On a conjecture of Fujita

Miguel A. Barja

0. Introduction

Let Y be a smooth projective variety. Let $\mathcal{M} = \mathcal{O}_Y(D)$ be an invertible sheaf. There are several notions of positivity for $\mathcal{M}$. $\mathcal{M}$ is said to be *nef* (or semipositive) if $DC \geq 0$ for every curve $C \subseteq Y$; $\mathcal{M}$ is said to be *strictly nef* (cf. [12]) if $DC > 0$. On the other hand $\mathcal{M}$ is *ample* if for any dimension k subvariety $E \subseteq Y$ we have $D^k E > 0$. Alternatively $\mathcal{M}$ is ample if and only if $\mathcal{M}^{\otimes r}$ is very ample for some $r \geq 1$. From this, the definition of semiampleness comes naturally. $\mathcal{M}$ is said to be *semiample* if $\mathcal{M}^{\otimes r}$ is generated by global sections. Note that the notion of semiampleness is not numerical (for instance, $\mathcal{M} = \mathcal{O}_Y$ is semiample).

Let $\mathcal{F}$ be a locally free sheaf on Y. Consider $\mathcal{M} = \mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}})$ the tautological line bundle on $\mathbb{P} = \mathbb{P}_Y(\mathcal{F})$. We can extend the above notions of positivity for line bundles to $\mathcal{F}$ via $\mathcal{M}$. Note that then $\mathcal{F}$ is semiample if and only if $S^r\mathcal{F}$ is generated by global sections on Y for some $r \geq 1$; indeed, just consider the natural isomorphism $H^0(Y, S^r\mathcal{F}) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(r))$.

Remember that given a smooth variety Z is equivalent to give a non constant map $f : Z \longrightarrow Y$ and a quotient line bundle $f^*\mathcal{F} \twoheadrightarrow \mathcal{L}$ on Z, to give a map $\varphi : Z \longrightarrow \mathbb{P}$ such that $\varphi^*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \mathcal{L}$ (cf. [9] II.7.12). Then, when Y is a curve we can give alternative equivalent definitions for the numerical properties. Observe that if Z is a curve then $\deg\mathcal{L} = \deg\varphi^*\mathcal{O}_{\mathbb{P}}(L_{\mathcal{F}}) = \varphi(Z).L_{\mathcal{F}}$.

Following [1] we can define the *lower degree* of $\mathcal{F}$ as

$$\ell d(\mathcal{F}) = \min\{\deg\mathcal{L} \mid \mathcal{L} \text{ quotient line bundle of } \mathcal{F}\}$$

and the *stable lower degree* of $\mathcal{F}$ as

$$s\ell d(\mathcal{F}) = \inf\left\{\frac{\ell d(f^*\mathcal{F})}{\deg f} \mid f : Z \longrightarrow Y \text{ a finite map of nonsingular curves}\right\}$$

Then it comes out immediately from the definition that $\mathcal{F}$ is nef if and only if $\ell d(\mathcal{F}) \geq 0$ and that $\mathcal{F}$ is strictly nef if and only if $\ell d(\mathcal{F}) > 0$. Moreover $\mathcal{F}$ is nef if and only if any quotient of $\mathcal{F}$ has nonnegative degree. Obviously if $\mathcal{F}$ is ample any quotient of $\mathcal{F}$ has positive degree. A not so trivial fact based on Seshadri criterion for ampleness is that $\mathcal{F}$ is ample if and only if $s\ell d(\mathcal{F}) > 0$ (cf. [1] 1.2).

We have the obvious relations ample $\Rightarrow$ semiample $\Rightarrow$ nef, ample $\Rightarrow$ strictly nef $\Rightarrow$ nef.

Let X, Y be smooth projective varieties of dimensions n and m respectively and $f : X \longrightarrow Y$ a fibration. Consider the relative dualizing sheaf $\omega_{X/Y}$ and the relative canonical divisor $K_{X/Y}$ ($\omega_{X/Y} = \mathcal{O}_X(K_{X/Y})$). It turns out that the sheaves $R^i f_* \omega_{X/Y}^{\otimes r}$ have certain properties of positivity. In general these sheaves are only torsion free sheaves but not locally free. The notion of nefness can be extended to torsion free sheaves and to quasi-projective varieties, and we get the so called *weakly positive* sheaves (cf. [14]). Under our hypothesis both notions coincide and so we will not define this new notion of positivity.

