ON THE HAMILTONIAN ANDRONOV-HOPF BIFURCATION

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In this contribution, we consider an specific type of Hamiltonian H_0 and two of its basic facts are analyzed: the existence of a one-parameter family of periodic orbits with a transition from stability to complex instability and the appearance of a two-parameter family of bifurcating tori.

The importance of this specific Hamiltonian is that it constitutes a versal deformation of such quasi-periodic unfolding: under some quite generic conditions, any Hamiltonian with three degrees of freedom and having a family of periodic orbits with a transition stable-complex unstable; can be reduced –through a canonical change–, to the sum of H_0 plus some remainder H_1, containing higher order terms.

At the end, we give some ideas about the proof of the persistence of the invariant tori when H_1 is added and the whole transformed Hamiltonian is taken into account.

Complex instability

Let H_0 be a real three degree of freedom Hamiltonian given by,

$$H_0(\theta_1, I_1, x, y) = \omega_1 I_1 + \omega_2 y \times x + \frac{1}{2}|y|_2^2 + \Lambda(|x|_2^2/2, I_1, y \times x), \tag{1}$$

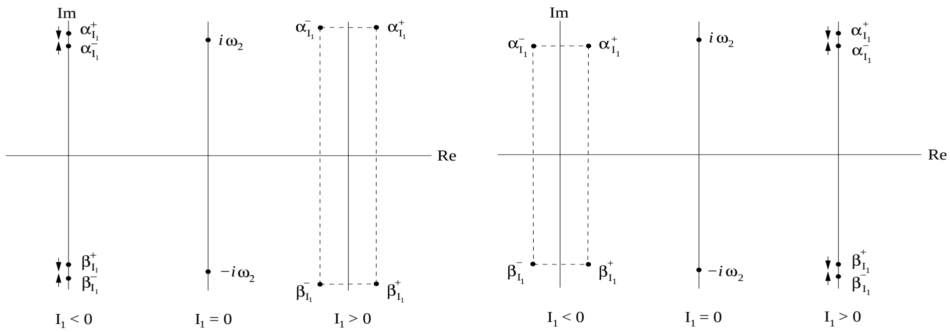
where $x^T = (x_1, x_2)$, $y^T = (y_1, y_2)$,

$$|x|_2^2 = x_1^2 + x_2^2, \quad |y|_2^2 = y_1^2 + y_2^2, \quad y \times x = y_1 x_2 - y_2 x_1,$$

and $\Lambda(u, v, w)$ is a polynomial of degree r beginning with terms of order two, so we can write,

$$\Lambda(u,v,w) = \frac{1}{2}(au^2 + bv^2 + cw^2) + \\ + duv + euw + fvw + \sum_{3 \le l+m+n \le r} \Lambda_{l,m,n} u^l v^m w^n. \tag{2}$$

Note that H_0 depends on an *action*, I_1, and on the *normal* positions (x_1, x_2) and momenta (y_1, y_2), but it does not depend on the angle conjugate to the action, θ_1, so we shall denote $H_0(I_1, x, y)$.



(a) $d < 0$ (b) $d > 0$

Figure 1: We note that when $I_1 = 0$, then $\alpha_0^- = \alpha_0^+ = i\omega_2$ and $\beta_0^- = \beta_0^+ = -i\omega_2$ (collision of characteristics exponents). Therefore, the family changes its linear character from stable to complex-unstable (when $d < 0$, fig. 1(a)), or vice-versa (when $d > 0$, fig. 1(b)).

It is straightforward to check that the Hamiltonian (1) has a one parameter family of periodic orbits which can be expressed as,

$$\mathcal{M}_{I_1} : \begin{cases} \theta_1 = (\omega_1 + \partial_2 \Lambda(0, I_1, 0))t + \theta_1^0, \\ I_1 = \text{const.}, \\ x_1 = x_2 = y_1 = y_2 = 0, \end{cases}$$

being the action I_1 the parameter of the family. We can compute the characteristic exponents in the normal directions to the periodic orbit, which turn out to be

