

Perturbation of Quadrics

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Abstract. The aim of this paper is to study what happens when a slight perturbation affects the coefficients of a quadratic equation defining a variety (a quadric) in \mathbb{R}^n . Structurally stable quadrics are those a small perturbation on the coefficients of the equation defining them does not give rise to a “different” (in some sense) set of points. In particular, we characterize structurally stable quadrics and give the “bifurcation diagrams” of the non-stable ones (showing which quadrics meet all of their neighbourhoods), when dealing with the “affine” and “metric” equivalence relations. This study can be applied to the case where a set of points, which constitute the set of solutions of a problem, is defined by a quadratic equation whose coefficients are given with parameter uncertainty.

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1. Introduction

The concept of structural stability was first introduced by A.A. Andronov and L. Pontryagin in the qualitative theory of dynamical systems (see [AP37]). Roughly speaking, a structurally stable element is an element whose “behaviour” does not change when suffering small perturbations. In the case of control systems this property has been widely studied by several authors. Here we will use the notion of structural stability with respect to an equivalence relation defined on a topological space X , as appears in [Wi80].

In the case where X is a differentiable manifold and the equivalence relation is induced by the action of a Lie group \mathcal{G} on X , so that the orbits are also manifolds, then the condition of structural stability is equivalent to the maximal dimension of the orbit.

V.I. Arnold considered the manifold of square matrices under the linear group action and presented explicitly miniversal deformations, which are the orthogonal linear varieties

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to the orbits. The techniques used in this paper will be those used in [Ar71]; that is to say, we will consider miniversal deformations and derive from them the calculus of the dimension of orbits (by explicitly calculating a basis of the orthogonal space to the orbit, with respect to a scalar product).

Our goal is to study what happens when applying a small perturbation to the coefficients of a quadratic equation which defines a quadric. That is to say, to know whether this small perturbation gives rise to a set of points of the same type or not (after defining an equivalence relation to identify “similar” sets of points).

A quadric in \mathbb{R}^n is generally thought as the set of solutions of an equation defined by a symmetric square real matrix of order $n + 1$. Then we identify the quadric with a matrix defining it and consider the differentiable manifold of symmetric square matrices.

Obviously, there exist equations expressed by means of different symmetric square matrices giving rise to the same set of points (for example, the sets of solutions of the equations given by the matrices A and λA , $\lambda \in \mathbb{R}^*$, are the same). Besides, there exist different matrices such that the corresponding equations have no real points satisfying them (in particular, they give rise to the same set of points: the empty set!). Nevertheless, for our study it is convenient to deal with the differentiable manifold \mathcal{S} of (all) symmetric square matrices.

The equivalence relations considered in \mathcal{S} will be that coming from the affine equivalence and from the metric equivalence of quadrics. These equivalence relations may be viewed as those induced by the action of suitable Lie groups.

This fact allows us to use Arnold’s techniques, that is to say, miniversal deformations. In particular, we can identify structurally stable matrices by a previous calculation of the dimension of their orbits.

In the case of non-structurally stable matrices giving rise to a non-empty set of points we will explicitly describe the corresponding bifurcation diagrams from the study of the partitions in \mathcal{S} with respect to these equivalence relations. This allows to present the quadrics which meet every neighbourhood of the given quadric, again identifying the quadric with a symmetric square matrix defining it.

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2. Preliminaries and notation

Throughout the paper we have tried to follow standard notations. The most commonly used symbols are the following ones.

\mathbb{R} will denote the field of real numbers.

\mathbb{R}^* is the set of all non-zero real numbers.

$\mathcal{M}_{n \times m}(\mathbb{R})$ is the set of $n \times m$ matrices with real coefficients.

Given a matrix $M \in \mathcal{M}_{n \times m}(\mathbb{R})$, M^t will stand for the transpose of M and $\text{rk } M$ will denote the rank of M .

In the particular case where $n = m$, we will simply denote the set of $n \times n$ matrices by $\mathcal{M}_n(\mathbb{R})$.

The trace of the matrix $M \in \mathcal{M}_n(\mathbb{R})$ will be denoted by $\text{tr } M$.