The following proposition gives a brief account of the properties we need.

Proposition 0.1 *Let $f : X \longrightarrow Y$ be a fiber space between smooth projective varieties of dimensions n and m respectively. Assume the branch locus of f is contained in a normal crossings divisor. Then*

(i) ([10] 2.6) $R^i f_* \omega_{X/Y}$ *and* $R^j f_* \mathcal{O}_X$ *are locally free.*

(ii) ([10] Relative duality) *For* $0 \leq i \leq d = n - m$, $(R^i f_* \omega_{X/Y})^* \cong R^{d-i} f_* \mathcal{O}_X$.

(iii) ([14]) *For* $k \geq 1$, $f_* \omega_{X/Y}^{\otimes k}$ *is nef.*

(iv) ([1]) *For $k \geq 2$, $m = 1$, $f_*\omega_{X/Y}^{\otimes k}$ is ample.*

(v) ([2], [3]) *If $m = 1$, we have a decomposition $\mathcal{E} = f_*\omega_{X/Y} = \mathcal{A} \oplus \mathcal{E}_1 \oplus \ldots \oplus \mathcal{E}_r$, where $\mathcal{A}$ is ample, for $1 \leq i \leq r$, $\mathcal{E}_i$ are stable, degree zero and, if $s = h^1(\omega_X)$, $\mathcal{E}_1 = \ldots = \mathcal{E}_s = \mathcal{O}_Y$, $\mathcal{E}_j \neq \mathcal{O}_Y$ for $s+1 \leq j \leq r$. Moreover, if $\mathcal{F}$ is a stable degree zero sheaf such that there exists a surjective map $\mathcal{E} \twoheadrightarrow \mathcal{F}$, then $\mathcal{F}$ is a direct summand of $\mathcal{E}$ and hence $\mathcal{F} = \mathcal{E}_i$ for some $i \in \{1, \ldots, r\}$.*

In [4] p. 600, Fujita propose the following

Conjecture. Given a fibration $f : X \longrightarrow Y$, is there a birational model $f' : X' \longrightarrow Y'$ such that $\mathcal{E}' = f_*\omega_{X'/Y'}$ is semiample?

Observe that given a fibration f we can always get a birational model $f' : X' \longrightarrow Y'$ with branch locus contained in a normal crossings divisor via Hironaka's Theorem.

Before giving an alternative interpretation in case Y is a curve let us state some well known results on semiample sheaves. A good reference is [6].

Proposition 0.2 ([5], [6]) *Let $\mathcal{F}$ be a locally free sheaf on a smooth variety Y.*

(i) If X is smooth and $g : X \longrightarrow Y$ dominating, then $\mathcal{F}$ is semiample if and only if $g^*\mathcal{F}$ is semiample.

(ii) If $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_k$, $\mathcal{F}$ is semiample if and only if $\mathcal{F}_i$ is semiample for $1 \leq i \leq k$.

(iii) If $\mathcal{F}$ is semiample then $\det(\mathcal{F})$ *is semiample.*

(iv) If $\mathcal{F}$ is semiample and $\mathrm{kod}(Y, \det(\mathcal{F})) = 0$ *then there exists an étale cover $g : \widetilde{Y} \longrightarrow Y$ such that $g^*\mathcal{F}$ is trivial.*

If Y is a curve, Fujita's conjecture is equivalent to ask whether the degree zero, non trivial summands $\{\mathcal{E}_i\}_{s+1 \leq i \leq r}$ become trivial after an étale base change.

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \longrightarrow & Y \end{array}$$

and hence $\widetilde{f}_*\omega_{\widetilde{X}/\widetilde{Y}} = \widetilde{\mathcal{A}} \oplus \mathcal{O}_{\widetilde{Y}}^{\oplus(q(\widetilde{X})-g(\widetilde{Y}))}$ with $\widetilde{\mathcal{A}}$ ample.

1. The conjecture of Fujita

We prove that Fujita's conjecture is true if Y is an elliptic curve. In fact we prove much more: for arbitrary X and Y any locally free quotient $\mathcal{F}$ of $\mathcal{E}$ with $\det(\mathcal{F}) \in Pic^0(Y)$ verifies that $\det(\mathcal{F})$ is a torsion line bundle. In particular, when Y is a curve $\det\mathcal{E}_i$ are torsion. If $\mathcal{E}_i$ has rank one this proves Fujita's conjecture for this piece. In any case note that according to Proposition 0.2 (ii) the torsion nature of $\det(\mathcal{E}_i)$ is a necessary condition in order Fujita's conjecture holds.

