Graphs of Non-crossing Perfect Matchings*

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Abstract

Let P_n be a set of $n = 2m$ points that are the vertices of a convex polygon, and let $\mathcal{M}_m$ be the graph having as vertices all the perfect matchings in the point set P_n whose edges are straight line segments and do not cross, and edges joining two perfect matchings M_1 and M_2 if $M_2 = M_1 - (a,b) - (c,d) + (a,d) + (b,c)$ for some points a,b,c,d of P_n. We prove the following results about $\mathcal{M}_m$: its diameter is $m-1$; it is bipartite for every m; the connectivity is equal to $m-1$; it has no Hamilton path for m odd, $m > 3$; and finally it has a Hamilton cycle for every m even, $m \geq 4$.

Keywords. Perfect matching. Non-crossing configuration. Gray code.

Introduction

Given a graph G, one can consider an associated graph $\mathcal{M}(G)$ whose vertices are the perfect matchings of G and where two perfect matchings are adjacent if their symmetric difference is a cycle C of G. In this case one says that C is an *alternating cycle* for the two matchings. This definition is closely related to the matching polytope $M(G)$ of G, a polytope whose vertices are the incidence vectors of all the matchings in G, since two perfect matchings are adjacent in $\mathcal{M}(G)$ if, and only if, they are adjacent in the graph of $M(G)$ [8].

The graphs $\mathcal{M}(G)$ have been studied in the past and some general results are known, like the fact that they are always Hamiltonian [9]. Particular attention has been paid to the case in which G is a plane bipartite graph, a situation of particular interest in the study of chemical compounds [10, 15]. Another noteworthy instance is when $G = K_{n,n}$, and in this case $M(K_{n,n})$ is the graph of the so called assignment polytope [2, 3].

In this paper we study a geometric version of the problem. Let P_n be a set of $n = 2m$ points that are the vertices of a convex polygon, and let us consider matchings in the point set P_n whose edges are straight line segments. A perfect matching in P_n is said to be *non-crossing* if no two of its edges intersect. The points of P_n are labeled, consequently two matchings are considered equal only if they have exactly the same set of edges. We define the graph $\mathcal{M}_m$ as the graph having as vertices the non-crossing perfect matchings of P_n and edges joining M_1 and M_2 if $M_2 = M_1 - (a,b) - (c,d) + (a,d) + (b,c)$ for some points a,b,c,d of P_n (see Fig. 1). Observe that in this case the symmetric difference of M_1 and M_2 is a cycle of length four. This definition is adopted so that two adjacent matchings differ as little as possible, namely only in two edges.

The graph $\mathcal{M}_4$ is depicted in Fig. 2. Some relevant properties can be observed: the graph is bipartite (this is indicated by black and white vertices), it is Hamiltonian, and every vertex

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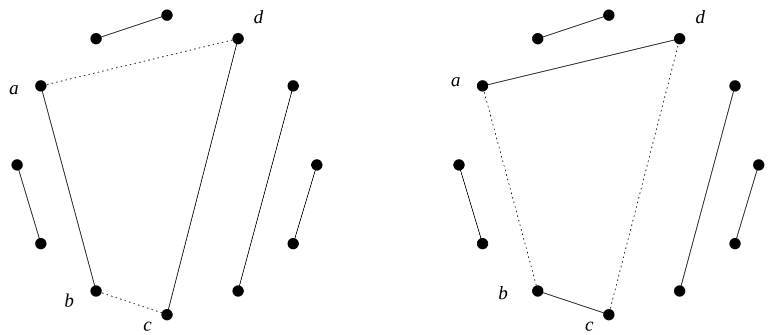

Figure 1: Adjacent matchings in $\mathcal{M}_6$.

has another vertex at maximum distance 3. Moreover, it has minimum degree 3 and it can be checked that it is 3-connected. The aim of this paper is to establish these properties in general.

In Section 1 we show how to obtain a shortest path in $\mathcal{M}_m$ between any two vertices; in fact, our results provide a linear time algorithm for finding shortest paths. As a corollary, we prove that every vertex of $\mathcal{M}_m$ has eccentricity equal to $m-1$ and, as a consequence, the diameter of $\mathcal{M}_m$ is $m-1$. In Section 2 we show that $\mathcal{M}_m$ is a bipartite graph for every m, and in Section 3 we prove that the minimum degree and the connectivity of $\mathcal{M}_m$ are equal to $m-1$. Finally, we show in Section 4 that $\mathcal{M}_m$ has no Hamilton path for m odd, $m > 3$, and, drawing on previous work by Ruskey and Proskurowski [11], we show that $\mathcal{M}_m$ is Hamiltonian for every m even, $m \geq 4$.

