Controllability of second order linear systems

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Abstract: - Let (A_1, A_2, B) be a triple of matrices representing two-order time-invariant linear systems, $\ddot{x} = A_1\dot{x} + A_2x + Bu$. Using linearization process we study the controllability of second order linear systems. We obtain sufficient conditions for controllability and we analyze the kind of systems verifying these conditions.

Key-Words: - Two-order linear systems, linearization, feedback, controllability.

1 Introduction

The study of second order systems has experienced a great deal, they are applied in engineering as well as in economic systems. They are used for example, in modelling of flexible beams [5].

A second order linear system is described by the following state space equation

$$\ddot{x} = A_1\dot{x} + A_2x + Bu, \tag{1}$$

where A_i are n-square complex matrices and B a rectangular complex matrix in adequate size. Applying the Laplace transform to the equation (1) with zero initial conditions we obtain the corresponding transfer condition

$$\widehat{x}(s) = (s^2I_n - sA_1 - A_2)^{-1}B\widehat{u}(s). \tag{2}$$

G. Antoniou in [1], gives an algorithm for computing the transfer function $T(s) = (s^2I_n - sA_1 - A_2)^{-1}B$.

In this paper, and using linearization process (see [4], for example), we present sufficient conditions for existence of a control w in such a way the sate can be driven from any position to any other in a prescribed period of time.

The structure of this paper is as follows.

In section 2, an equivalence relation over the space of second order linear systems is defined and it induce an equivalence over the space of linearized systems. We observe that the equivalent linearized systems are feedback equivalent as linear systems, but the converse is not true.

In section 3, the controllability analysis relating controllability of second order linear systems and controllability of linear systems associated is presented.

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Finally in section 4, sufficient conditions to ensure controllability of second order systems, in terms of matrices defining the systems are presented. Also, we describe the collection of systems verifying these conditions finding pairs of matrices with several prescribed blocks, as well as several prescribed invariants for prescribed blocks of pairs of matrices. In recent years several results are obtained describing invariants of matrices with several prescribed blocks as for example [2], among others.

In the sequel, we denote by $M_{r\times s}(\mathbb{C})$ the space of complex matrices having r rows and s columns, and in the case which $r = s$ we write $M_r(\mathbb{C})$. In order to simplify notations, we denote second order linear systems by triple of matrices (A_1, A_2, B), and the space of all triples of matrices by $\mathcal{M} = \{(A_1, A_2, B) \mid A_1, A_2 \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})\}$.

We identify pairs (A, B) and triples of matrices (A, B, C) with rectangular matrices $\begin{pmatrix} A & B \end{pmatrix}$ and $\begin{pmatrix} A & B & C \end{pmatrix}$ in order to use the matrix representation of the transformations.

2 Orbits of linearized systems

Let $\ddot{x} = A_1\dot{x} + A_2x + Bu$ be a second order linear system as in the introduction, the standard transformations that can be applied are

1. basis change in the state space: $(A_1, A_2, B) \to (P^{-1}A_1P, P^{-1}A_2P, P^{-1}B)$,
2. basis change in the input space: $(A_1, A_2, B) \to (A_1, A_2, BQ)$,
3. feedback: $(A_1, A_2, B) \to (A_1, A_2 + BF_2, B)$,
4. and derivative feedback: $(A_1, A_2, B) \to (A_1 + BF_1, A_2, B)$.

Then, the initial equation is transformed to

$$\ddot{x} = (P^{-1}A_1P + P^{-1}BF_1)\dot{x} + (P^{-1}A_2P + P^{-1}BF_2)x + P^{-1}BQu \tag{3}$$

and we get the following definition of equivalence for second order linear systems

Definition 2.1. *Two second order linear systems (A_1', A_2', B'), $(A_1'', A_2'', B'') \in \mathcal{M}$, are equivalent if and only if there exist matrices $P \in Gl(n;\mathbb{C})$, $Q \in Gl(m;\mathbb{C})$ and $F_1, F_2 \in M_{m\times n}(\mathbb{C})$ such that these equalities*

$$\begin{array}{ll} A_1'' & = P^{-1}A_1'P + P^{-1}B'F_1 \\ A_2'' & = P^{-1}A_2'P + P^{-1}B'F_2 \\ B'' & = P^{-1}B'Q \end{array} \tag{4}$$

hold.