Let $\mathcal{H} = \mathcal{O}_Y(H)$ be an ample line bundle on Y. Recall that a locally free sheaf $\mathcal{F}$ on Y is called $\mathcal{H}$-stable if for any $\mathcal{G} \subseteq \mathcal{F}$ we have

$$\frac{H^{m-1}c_1(\mathcal{G})}{\mathrm{rk}\mathcal{G}} < \frac{H^{m-1}c_1(\mathcal{F})}{\mathrm{rk}\mathcal{F}}$$

Clearly $\mathcal{F}$ is $\mathcal{H}$-stable if and only if it is $\mathcal{H}^{\otimes r}$-stable for any $r \geq 1$ so we can assume $\mathcal{H}$ is very ample. Then if $C = H_1 \cap \ldots \cap H_{m-1}$ $(H_i \sim H)$ is a smooth curve, it is equivalent to say that $\mathcal{F}$ is $\mathcal{H}$-stable to say that $\mathcal{F}_{|C}$ is stable in the usual sense.

Proposition 1.1 *Let Y be a smooth curve, X a smooth projective variety of dimension n and $f : X \longrightarrow Y$ a fibration. Let $\mathcal{F}$ be a stable, degree zero, locally free sheaf on Y. Then*

(i) A map $\mathcal{E} = f_*\omega_{X/Y} \longrightarrow \mathcal{F}$ is non trivial if and only if it is surjective

(ii) There exists a non zero map $\mathcal{E} = f_*\omega_{X/Y} \longrightarrow \mathcal{F}$ if and only if $h^0(Y, (R^{n-1}f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0$

(iii) For any $1 \leq i \leq n-1$ there exists a finite number of stable, degree zero vector bundles $\mathcal{F}$ on Y such that $h^0(Y, (R^i f_*\mathcal{O}_X) \otimes \mathcal{F}) \neq 0$

PROOF:

(i) Since $\mathcal{E}$ is nef by Proposition 0.1, the image of $\mathcal{E} \longrightarrow \mathcal{F}$ has nonnegative degree. Since $\mathcal{F}$ is stable and of degree zero, $\mathcal{F}$ can not have any proper subsheaf of nonpositive degree. Hence if the map is non trivial it is surjective.

(ii) By relative duality we have

$$\mathrm{Hom}_Y(\mathcal{E},\mathcal{F}) \cong H^0(Y,\mathcal{E}^*\otimes\mathcal{F}) = H^0(Y,(R^{n-1}f_*\mathcal{O}_X)\otimes\mathcal{F})$$

(iii) Assume first $i = n-1$. From (i) and (ii) we obtain that if $h^0(Y,(R^{n-1}f_*\mathcal{O}_X)\otimes\mathcal{F}) \neq 0$, then there is an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$. According to Fujita's decomposition in Proposition 0.1 (v), we know that there exists i such that $\mathcal{E}_i \cong \mathcal{F}$.

Assume now $1 \leq i \leq n-2$. If we consider Z a general linear section of codimension i in X and $g : Z \longrightarrow Y$ the induced fibration, if follows from [10] 2.34 that $R^{n-1-i}f_*\omega_{X/Y}$ is a direct summand of $g_*\omega_{Z/Y}$. Hence by relative duality

$$\begin{aligned}\mathrm{Hom}_Y(R^{n-1-i}f_*\omega_{X/Y},\mathcal{F}) &= H^0(Y,(R^{n-1-i}f_*\omega_{X/Y})^*\otimes\mathcal{F}) = \\ &= H^0(Y,(R^i f_*\mathcal{O}_X)\otimes\mathcal{F}) \neq 0\end{aligned}$$

and hence there exists a non trivial map

$$g_*\omega_{Z/Y} \twoheadrightarrow R^{n-1-i}f_*\omega_{X/Y} \longrightarrow \mathcal{F}$$

and the argument finishes arguing as in the case $i = n-1$. ■

Then we can state the main result.

Theorem 1.2 *Let X, Y be smooth projective varieties of dimension n and m respectively. Let $f : X \longrightarrow Y$ be a fibration with branch locus contained in a normal crossings divisor of Y.*

Fix a very ample line bundle $\mathcal{H} = \mathcal{O}_Y(H)$ on Y. Let $\mathcal{E} = f_*\omega_{X/Y}$ and let $\mathcal{F}$ be a $\mathcal{H}$-stable locally sheaf on Y such that $det(\mathcal{F}) \in Pic^0(Y)$.

