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An Ellipsoidal Billiard with a Quadratic Potential*

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There exists an infinite hierarchy of integrable generalizations of the geodesic flow on an n-dimensional ellipsoid. These generalizations describe the motion of a point in the force fields of certain polynomial potentials. In the limit as one of semiaxes of the ellipsoid tends to zero, one obtains integrable mappings corresponding to billiards with polynomial potentials inside an $(n-1)$-dimensional ellipsoid.

In this paper, for the first time we give explicit expressions for the ellipsoidal billiard with a quadratic (Hooke) potential, its representation in Lax form, and a theta function solution. We also indicate the generating function of the restriction of the potential billiard map to a level set of an energy type integral. The method we use to obtain theta function solutions is different from those applied earlier and is based on the calculation of limit values of meromorphic functions on generalized Jacobians.

Introduction

One of the best-known discrete integrable systems is the billiard inside an $(n-1)$-dimensional ellipsoid (more generally, a quadric)

$$Q = \left\{ \frac{X_1^2}{a_1} + \dots + \frac{X_n^2}{a_n} = 1 \right\} \in \mathbb{R}^n, \qquad \mathbb{R}^n \ni (X_1, \dots, X_n),\ 0 < a_1 < \dots < a_n,$$

with elastic reflections on Q. Following Birkhoff [1], one can treat the billiard as the limit of the geodesic flow on the n-dimensional ellipsoid

$$\widetilde{Q} = \left\{ \frac{X_1^2}{a_1} + \dots + \frac{X_{n+1}^2}{a_{n+1}} = 1 \right\}, \qquad 0 < a_{n+1} < a_1 < \dots < a_n, \tag{1}$$

as one of the semiaxes (a_{n+1}) tends to zero. In this case, the straight line trajectories of the ellipsoidal billiard inherit the remarkable property of geodesics on $\widetilde{Q}$ given by the Chasles theorem: the continuations of the trajectories before and after impacts are simultaneously tangent to $n-1$ quadrics confocal to Q. The parameters of these quadrics are first integrals of the discrete system.

Veselov [2] described the billiard map in terms of discrete Lagrangian formalism and showed that its complex invariant manifolds are open subsets of coverings of hyperelliptic Jacobians and that the restriction of the map to such manifolds is represented by shifts by a constant vector. Explicit theta function solutions for the billiard were obtained in [2] by applying spectral theory of difference operators and in [3, 14] by making use of the Lax representation and the method of factorization of matrix polynomials.

Later, the author [11] recovered these solutions as the degeneration of theta function solutions for geodesics on $\widetilde{Q}$ in the limit as $a_{n+1} \to 0$. He also obtained biasymptotic solutions describing spatial billiard trajectories tending to oscillations along the major axis of the ellipsoid as well as generic theta function solutions for the billiard in a domain bounded by two confocal ellipsoids.

On the other hand, as was noticed by Jacobi himself and later by many other authors (e.g., see [16]), there exists an infinite hierarchy of integrable generalizations of this problem. These generalizations describe the motion on $\widetilde{Q}$ in the force field of some basis polynomial potentials $\mathscr{V}_p(X_1, \dots, X_{n+1})$ of degree $2p$, $p \in \mathbb{N}$, and their linear combinations. The simplest integrable

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potential is the Hooke potential, or the potential of an ideal elastic string joining the center of the ellipsoid $\widetilde{Q}$ with a point mass on $\widetilde{Q}$:

$$\mathscr{V}_1(X) = \sigma(X_1^2 + \dots + X_{n+1}^2)/2, \qquad \sigma = \text{const}\,.$$

In the limit as $a_{n+1} \to 0$ the motion on $\widetilde{Q}$ under the Hooke force passes into the motion inside Q under the potential $\mathscr{V} = \sigma(X_1^2 + \dots + X_n^2)/2$. However, in contrast with the cases $\sigma = 0$ and $\sigma < 0$, for $\sigma > 0$ (an attractive Hooke potential), the total energy h must be sufficiently large for the trajectory to reach Q. Namely,

$$h + \sigma(X_1^2 + \dots + X_n^2)/2 > \varepsilon > 0 \tag{2}$$

inside Q for some positive constant ε. If condition (2) is satisfied, then the motion on $\widetilde{Q}$ passes into a motion inside Q with impacts and reflection on Q. One can show that these reflections are still elastic. Thus, we arrive at an ellipsoidal billiard with the Hooke potential described by the mapping $\mathscr{B}$: $(x, v) \to (\tilde{x}, \tilde{v})$, where $x \in Q$, $v \in \mathbb{R}^n$ are the coordinates of an impact point on Q and the *outgoing* velocity at this point, respectively, and $\tilde{x}$, $\tilde{v}$ are the same vectors at the next impact point.

As was noted in [12], this system, as well as the billiard limits of the motion on $\widetilde{Q}$ with the higher-order potentials $\mathscr{V}_p(X_1, \dots, X_n, X_{n+1})$, is completely integrable.

