590, 41.4977, 239.6486, 41.4977)$. Therefore we conclude that the $1 + 4$ body problem has three and only three central configurations.

6. The case $n = 4$. Analytic approach

The equations of the central configurations for the $1 + 4$ body problem are

$$\begin{aligned} f(\theta_1) + f(\theta_1 + \theta_2) + f(\theta_1 + \theta_2 + \theta_3) &= 0, & (8) \\ f(\theta_2) + f(\theta_2 + \theta_3) + f(\theta_2 + \theta_3 + \theta_4) &= 0, & (9) \\ f(\theta_3) + f(\theta_3 + \theta_4) + f(\theta_3 + \theta_4 + \theta_1) &= 0, & (10) \\ f(\theta_4) + f(\theta_4 + \theta_1) + f(\theta_4 + \theta_1 + \theta_2) &= 0, & (11) \\ \theta_1 + \theta_2 + \theta_3 + \theta_4 &= 2\pi. \end{aligned}$$

Proposition 11. *If a central configuration of the planar $1 + 4$ body problem has two infinitesimal bodies diametrically opposite, then the central configuration is symmetric*

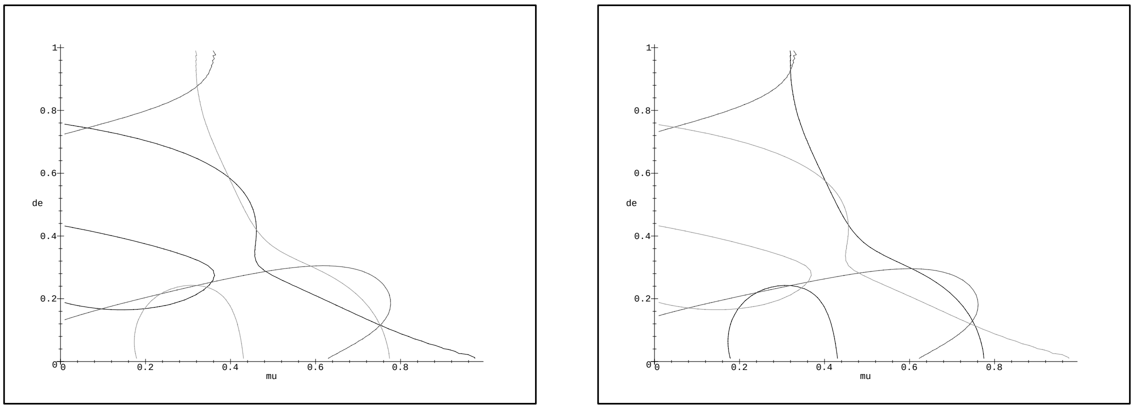

Figure 7: $\lambda = 0.5267$, $\lambda = 0.53$

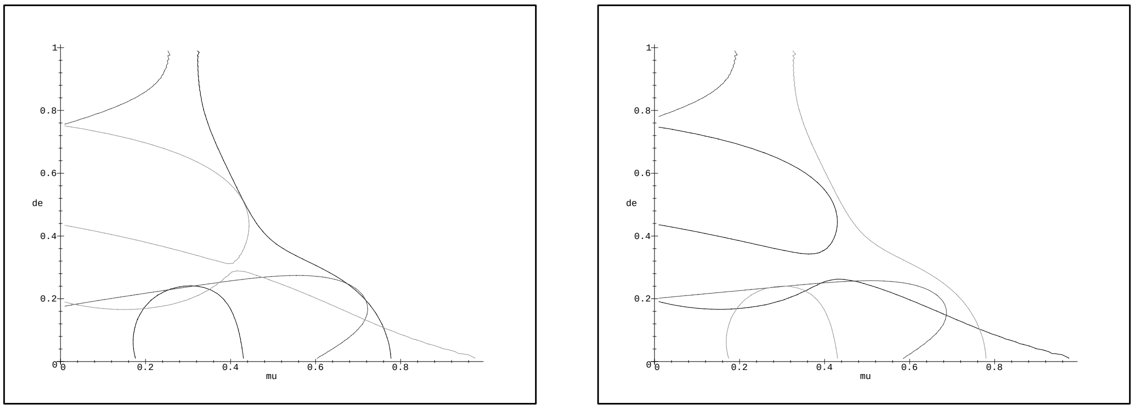

Figure 8: $\lambda = 0.54$, $\lambda = 0.55$

with respect to the straight line determined by these two bodies.

Proof. We divide the proof into two cases.

Case 1: $\theta_1 + \theta_2 + \theta_3 = \pi$, and $\theta_4 = \pi$.