$$\alpha_{I_1}^{\pm} = i(\omega_2 + fI_1) \pm \sqrt{-dI_1 + \mathcal{O}_2} + \mathcal{O}_2, \tag{3}$$

$$\beta_{I_1}^{\pm} = -i(\omega_2 + fI_1) \pm \sqrt{-dI_1 + \mathcal{O}_2} + \mathcal{O}_2, \tag{4}$$

where $\mathcal{O}_2 = \mathcal{O}_2(I_1)$, and the coefficients d and f which appear in the formulas are those of the development of Λ, (2). Thus, if I_1 is taken to be small enough, the character of the exponents depend mainly on the sign of the product $-dI_1$ inside the square roots, so the evolution of the stability of the family is as shown in figure 1.

The interest of Hamiltonian (1) relies on the following result.

Theorem 1 Let $\mathcal{H}$, be a 3-degree of freedom Hamiltonian and $\{\mathcal{M}_\sigma\}_{\sigma\in\mathbb{R}}$ a non-degenerate one-parameter family of periodic orbits of the corresponding Hamiltonian system. Suppose, in addition, that this family undergoes a transition stable-complex unstable as the one described in figure 1, for some value of the parameter (say $\sigma = 0$). Let ω_1 and ω_2 be the angular frequency and the absolute value of the characteristic exponent for $\mathcal{M}_0$, the *critical* (*degenerate, resonant*) periodic orbit. Then, if $\omega_1/\omega_2 \notin \mathbb{Q}$, it is always possible, to reduce *formally* the initial Hamiltonian $\mathcal{H}$ to a normal form given by (1), with Λ as in (2), but a power series instead of a polynomial.

For a proof, see [3]. Nevertheless, though it is expressed formally (as an infinite process), in the practice, the nonlinear reduction of the initial Hamiltonian is carried on only up to some suitable degree r, so the transformed Hamiltonian, H, can be casted into,

$$H(I_1, \theta_1, x, y) = H_0(I_1, x, y) + H_1(\theta_1, I_1, x, y), \qquad (5)$$

where H_1 is analytic, 2π-periodic on the angle θ_1, and holds the terms of degree higher than r, so it can be thought of as a perturbation of H_0.

The unfolding of 2D invariant tori

We are interested in the quasi-periodic bifurcation phenomena linked with the transition stable-complex unstable. We will describe such phenomena using the normal form H_0 (and we shall skip the remainder H_1 off).

The existence of the quasiperiodic solutions we are looking for is more easily shown if we first change to (canonical) polar coordinates,

$$x_1 = \sqrt{2r}\cos\theta_2, \quad y_1 = -\frac{I_2}{\sqrt{2r}}\sin\theta_2 + p_r\sqrt{2r}\cos\theta_2,$$
$$x_2 = -\sqrt{2r}\sin\theta_2, \quad y_2 = -\frac{I_2}{\sqrt{2r}}\cos\theta_2 - p_r\sqrt{2r}\sin\theta_2.$$

This introduces a second action I_2, together with its conjugate angle, θ_2; while r and p_r are the new normal coordinates and momenta. With these polar coordinates, the Hamiltonian H_0 takes the form,

$$H_0(r, I_1, I_2, p_r) = \omega_1 I_1 + \omega_2 I_2 + r p_r^2 + \frac{I_2^2}{4r} + \Lambda(r, I_1, I_2), \qquad (6)$$

(we use the same notation for the transformed Hamiltonian). From its corresponding Hamiltonian equations, it is seen that this system generically presents a bifurcating two parametric family of quasi-periodic solutions. We specify this result in the following.