Remark 1. In contrast with the classical (geodesic) billiard, in a potential billiard the velocity vector between the impacts is not constant. Another new feature of the latter billiard is that for any subsequent impact points $x, \tilde{x} \in Q$ there is a whole one-parameter family of trajectories with different energy passing through these points, namely, arcs of conics (for $\sigma > 0$, ellipses) lying in the plane Π passing through x, $\tilde{x}$, and the origin in $\mathbb{R}^n$. This family forms a *pencil* of conics with base points x, $\tilde{x}$ (the case of a billiard in $\mathbb{R}^2$ is depicted in Fig. 1). In particular, for $\sigma > 0$, the pencil includes the ellipse $\Pi \cap Q$ and the straight line segment joining x with $\tilde{x}$, which represents a trajectory with infinite energy.

One can show that after the reflection at $\tilde{x}$ these trajectories no longer form a pencil of conics; i.e., the next impact point depends not only on x, $\tilde{x}$, but also on the energy. In other words, two subsequent impact points on Q do not determine the entire sequence of impact points. This implies that, in contrast with the geodesic billiard, the discrete time Lagrangian formalism developed in [2] is applicable to a potential billiard only under some constraints on the initial conditions.

In this paper, for the first time we give explicit expressions for the map $\mathscr{B}$, its representation in Lax form, the generating function (a discrete Lagrangian) of the restriction of $\mathscr{B}$ to a level set of an energy type integral, and a theta function solution. We show that adding the quadratic potential to the billiard changes the geometric and symplectic properties of this discrete system dramatically.

The method we use to obtain theta-functional solutions is different from those applied in [2, 3] and is based on the calculation of limit values of meromorphic functions on generalized Jacobians.

1. The Mapping $\mathscr{B}$, the Lax Representation, and First Integrals

Theorem 1. *The mapping $\mathscr{B}$ describing the billiard with a quadratic potential has the form*

$$\begin{aligned}
\tilde{x} &= -\frac{1}{\nu}[(\sigma - (v, a^{-1}v))x + 2(x, a^{-1}v)v], \\
\tilde{v} &= -\frac{1}{\nu}[(\sigma - (v, a^{-1}v))v - 2\sigma(x, a^{-1}v)x] + \mu a^{-1}\tilde{x} \\
&= -\frac{1}{\nu}[(\sigma - (v, a^{-1}v))(v + \mu a^{-1}x) + 2(x, a^{-1}v)(\mu a^{-1}v - \sigma x)], \\
\nu &= \sqrt{4\sigma(x, a^{-1}v)^2 + (\sigma - (v, a^{-1}v))^2}, \qquad \mu = \frac{2(\tilde{v}, a^{-1}\tilde{x})}{(\tilde{x}, a^{-2}\tilde{x})}.
\end{aligned} \tag{3}$$

Note that these relations determine the mapping uniquely and, in the limit as $\sigma \to 0$, it passes into the standard billiard mapping [2, 14]

$$\tilde{x} = x - \frac{2(x, a^{-1}v)}{(v, a^{-1}v)}\, v, \qquad \tilde{v} = v + \frac{2(\tilde{v}, a^{-1}\tilde{x})}{(\tilde{x}, a^{-2}\tilde{x})}\, a^{-1}\tilde{x}. \tag{4}$$

As follows from (3), (4), both mappings have the integral

$$(v, a^{-1}x) = (\tilde{v}, a^{-1}\tilde{x}). \tag{5}$$

Sketch of proof. Consider a segment of the billiard trajectory between subsequent impacts on Q assuming that $\sigma > 0$ (the case $\sigma < 0$ can be considered in a similar way). Let (x_0, v_0) and $(\tilde{x}, v')$ be the position and velocity vectors of the mass point at the initial point and the end point of the segment, respectively. The continuation of the segment in $\mathbb{R}^n$ is the ellipse given in parametric form by

$$X(\phi) = \cos(\phi)x + \frac{1}{\sqrt{\sigma}}\sin(\phi)v, \tag{6}$$

$$\phi = \sqrt{\sigma}\, t,$$

$$V(\phi) = \cos(\phi)v - \sin(\phi)\sqrt{\sigma}x. \tag{7}$$

Substituting (6) into the equation $(X, a^{-1}X) = 1$ of the ellipsoid, we obtain the parameter ϕ, the coordinates, and the velocity at the end point of the segment. Solving this equation, we obtain

$$\begin{aligned} \tilde{x} &= -\frac{(\sigma - (v, a^{-1}v))x + 2(x, a^{-1}v)v}{\sqrt{4\sigma(x, a^{-1}v)^2 + (\sigma - (v, a^{-1}v))^2}}, \\ v' &= -\frac{-2\sigma(x, a^{-1}v)x + (\sigma - (v, a^{-1}v))v}{\sqrt{4\sigma(x, a^{-1}v)^2 + (\sigma - (v, a^{-1}v))^2}}. \end{aligned} \tag{8}$$

On the other hand, since the reflection at the point $\tilde{x}$ is elastic, we have

$$\tilde{v} - v' = \mu a^{-1}\tilde{x}, \qquad \mu = \frac{(\tilde{v}, a^{-1}\tilde{x}) - (v', a^{-1}\tilde{x})}{(\tilde{x}, a^{-2}\tilde{x})} = \frac{2(\tilde{v}, a^{-1}\tilde{x})}{(\tilde{x}, a^{-2}\tilde{x})}. \tag{9}$$

As a result, from (8) and (9) we obtain (3).