Since $f(\theta_1 + \theta_2 + \theta_3) = 0$, from (8) we obtain that $f(\theta_1 + \theta_2) = -f(\theta_1)$, and then from the plot of f we have that $\theta_1 < \pi/3$. The maximum positive value for $f(\theta_1 + \theta_2)$ is 0.7265.., so if $\theta_1 < 0.8166..$, i.e. $f(\theta_1) < -0.7265..$, then (8) cannot hold. When $\theta_1 = 0.8166..$ we have that $\theta_2 = 1.0746..$ and (11) does not hold because $f(\theta_4) = 0$ and $f(\theta_4 + \theta_1) < 0$ and $f(\theta_4 + \theta_1 + \theta_2) < 0$. When $0.8166.. < \theta_1 < \pi/3$ we have two possible values for $\theta_1 + \theta_2$ in order to satisfy (8). Suppose that $\theta_1 + \theta_2 \leq 1.8911..$, i.e. $\theta_1 + \theta_2$ is smaller than the local maximum of f, that is a contradiction with (11) since $f(\theta_4) = 0$ and $f(\theta_4 + \theta_1) < 0$ and $f(\theta_4 + \theta_1 + \theta_2) < 0$. Now assume $\theta_1 + \theta_2 > 1.8911...$ In this case if $\theta_1 + \theta_2 \leq 2\pi/3$ we have again, using the same argument, contradiction with (11), and if $\theta_1 + \theta_2 > 2\pi/3$ we have contradiction with (10) due to the fact that $\theta_3 < \pi/3$ and so all terms in (10) are negative. So, Case 1 does not occur.

Case 2: $\theta_1 + \theta_2 = \pi$ and $\theta_3 + \theta_4 = \pi$.