If there exists a non-trivial map $\mathcal{E} \longrightarrow \mathcal{F}$, then $det(\mathcal{F})$ is torsion.

PROOF: First of all note that it is enough to prove the theorem for $m = 1$. Indeed, let $Z \in |H|$ be a general smooth member. By Bertini's Theorem $T = f^*(Z) = X \times_Y Z$ is again smooth. Let $g = f_{|T} : T \longrightarrow Z$. By adjunction we have that $g_*\omega_{T/Z} = i^*(f_*\omega_{X/Y})$, where $i : Z \hookrightarrow X$ is the natural inclusion.

Since $\mathcal{F}$ is locally free, $\mathrm{Im}(\mathcal{E} \longrightarrow \mathcal{F})$ is torsion free and non trivial, hence the induced map $g_*\omega_{T/Z} = i^*(f_*\omega_{X/Y}) \longrightarrow i^*\mathcal{F}$ is non-trivial; we also have $\det(i^*\mathcal{F}) = i^*(\det\mathcal{F}) \in Pic^0(Z)$ and $i^*(\mathcal{F})$ is $\mathcal{H}$-stable on Z. By induction we have for some $r \in \mathbb{N}$ $(i^*\mathcal{L})^{\otimes r} = i^*(\mathcal{L}^{\otimes r}) = \mathcal{O}_Z$. Kodaira's vanishing gives $h^0(Y, \mathcal{L}^{\otimes r}) = h^0(Z, i^*(\mathcal{L}^{\otimes r})) = 1$ and hence $\mathcal{L}^{\otimes r} = \mathcal{O}_Y$; indeed, consider the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) \longrightarrow H^0(Y, \mathcal{L}^{\otimes r}) \longrightarrow$$

$$\longrightarrow H^0(Z, i^*(\mathcal{L}^{\otimes r})) \longrightarrow H^1(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r})$$

and that

$$H^0(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) = H^1(Y, \mathcal{O}_Y(-Z) \otimes \mathcal{L}^{\otimes r}) = 0$$

since $\mathcal{O}_Y(Z) \otimes \mathcal{L}^{-\otimes r}$ is ample ($\mathcal{O}_Y(Z)$ is ample, $\mathcal{L}$ is numerically trivial and ampleness is a numerical condition).

From now on we assume Y to be a smooth curve of genus b and $\mathcal{F}$ a stable, degree zero locally free sheaf on Y. Let $d = n - m = n - 1$ the relative dimension of f.

Consider, from Leray's spectral sequence $E_2^{p,q} = H^p(R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(f_*\mathcal{F})$

$$0 \longrightarrow H^1(Y, (R^{d-1} f_*\mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow H^{n-1}(X, f^*(\mathcal{F})) \longrightarrow$$

$$\longrightarrow H^0(Y, (R^d f_*\mathcal{O}_X) \otimes \mathcal{F}) \longrightarrow 0 \qquad (1.1)$$

By Proposition 1.3, $h^0(Y, (R^{d-1} f_*\mathcal{O}_X) \otimes \mathcal{F}) = h^0(Y, (R^d f_*\mathcal{O}_X) \otimes \mathcal{F}) = 0$ except for a finite number of such $\mathcal{F}$. We have then that $h^{n-1}(X, f^*(\mathcal{F}))$ is constant, say a, except for a finite number of $\mathcal{F}$. Let $s = \mathrm{rank}\mathcal{F}$.

If $s = 1$ then Proposition 1.1 asserts that $\mathcal{F} \in A = \{\mathcal{M} \in Pic^0(Y) \mid h^{n-1}(X, f^*(\mathcal{M})) \geq a + 1\}$ which is finite. Then we can apply the remarkable result of Simpson in [13]: the irreducible components of A are translations of tori by torsion points. Finiteness of A implies that in fact its points are torsion.