It turns out that up to the action of the group generated by the reflections $(x_i, v_i) \to (-x_i, -v_i)$, $i = 1, \dots, n$, the potential billiard map $\mathscr{B}$ is equivalent to the following 2×2 matrix equations with parameter $\lambda \in \mathbb{C}$:

$$\widetilde{L}(\lambda) = M(\lambda)L(\lambda)M^{-1}(\lambda),$$

$$L(\lambda) = \begin{pmatrix} q_\lambda(x, v) & q_\lambda(v, v) - \sigma \\ -q_\lambda(x, x) + 1 & -q_\lambda(x, v) \end{pmatrix}, \qquad q_\lambda(x, y) = \sum_{i=1}^{n} \frac{x_i y_i}{\lambda - a_i}, \tag{10}$$

$$M(\lambda) = \begin{pmatrix} [\sigma - (v, a^{-1}v)]\lambda + 2(x, a^{-1}v)\mu & 2\sigma(x, a^{-1}v)\lambda - [\sigma - (v, a^{-1}v)]\mu \\ -2(x, a^{-1}v)\lambda & [\sigma - (v, a^{-1}v)]\,\lambda \end{pmatrix},$$

where $\widetilde{L}(\lambda)$ depends on $\tilde{x}$, $\tilde{v}$ in the same manner as $L(\lambda)$ depends on x, v. Note that $\det M(\lambda) = \lambda\nu^2$, where the factor ν is defined in (3). The verification of (10) is straghtforward.

Following the conventional terminology, in the sequel we refer to (10) as a *discrete Lax pair*.

For $\sigma = 0$, the representation (10) is reduced to a 2×2 Lax pair for the classical billiard (4).

The equation $|L(\lambda) - wI| = 0$ defines a hyperelliptic spectral curve Γ of genus $n - 1$, and the coefficients of the polynomial $(\lambda - a_1)\cdots(\lambda - a_n)\det L(\lambda)$ give n independent integrals of the potential billiard map. Under the condition $(x, a^{-1}x) = 1$, the free term of this polynomial coincides with the integral (5). By consequtively setting $\lambda = a_1, \dots, \lambda = a_n$ in $\det L(\lambda)$ and by calculating the residue, we obtain the integrals

$$\sum_{j \neq i} \frac{(x_i v_j - x_j v_i)^2}{a_i - a_j} + v_i^2 + \sigma x_i^2, \qquad i = 1, \dots, n, \tag{11}$$

whose sum is equal to $(v,v)+\sigma(x,x)$, which is an analog of the energy integral for the motion with the Hooke potential.

Remark 2. The Lax matrix $L(\lambda)$ is dual to the $n\times n$ matrix

$$\mathscr{L}(s)=(s^2+\sigma)a+s(x\otimes v-v\otimes x)+v\otimes v+\sigma x\otimes x\,,\qquad s\in\mathbb{C},$$

in the sense that the spectral curves $|L(\lambda)-w\,\mathbf{I}|=0$ and $|\mathscr{L}(s)-\lambda\,\mathbf{I}|=0$ are birationally equivalent, the parameter λ being the eigenvalue parameter for $\mathscr{L}(s)$ (the Weinstein–Aronszajn formula; e.g., see [5]). For $\sigma=0$ and $s=1$, the matrix $\mathscr{L}(s)$ forms the $n\times n$ Lax representation of the geodesic flow on Q, which was first indicated by Moser [13]. Its factorization was applied in [14] to the construction of a discrete $n\times n$ Lax pair for the classical billiard inside Q. The factorization of the dual Lax matrix $\mathscr{L}(s)$ is an open problem yet.

Proposition 1. *The characteristic equation*

$$\det L(\lambda)\equiv q_\lambda(x,x)q_\lambda(v,v)-q_\lambda^2(x,v)-\sigma q_\lambda(x,x)-q_\lambda(v,v)+\sigma=0 \tag{12}$$

represents the condition that the ellipse formed by the initial vectors (x_0,v_0) *is tangent to the quadric* $Q_\lambda=\{q_\lambda(X,X)=1\}$.