From the graph of f we can consider three subcases according with θ_1 belongs to

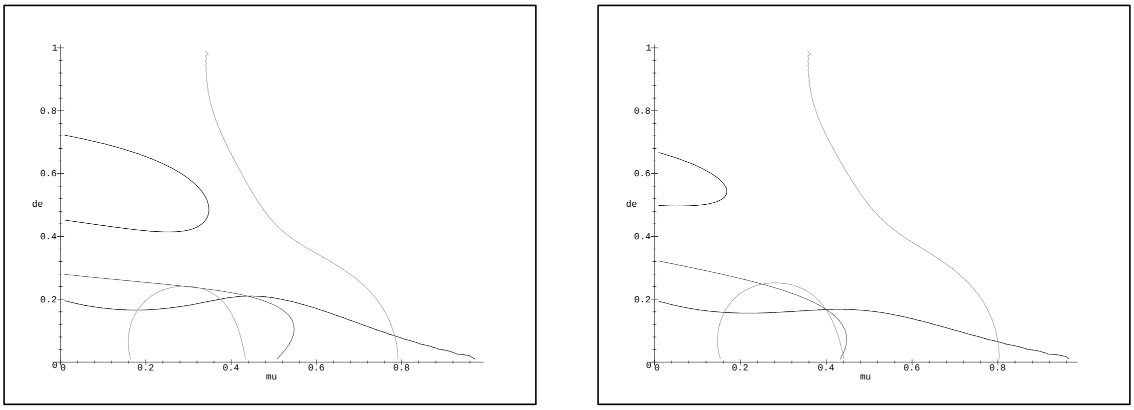

Figure 9: $\lambda = 0.6$, $\lambda = 0.6657$

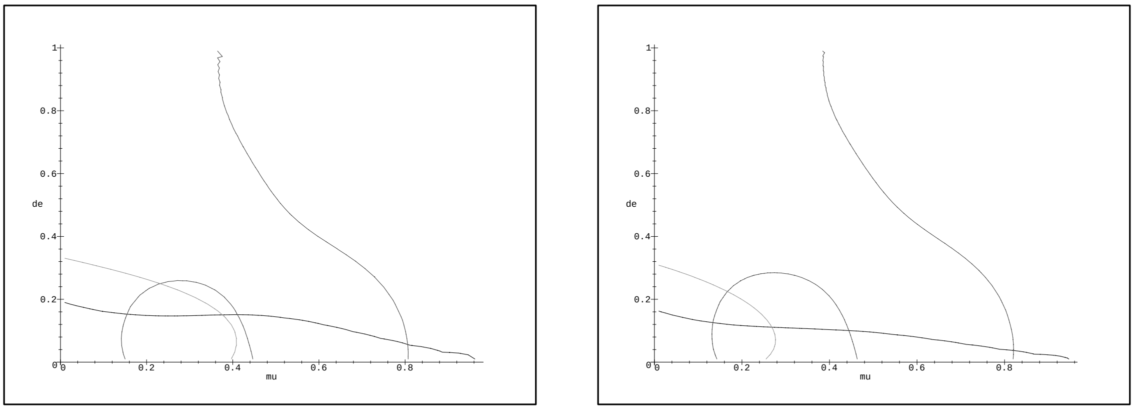

Figure 10: $\lambda = 0.7$, $\lambda = 0.8$

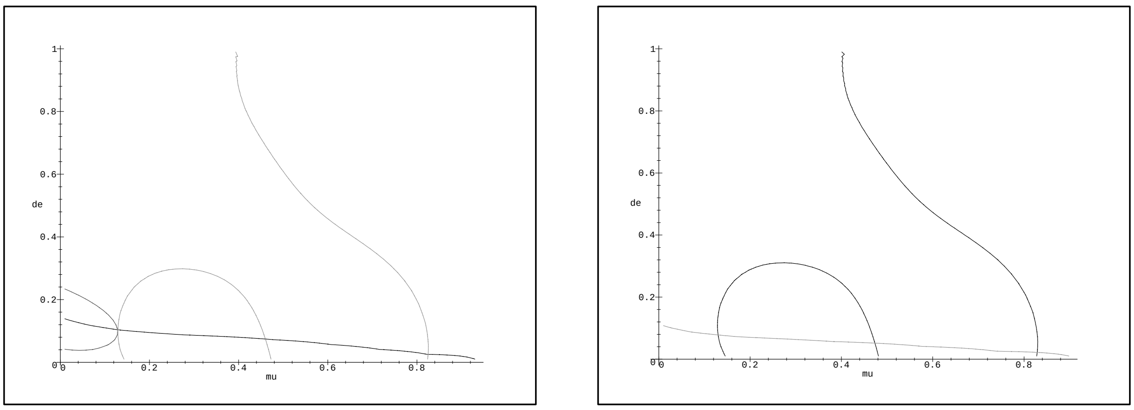

Figure 11: $\lambda = 0.8524$, $\lambda = 0.9$

$(0, \frac{\pi}{3})$, $(\frac{\pi}{3}, \pi)$ or it is equal to $\frac{\pi}{3}$. Assume that $\theta_1 \in (0, \frac{\pi}{3})$. If $\theta_3 \leq 2\pi/3$, then (8) does not hold due $f(\theta_1) < 0$, $f(\theta_1 + \theta_2) = 0$ and $f(\theta_1 + \theta_2 + \theta_3) \leq 0$. If $\theta_3 > 2\pi/3$ from (8) we have $f(\theta_1 + \theta_2 + \theta_3) = -f(\theta_1)$, this implies $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$. So, the proposition follows under these assumptions.

Now suppose that $\theta_1 \in (\frac{\pi}{3}, \pi)$. If $\theta_1 < 1.8911..$ we have two possible values for θ_3 in order to satisfy (8); $\pi < \theta_1 + \theta_2 + \theta_3 < 4.392..$ (see Proposition 3) and $4.392.. < \theta_1 + \theta_2 + \theta_3 < 5\pi/3$. The last one implies by the symmetry ($\theta_1 = \theta_4$ and $\theta_2 = \theta_3$), and consequently the proposition holds. When $\pi < \theta_1 + \theta_2 + \theta_3 < 4.392..$ we have that $\theta_3 < 1.25..$ and $4\pi/3 < \theta_3 + \theta_4 + \theta_1 < 5.0326...$ Then $-0.69.. < f(\theta_3 + \theta_4 + \theta_1) < -0.35$ and $f(\theta_3) < 0.35..$, so (10) does not hold.

The proof when $\theta_1 > 1.8911$ is similar.

Finally, when $\theta_1 = \pi/3$ in order to verify (8) we have that $\theta_3 = 2\pi/3$, so the proposition follows. □

Proposition 12. *There are exactly two central configurations of the planar* $1+4$ *body problem under the hypothesis of Proposition 11.*

Proof. We can assume that $\theta_2 = \theta_3 = \theta \leq \frac{\pi}{2}$ and $\theta_1 = \theta_4 = \pi - \theta$. Then the equations of central configurations become

$$\begin{aligned} f(\pi - \theta) + f(\pi) + f(\pi + \theta) &= 0, & (12)\\ f(\theta) + f(2\theta) + f(\pi + \theta) &= 0, & (13)\\ f(\theta) + f(\pi) + f(2\pi - \theta) &= 0, & (14)\\ f(\pi - \theta) + f(2\pi - 2\theta) + f(2\pi - \theta) &= 0. & (15) \end{aligned}$$

Clearly Equations (12) and (14) always hold (see Proposition 3), and Equations (13) and (15) are the same equation (see again Proposition 3).