It is straightforward that this relation is an equivalence relation.

Remark 2.1. Given two equivalent triples (A_1', A_2', B'), $(A_1'', A_2'', B'') \in \mathcal{M}$, the pairs of matrices (A_1', B'), and (A_2', B'), are feedback equivalent to (A_1'', B'') and (A_2'', B'') respectively.

The problem of finding a canonical reduced form for triples of matrices under this equivalence relation is an open problem. In order to obtain some structural invariants we consider the linearization process.

Remember that (see [4] for details), if we consider $X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, we can rewrite the second order linear system (1) as the following linear system

$$\dot{X} = \mathbb{A}X + \mathbb{B}u, \tag{5}$$

where $\mathbb{A}$ and $\mathbb{B}$ are the matrices:

$$\mathbb{A} = \begin{pmatrix} 0 & I_n \\ A_2 & A_1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}. \tag{6}$$

The expression of second order linear systems as a linear systems permit to consider feedback equivalence. Remember that (see [3], for example), two linear systems $(\mathbb{A}_1, \mathbb{B}_1)$ and $(\mathbb{A}_2, \mathbb{B}_2)$ are called feedback equivalent if and only if there exist $(\mathbb{P}, \mathbb{Q}, \mathbb{F})$ in the full feedback group $\mathcal{G} = \{(\mathbb{P}, \mathbb{Q}, \mathbb{F}) \mid \mathbb{P} \in Gl(2n; \mathbb{C}), \mathbb{Q} \in Gl(m; \mathbb{C}), \mathbb{F} \in M_{m\times 2n}(\mathbb{C})$ such that

$$(\mathbb{A}_2, \mathbb{B}_2) = \mathbb{P}^{-1}(\mathbb{A}_1, \mathbb{B}_1) \begin{pmatrix} \mathbb{P} & 0 \\ \mathbb{F} & \mathbb{Q} \end{pmatrix}.$$

It is easy to prove the following proposition.

Proposition 2.1. *Let (A_1, A_2, B) and (A'_1, A'_2, B') be two equivalent systems. Then, the linearized systems are feedback equivalent.*

The converse is not true as we can see in the following example.

Example 2.1. Let (A_1, A_2, B) with $A_1 = 0 \in M_2(\mathbb{C})$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $B = 0 \in M_{2\times 1}(\mathbb{C})$ and (A'_1, A'_2, B') with $A'_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A'_2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$, $B' = 0 \in M_{2\times 1}(\mathbb{C})$. The systems are not equivalent but the linearized systems are feedback equivalent.

Proposition (2.1) ensures that all structural invariants of a linearized system as a pair under feedback equivalence are invariants for the initial triple, but the set is not a complete system of invariants.

In order to preserve the form (6) for equivalent linear systems, in the sense that the only equivalent pairs are those that are linearized of some equivalent second order linear system, we need to restrict to the subgroup $\mathcal{G}_2 \subset \mathcal{G}$ formed by matrices $(\mathbb{P}, \mathbb{Q}, \mathbb{R}) \in \mathcal{G}$ with $\mathbb{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$, $P \in Gl(n, \mathbb{C})$. So, we consider the following equivalence relation.

Definition 2.2. *Two pairs* $\left(\begin{pmatrix} 0 & I_n \\ A'_2 & A'_1 \end{pmatrix}, \begin{pmatrix} 0 \\ B' \end{pmatrix}\right)$ *and* $\left(\begin{pmatrix} 0 & I_n \\ A''_2 & A''_1 \end{pmatrix}, \begin{pmatrix} 0 \\ B'' \end{pmatrix}\right)$ *are equivalent if and only if, there exists* $(\mathbb{P}, \mathbb{Q}, \mathbb{F}) \in \mathcal{G}_2$, *such that*

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_n & 0 \\ A'_2 & A'_1 & B' \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_2 & F_1 & Q \end{pmatrix} = \begin{pmatrix} 0 & I_n & 0 \\ A''_2 & A''_1 & B'' \end{pmatrix}.$$