Proposition 2 If the coefficient a in the expansion of Λ is different from zero, there exists a real analytic function, $\psi(J_1, J_2)$, defined in a neighborhood of $(0,0)$, such that the parameterization

$$\mathcal{T}_{J_1,J_2}: \begin{cases} \theta_i = \Omega_i(J_1,J_2)t + \theta_i^0, \quad i=1,2 \\ r = \psi(J_1,J_2), \\ I_1 = J_1, \\ I_2 = 2J_2\psi(J_1,J_2), \\ p_r = 0, \end{cases} \tag{7}$$

with

$$\Omega_1(J_1,J_2) = \omega_1 + \partial_2\Lambda|_{\mathcal{T}_{J_1,J_2}} \tag{8}$$

$$\Omega_2(J_1,J_2) = \omega_2 + J_2 + \partial_3\Lambda|_{\mathcal{T}_{J_1,J_2}} \tag{9}$$

defines a (two-parameter) family of quasi-periodic solutions of the Hamiltonian (6). Moreover, if

$$-\frac{d}{a}J_1 > 0 \quad \text{and} \quad |J_2| \leq |J_1|^{2/3}, \tag{10}$$

for J_1 small enough, then the solutions (7) correspond to a 2-parameter family of real 2D-invariant tori which are non-degenerated, provided

$$b - \frac{d^2}{a} \neq 0 \tag{11}$$

Assuming the reality conditions (10), we can investigate the stability of the real invariant tori through their normal eigenvalues (characteristic exponents). They result to be

$$\gamma^{\pm}_{J_1,J_2} = \pm\sqrt{2dJ_1 + \mathcal{O}_2(J_1,J_2)}. \tag{12}$$

The formulas (12), (3) and (4), yield to the next proposition.

Proposition 3 Under the conditions of proposition 2, the type of the bifurcation is determined by the sign of the coefficient a of Λ in (2). More precisely,

Case 1. If $a < 0$: inverse bifurcation (hyperbolic invariant tori unfold around stable orbits).

Case 2. If $a > 0$: direct bifurcation (elliptic invariant tori unfold around complex unstable orbits).

The proofs of propositions 2 and 3 can be found at [4]. This bifurcation resembles the classical Andronov-Hopf's one in the sense that stable (elliptic) objects unfold around lower dimensional unstable ones, and conversely, unstable (hyperbolic) higher dimensional manifolds may appear around stable objects.

Preservation of the invariant tori: main ideas

The unfolding shown by proposition 2, and also the normal character of the bifurcating tori have been obtained and analyzed from the normal form H_0 only. The next step is to investigate the persistence of these invariant tori when the remainder H_1 (supposed to be an small perturbation), is added and the full Hamiltonian is considered. To do so, we apply basically the Kolmogorov method adapted to lower dimensional tori (see [2]).

Here, we shall briefly outline the method followed. First, we select a "good" (from the point of view of Diophantine conditions on the frequencies Ω_1, Ω_2 and on the normal eigenvalues $\gamma^{\pm}$) invariant torus of the family, say $\mathcal{T}_{J_1,J_2}$ with J_1, J_2 fixed, and expand the initial Hamiltonian, $H^{(0)} = H$ around it; so we introduce new actions $I'^T = (I'_1, I'_2)$, and a new normal coordinates $z^T = (x, y)$ through,

$$r = \psi + x - \frac{\psi}{\gamma^+} y, \quad p_r = \frac{\gamma^+}{2\psi} x + \frac{1}{2} y, \quad I_1 = J_1 + I'_1, \quad I_2 = 2J_2\psi + I'_2,$$

where, $\psi = \psi(J_1, J_2)$ and γ^+ is the positive determination of the characteristic exponents (12). Note that the selected initial torus corresponds then to $I'_1 = I'_2 = x = y = 0$). From now on, we drop the primes from the new actions and, with this coordinates, the above mentioned expansion can be expressed as,

$$\begin{aligned} H^{(0)}(\theta, x, I, y) = \bar{a} + \Omega^T \cdot I + \frac{1}{2} I^T \mathcal{C}^{(0)} I + \frac{1}{2} z^T \mathcal{B}^{(0)*} z + I^T \mathcal{E}^* z + H_*^{(0)} + \\ + \tilde{a}(\theta) + b^T(\theta) \cdot z + \tilde{c}^T(\theta) \cdot I + \\ + \frac{1}{2} I^T \widetilde{C}(\theta) I + \frac{1}{2} z^T \widetilde{B}(\theta) z + \frac{1}{2} I^T \widetilde{E}(\theta) z, \end{aligned} \quad (13)$$