Indeed, this condition is equivalent to the following pair of equations for the parameter ϕ^* at the point of tangency:

$$q_\lambda(X(\phi^*),X(\phi^*))=1,\qquad \frac{d}{d\phi}\,q_\lambda(X(\phi),X(\phi))\bigg|_{\phi=\phi^*}=0.$$

Applying the parameterization (6), one eliminates ϕ^* and, after some calculations, arrives at the equation

$$q_\lambda^2(x,v)[(q_\lambda(x,x)-q_\lambda(v,v))^2+4q_\lambda^2(x,v)]\det L(\lambda)=0 \tag{13}$$

for λ. Note that the bilinear form $q_\lambda(x_0,v_0)$ vanishes if and only if the quadric Q_λ coincides with the ellipsoid Q and the initial velocity vector v_0 is tangent to it. This case corresponds to the trivial periodic motion along some ellipse on Q, and hence we exclude it from our considerations. Next, the expression in square brackets in [13] is always nonzero. As a result, we conclude that it is the equation $\det L(\lambda)=0$ alone that gives the above tangency condition, which completes the proof.

Since the coefficients of the polynomial $(\lambda-a_1)\cdots(\lambda-a_n)\det L(\lambda)$ are integrals of the potential billiard map and the degree of $R(\lambda)$ is equal to n, from Proposition 1 we obtain the following assertion.

Corollary. *The ellipses that continue segments of trajectories of the billiard with the Hooke potential in* $Q\subset\mathbb{R}^n$ *are tangent to exactly* n *fixed quadrics confocal to* Q. *The parameters of the quadrics are the roots of* $R(\lambda)$.

This corollary generalizes the well-known property of the potential-free billiard in $Q\subset\mathbb{R}^n$, whose trajectories (or their continuations) are tangent to $n-1$ fixed confocal quadrics. The case of a planar potential billiard in $\mathbb{R}^2$ and $\sigma>0$ is illustrated in Figure 1, where the ellipses that continue segments of billiard trajectories are shown by dashed lines.

2. The Poisson Property

By the Maupertuis principle, for a given value of total energy satisfying condition (2), the motion in the potential field inside Q is reduced to *a geodesic* motion with some nondegenerate metric. Thus, it is natural to conjecture that for a given value of the integral $(v,v)+\sigma(x,x)$ of the mapping $\mathscr{B}$, the impact points x and $\tilde{x}$ already determine the entire sequence of impact points uniquely.

Theorem 2. (1) *The even-dimensional variety* $\mathscr{I}_h\subset\mathbb{R}^{2n}=(x,v)$ *that is the joint level of the integrals* $f_1=(x,a^{-1}x)=1$, $f_2=(v,v)+\sigma(x,x)=h$ *is a symplectic manifold. The restriction of* $\mathscr{B}$ *to* $\mathscr{I}_h$ *preserves the symplectic structure and is given in the canonical excessive coordinates* x, v *by the expressions*

$$v=\frac{\partial S_h(x,\tilde{x})}{\partial x},\qquad \tilde{v}=-\frac{\partial S_h(x,\tilde{x})}{\partial\tilde{x}} \tag{14}$$

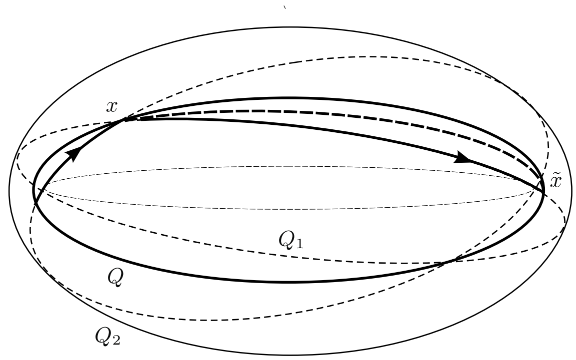

Fig. 1. The ellipses continuing a trajectory before and after the impact at the point x are tangent to the confocal quadrics Q_1 and Q_2

with the generating function (a discrete analog of Lagrangian)

$$S_h(x,\tilde{x}) = \frac{h}{2\sqrt{\sigma}}\,\hat{\phi} + \frac{h}{2\sqrt{\sigma}}\sin\hat{\phi}\cos\hat{\phi} - \sqrt{\sigma}\sin\hat{\phi}(x,\tilde{x})$$

on $Q \times Q$, where the value $\hat{\phi}$ of the parameter ϕ at the end point of the segment of the billiard trajectory is given by

$$\cos\hat{\phi} = \frac{1}{h}\,\sigma(x,\tilde{x}) + \frac{1}{h}\sqrt{\sigma^2(x,\tilde{x})^2 - \sigma h((x,x)+(\tilde{x},\tilde{x})) + h^2}\,. \tag{15}$$

(2) *The mapping* (3) *preserves the degenerate Poisson structure given by the bracket*

$$\{x_i,x_j\} = 0, \qquad \{x_i,v_j\} = \delta_{ij} - \frac{v_i a_j^{-1} x_j}{(x,a^{-1}v)}, \qquad \{v_i,v_j\} = \sigma\,\frac{(a_i^{-1}-a_j^{-1})x_i x_j}{(x,a^{-1}v)} \tag{16}$$

on the space (x,v), and the integrals (11) *commute with respect to the bracket.*

This theorem implies that the restriction of $\mathscr{B}$ to every manifold $\mathscr{I}_h$, which a symplectic leaf of the degenerate bracket (16), is integrable by virtue of the discrete analog of the Liouville theorem (see [2]). Hence the map is integrable on the entire space (x,v) and its generic invariant manifolds are $(n-1)$-dimensional tori.