Consider the function $g(\theta) = f(\theta) + f(2\theta) + f(\pi + \theta)$. It is easy to check that $g(x)$ has the three zeros : $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{2}$ and $\theta = 2\pi/3$. Since $\frac{d}{d\theta^3} g(\theta) > 0$ for all $\theta \in (0, \pi)$, by the Rolle Theorem, cannot have more than 3 zeros. So, we have exactly the following two solutions when $0 < \theta \leq \frac{\pi}{2}$.

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{2},$$

and

$$\theta_1 = \theta_4 = 2\pi/3 \quad \text{and} \quad \theta_2 = \theta_3 = \frac{\pi}{3}.$$

□

Now, we look for symmetric central configurations without two infinitesimal bodies diametrically opposite. Notice that, from Definition 5, it is equivalent to have $\theta_1 = \theta_3$.

Proposition 13. *There is exactly one central configuration in the planar* $1+4$ *body problem with* $\theta_1 = \theta_3$*.*

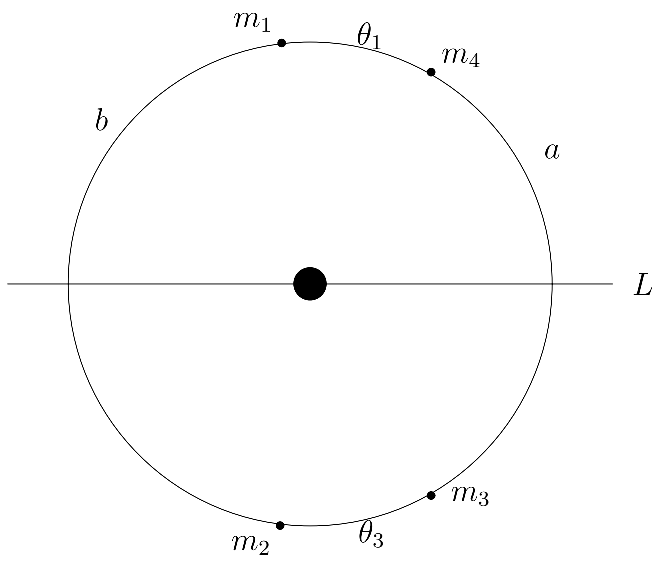

Figure 12: Variables a and b on a symmetric configuration without two infinitesimal bodies diametrically opposite.

The proof is based on results about polynomials that we introduce now.

Let the roots of a polynomial $P(x)$ of degree n with leading coefficient one be denoted by a_i, $i = 1, 2, \ldots, n$ and those of a polynomial $Q(x)$ of degree m with leading coefficient one be denoted by b_j, $j = 1, 2, \ldots, m$. The *resultant* of P and Q, $\mathrm{Res}[P, Q]$ is the expression formed by the product of all the differences $a_i - b_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. In order to see how to compute $\mathrm{Res}[P, Q]$, see for instance [15] and [28].

The main property of the resultant is that if P and Q have a common solution then necessarily $\mathrm{Res}[P, Q] = 0$. For polynomials of two variables, say $P(X, Y)$ and $Q(X, Y)$, they can be considered as polynomials in X with polynomial coefficients in Y, then the resultant with respect to X, $\mathrm{Res}[P, Q, X]$ is a polynomial in the variable Y with the following property. If $P(X, Y)$ and $Q(X, Y)$ have a common solution (X_0, Y_0) then $\mathrm{Res}[P, Q, X](Y_0) = 0$. Similarly, $\mathrm{Res}[P, Q, Y](X_0) = 0$.

Proof. Consider the symmetric axis L that divides θ_4 and θ_2 in two equal angles. Let b be the angle from the axis to m_1 as it is indicated in Figure 12, and let a be the angle from m_4 to the axis, also as it is indicated from Figure 12.

Then the equations (8–11) for the central configurations in the variables (a, b) reduce to

$$\begin{aligned} g_1(a, b) &= 0, \qquad (16)\\ g_2(a, b) &= 0, \qquad (17)\end{aligned}$$

where

$$\begin{aligned} g_1(a, b) &= f(\pi - a - b) + f(\pi + a - b) + f(2\pi - 2b),\\ g_2(a, b) &= f(2a) + f(\pi + a - b) + f(\pi + a + b).\end{aligned}$$