Example 2.2. Let $(\mathbb{A},\mathbb{B}) = \left(\begin{pmatrix} 0 & I_2 \\ A_2 & A_1 \end{pmatrix}, \begin{pmatrix} 0 \\ B \end{pmatrix} \right)$ be a pair with $A_1 = A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The collection of pairs of matrices equivalent to it under the relation (2.2), is $\mathcal{O}(\mathbb{A},\mathbb{B}) = \{(\mathbb{X},\mathbb{Y})\}$ where $(\mathbb{X},\mathbb{Y}) = \left(\begin{pmatrix} 0 & I_2 \\ X_2 & X_1 \end{pmatrix}, \begin{pmatrix} 0 \\ Y \end{pmatrix} \right)$ being a pair with

$$X_1 = \begin{pmatrix} \dfrac{x_3x_4 - x_2x_8}{x_1x_4 - x_2x_3} & \dfrac{x_4^2 - x_2x_9}{x_1x_4 - x_2x_3} \\ \dfrac{-x_3^2 + x_1x_8}{x_1x_4 - x_2x_3} & \dfrac{-x_3x_4 + x_1x_9}{x_1x_4 - x_2x_3} \end{pmatrix}, \; X_2 = \begin{pmatrix} \dfrac{x_3x_4 - x_2x_6}{x_1x_4 - x_2x_3} & \dfrac{x_4^2 - x_2x_7}{x_1x_4 - x_2x_3} \\ \dfrac{-x_3^2 + x_1x_6}{x_1x_4 - x_2x_3} & \dfrac{-x_3x_4 + x_1x_7}{x_1x_4 - x_2x_3} \end{pmatrix},$$

and

$$Y = \begin{pmatrix} \dfrac{-x_2x_5}{x_1x_4 - x_2x_3} \\ \dfrac{x_1x_5}{x_1x_4 - x_2x_3} \end{pmatrix}.$$

for all $(x_1,\dots,x_9) \in \mathbb{C}^9$ with $x_1x_4 - x_2x_3 \neq 0$

So, we have the following proposition

Proposition 2.2. *Two second order linear systems (A'_1, A'_2, B'), $(A''_1, A''_2, B'') \in \mathcal{M}$, are equivalent if and only if the associated linearized systems $(\mathbb{A}',\mathbb{B}')$, $(\mathbb{A}'',\mathbb{B}'')$ are $\mathcal{G}_2$-equivalent.*

3 Controllability

In this section we will go to study controllability of second order linear systems using the controllability character of the linearized systems (5).

We recall that a second order linear system is called controllable if, for any $t_1 > 0$, $x(0), \dot{x}(0) \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, there exists a control $u(t)$ such that $x(t_1) = w$. This definition is a natural generalization of controllability concept in the first order linear systems.

Taking into account that $x(t)$ is a solution of the second order linear system if and only if $\binom{x(t)}{\dot{x}(t)}$ is a solution of the associated linearized system, the second order linear system is controllable if and only if the linearized system is controllable.

So, we can use results about controllability of linear systems, in particular (see for example [3]) we have that the pair $(\mathbb{A},\mathbb{B})$ (linearized of (A_1, A_2, B)) is controllable if and only if

$$\operatorname{rank} \begin{pmatrix} sI_{2n} - \mathbb{A} & \mathbb{B} \end{pmatrix} = 2n \quad \forall s \in \mathbb{C}, \tag{7}$$

or equivalently, if and only if

$$\operatorname{rank} \begin{pmatrix} \mathbb{B} & \mathbb{A}\mathbb{B} & \dots & \mathbb{A}^{2n}\mathbb{B} \end{pmatrix} = 2n. \tag{8}$$

Making elementary transformations in the matrix (7), we can analyze the controllability directly from the matrices defining the second order linear system, obtaining the following characterization.