Remark 3. Note that according to (15), the angle $\hat{\phi}$ tends to zero as $\sigma \to 0$. In this case, owing to the relation $\phi = \sqrt{\sigma}t$ in (6) and the natural condition $h = (v,v) = 1$, the generating function $S_h(x,\tilde{x})$ is equal to $t = |\tilde{x}-x|$, i.e., the discrete Lagrangian of the potential-free billiard.

On the other hand, to find a continuous limit of the generating function in the general case, we set $\tilde{x} = x + \dot{x}\Delta t + O((\Delta t)^2)$, where $\Delta t \ll 1$. Substituting this into (15), we obtain the expansions

$$\cos\hat{\phi} = 1 - \frac{\sigma(\dot{x},\dot{x})(\Delta t)^2}{h-\sigma(x,x)} + O((\Delta t)^3), \qquad \hat{\phi} = \frac{\sqrt{2\sigma}|\dot{x}|\Delta t}{\sqrt{h-\sigma(x,x)}} + O((\Delta t)^3),$$

which lead to the following expansion of S_h:

$$\sin\hat{\phi}\left(\frac{h}{\sqrt{\sigma}}\cos\hat{\phi} - \sqrt{\sigma}(x,x)\right) + O((\Delta t)^3) = \sqrt{h-\sigma(x,x)}\sqrt{2(\dot{x},\dot{x})}\Delta t + O((\Delta t)^3).$$

The first term on the right-hand side coincides with the differential of the Maupertuis action (or the so-called *truncated* action) for the motion of a point on the ellipsoid Q with the Hooke potential and given energy h, where $\dot{x}$ denotes the point velocity.

We point out that Theorem 2 *does not imply* that the potential billiard map $\mathscr{B}$ is a discrete Lagrangian system. It only asserts that *its restriction* to every level set of the energy type integral is described by a generating function. For the map on the whole phase space (x,v), such a function (which must depend only on the original and new coordinates) does not exist (see Remark 1).

Proof of Theorem 2. (1) By virtue of the parametrizations (6) and (7), we have

$$v = \sqrt{\sigma}\,\frac{\tilde{x} - \cos\hat{\phi}\,x}{\sin\hat{\phi}}, \qquad v' = -\sqrt{\sigma}\,\frac{x - \cos\hat{\phi}\,\tilde{x}}{\sin\hat{\phi}}, \tag{17}$$

where $\hat{\phi}$ is the value of ϕ at the point $\tilde{x}$ and $v' = V(\hat{\phi})$. To express $\hat{\phi}$ via x, $\tilde{x}$, and h, we take the square of both sides of (7) and, applying the first relation in (17) and the formula $(v,v) = h - \sigma(x,x)$, arrive at a second-order equation for $\cos\hat{\phi}$. One of its solutions is given by (15), which is chosen from the condition $\lim_{\sigma\to 0}\cos\hat{\phi} = 1$.

Note that the restrictions of $\tilde{v}$ and v' to $\mathscr{I}_h$ coincide by (9). Using this fact, as well as formulas (15) and the relation $\sin\hat{\phi} = \sqrt{1-\cos^2\hat{\phi}}$, we see that the expressions (17) become equivalent to (14).

The mapping given by (14) preserves the standard symplectic 2-form $\sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}v_i$ in $\mathbb{R}^{2n} = (x,v)$, and for the canonical Poisson bracket in this space we have $\{f_1, f_2\} = (x, a^{-1}v) \neq 0$. As a result, the submanifold $\mathscr{I}_h$ is symplectic and the restriction of the map (14) to it is also symplectic.

(2) Calculating the restriction of the standard Poisson bracket in $\mathbb{R}^{2n}$ to $\mathscr{I}_h$ by the Dirac procedure, we arrive at (16). Then the commutativity of the integrals (11) is verified by straightforward calculations.

3. Linearization on the Generalized Jacobian

Now consider the limit of the flow with the Hooke potential on the n-dimensional ellipsoid $\widetilde{Q} \subset \mathbb{R}^{n+1}$ as $a_{n+1} \to 0$ from the algebraic-geometric point of view.

The motion on $\widetilde{Q}$ is known to separate in the ellipsoidal coordinates $\lambda_1, \dots, \lambda_n$; namely, the total energy acquires a Stäckel form (e.g., see [16]). After the passage to a new parameter τ such that

$$dt = \lambda_1 \cdots \lambda_n \, d\tau, \tag{18}$$

the motion is linearized on the Jacobi variety of the hyperelliptic curve of genus n

$$\widetilde{\Gamma} = \{\mu^2 = \lambda(\lambda - a_1)\cdots(\lambda - a_{n+1})[c_0(\lambda - c_1)\cdots(\lambda - c_{n-1}) - \sigma\lambda^{n+1}]\},$$

where $c_0, \dots, c_{n-1}$ are constants of motion, which are positive in the real case.