In order to write the above trigonometric equations in polynomial form, first use the built–in Mathematica function TrigExpand that splits up sums and integer multiples that appear in arguments of trigonometric functions, and then expands out products of trigonometric functions into sums of powers, using trigonometric identities when possible. Doing the change of variables $x = \sin(\frac{a}{2})$, $y = \cos(\frac{b}{2})$ we have that

$$\begin{aligned} g_1(x,y) &= -128\,x^6\,y^3\,(1-y^2)^{\frac{3}{2}} \\ &+ x^4\left(1-2\,y^2+256\,y^3\,\sqrt{1-y^2}-384\,y^5\,\sqrt{1-y^2}+128\,y^7\,\sqrt{1-y^2}\right) \\ &- (-1+y^2)^2\left(-1+2\,y^2+8\,\sqrt{1-x^2}\,y^3-128\,y^5\,\sqrt{1-y^2}+128\,y^7\,\sqrt{1-y^2}\right) \\ &- 2\,x^2\,(-1+y^2)\left(-1+2\,y^2+64\,y^7\,\sqrt{1-y^2}-4\,y^3\left(\sqrt{1-x^2}+16\,\sqrt{1-y^2}\right)\right) \end{aligned} \tag{18}$$

$$\begin{aligned} g_2(x,y) &= 2\,x^6+128\,x^{11}\,\sqrt{1-x^2}+128\,x^9\,\sqrt{1-x^2}\,(-3+y^2)-(-1+y^2)^2+x^4\,(-5+4\,y^2) \\ &+ 2\,x^2\,(2-3\,y^2+y^4)-8\,x^7\left(-48\,\sqrt{1-x^2}+16\,\sqrt{1-x^2}\,y^2+16\,\sqrt{1-x^2}\,y^4-\sqrt{1-y^2}\right) \\ &+ 8\,x^3\left(-32\,\sqrt{1-x^2}\,y^4+16\,\sqrt{1-x^2}\,y^6+\sqrt{1-y^2}+y^2\left(16\,\sqrt{1-x^2}+\sqrt{1-y^2}\right)\right) \\ &- 8\,x^5\left(-48\,\sqrt{1-x^2}\,y^4+16\,\sqrt{1-x^2}\,y^6+2\left(8\,\sqrt{1-x^2}+\sqrt{1-y^2}\right)\right. \\ &+ \left.y^2\left(16\,\sqrt{1-x^2}+\sqrt{1-y^2}\right)\right) \end{aligned} \tag{19}$$

Second step is to eliminate the terms $\sqrt{1-x^2}$ and $\sqrt{1-y^2}$ and so create fictitious solutions. Then $g_1(x,y)$ and $g_2(x,y)$ have the following polynomial expressions

$$\begin{aligned} g_1(x,y) = {} & -64\,(x-y)\,(x+y)\,(-1+x^2+y^2) \\ & (-x^4+3\,x^6+253\,x^8-1535\,x^{10}+3840\,x^{12}-5120\,x^{14}+3840\,x^{16} \\ & -1536\,x^{18}+256\,x^{20}-x^2\,y^2+5\,x^4\,y^2-268\,x^6\,y^2-243\,x^8\,y^2+6139\,x^{10}\,y^2 \\ & -17920\,x^{12}\,y^2+24320\,x^{14}\,y^2-17664\,x^{16}\,y^2+6656\,x^{18}\,y^2-1024\,x^{20}\,y^2+64\,x^3\,y^3 \\ & -288\,x^5\,y^3+416\,x^7\,y^3-160\,x^9\,y^3-96\,x^{11}\,y^3+64\,x^{13}\,y^3-y^4+5\,x^2\,y^4-9\,x^4\,y^4 \\ & +1803\,x^6\,y^4-5134\,x^8\,y^4-1528\,x^{10}\,y^4+22528\,x^{12}\,y^4-35072\,x^{14}\,y^4+24576\,x^{16}\,y^4 \\ & -8192\,x^{18}\,y^4+1024\,x^{20}\,y^4-288\,x^3\,y^5+1280\,x^5\,y^5-1856\,x^7\,y^5+832\,x^9\,y^5+224\,x^{11}\,y^5 \\ & -192\,x^{13}\,y^5+3\,y^6-268\,x^2\,y^6+1803\,x^4\,y^6-9722\,x^6\,y^6+24572\,x^8\,y^6-26372\,x^{10}\,y^6 \\ & +5376\,x^{12}\,y^6+11776\,x^{14}\,y^6-9216\,x^{16}\,y^6+2048\,x^{18}\,y^6+416\,x^3\,y^7-1856\,x^5\,y^7 \\ & +2752\,x^7\,y^7-1440\,x^9\,y^7+128\,x^{13}\,y^7+253\,y^8-243\,x^2\,y^8-5134\,x^4\,y^8+24572\,x^6\,y^8 \\ & -48632\,x^8\,y^8+48896\,x^{10}\,y^8-24832\,x^{12}\,y^8+5120\,x^{14}\,y^8-160\,x^3\,y^9+832\,x^5\,y^9 \\ & -1440\,x^7\,y^9+896\,x^9\,y^9-128\,x^{11}\,y^9-1535\,y^{10}+6139\,x^2\,y^{10}-1528\,x^4\,y^{10} \\ & -26372\,x^6\,y^{10}+48896\,x^8\,y^{10}-36864\,x^{10}\,y^{10}+13312\,x^{12}\,y^{10}-2048\,x^{14}\,y^{10}-96\,x^3\,y^{11} \\ & +224\,x^5\,y^{11}-128\,x^9\,y^{11}+3840\,y^{12}-17920\,x^2\,y^{12}+22528\,x^4\,y^{12}+5376\,x^6\,y^{12} \\ & -24832\,x^8\,y^{12}+13312\,x^{10}\,y^{12}-2048\,x^{12}\,y^{12}+64\,x^3\,y^{13}-192\,x^5\,y^{13}+128\,x^7\,y^{13} \\ & -5120\,y^{14}+24320\,x^2\,y^{14}-35072\,x^4\,y^{14}+11776\,x^6\,y^{14}+5120\,x^8\,y^{14}-2048\,x^{10}\,y^{14} \\ & +3840\,y^{16}-17664\,x^2\,y^{16}+24576\,x^4\,y^{16}-9216\,x^6\,y^{16}-1536\,y^{18}+6656\,x^2\,y^{18} \\ & -8192\,x^4\,y^{18}+2048\,x^6\,y^{18}+256\,y^{20}-1024\,x^2\,y^{20}+1024\,x^4\,y^{20}) \end{aligned}$$