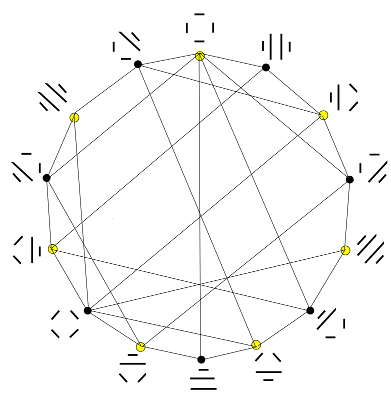

Figure 2: The graph $\mathcal{M}_4$.

A result we wish to emphasize is that shortest paths can be easily constructed, which gives as a corollary the determination of the exact value of the diameter for every m. These problems are usually difficult and have been only partly solved in related graphs like graphs of triangulations [7, 14], and graphs of non-crossing spanning trees [1, 5].

On the other hand, it is well-known that $\mathcal{M}_m$ has C_m vertices, where $C_m = \frac{1}{m+1}\binom{2m}{m}$ is a Catalan number. In the paper we make use of several bijections between matchings in $\mathcal{M}_m$

and other combinatorial objects counted by the Catalan numbers, like plane trees and balanced binary strings. In addition, in our study of $\mathcal{M}_m$ we have found a bijection between matchings in $\mathcal{M}_m$ an certain permutations of length m. This fact has an interesting combinatorial consequence: it produces a new family of permutations counted by the Catalan numbers, not defined in terms of one forbidden subsequence of length three (see [13]).

From now on a non-crossing perfect matching will be called simply a *matching*. The graph $\mathcal{M}_m$ does not depend on the exact position of the points of P_n and, to fix ideas, we take P_n as the set of vertices of a regular n-gon. Edges corresponding to the sides of the polygon are called *boundary* edges.

1 Shortest paths, eccentricities and diameter

We recall that the distance $d(u, v)$ between two vertices u and v in a graph is the length r of a shortest path $u = w_0 \sim w_1 \sim \cdots \sim w_r = v$ connecting them, where $\sim$ denotes adjacency. The eccentricity of a vertex v in a graph is the maximum of the distances between v and any other vertex of the graph, and the diameter is the maximum of the eccentricities.

In this section we show a simple method for finding shortest paths between any two vertices in $\mathcal{M}_m$ and, as a consequence, we obtain the exact value of the diameter. Before that we need to introduce some definitions and notations. If e is an edge not belonging to a matching M in $\mathcal{M}_m$, but one of its neighbors M^* contains e, we say that M^* is *obtained by insertion of* e *into* M, and we use the notation $M * e$ for the matching M^*. If e belongs to M we simply define $M * e = M$.

Observe that the insertion of an arbitrary edge e into a given matching is not always possible, but it is certainly feasible when e is a boundary edge. Also, if $M \sim M'$ and e is one of the edges exchanged, then $M * e = M' * e$. If $M \sim M'$ but e is not exchanged, then it can be checked that $M * e \sim M' * e$.

The insertion of a sequence of edges is recursively defined by

$$M * (e_1, e_2, \ldots, e_r) = (M * (e_1, e_2, \ldots, e_{r-1})) * e_r.$$

Lemma 1.1 *Let M' and M'' be in $\mathcal{M}_m$ and suppose they share the boundary edge e. Then all the matchings in any shortest path between M' and M'' contain e.*

Proof. Let $d(M', M'') = d$ and let

$$M' = M_0 \sim M_1 \sim \ldots \sim M_d = M''$$

be a shortest path between M' and M''. Suppose that e does not belong to some matching in the path and let M_i be the first one with this property. Consider now the sequence

$$M' = M_0 * e \sim M_1 * e \sim \ldots \sim M_d * e = M''.$$

By the previous remarks, consecutive matchings in this sequence are either equal or adjacent, but $M_{i-1} * e = M_i * e$, and this gives a path of length smaller than d, a contradiction. □

Let M be in $\mathcal{M}_m$. Let us consider the set E_1 of boundary edges in M, and remove the endpoints of E_1 both from P_n and from M. This gives a new point set with a new matching, and we can similarly define the current boundary edges E_2 and repeat the process. A *disassembly sequence* for the matching M is any sequence $(e_1, \ldots, e_m)$ consisting of all the edges in M, with the property that all the edges in E_1 come first, next all the edges in E_2, and so on.