Proposition 3.1. *The second order linear system* (A_1, A_2, B)*, is controllable if and only if*

$$\text{rank}\begin{pmatrix} s^2I_n - sA_1 - A_2 & B \end{pmatrix} = n \quad \forall s \in \mathbb{C}$$

Proof. making elementary block transformations on the matrix (7) we have

$$\begin{aligned} r &= \text{rank}\begin{pmatrix} sI_n & -I_n & 0 \\ -A_2 & sI_n - A_1 & B \end{pmatrix} \\ &= \text{rank}\begin{pmatrix} 0 & I_n & 0 \\ s^2I_n - sA_1 - A_2 & 0 & B \end{pmatrix} = \\ &= n + \text{rank}\begin{pmatrix} s^2I_n - sA_1 - A_2 & B \end{pmatrix}. \end{aligned} \tag{9}$$

Then, $r = 2n, \quad \forall s \in \mathbb{C}$ if and only if the proposition holds.

□

The proposition (3.1) permit us to define controllability in the following manner.

Definition 3.1. *We say that the second order linear system* (A_1, A_2, B)*, is controllable if and only if,*

$$\text{rank}\begin{pmatrix} s^2I_n - sA_1 - A_2 & B \end{pmatrix} = n \quad \forall s \in \mathbb{C}. \tag{10}$$

It is well know that, the controllability of a linear system is invariant under feedback equivalence, then the controllability of linearized systems is invariant under $\mathcal{G}_2$-equivalence. So, the controllability of second order linear systems is invariant under equivalence relation considered. In fact we have the following proposition

Proposition 3.2. *The*

$$\mathit{rank}\begin{pmatrix} s^2I_n - sA_1 - A_2 & B \end{pmatrix}$$

is invariant under equivalence defined above.

Proof. Let (A_1', A_2', B') and (A_1'', A_2'', B'') be two equivalent triples. Then, there exist $(\mathbb{P}, \mathbb{Q}, \mathbb{F}) \in \mathcal{G}_2$ such that

$$\begin{aligned} A_1'' &= P^{-1}A_1'P + P^{-1}B'F_1 \\ A_2'' &= P^{-1}A_2'P + P^{-1}B'F_2 \\ B'' &= P^{-1}B'Q. \end{aligned}$$

Then,

$$\text{rank}\begin{pmatrix} s^2I_n - sA_1'' - A_2'' & B'' \end{pmatrix} =$$
$$= \text{rank}\left((P^{-1}\begin{pmatrix} s^2I_n - sA_1' - A_2' & B' \end{pmatrix}\begin{pmatrix} P & 0 \\ -sF_1 - F_2 & Q \end{pmatrix} \right)$$

□

The stability of linear systems $(\mathbb{A},\mathbb{B})$ can be determined directly from the eigenvalues of matrix $\mathbb{A}$. When the system is controllable the eigenvalues can be modified conveniently by feedback $\mathbb{A}+\mathbb{B}\mathbb{F}$, in order to stabilize the system.

It is not difficult to compute that the poles of the transfer function (2) of the second order linear system (A_1, A_2, B) are the eigenvalues of the matrix $\mathbb{A}$ in the linearized system $(\mathbb{A},\mathbb{B})$, as well as that the poles of the transfer function of the second order linear system (A_1+BF_1, A_2+BF_2, B) are also, the eigenvalues of $(\mathbb{A}+\mathbb{B}\mathbb{F},\mathbb{B})$, with $\mathbb{F}=\begin{pmatrix} F_2 & F_1 \end{pmatrix}$.

As a consequence we have the following proposition.

Proposition 3.3. *Let (A_1, A_2, B) be a controllable second order linear system. Then the system can be stabilized by means a feedback and derivative feedback.*

4 Sufficient conditions for controllability

Now we present conditions for matrices A_1, A_2, B ensuring the controllability of the system.