and

$$
\begin{aligned}
g_1(x,y) \;=\;& 1 - 8\,x^2 + 28\,x^4 - 56\,x^6 + 70\,x^8 - 56\,x^{10} + 28\,x^{12} - 8\,x^{14} + x^{16} - 16\,y^2 \\
&+120\,x^2\,y^2 - 392\,x^4\,y^2 + 728\,x^6\,y^2 - 840\,x^8\,y^2 + 616\,x^{10}\,y^2 - 280\,x^{12}\,y^2 \\
&+72\,x^{14}\,y^2 - 8\,x^{16}\,y^2 + 116\,y^4 - 808\,x^2\,y^4 + 2436\,x^4\,y^4 - 4144\,x^6\,y^4 \\
&+4340\,x^8\,y^4 - 2856\,x^{10}\,y^4 + 1148\,x^{12}\,y^4 - 256\,x^{14}\,y^4 + 24\,x^{16}\,y^4 - 632\,y^6 \\
&+3608\,x^2\,y^6 - 41744\,x^4\,y^6 + 275056\,x^6\,y^6 - 929464\,x^8\,y^6 + 1842360\,x^{10}\,y^6 \\
&-2296608\,x^{12}\,y^6 + 1835584\,x^{14}\,y^6 - 917536\,x^{16}\,y^6 + 262144\,x^{18}\,y^6 - 32768\,x^{20}\,y^6 \\
&+2998\,y^8 + 52712\,x^2\,y^8 - 10428\,x^4\,y^8 - 1857176\,x^6\,y^8 + 8274374\,x^8\,y^8 \\
&-17444096\,x^{10}\,y^8 + 21566816\,x^{12}\,y^8 - 16516224\,x^{14}\,y^8 + 7733264\,x^{16}\,y^8 - 2031616\,x^{18}\,y^8 \\
&+229376\,x^{20}\,y^8 - 43944\,y^{10} - 683736\,x^2\,y^{10} + 3006264\,x^4\,y^{10} + 1325944\,x^6\,y^{10} \\
&-27988336\,x^8\,y^{10} + 68823744\,x^{10}\,y^{10} - 85794944\,x^{12}\,y^{10} + 62654208\,x^{14}\,y^{10} \\
&-27033600\,x^{16}\,y^{10} + 6356992\,x^{18}\,y^{10} - 622592\,x^{20}\,y^{10} + 525612\,y^{12} \\
&+3005384\,x^2\,y^{12} - 22717444\,x^4\,y^{12} + 34601728\,x^6\,y^{12} + 298881488\,x^8\,y^{12} \\
&-2300179584\,x^{10}\,y^{12} + 7714320576\,x^{12}\,y^{12} - 15160837632\,x^{14}\,y^{12} + 18835013632\,x^{16}\,y^{12} \\
&-15040577536\,x^{18}\,y^{12} + 7517011968\,x^{20}\,y^{12} - 2147483648\,x^{22}\,y^{12} + 268435456\,x^{24}\,y^{12} \\
&-3464552\,y^{14} - 502776\,x^2\,y^{14} + 79340128\,x^4\,y^{14} - 1273441344\,x^6\,y^{14} \\
&+4932346944\,x^8\,y^{14} - 1932968192\,x^{10}\,y^{14} - 30394372608\,x^{12}\,y^{14} + 90359202304\,x^{14}\,y^{14} \\
&-127799361536\,x^{16}\,y^{14} + 105225322496\,x^{18}\,y^{14} - 51540131840\,x^{20}\,y^{14} + 13958643712\,x^{22}\,y^{14} \\
&-1610612736\,x^{24}\,y^{14} + 12150337\,y^{16} - 49721600\,x^2\,y^{16} + 1489171936\,x^4\,y^{16} \\
&+2727684736\,x^6\,y^{16} - 42685595552\,x^8\,y^{16} + 102904350208\,x^{10}\,y^{16} - 52133927936\,x^{12}\,y^{16} \\
&-165530042368\,x^{14}\,y^{16} + 349475340288\,x^{16}\,y^{16} - 313515835392\,x^{18}\,y^{16} + 151934599168\,x^{20}\,y^{16} \\
&-38654705664\,x^{22}\,y^{16} + 4026531840\,x^{24}\,y^{16} - 16880072\,y^{18} - 840023232\,x^2\,y^{18} \\
&-14048119424\,x^4\,y^{18} + 38703687936\,x^6\,y^{18} + 92593222144\,x^8\,y^{18} - 453161779712\,x^{10}\,y^{18} \\