Observe that a disassembly sequence $(e_1, \ldots, e_m)$ for a matching M can be inserted into any other matching M', and that $M' * (e_1, \ldots, e_m) = M$.

Theorem 1.2 *Let* $(e_1, \ldots, e_m)$ *be a disassembly sequence for a matching* M*, and let* M' *be any other matching. Then*

$$M' \sim M' * e_1 \sim M' * (e_1, e_2) \sim \ldots \sim M' * (e_1, \ldots, e_m) = M$$

is a shortest path between M' *and* M*, where it is understood that some of the adjacencies in the above expression can be equalities.*

Proof. Let

$$M' = M_0 \sim M_1 \sim \ldots \sim M_d = M$$

be a shortest path P' between M' and M.

Let us assume first that e_1 does not belong to M', and prove that there is a shortest path from M' to M in which the first step is the insertion of e_1. As e_1 belongs to M but not to M', there is an $i > 0$ such that M_i is the first matching in the sequence containing e_1. In the sequence

$$M' \sim M_0 * e_1 \sim M_1 * e_1 \sim \ldots \sim M_d * e_1 = M,$$

the equality $M' = M_0$ from the original path has been replaced by the proper adjacency $M' \sim M_0 * e_1$, but the adjacency $M_{i-1} \sim M_i$ has been replaced by the equality $M_{i-1} * e_1 = M_i * e_1$, therefore the length remains unchanged and we still have a shortest path.

If e_1 belongs to M', the first step in the path P' is simply an equality. Henceforth we see that in any case there is a path from M' to M starting by the insertion of e_1. By Lemma 1.1 we know that all matchings in shortest paths from $M' * e_1$ to M contain e_1. The same argument shows that there is a path from $M' * e_1$ to M starting by the insertion of e_2, and the repetition of the process proves the claim. □

Remark that the above result provides in fact a linear time algorithm for finding shortest paths. Also, since the shortest path involves no exchange when an edge already in M is to be inserted, we get the following result.

Corollary 1.3 *If two matchings* M *and* M' *in* $\mathcal{M}_m$ *have exactly* k *edges in common, then*

$$d(M, M') \leq m - k - 1.$$

This implies that the diameter is at most $m - 1$. We show now that this is in fact the exact value of the diameter.

The union $M_1 \cup M_2$ of two matchings in $\mathcal{M}_m$ is the union $C_1 \oplus \ldots \oplus C_k$ of edge disjoint cycles, in which the edges from M_1 and the edges from M_2 alternate in each cycle (an edge e in both M_1 and M_2 gives a "double" edge in $M_1 \cup M_2$, which we consider as a cycle of length 2). This decomposition is closely related to the distance between the two matchings:

Theorem 1.4 *If* M_1 *and* M_2 *are such that* $M_1 \cup M_2 = C_1 \oplus \ldots \oplus C_k$ *then*

$$d(M_1, M_2) = \frac{1}{2} \sum_{i=1}^{k} (length(C_i) - 2).$$

Proof. The result is obvious for $m = 2$, for $m > 2$ we distinguish two cases.

If M_1 and M_2 share a boundary edge e, we remove the endpoints of e from the point set P_n, and from both M_1 and M_2, then apply induction to the resulting point set and matchings. Otherwise, let e be a boundary edge in M_2 but not in M_1. We know from Theorem 1.2 that there is a shortest path between M_1 and M_2 in which we first move from M_1 to $M_1 * e$. The cycle in $M_1 \cup M_2$ containing e becomes in $(M_1 * e) \cup M_2$ the double edge e plus a cycle with length equal to $length(C) - 2$; we now apply the previous case to $M_1 * e$ and M_2. □

The preceding result is now applied to a special situation, which gives as a consequence the lower bound for the diameter. Define the *rotation* of an edge (i, j) as the edge $(i + 1, j + 1)$, addition being modulo $2m$. Given a matching $M \in \mathcal{M}_m$, its rotation M^* is the matching consisting of all the edges of M rotated.

Theorem 1.5 *Let $M \in \mathcal{M}_m$ and let M^* be its rotation. Then $d(M, M^*) = m - 1$.*

Proof. Let r be an oriented line, non crossed by any edge in M, and let $i, i + 1, \ldots, i + p$ be the points in P_n to the left of r. We claim that the union of the edges in M to the left of r, together with its rotations, is an alternating path starting at i, going through all the points to the left of r, and ending in $i + p + 1$ (Fig. 3a). Observe that applying the claim to the edges to the right of r we get a similar path from $i + p + 1$ to i. The concatenation of the two paths shows that $M \cup M^*$ is a single cycle of length $n = 2m$, and Theorem 1.4 implies the result.