$Gl_n(\mathbb{R})$ is the set of invertible matrices in $\mathcal{M}_n(\mathbb{R})$, that is to say, the general linear group of order n .

$O_n(\mathbb{R})$ is the set of matrices M in $\mathcal{M}_n(\mathbb{R})$ such that $M^t M = I_n$, that is to say, the orthogonal group of order n .

\mathcal{S}_{n+1} is the set of symmetric square matrices of order n .

\mathcal{A}_n is the set of skew-symmetric square matrices of order n .

If X is a differentiable manifold and $x \in X$, $T_x X$ is the tangent space of X at x .

If G is a group acting on X , $\mathcal{O}(x)$ denotes the orbit of x in X under the action of G .

Let F be a vector subspace in \mathbb{R}^n . Then F^\perp denotes the orthogonal complement to F with respect to a scalar product $\langle \cdot, \cdot \rangle$ defined in \mathbb{R}^n .

Finally, if X, Y are differentiable manifolds and $\varphi : X \rightarrow Y$ is a differentiable mapping, $d\varphi$ denotes the differential of φ .

We list now briefly a few of the basic definitions about quadrics which will be mentioned later.

A *quadric* Q in \mathbb{R}^n is generally thought as a set of points in \mathbb{R}^n with coordinates (x_1, \dots, x_n) satisfying, in a given system of coordinates, an equation of the form

$$x^t A_\infty x + 2B^t x + c = 0$$

where $x = (x_1 \dots x_n)^t$, $A_\infty \in \mathcal{M}_n(\mathbb{R})$ is a symmetric matrix, $B \in \mathcal{M}_{n \times 1}(\mathbb{R})$ and $c \in \mathbb{R}$. Equivalently, this equation can be written in the form

$$X^t A X = 0$$

with $A = \begin{pmatrix} A_\infty & B \\ B^t & c \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{R})$ and $X = (x_1 \dots x_n \ 1)^t$. (Note that A is also a symmetric matrix).

In the case where $n = 2$ they are classically called *conics*.

It is possible that the set of solutions of such an equation is the empty set. Actually, different equations of this type give rise to the empty set. For example, when considering the equations $1 = 0$, $-x_1^2 = 1$, etc.

Besides, it is obvious that, for all $\lambda \in \mathbb{R}^*$, the symmetric square matrix λA defines the same set of points than A .

Let $X^t A X = 0$ be the equation of a quadric in a given system of coordinates. Let us consider another system of coordinates with origin in $P = (p_1 \dots p_n)^t$ and axis directions given by the vectors in \mathbb{R}^n whose coordinates are the columns of a matrix $S \in Gl_n(\mathbb{R})$. If $(\bar{x}_1 \dots \bar{x}_n)^t$ are the coordinates of the point x in the new system of coordinates and $\bar{X} = (\bar{x}_1 \dots \bar{x}_n \ 1)^t$, then the equation of the quadric in the new system of coordinates is $\bar{X}^t A' \bar{X} = 0$, with $A' = \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}$. In the case where $S \in O_n(\mathbb{R})$ the metric concepts attached to the quadric are preserved.

A quadric Q defined by the equation $X^t A X = 0$ is said to be *degenerate* if $rk(A) < n+1$ and *non-degenerate* otherwise.

The *centers* of a quadric Q may be defined as the set of points in \mathbb{R}^n whose coordinates are solutions of the system $A_\infty Z = -B$. Note that Q is a quadric with center if $rk(A_\infty) = rk(A_\infty|B)$ and that Q has an unique center if $rk(A_\infty)(= rk(A_\infty|B)) = n$. The centers thus defined coincide with the set of centers of symmetry of Q .

The concepts of affine and metric equivalence of quadrics are as follows.

Definition 1. Let Q, Q' be the quadrics defined by the equations $X^t A X = 0$ and $X^t A' X = 0$, respectively. Then Q and Q' are called *affine equivalent* (and we write

$Q \sim Q'$) if there exists a matrix $\begin{pmatrix} S & P \\ 0 & I \end{pmatrix}$ with $S \in Gl_n(\mathbb{R})$, $P \in \mathcal{M}_{n \times 1}(\mathbb{R})$ and $\lambda \in \mathbb{R}^*$ such that $A' = \lambda \begin{pmatrix} S & P \\ 0 & I \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & I \end{pmatrix}$.