with $\theta^T = (\theta_1, \theta_2)$ and $H_*^{(0)}$ holding the terms of degree greater than 2 in (x, I_1, I_2, y). If we define,

$$\mathcal{H}_0(x, I_1, I_2, y) = H_0 \left(\psi + x - \frac{\psi}{\gamma^+} y, J_1 + I_1, 2J_2\psi + I_2, \frac{\gamma^+}{2\psi} x + \frac{1}{2} y \right)$$

and in the same way $\mathcal{H}_1(\theta_1, \theta_2, x, I_1, I_2, y)$, from the perturbative term H_1 in (5), then the different coefficients which appear in the expansion (13) can be identified as $\bar{a} = \mathcal{H}_0(0,0,0,0)$, $\tilde{a}(\theta) = \mathcal{H}_1(\theta, 0, 0, 0, 0)$, and so on.

Actually, these coefficients depend also on the parameters (J_1, J_2), but as they are held fixed, this dependence is not explicitly written.

Now, we include the term $\frac{1}{2}I^T\tilde{C}(\theta)I$ in H_0^* and define,

$$\hat{H}^{(0)} = \tilde{a}(\theta) + b^T(\theta)z + \tilde{c}^T(\theta)\cdot I + \frac{1}{2}z^T\tilde{B}(\theta)z + \frac{1}{2}I^T\tilde{E}(\theta)z,$$

and therefore, $H^{(0)}$ may be written shortly as,

$$H^{(0)} = \bar{a} + \Omega^T\cdot I + \frac{1}{2}I^T\mathcal{C}^{(0)}I + \frac{1}{2}z^T\mathcal{B}^{(0)*}z + I^T\mathcal{E}^*z + H_*^{(0)} + \hat{H}^{(0)} \qquad (14)$$

Remark 4 We note that, in (14), the term $\hat{H}^{(0)}$ gathers all the terms which avoid the existence of the initial invariant torus in the following sense: if it were not present, $x = I_1 = I_2 = y = 0$ would still be a solution of the complete Hamiltonian system, winding a two dimensional –and reducible in the normal directions–, invariant torus. In other words, the selected torus of the unperturbed system would then be preserved.

Thus, the idea is to eliminate, by a sequence of canonical changes, the successive terms $\hat{H}^{(i)}$, which appear at every step of the reduction process.

The generating function S, of these changes must be taken of the form,

$$S = \xi^T\cdot\theta + d(\theta) + e^T(\theta)\cdot z + f^T(\theta)\cdot I + \frac{1}{2}z^TG(\theta)z + I^TF(\theta)z. \qquad (15)$$

The averages of f and g with respect θ are $\overline{d(\theta)} = 0$, $\overline{f(\theta)} = 0$. $G(\theta)$ is a symmetric matrix verifying $\overline{G_{1,2}(\theta)} = 0$ and $\xi^T = (\xi_1, \xi_2)$ is a parameter to be adjusted conveniently. From this expression for the generating function, it is possible to set and solve the corresponding homological equations to find the unknowns ξ, d, e, f, G, F and check out that, at each step the norm of the corresponding $\hat{H}^{(i)}$ decays quadratically, so the perturbative scheme is quadratically convergent and thus, the invariant torus is preserved "but slightly deformed". To prove this convergence, we follow the ideas about preservation of lower dimensional tori given in [2].

Acknowledgments. The first author acknowledges the Centre de Recerca Matemàtica (CRM) where this research was carried out. This work has been partially supported by the Catalan grant CIRIT number 2000SGR-00027, by the Spanish grant number BFM2000-0623 and by the INTAS grant 00-221.

References

[1] T. J. Bridges, R. H. Cushman, and R. S. Mackay, *Fields Ins. Comm.*, 1995, **4**, 61–79.

[2] A. Jorba and J. Villanueva, *Nonlinearity*, 1997, **10**, 783–882.

[3] M. Ollé, J. R. Pacha, and J. Villanueva, preprint (submitted to proceedings of IV International symposium /workshop HAMSYS 2001).

[4] M. Ollé, J. R. Pacha, and J. Villanueva, preprint (to be submitted to Celestial Mech. and Dynamical Astron.).