The claim is obvious when $p = 1$, and we proceed by induction on p. If $(i, i + p)$ is an edge in M, we can take a line parallel to r leaving exactly the points $i + 1, \ldots, i + p - 1$ to its left; by induction this gives by rotation a path from $i + 1$ to $i + p$, which can be concatenated with the edge $(i, i + p)$ and its rotation $(i + 1, i + p + 1)$. If $(i, i + p)$ is not an edge in M we can find a number q and two oriented lines r' and r'', non crossed by any edge of M, such that r' has exactly the points $i, \ldots, i + q$ to its left and r'' the points $i + q + 1, \ldots, i + p$ (Fig. 3b). Now we apply induction to these two portions and concatenate the two resulting paths. □

Corollary 1.6 *The eccentricity of every vertex in $\mathcal{M}_m$ is equal to $m - 1$, and the diameter of $\mathcal{M}_m$ is equal to $m - 1$.*

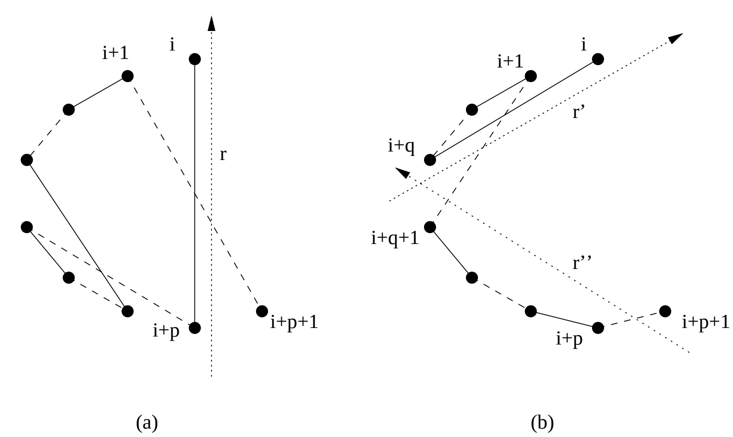

Figure 3: Edges in M are continuous; edges in the rotate matching M^* are dashed.

2 $\mathcal{M}_m$ is a bipartite graph

In this section we show that $\mathcal{M}_m$ is a bipartite graph for all m. To this end we use a classical bijection between non-crossing perfect matchings and plane trees. Given a matching M in $\mathcal{M}_m$ define a plane tree t_M with $m+1$ nodes as follows: there is a node of t_M for every edge of M, plus a root ρ that is placed outside the edge $(1, 2m)$. Join ρ to all the nodes corresponding to edges visible from ρ and proceed recursively (see Fig. 4, where the edges of t_M are dashed and the root is in grey).

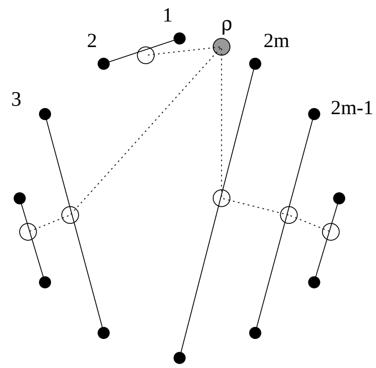

Figure 4: Plane tree associated to a matching.

Recall that the path length of a tree is the sum of the heights of all its nodes. Let us denote the path length of a tree t by $\pi(t)$. For instance, $\pi(t_M) = 10$ in the matching of Fig. 4.

Lemma 2.1 *If M and M' are adjacent in $\mathcal{M}_m$, then the path lengths of t_M and $t_{M'}$ have different parity.*

Proof. Suppose $M' = M - ab - cd + ad + bc$, and suppose the node in t_M corresponding to (c, d) is a descendant of the node corresponding to (a, b). If we let C be the set of vertices of P_n to the right of the edge (c, d), then it is straightforward to check that

$$\pi(t_M) - \pi(t_{M'}) = 1 + 2|C|.$$

The other cases are treated similarly. □

As a corollary we have:

Corollary 2.2 *The graph $\mathcal{M}_m$ is bipartite for every m.*

Let us mention a different proof of this result. Label the points of P_n as $\{1, 1', 2, 2', \ldots, m, m'\}$ in circular order, and let M be any (non-crossing perfect) matching. Then, because of parity considerations, any point i in $\{1, \ldots, m\}$ has to be matched in M with a point j' in $\{1,', \ldots, m'\}$. If we set $\sigma i = j$, this defines a permutation $\sigma = \sigma(M)$ of n letters. It is easy to check that if M and M' are adjacent in $\mathcal{M}_m$ then $\sigma(M)$ and $\sigma(M')$ differ in a single trasposition (but not conversely), hence they have different parity. This establishes the bipartition.