&+585645735936\,x^{12}\,y^{18} - 120057495552\,x^{14}\,y^{18} - 442330775552\,x^{16}\,y^{18} + 510007181312\,x^{18}\,y^{18} \\
&-250181844992\,x^{20}\,y^{18} + 59055800320\,x^{22}\,y^{18} - 5368709120\,x^{24}\,y^{18} + 233443032\,y^{20} \\
&+12322659968\,x^2\,y^{20} + 43244971200\,x^4\,y^{20} - 267367033344\,x^6\,y^{20} + 143672879104\,x^8\,y^{20} \\
&+855112884224\,x^{10}\,y^{20} - 1660477698048\,x^{12}\,y^{20} + 1049907691520\,x^{14}\,y^{20} + 97949188096\,x^{16}\,y^{20} \\
&-474583400448\,x^{18}\,y^{20} + 249644974080\,x^{20}\,y^{20} - 53687091200\,x^{22}\,y^{20} + 4026531840\,x^{24}\,y^{20} \\
&-3531262240\,y^{22} - 68982658816\,x^2\,y^{22} - 8371166720\,x^4\,y^{22} + 790927441408\,x^6\,y^{22} \\
&-1195522703360\,x^8\,y^{22} - 483877863424\,x^{10}\,y^{22} + 2464930594816\,x^{12}\,y^{22} - 2089355837440\,x^{14}\,y^{22} \\
&+494464401408\,x^{16}\,y^{22} + 224409944064\,x^{18}\,y^{22} - 152471339008\,x^{20}\,y^{22} + 28991029248\,x^{22}\,y^{22} \\
&-1610612736\,x^{24}\,y^{22} + 23867609616\,y^{24} + 222675806720\,x^2\,y^{24} - 345808296960\,x^4\,y^{24} \\
&-1276101984256\,x^6\,y^{24} + 2996536696832\,x^8\,y^{24} - 1159005732864\,x^{10}\,y^{24} - 1999170699264\,x^{12}\,y^{24} \\
&+2269835689984\,x^{14}\,y^{24} - 770141323264\,x^{16}\,y^{24} - 8589934592\,x^{18}\,y^{24} + 54223962112\,x^{20}\,y^{24} \\
&-8589934592\,x^{22}\,y^{24} + 268435456\,x^{24}\,y^{24} - 96850870784\,y^{26} - 460498812416\,x^2\,y^{26} \\
&+1261452951552\,x^4\,y^{26} + 957643735040\,x^6\,y^{26} - 4309241561088\,x^8\,y^{26} + 3053380435968\,x^{10}\,y^{26} \\
&+571209678848\,x^{12}\,y^{26} - 1488197779456\,x^{14}\,y^{26} + 565861941248\,x^{16}\,y^{26} - 46170898432\,x^{18}\,y^{26} \\
&-9663676416\,x^{20}\,y^{26} + 1073741824\,x^{22}\,y^{26} + 267801694208\,y^{28} + 614528638976\,x^2\,y^{28} \\
&-2496661385216\,x^4\,y^{28} + 424465661952\,x^6\,y^{28} + 3901501997056\,x^8\,y^{28} - 3575459610624\,x^{10}\,y^{28} \\
&+503584915456\,x^{12}\,y^{28} + 571230650368\,x^{14}\,y^{28} - 232733540352\,x^{16}\,y^{28} + 21474836480\,x^{18}\,y^{28} \\
&+536870912\,x^{20}\,y^{28} - 536744919040\,y^{30} - 461097992192\,x^2\,y^{30} + 3259338588160\,x^4\,y^{30} \\
&-1948515303424\,x^6\,y^{30} - 2154437804032\,x^8\,y^{30} + 2527575670784\,x^{10}\,y^{30} - 631360192512\,x^{12}\,y^{30} \\
&-105226698752\,x^{14}\,y^{30} + 51002736640\,x^{16}\,y^{30} - 3221225472\,x^{18}\,y^{30} + 805774327808\,y^{32} \\
&+295698432\,x^2\,y^{32} - 2958572257280\,x^4\,y^{32} + 2433019281408\,x^6\,y^{32} + 553245474816\,x^8\,y^{32}
\end{aligned}
$$