Lemma 4.1. *Let (A_1, A_2, B) be a second order linear system with $A_1 = A_2$ and (A_1, B) a controllable pair in its Kronecker canonical reduced form. Then it is controllable.*

Proof. Taking into account that the pair (A_1, B) is in its Kronecker canonical reduced form (see [3], for example), it is (N, B) with

$$\begin{aligned} &N=\begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_m \end{pmatrix},\ N_i=\begin{pmatrix} 0 & I_{k_i-1}\\ 0 & 0\end{pmatrix}\in M_{k_i}(\mathbb{C}),\\ &1\le i\le m,\ k_1+\dots+k_m=n,\ k_1\ge\dots\ge k_m,\\ &B=\begin{pmatrix} E_1 & & \\ & \ddots & \\ & & E_m \end{pmatrix},\ E_i=(0,\dots,0,1)^t\in M_{k_i\times 1}(\mathbb{C}),\\ &1\le i\le m, \end{aligned}$$

then

$$\begin{aligned} &\operatorname{rank}\begin{pmatrix} s^2I_n-sN-N & B\end{pmatrix}=\\ &=\operatorname{rank}\begin{pmatrix} s^2I_{k_1}-sN_1-N_1 & & & E_1 & & \\ & \ddots & & & \ddots & \\ & & s^2I_{k_m}-sN_m-N_m & & & E_m\end{pmatrix}=\\ &=\operatorname{rank}\begin{pmatrix} s^2I_{k_1}-sN_1-N_1 & E_1\end{pmatrix}+\dots+\operatorname{rank}\begin{pmatrix} s^2I_{k_m}-sN_m-N_m & E_m\end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} &\begin{pmatrix} s^2I_{k_i}-sN_i-N_i & E_i\end{pmatrix}=\\ &=\begin{pmatrix} s^2 & -s-1 & 0 & \dots & 0\\ \vdots & \ddots & & & \\ 0 & \dots & s^2 & -s-1 & 0\\ 0 & \dots & 0 & s^2 & 1\end{pmatrix}\in M_{k_i\times(k_i+1)},\ 1\le i\le m \end{aligned}$$

and obviously,

$$\begin{aligned}
&\operatorname{rank}\begin{pmatrix} s^2I_{k_i}-sN_i-N_i & E_i\end{pmatrix}=k_i\,,\ 1\leq i\leq m\,,\ \text{if } s\neq 0\,,\\
&\operatorname{rank}\begin{pmatrix} s^2I_{k_i}-sN_i-N_i & E_i\end{pmatrix}=k_i\,,\ 1\leq i\leq m\,,\ \text{if } s=0\,.
\end{aligned}$$

Then

$$\operatorname{rank}\begin{pmatrix} s^2I_n-sN-N & B\end{pmatrix}=k_1+\cdots+k_m=n \quad \forall s\in\mathbb{C}$$

□

Proposition 4.1. *Let (A_1,A_2,B) be a second order linear system with $A_1=A_2$ and the pair (A_1,B) being controllable. Then the given system is controllable.*

Proof. Taking into account that (A,B) is controllable, there exist $P\in Gl(n;\mathbf{C})$, $Q\in Gl(m;\mathbf{C})$ and $F\in M_{m\times n}(\mathbf{C})$, such that

$$P^{-1}\left(A_1,B\right)\begin{pmatrix}P&0\\F&Q\end{pmatrix}=\left(N,E\right).$$

So, the triple (A_1,A_2,B) can be reduced to the triple (N,N,E) making

$$P^{-1}\left(A_1,A_2,B\right)\begin{pmatrix}P&0&0\\0&P&0\\F&F&Q\end{pmatrix}=\left(N,N,E\right).$$

Now, taking into account the invariance under equivalence and applying the above lemma we obtain the result. □

Notice that if (A_1,A_2,B) is in such a way that (A_1,B) and (A_2,B) are controllable we cannot deduce the controllability of the triple,

Example 4.1. Let (A_1,A_2,B) be a second order linear system in $\mathcal{M}$ with

$$A_1=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ A_2=\begin{pmatrix}1&1\\0&0\end{pmatrix},\ B=\begin{pmatrix}0\\1\end{pmatrix}$$

Clearly (A_1,B) and (A_2,B) are controllable, but the triple (A_1,A_2,B) is not controllable

$$\operatorname{rank}\begin{pmatrix} s^2I_2-sA_1-A_2 & B\end{pmatrix}=1,\ \text{for } s=-1.$$