Definition 2. Let Q, Q' be the quadrics defined by the equation $X^t A X = 0$ and $X^t A' X = 0$, respectively. Then Q and Q' are called *metrically equivalent* (and we write $Q \equiv Q'$) if there exists a matrix $\begin{pmatrix} S & P \\ 0 & I \end{pmatrix}$ with $S \in O_n(\mathbb{R})$, $P \in \mathcal{M}_{n \times 1}(\mathbb{R})$ and $\lambda \in \mathbb{R}^*$ such that $A' = \lambda \begin{pmatrix} S & P \\ 0 & I \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & I \end{pmatrix}$.

We have the following Classification Theorem (see, for instance, [Pu95], [Xa77]).

Theorem 1. Let Q be the quadric defined by the equation $X^t A X = 0$. Let us denote by $s(A)$ and $s(A_\infty)$ the minimum between the number of positive eigenvalues and the number of negative eigenvalues of the matrices A and A_∞ , respectively. Then $rk(A)$, $rk(A_\infty)$, $s(A)$, $s(A_\infty)$ is a complete system of invariants of the affine class of Q . Moreover Q is affine equivalent to the quadric defined by the equation

$$x_1^2 + \dots + x_{r_0}^2 - x_{r_0+1}^2 - \dots - x_r^2 = 1 \text{ or } 0$$

if Q has center(s), and

$$x_1^2 + \dots + x_{r_0}^2 - x_{r_0+1}^2 - \dots - x_r^2 + 2x_{r+1} = 0$$

if Q does have no center.

Besides, Q is metrically equivalent to the quadric defined by the equation

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = 1 \text{ or } -1 \text{ or } 0$$

if Q has center(s), and of the form

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 + 2\mu x_{r+1} = 0$$

if Q does have no center, where $\lambda_1, \dots, \lambda_r$ are the non-zero eigenvalues of A_∞ .

These equations are called *reduced affine equation* (respectively, *reduced metric equation*) of Q .

Note that $r = rk(A_\infty)$ and $r_0 = \max \left\{ \frac{r(A_\infty) + s(A_\infty)}{2}, \frac{r(A_\infty) - s(A_\infty)}{2} \right\}$.

3. Geometrical study

We denote by \mathcal{S}_{n+1} the differentiable manifold of symmetric square matrices of order $n + 1$ with real entries.

The definitions in the preceding Section lead up to consider the following equivalence relations in \mathcal{S}_{n+1} .

Definition 3. Two matrices A and A' in \mathcal{S}_{n+1} are said to be *affine equivalent* (and we write $A \sim A'$) if there exists a matrix $\begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}$ with $S \in Gl_n(\mathbb{R})$, $P \in \mathcal{M}_{n \times 1}(\mathbb{R})$ and $\lambda \in \mathbb{R}^*$ such that $A' = \lambda \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}$.

Definition 4. Two matrices A and A' in \mathcal{S}_{n+1} are said to be *metrically equivalent* (and we write $A \equiv A'$) if there exists a matrix $\begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}$ with $S \in O_n(\mathbb{R})$, $P \in \mathcal{M}_{n \times 1}(\mathbb{R})$ and $\lambda \in \mathbb{R}^*$ such that $A' = \lambda \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}$.