This proof has an interesting consequence. It is well known that the number of non-crossing perfect matchings of P_n is the Catalan number $\frac{1}{m+1}\binom{2m}{m}$. The above correspondence then gives a family of permutations that is counted by the Catalan numbers. All the families of permutations we know with this property are defined in terms of one forbidden subsequence of length three [13]. Our family is not of this kind since the only forbidden subsequence for $m = 3$ is 231 but it appears in the permutation 4231 that does come from a matching. Hence we have found what appears to be a new family of permutations counted by the Catalan numbers.

3 Connectivity

We begin this section with a simple lemma.

Lemma 3.1 *The minimum degree of* $\mathcal{M}_m$ *is equal to* $m-1$.

Proof. Let M be in $\mathcal{M}_m$, and let G be the graph having a vertex for every edge in M, and where two edges uv and $u'v'$ are adjacent if they *see* each other, i.e., if there is a straight-line segment connecting them and touching no other segment. Then the degree of M in $\mathcal{M}_m$ is equal to the number of edges in G. But G has m vertices and is clearly connected, so that it has at least $m-1$ edges. □

We are now ready for the main result.

Theorem 3.2 *The connectivity of* $\mathcal{M}_m$ *is equal to* $m-1$.

Proof. By the above lemma we only need to show that $\mathcal{M}_m$ is $(m-1)$-connected. The proof is by induction on m, starting with the case $m = 3$, which is clear since $\mathcal{M}_3$ is the graph $K_{2,3}$. Suppose then $m \geq 4$.

By Menger's theorem, given M' and M'' in $\mathcal{M}_m$ it is enough to prove that there are $m-1$ paths from M' to M'' internally disjoint. We consider three cases.

1) M' and M'' have a common boundary edge $e = (i, i+1)$. Removing the endpoints of e, by induction there are $m-2$ internally disjoint paths from M' to M'' all of them containing e. If $e' = (i+1, i+2)$, we know there exists a path $\mathcal{P}$ from $M' * e'$ to $M'' * e'$ such that all its vertices contain e'. Since e and e' cannot both be contained in a matching, concatenating $\mathcal{P}$ with the adjacencies $M' \sim M' * e'$ and $M'' * e' \sim M''$ we obtain a path from M' to M'' internally disjoint with the previous ones.

2) M' and M'' have no common boundary edge and none of them has all its edges on the boundary. Then, without of loss of generality, we can assume that there exist boundary edges $e = (i, i+1)$ and $e' = (i+1, i+2)$ such that

$$e \in M'', e \notin M', \qquad e' \notin M', e' \notin M''.$$

By Lemma 3.1, M' has at least $m-1$ different neighbors $M_1, \ldots, M_{m-1}$. Only one of them contains e, assume it is M_1, and only one of them contains e', assume it is M_2. A simple check shows that the matchings $M_2 * e, \ldots, M_{m-1} * e$ are all different. Observe also that

$$M_1 * e' \neq M_i, \qquad i = 2, \ldots, m-1,$$

since $\mathcal{M}_m$ does not have triangles (Corollary 2.2), and that

$$M_1 * e' \neq M_i * e, \qquad i = 2, \ldots, m-1,$$

since e and e' cannot both be in a matching.

By induction, and by a consequence of Menger's theorem (see [4, p.82]), there are $m-2$ internally disjoint paths between M'' and $M_2 * e, \ldots, M_{m-1} * e$, all of them containing e. By induction again, there are $m-2$ internally disjoint paths between $M_1 * e'$ and $M'' * e'$, all of them containing e'. Take one of them not containing M_2 (by assumption $m-2>1$) and concatenate it with $M' \sim M_1 \sim M_1 * e'$ and $M'' * e' \sim M''$; in this way we have an additional path from M' and M'' internally disjoint with the former ones.