$$
\begin{aligned}
&-1125281431552\,x^{10}\,y^{32} + 308163903488\,x^{12}\,y^{32} - 4563402752\,x^{16}\,y^{32} - 921156714496\,y^{34}\\
&+460524879872\,x^{2}\,y^{34} + 1877961867264\,x^{4}\,y^{34} - 1783476781056\,x^{6}\,y^{34} + 104152956928\,x^{8}\,y^{34}\\
&+302795194368\,x^{10}\,y^{34} - 75161927680\,x^{12}\,y^{34} + 2147483648\,x^{14}\,y^{34} + 806088736768\,y^{36}\\
&-614157254656\,x^{2}\,y^{36} - 809064464384\,x^{4}\,y^{36} + 839666106368\,x^{6}\,y^{36} - 130728067072\,x^{8}\,y^{36}\\
&-42949672960\,x^{10}\,y^{36} + 7516192768\,x^{12}\,y^{36} - 537405685760\,y^{38} + 460633145344\,x^{2}\,y^{38}\\
&+216895848448\,x^{4}\,y^{38} - 250181844992\,x^{6}\,y^{38} + 40265318400\,x^{8}\,y^{38} + 2147483648\,x^{10}\,y^{38}\\
&+268703891456\,y^{40} - 223338299392\,x^{2}\,y^{40} - 27380416512\,x^{4}\,y^{40} + 42949672960\,x^{6}\,y^{40}\\
&-4563402752\,x^{8}\,y^{40} - 97710505984\,y^{42} + 69793218560\,x^{2}\,y^{42} - 1073741824\,x^{4}\,y^{42}\\
&-3221225472\,x^{6}\,y^{42} + 24427626496\,y^{44} - 12884901888\,x^{2}\,y^{44} + 536870912\,x^{4}\,y^{44}\\
&-3758096384\,y^{46} + 1073741824\,x^{2}\,y^{46} + 268435456\,y^{48}
\end{aligned}
$$

The resultant with respect to x (respectively y), $\mathrm{Res}[g_1, g_2, x]$ (respectively $\mathrm{Res}[g_1, g_2, y]$) are polynomials in the variable y (respectively x) of degree 1012. More detailed,

$$
\mathrm{Res}[g_1, g_2, x] = 79228162514264337593543950336(-1+y)^{48}y^{96}(1+y)^{48}(-1+2y^2)^{24}U(y)W(y)
$$

where $U(y)$ and $W(y)$ are polynomials of degree 312 and 360 respectively. And

$$
\begin{aligned}
\mathrm{Res}[g_1, g_2, y] \;=\;& 2964277484475294602843417216222410441043711607 44\\
& 0398439410114150602576118782 36(-1+x)^{48}x^{96}(1+x)^{48}\\
& (7 - 99x^2 + 480x^4 - 1276x^6 + 1920x^8 - 1536x^{10} + 512x^{12})^4 V(x)Z(x)
\end{aligned}
$$

where $V(x)$ and $Z(x)$ are polynomials of degree 312 and 360 respectively.

The only couple of real roots of $\mathrm{Res}[g_1, g_2, y]$ and $\mathrm{Res}[g_1, g_2, x]$ that are roots of equations (18) and (19) are $(x, y) = (0.8625.., 0.1622..)$ and $(x, y) = (0.1622.., 0.8625..)$ that give rise to the same central configuration. □

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1 Dept. Matemàtica Aplicada III, Universitat Politècnica de Catalunya, 08240 Manresa, Barcelona, Spain. e-mail: cors@eupm.upc.es

2 Dept. Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain. e-mail: jllibre@mat.uab.es

3 Dept. Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain. e-mail: merce.olle@upc.es