But, we have the following proposition

Proposition 4.2. *Let (A_1,A_2,B) be a second order linear system in $\mathcal{M}$ such that, the pairs (A_1,B) and (A_2,B) are controllable and they can be reduced simultaneously modulo feedback to the same Kronecker canonical reduced form. Then (A_1,A_2,B) is controllable.*

Proof. Let $P \in Gl(n;\mathbf{C})$, $Q \in Gl(m;\mathbf{C})$ and $F_1, F_2 \in M_{m\times n}(\mathbf{C})$ be matrices such that

$$P^{-1}\begin{pmatrix} A_1 & B \end{pmatrix}\begin{pmatrix} P & 0 \\ F_1 & Q \end{pmatrix} = \begin{pmatrix} N & E \end{pmatrix},$$
$$P^{-1}\begin{pmatrix} A_2 & B \end{pmatrix}\begin{pmatrix} P & 0 \\ F_2 & Q \end{pmatrix} = \begin{pmatrix} N & E \end{pmatrix},$$

then

$$P^{-1}\begin{pmatrix} A_1 & A_2 & B \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_1 & F_2 & Q \end{pmatrix} = \begin{pmatrix} N & N & E \end{pmatrix}.$$

Now we can apply lemma before. □

We want to remark that the controllability of the pairs (A_1, B) and (A_2, B) is not a necessary condition for controllability of the system (A_1, A_2, B). To prove that it suffices an example

Example 4.2. Let $A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It is clear that (A_1, B) is controllable, (A_2, B) is uncontrollable and

$$\operatorname{rank}\begin{pmatrix} s^2 I_2 - sA_1 - A_2 & B \end{pmatrix} = 2, \text{ for all } s \in \mathbb{C}.$$

To find the collection of triples of matrices $(A_1, A_2, B) \in \mathcal{M}$ equivalent to a triple in the form (A', A', B'), is equivalent to find pairs of matrices with several prescribed blocks in the matrices as well as several prescribed invariants for prescribed blocks of pairs of matrices. That is to say, to find all pairs of matrices

$$\left(\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\right) \in M_{2n}(\mathbb{C}) \times M_{2n\times m}(\mathbb{C}), \tag{11}$$

with $A_{11} = 0$, $A_{12} = I_n$, $B_1 = 0$, and the pairs of matrices (A_{21}, B_2) and (A_{22}, B_2) having the same prescribed complete set of invariants.

We have the following proposition.

Proposition 4.3. *A second order linear system of matrices $(A_1, A_2, B) \in \mathcal{M}$ is equivalent to a system in the form (A', A', B'), if and only if*

$$rank\begin{pmatrix} A_2 - A_1 & B \end{pmatrix} = rank\, B \tag{12}$$

Proof. Suppose that (A_1, A_2, B) is equivalent to (A', A', B'), then there exist matrices $P \in GL(n;\mathbb{C})$, $Q \in Gl(m;\mathbb{C})$ and $F_1, F_2 \in M_{m\times n}(\mathbb{C})$ such that

$$(A_1, A_2, B) = (P^{-1}A'P + P^{-1}B'F_1, P^{-1}A'P + P^{-1}B'F_2, P^{-1}B'Q).$$

So

$$\operatorname{rank}\begin{pmatrix} A_2 - A_1 & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} BQ^{-1}F_2 - BQ^{-1}F_1 & B \end{pmatrix} = \operatorname{rank} B$$

Conversely. If rank $\begin{pmatrix} A_2 - A_1 & B \end{pmatrix} = \operatorname{rank} B$, then there exists a matrix $F \in M_{m\times n}(\mathbb{C})$ such that $A_2 - A_1 = BF$ equivalently,

$$\begin{pmatrix} 0 & I & 0 \\ A_2 & A_1 & B \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -F & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ A_1 & A_1 & B \end{pmatrix}.$$

$\square$

As example, we compute the collection of matrices $A_1 \in M_n(\mathbb{C})$ such that the systems $(N, A_1, E) \in \mathcal{M}$, with $m = 1$, are equivalent to (N, N, E). For that it suffices to compute the matrix A_1 such that rank $\begin{pmatrix} N - A_1 & E \end{pmatrix} = 1$, and it is if and only if the matrix A_1 is a companion matrix.

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