For $\sigma = 0$ (a geodesic flow on $\widetilde{Q}$), c_0 is the value of the integral $(\dot{X}, \dot{X})$, and the constants $c_1, \dots, c_{n-1}$ admit a clear geometric interpretation: the tangent line to a geodesic is also tangent to the confocal quadrics $\widetilde{Q}(c_1), \dots, \widetilde{Q}(c_{n-1})$ (the Chasles theorem). For this case, the parameter τ was first introduced in [15].

In the limit as $a_{n+1} \to 0$, the curve $\widetilde{\Gamma}$ becomes singular (a double point appears at $\lambda = 0$). Its regularization is the following hyperelliptic curve of genus $g = n - 1$:

$$\begin{gathered} \Gamma = \{w^2 = \rho(\lambda)\}, \\ \rho(\lambda) = -(\lambda - a_1)\cdots(\lambda - a_n)[c_0(\lambda - c_1)\cdots(\lambda - c_{n-1}) - \sigma\lambda^{n+1}], \end{gathered} \tag{19}$$

which coincides with the spectral curve of the Lax pair (10). We equip this curve with the pair of distinguished points $E_\pm = (0, \pm\sqrt{\rho(0)})$, which arise from the double point on $\widetilde{\Gamma}$.

As a result of the regularization, n independent holomorphic differentials on $\widetilde{\Gamma}$ transform into linear combinations of some independent holomorphic differentials $\omega = (\omega_1, \dots, \omega_{n-1})$ on Γ and the normalized differential of the third kind $\Omega_{E_\pm}$ with a pair of simple poles at E_- and E_+. According to [4, 9, 10], in the above limit the n-dimensional Jacobian of $\widetilde{\Gamma}$ transforms into the *generalized Jacobian* $\operatorname{Jac}(\Gamma, E_\pm)$, the quotient variety of $\mathbb{C}^n$ by the lattice Λ_{2n-1} generated by the $2n-1$ independent vectors of periods of the differentials $\omega_1, \dots, \omega_{n-1}, \Omega_{E_\pm}$. The variety $\operatorname{Jac}(\Gamma, E_\pm)$ is an algebraic extension of the classical $(n-1)$-dimensional Jacobian $\operatorname{Jac}(\Gamma)$ by the group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$.

On the other hand, in the limit the coordinates $\lambda_1, \dots, \lambda_n$ on $\widetilde{Q}$ are transformed into the elliptic coordinates in $\mathbb{R}^n \ni (X_1, \dots, X_n)$ such that

$$X_i^2 = \frac{(a_i - \lambda_1)\cdots(a_i - \lambda_n)}{\prod_{j\neq i}(a_i - a_j)}, \qquad i = 1, \dots, n. \tag{20}$$

The quadratures that linearize the motion on the ellipsoid $\widetilde{Q}$ under the action of the potential $\mathscr{V}_1$ pass into the *generalized Abel–Jacobi map*

$$\int_{P_0}^{P_1} \omega + \dots + \int_{P_0}^{P_n} \omega = z, \qquad \int_{P_0}^{P_1} \Omega_{E_\pm} + \dots + \int_{P_0}^{P_n} \Omega_{E_\pm} = Z, \tag{21}$$

$$P_i = (\lambda_i, w_i), \quad z = \text{const}, \quad Z = \sqrt{\rho(0)}\,\tau + \text{const}, \tag{22}$$

where $z \in \mathbb{C}^{n-1}$ and $Z \in \mathbb{C}$ are coordinates on the universal covering of $\operatorname{Jac}(\Gamma)$ and on $\mathbb{C}^*$, respectively, such that $\{e^Z\} = \mathbb{C}^*$. Next, P_0 is an arbitrary base point on $\Gamma \setminus \{E_-, E_+\}$, and the new parameter τ is defined in (18). In this case, regardless of the sign of σ, one has $\sqrt{\rho(0)} \in \mathbb{R}$. In the sequel, we assume that P_0 is a Weierstrass point on Γ.

The quadratures (21), (22) describe the motion under the Hooke potential inside Q between subsequent impacts. In view of (22), this corresponds to a straight line flow on $\operatorname{Jac}(\Gamma, E_\pm)$ directed along the real line in $\mathbb{C}^*$; namely, Z and τ vary from $-\infty$ to ∞. (Note that in view of (18), the infinite time τ between subsequent impacts corresponds to a finite interval in the original time t).

According to (21), for $Z = \pm\infty$ one of the points P_i on Γ (without loss of generality, we assume that $i = n$) coincides with one of the poles $E_\pm$ of the differential $\Omega_{E_\pm}$. Hence, for given constants of integrals of motion, the coordinates of impact points on Q are described by the divisor of points $P_1, \dots, P_{n-1}$ on Γ, that is, by a point in the ordinary Jacobian $\operatorname{Jac}(\Gamma)$ (for a given base point P_0). These coordinates are also explicitly given by (20) with $\lambda_n = 0$. In view of (22), for the motion on $\operatorname{Jac}(\Gamma, E_\pm)$ the variables z do not change, and hence two subsequent impact points are described by divisors that differ by the vector $q = \int_{E_-}^{E_+} \omega \in \mathbb{C}^{n-1}$. By the Riemann bilinear relations, q is also the vector of b-periods of $\Omega_{E_\pm}$ on Γ. On the other hand, the end point of a segment of the billiard trajectory corresponding to $Z = \infty$ is also the starting point of the next segment with $Z = -\infty$ and a certain phase vector z. By induction, we arrive at the following result.