3) The only case left is $M' = \{e_1, e_3, \ldots, e_{2m-1}\}$ and $M'' = \{e_2, e_4, \ldots, e_{2m}\}$, where $e_i = (i, i+1)$, $i = 1, \ldots, 2m$. Let M_i be the matching obtained from M' by exchanging e_1 and e_{2i-1}, that is,

$$M_i = M' - (1,2) - (2i-1, 2i) + (2, 2i-1) + (2i, 1), \qquad i = 2, \ldots, m-1.$$

These are $m-2$ matchings adjacents to M', none of them containing e_1 nor e_{2m}. Now $M_2 * e_{2m}, \ldots, M_{m-1} * e_{2m}$ are all different and none of them contains e_1; as in the former case, there are $m-2$ internally disjoint paths connecting them to M''. Now take a path from M' to $M'' * e_1$, all of whose vertices contain e_1, and concatenate it with $M'' * e_1 \sim M''$, obtaining $m-1$ disjoint paths form M' to M''. □

4 Hamiltonian properties

Let E_m be the number of matchings M in $\mathcal{M}_m$ such that the path length of t_M is even, and let O_m be the number of matchings in $\mathcal{M}_m$ such that the path length of t_M is odd. We next show that $|E_m - O_m| > 1$ for m odd and, as a consequence of Lemma 2.1, that $\mathcal{M}_m$ has no Hamilton path. We make use of the fact that the generating function $C(z) = \sum_{m \geq 0} C_m z^m$ of the Catalan numbers satisfies the quadratic equation

$$zC(z)^2 - C(z) + 1 = 0.$$

Lemma 4.1 *$|E_m - O_m| > 1$ for m odd, $m > 3$.*

Proof. Let $t_{m,k}$ be the number of plane trees with m nodes and path length k, and let $Q(u,z) = \sum t_{m,k} u^k z^m$ be the corresponding bivariate generating function. It is a standard fact [12] that Q satisfies the functional equation

$$Q(u,z) = \frac{1}{1 - Q(u, zu)}.$$

We are interested in $D(z) = \sum d_m z^m = Q(-1, z)$, since the coefficient of z^m in this series is the difference between the number of plane trees with even path length and odd path length. Taken into account that $E_m - O_m = d_{m+1}$ we only need to consider the even part of $D(z)$

$$P(z) = \frac{D(z) + D(-z)}{2}.$$

Straightforward manipulation of the equations $Q(1,z) = 1/(1 - Q(1,z)$ and $Q(-1,z) = 1/(1 - Q(-1,-z))$ gives

$$P(z)^2 - P(z) - z^2 = 0.$$

Comparing this with the equation for $C(z)$ we see that $P(z) = z^2 C(z)^2$. This implies that

$$d_m = (-1)^{m/2} C_{m/2-1}$$

for m even, and the result follows. □

As a corollary we have:

Theorem 4.2 *The graph $\mathcal{M}_m$ has no Hamilton path for m odd, $m > 3$.*

For m even, the same proof as before gives $E_m - O_m = 0$, thus the necessary condition for the existence of a Hamilton cycle is fulfilled. Our last result is that Hamilton cycles do exist in this case.

To this end we use yet another bijection, namely between matchings of $\mathcal{M}_m$ and the set B_m of binary strings consisting of m zeros and m ones, and having the prefix property, i.e., in every prefix the number of ones it at least the number of zeros. The bijection is illustrated in Fig. 5, where a matching is represented linearly with curved arcs, and an arc going up corresponds to a 1 and an arc going down to a 0. It is immediate that the resulting string has the prefix property and that this defines indeed a bijection between $V(\mathcal{M}_m)$ and B_m.

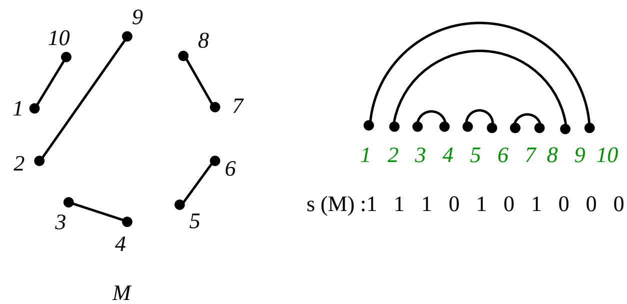

Figure 5: Binary string associated to a matching.