Assume $s \geq 2$. Following Viehweg ([14], [11] 4.11) we can consider $X^{(s)}$ a resolution of the component of $X^s = X \times_Y \overset{s)}{\ldots} \times_Y X$ dominating Y and $f^{(s)} : X^{(s)} \longrightarrow Y$ the induced fibration. We have then an inclusion

$$\left(f_*^{(s)}\omega_{X^{(s)}/Y}\right)^{**} \hookrightarrow \left(\overset{s}{\otimes} f_*\omega_{X/Y}\right)^{**}$$

Note that since $\dim Y = 1$ both are locally free sheaves (hence reflexive) and so we have an inclusion of vector bundles of the same rank

$$j : f_*^{(s)}\omega_{X^{(s)}/Y} \hookrightarrow \overset{s}{\otimes} f_*\omega_{X/Y}$$

Now consider the projection π

$$\overset{s}{\otimes} f_*\omega_{X/Y} \twoheadrightarrow \overset{s}{\otimes} \mathcal{F} \twoheadrightarrow \det\mathcal{F}$$

Note that $\pi \circ j$ is non trivial since the first two vector bundles have the same rank and j is injective, and then we can apply the argument of the rank one case. ∎

Remark 1.3 A similar result is given independently in [15] for the very particular case of a surface S of Albanese dimension 1, where $q(S) = g(Alb(S)) = 1$ and the canonical map of S is composed with the Albanese fibration.

Corollary 1.4 *Let $f : X \longrightarrow B$ be a fibration of a smooth projective variety X onto an smooth curve of genus b.*

If $b \leq 1$ then $f_*\omega_{X/B}$ is semiample.

PROOF: We just need to apply that on an elliptic curve any stable degree zero sheaf has rank one. ∎

If $Y = B$ is any smooth curve, Theorem 1.2 says that Fujita's conjecture is true for any rank one degree zero summand of $\mathcal{E}$. Hence the only open question is whether the degree zero summands of rank at least two in Fujita's decomposition are semiample. We can not prove this but we prove that if they are not semiample they are more positive than being nef, they are *strictly nef.* More concretely

Corollary 1.5 *Let $f : X \longrightarrow B$ be as above. Let $\mathcal{F}$ be a stable degree zero vector bundle on B.*

If there exists a non-trivial map $\mathcal{E} = f_*\omega_{X/B} \longrightarrow \mathcal{F}$ then there is a base change $\sigma : \widetilde{B} \longrightarrow B$ such that $\sigma^*\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{O}_{\widetilde{B}}^{\oplus r}$, where $\mathcal{F}_0$ is a strictly nef vector bundle with trivial determinant.

Proof: Assume $\mathcal{F}$ is not strictly nef. Then there exists a smooth curve C and a map $\sigma : C \longrightarrow B$ such that $\sigma^*\mathcal{F} \twoheadrightarrow \mathcal{L}$, $\deg\mathcal{L} = 0$, where $\mathcal{L} = \widehat{\sigma}^*\mathcal{O}_{\mathbb{P}}(1)$ and $\widehat{\sigma}$ is a map $\widehat{\sigma} : C \longrightarrow \mathbb{P} = \mathbb{P}_B(\mathcal{F})$.

Since σ is flat we have ([11] 4.10)

$$0 \longrightarrow \widetilde{f}_*\omega_{\widetilde{S}/C} \longrightarrow \sigma^* f_*\omega_{S/B}$$

where $\widetilde{S}$ is a desingularization of $S \times_B C$.

Since $\sigma^*\mathcal{E} \twoheadrightarrow \sigma^*\mathcal{F} \twoheadrightarrow \mathcal{L}$ and the induced map $\widetilde{f}\omega_{\widetilde{S}/C} \longrightarrow \mathcal{L}$ can not be zero ($\widetilde{f}_*\omega_{\widetilde{S}/C}$ and $\sigma^* f_*\omega_{S/B}$ are of the same rank), by Theorem 1.2 $\mathcal{L}$ is torsion and hence, up to a new base change, we can assume is trivial.

Then we argue by induction on the rank of $\mathcal{F}$. ■

Remark 1.6 If $\mathcal{F}_0$ is as in the previous result, note that for every $\widehat{\sigma} : C \longrightarrow \mathbb{P}_B(\mathcal{F}_0)$, $\mathcal{O}_{\mathbb{P}}(1)\widehat{\sigma}(C) > 0$ but $\mathcal{F}_0$ is not ample. In our case $s\ell d(\mathcal{F}_0) = 0$ but we can not achieve this infimum.

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Miguel A. Barja

Departament de Matemàtica Aplicada I

Universitat Politècnica de Catalunya

Avda. Diagonal, 647

08028-Barcelona (SPAIN)

e-mail: barja@ma1.upc.es