Theorem 3. *The restriction of the billiard map $\mathscr{B}$ to each regular Jacobian of the curve Γ is represented by the shift by the vector q.*

In the sequel, this property will be applied to construct theta function solutions for the billiard with a quadratic potential.

We note that for the motion without the potential ($\sigma = 0$), the curve Γ has an odd order and just one infinite point; however, the above considerations and Theorem 3 remain valid.

4. The Theta Function Solution

Let $x(N)$, $v(N)$ be the result of the Nth iteration of $\mathscr{B}$. Their explicit theta function expresions can be obtained in different ways, in particular, by calculating the vector Baker–Akhiezer function for the operator $L(\lambda)$ in (10). In the case of a hyperelliptic spectral curve, one can follow a simpler approach based on the result obtained in [11] (for further references, we state it as the following theorem, using the notation of the present paper).

Theorem 4. *The inversion of the Abel–Jacobi mapping* (21) *associated with the curve* (19) *gives the relations*

$$\frac{(a_i - \lambda_1)\cdots(a_i - \lambda_n)}{\prod_{j\neq i}(a_i - a_j)} = \kappa_i^2 \frac{\tilde\theta^2[\delta + \eta_{(i)}](z, Z)}{\tilde\theta[\delta](z - \xi/2, Z - S/2)\,\tilde\theta[\delta](z + \xi/2, Z + S/2)}, \qquad i = 1, \dots, n, \tag{23}$$

$$\lambda_1 \cdots \lambda_n = \kappa_0^2 \frac{\theta^2[\delta](z)}{\tilde\theta[\delta](z-\xi/2, Z-S/2)\,\tilde\theta[\delta](z+\xi/2, Z+S/2)}, \tag{24}$$
$$z \in \mathbb{C}^g, \quad Z \in \mathbb{C}, \quad \xi = \int_{\infty_-}^{\infty_+} \omega, \quad S = \int_{\infty_-}^{\infty_+} \Omega_{E_\pm},$$

where $\theta[\delta+\eta_{(i)}](z)$ and $\theta[\delta](z)$ are the ordinary theta functions with the Riemann matrix B related to this curve and half-integer theta-characteristics $\delta = (\delta'', \delta')^T$, $\eta_{(i)} = (\eta''_{(i)}, \eta'_{(i)})^T \in \mathbb{R}^{2g}/2\mathbb{R}^{2g}$ such that

$$2\pi i \eta''_{(i)} + B\eta'_{(i)} = \int_{P_0}^{(a_i,0)} \omega, \qquad 2\pi i \delta'' + B\delta' = \mathscr{K} \tag{25}$$

modulo the period lattice of Γ, where $\mathscr{K}$ is the vector of Riemann constants. Furthermore,

$$\tilde\theta[\delta+\eta_{(i)}](z, Z) = e^{-Z/2}\theta[\delta+\eta_{(i)}]\Big(z-\frac{q}{2}\Big) + e^{Z/2}\theta[\delta+\eta_{(i)}]\Big(z+\frac{q}{2}\Big), \qquad q = \int_{E_-}^{E_+} \omega \tag{26}$$

are generalized theta-functions with the same characteristics and κ_i, κ_0 are constant factors depending on the moduli of Γ alone.

Relations (23) and (24) generalize similar theta function expressions (so-called *Wurzelfunktionen*) that were found by Jacobi for the case of ordinary hyperelliptic Jacobians (e.g., see [6, 7]). The definition and a description of properties of generalized theta functions can be found in [8, 10, 11].

Now comparing the left-hand sides of (23) with the expressions for the elliptic coordinates (20) and taking account of (22), we conclude that the motion of the point between subsequent impacts is parametrized as

$$X_i(Z) = \kappa_i \frac{\tilde\theta[\delta+\eta_{(i)}](z, Z)}{\sqrt{\tilde\theta[\delta](z-\xi/2, Z-S/2)\,\tilde\theta[\delta](z+\xi/2, Z+S/2)}}, \tag{27}$$

where Z ranges over the real line and the constant phase vector z defines the position of the trajectory inside Q.