Let $b(M)$ be the binary string associated with the matching M. Then it is easy to check that two matchings M and M' are adjacent in $\mathcal{M}_m$ if, and only if,

$$b(M) = x1y0z \quad \text{and} \quad b(M') = x0y1z$$

for some binary word y having the prefix property. If we impose that y is empty in the above adjacency rule, that is, if we only allow the interchange of a zero with an adjacent one, we obtain a spanning subgraph S_m of $\mathcal{M}_m$. This graph was studied in [11], where the authors showed how to construct a Hamilton path when m is even. This path cannot be extended to a Hamilton cycle, since the string 1^m0^m has degree one in S_m. Our goal is to modify this construction conveniently in order to obtain a Hamilton cycle in the larger graph $\mathcal{M}_m$.

We sketch the construction in [11] and direct the reader to this reference for full details. Given $s = x10^k \in B_m$, let $\overline{s} = x1\underline{110^{k+2}}$ be the *set* of all strings in B_{m+2} beginning with $x1$. Let us call $\overline{s}$ the *generalized vertex* of s. The basic idea is to take two adjacent vertices s and t in $\mathcal{M}_m$ and recursively construct a Hamilton path spanning all the vertices in $\overline{s} \cup \overline{t} \subset \mathcal{M}_{m+2}$. For example, Fig. 6 shows the generalized vertices $11101\underline{110^5}$ and $1^4\underline{110^6}$, and a path connecting all the vertices.

Generalized vertices always have this grid structure in two dimensions. The two degrees of freedom correspond to the two "free" ones that can be moved. The paths connecting pairs of generalized vertices can be seen as a three dimensional object. We use six kinds of paths, described in Table 1. The directions in the last column correspond to the three possible directions in 3-space.

The construction of the Hamilton cycle for m even is by induction on m, starting with the case $m = 4$ (see Fig. 2). We start inductively with a Hamilton *path* $\mathcal{P}_{m-2}$ in $\mathcal{M}_{m-2}$, then we substitute each vertex $s = x10^t$ by its generalized vertex $\overline{s}x1\underline{110^{t+2}}$ and we build Hamilton paths between consecutive generalized vertices according to Table 1. In this way we construct a Hamilton path $\mathcal{P}_m$ in $\mathcal{M}_m$. The only caveat is that in order to get a Hamilton cycle, the last path of type M has to be replaced by a path of type N. This illustrated in Fig. 7 for $m = 6$, where the cycle has to be read by columns.

Hence we conclude:

Theorem 4.3 *The graph $\mathcal{M}_m$ is Hamiltonian for m even, $m \geq 4$.*

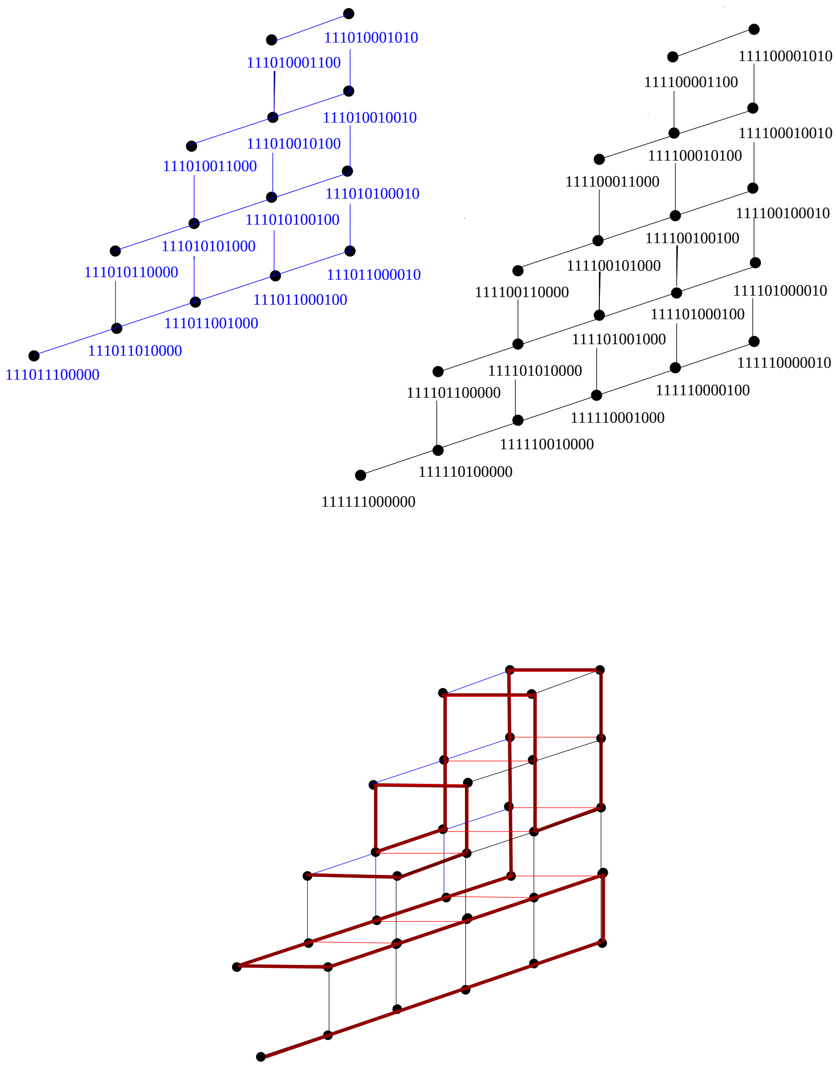

Figure 6: Top: generalized vertices. Bottom: Hamilton path connecting them.

Path Types					
Path Type	Pair of g-vertices	Initial	Final	Conditions	Direction
$A(t)$	$x1\underline{110^t} - x01\underline{110^{t-1}}$	$x1110^t$	$x010110^{t-2}$	t even	$\nearrow$
$X(t)$	$x1\underline{110^t} - y1\underline{110^t}$	$x10^{t-2}1100$	$y10^{t-2}1100$		$\leftrightarrow, \updownarrow$
$Y(t)$	$x1\underline{110^t} - y1\underline{110^t}$	$x10110^{t-1}$	$y10110^{t-1}$		$\leftrightarrow$
$Z(t)$	$x1\underline{110^t} - y1\underline{110^t}$	$x10^{t-2}1100$	$y1110^t$		$\leftrightarrow, \updownarrow$
$M(t)$	$x1\underline{110^t} - y1\underline{110^t}$	$x10110^{t-1}$	$y1110^t$		$\leftrightarrow$
$N(t)$	$x1\underline{110^t} - y1\underline{110^t}$	$x10110^{t-1}$	$y10^{t-2}1010$	t odd	$\leftrightarrow, \updownarrow$

Table 1: Types of paths connecting generalized vertices.

A	A	X	X	X	Z	N
111111000000	111001110000	110100101100	101010101100	101010101100	101101001100	110110110000
111110100000	111001101000	110100101010	101010101010	101010101010	110101001100	101110110000
111110010000	111001100100	110100110010	101010110010	101010110010	110101001010	101111010000
111110001000	111001100010	110100110100	101010110100	101010110100	101101001010	101111100000
111110000100	111001010010	110100111000	101010111000	101010111000	101101010010	110111100000
111110000010	111001010100	110010111000	101100111000	101100111000	101101010100	110111010000
111101000010	111001011000	110010110100	101100110100	101100110100	101101011000	110111001000
111101000010	111000111000	110010110010	101100110010	101100110010	110101011000	110110101000
111101000100	111000110100	110010101010	101100101010	101100101010	110101010100	110110011000
111101001000	111000110100	110010101100	101100101100	101100101100	110101010010	101110011000
111101010000	111000101010				110101100010	101110101000
111101100000	111001001010				101101100010	101110101000
111011100000	111001001100				101101100100	101111000100
111011010000	111000101100				110101100100	101110100100
111011001000					110101101000	101110010100
111011000100					101101101000	101110001100
111011000010					101101110000	110110001100
111010100010					110101110000	110110010100
111010010010						110110100100
111010001010						110111000100
111100001010						110111000010
111100010010						110110100010
111100100010						110110010010
111100100100						101110001010
111100010100						101110010010
111100001100						101110100010
111010001100						101111000010
111010010100						
111010100100						
111010101000						
111010011000						
111100011000						
111100101000						
111100110000						
111010110000						

Figure 7: A Hamilton cycle in $\mathcal{M}_6$.

5 Concluding remarks

We have established a number of significant properties of the graphs $\mathcal{M}_m$. An open problem is to compute the group of automorphisms. Without loss of generality we can assume that P_{2m} is the set of vertices of a regular $2m$-gon; then every symmetry of the polygon acts on the set of matchings and induces an automorphism of $\mathcal{M}_m$. We conjecture that these are the only automorphisms of the graph.

Finally, a natural line of research is to extend the study of graphs of non-crossing matchings to the case of sets of points in the plane that are not necessarily vertices of a convex polygon. Preliminary results have been obtained recently in this direction [6].

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