Moreover, differentiating (27) by Z and using the expressions (26), (24), we obtain a parametrization of the velocity of the point between these impacts:

$$V_i(Z) = \frac{\sqrt{\rho(0)}}{\lambda_1 \cdots \lambda_n} \frac{dX_i}{dZ} = \kappa'_i \frac{e^{Z/2}F_+(z) + e^{-Z/2}F_-(z)}{\theta^2[\delta](z)\sqrt{\tilde\theta[\delta](z-\xi/2, Z-S/2)\,\tilde\theta[\delta](z+\xi/2, Z+S/2)}}, \tag{28}$$
$$\kappa'_i = \sqrt{\rho(0)}\kappa_i/\kappa_0,$$

where

$$\begin{aligned}
F_+(z) = {} & e^{-S/2}\theta[\delta+\eta_{(i)}](z+q/2)\,\theta[\delta](z+q/2-\xi/2)\,\theta[\delta](z-q/2+\xi/2) \\
& + e^{S/2}\theta[\delta+\eta_{(i)}](z+q/2)\,\theta[\delta](z-q/2-\xi/2)\,\theta[\delta](z+q/2+\xi/2) \\
& - 2\theta[\delta+\eta_{(i)}](z-q/2)\,\theta[\delta](z+q/2-\xi/2)\,\theta[\delta](z+q/2+\xi/2), \\
F_-(z) = {} & -e^{-S/2}\theta[\delta+\eta_{(i)}](z-q/2)\,\theta[\delta](z+q/2-\xi/2)\,\theta[\delta](z-q/2+\xi/2) \\
& - e^{S/2}\theta[\delta+\eta_{(i)}](z-q/2)\,\theta[\delta](z-q/2-\xi/2)\,\theta[\delta](z+q/2+\xi/2) \\
& + 2\theta[\delta+\eta_{(i)}](z+q/2)\,\theta[\delta](z-q/2-\xi/2)\,\theta[\delta](z-q/2+\xi/2).
\end{aligned}$$

Next, setting consequently $Z = -\infty$ and $Z = \infty$ in (27), (28), we obtain the coordinates and the velocity of the mass point at the beginning and the end of the segment of the trajectory, which now depend only on the phase z and which are meromorphic functions on a certain *ramified* covering of the curve Γ.

As follows from (27), the theta function expressions for the coordinates of subsequent impact points differ only by the shift $z \to z + q$, which is consistent with Theorem 3. According to this theorem, the same holds for the components of the velocity v. By induction, we obtain the following result.

Theorem 5. *The coordinates and the outgoing velocity at impact points, as well as the velocity $v'(N)$ at the end point of the Nth segment of the billiard trajectory, have the form*

$$
\begin{aligned}
x_i(N) &= \kappa_i \frac{\theta[\delta+\eta_{(i)}](z_N)}{\sqrt{\theta[\delta](z_N-\zeta/2)\,\theta[\delta](z_N+\zeta/2)}},\\
v_i(N) &= \kappa_i' \frac{F_-(z_N+q/2)}{\theta^2[\delta](z_N+q/2)\sqrt{\theta[\delta](z_N-\zeta/2)\,\theta[\delta](z_N+\zeta/2)}},\\
v_i'(N) &= \kappa_i' \frac{F_+(z_N+q/2)}{\theta^2[\delta](z_N+q/2)\sqrt{\theta[\delta](z_N-\zeta/2)\,\theta[\delta](z_N+\zeta/2)}},\\
z_N &= z - q/2 + qN \in \mathbb{C}^{n-1}, \qquad i=1,\dots,n,
\end{aligned}
\tag{29}
$$

where z is a constant phase vector of the trajectory and the characteristics δ and $\eta_{(i)}$ are defined in (25)*.*

To determine the factors κ_i, in (21) we set

$$\{P_1,\dots,P_n\} = \{(a_1,0),\dots,(a_n,0),(0,\sqrt{\rho(0)})\} \setminus (a_i,0).$$

In this case, $Z=\infty$ and the left hand side of the ith relation in (23) equals 1. Then, using the definition of theta functions with characteristics, as well as formula (26), we find

$$\kappa_i = \varepsilon_i \frac{\theta[\delta+\hat{\eta}+\eta_{(i)}](\zeta/2)}{\theta[\delta+\hat{\eta}](0)}, \qquad \hat{\eta} = \sum_{s=1}^{n} \eta_{(s)} \bmod \mathbb{Z}^{2g},$$

where the ε_i are roots of unity.

In conclusion, we note that for the classical potential-free billiard ($\sigma=0$), the solutions obtained in [2, 11] have the form

$$x_i(N) = \varkappa_i \frac{\theta[\delta+\eta_{(i)}](z_N)}{\theta[\delta](z_N)}, \quad v_i(N) = \varkappa_i' \frac{\theta[\delta+\eta_{(i)}](z_N+q/2)}{\theta[\delta](z_N+q/2)}, \qquad N\in\mathbb{N}, \tag{30}$$

with some constants $\varkappa_i$, $\varkappa_i'$. Since in this case the trajectory consists of straight line segments, one has $v'(N)=v(N)$. As a result, in contrast with (29), for $\sigma=0$ the zeros (poles) of $x(N)$ and $v(N)$ viewed as meromorphic functions on a covering of $\mathrm{Jac}(\Gamma)$ differ only by the shift by $q/2$.

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