Let $\mathcal{G}_1, \mathcal{G}_2$ be the Lie groups

$$\mathcal{G}_1 = \mathbb{R}^* \times Gl_n(\mathbb{R}) \times \mathbb{R}^n, \quad \mathcal{G}_2 = \mathbb{R}^* \times O_n(\mathbb{R}) \times \mathbb{R}^n$$

where the group operations are the natural ones,

$$(\lambda_1, S_1, P_1) \circ (\lambda_2, S_2, P_2) = (\lambda_1 \lambda_2, S_1 S_2, P_1 + P_2) \quad \text{in } \mathcal{G}_1$$

$$(\lambda_1, S_1, P_1) \circ (\lambda_2, S_2, P_2) = (\lambda_1 \lambda_2, S_1 S_2, P_1 + P_2) \quad \text{in } \mathcal{G}_2$$

The starting point in the geometrical approach to the problem of studying the effects of small perturbations on the coefficients of the matrices is, in order to use Arnold's techniques, to view the equivalence classes with respect to the equivalence relations in \mathcal{S}_{n+1} defined above as the orbits of the following group actions on \mathcal{S}_{n+1} :

$$\begin{aligned} \alpha_i : \mathcal{G}_i \times \mathcal{S}_{n+1} &\longrightarrow \mathcal{S}_{n+1} \\ ((\lambda, S, P), A) &\longrightarrow \lambda \begin{pmatrix} S^t A_\infty S & S^t A_\infty P + S^t B \\ P^t A_\infty S + B^t S & P^t A_\infty P + 2B^t P + c \end{pmatrix} = \lambda \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix}^t A \begin{pmatrix} S & P \\ 0 & 1 \end{pmatrix} \end{aligned}$$

for $i = 1, 2$, respectively.

Note that in our set-up the orbits are differentiable manifolds ([Hu81]).

Miniversal deformations is the main tool we will make use of in our study. We briefly recall the basic definitions of deformations and versality (see, for example, [Ar 71] and [Ta81] for further details).

Let X be a manifold, $x \in X$ and \mathcal{G} a Lie group acting on X via an action

$$\begin{aligned} \alpha : \mathcal{G} \times X &\longrightarrow X \\ (g, x) &\longrightarrow \alpha(g, x) \end{aligned}$$

Definition 5. A *deformation* of x is a differentiable mapping $\varphi : \mathcal{U} \longrightarrow X$, with \mathcal{U} an open neighbourhood of the origin in \mathbb{R}^l , such that $\varphi(0) = x$.

Definition 6. A deformation $\varphi : \mathcal{U} \longrightarrow X$ of x is called *versal* at 0 if for any other deformation of x , $\psi : \mathcal{V} \longrightarrow X$, there exists an open set $\mathcal{V}' \subseteq \mathcal{V}$ with $0 \in \mathcal{V}'$, a differentiable mapping $\beta : \mathcal{V}' \longrightarrow \mathcal{U}$ with $\beta(0) = 0$ and a deformation of the identity $I \in \mathcal{G}$, $\theta : \mathcal{V}' \longrightarrow \mathcal{G}$, such that $\psi(\mu) = \alpha(\theta(\mu), \varphi(\beta(\mu)))$ for all $\mu \in \mathcal{V}'$.

A versal deformation with minimum number of parameters l is called a *miniversal deformation*.

To find versal deformations, the key point is the following result, proved by V. I. Arnold in the case where $\mathcal{G} = Gl_n(\mathbb{R})$ acting on $X = \mathcal{M}_n(\mathbb{R})$ in [Ar71], whose generalization was given by A. Tannenbaum in the case of a Lie group acting on a manifold (see [Ta81]).

Theorem 2. A differentiable family $\varphi : \mathcal{U} \longrightarrow X$, where \mathcal{U} is an open neighbourhood of the origin in \mathbb{R}^l , is a versal deformation of $\varphi(0) = x$ if and only if it is transversal to the orbit $\mathcal{O}(x)$ at x , that is to say, $T_x X = T_x \mathcal{O}(x) \oplus d\varphi T_0(\mathcal{U})$.

As an immediate consequence, we obtain a miniversal deformation.

Corollary 1. Let us fix any scalar product in X . Let l be the dimension of $T_x \mathcal{O}(x)^\perp$, $\{v_1, \dots, v_l\}$ a basis of $T_x \mathcal{O}(x)^\perp$ and \mathcal{U} an open neighbourhood of the origin in \mathbb{R}^l . Then for any $x \in X$, the manifold $x + T_x \mathcal{O}(x)^\perp$ defines a miniversal deformation of x . That is to say, the mapping $\varphi : \mathcal{U} \subseteq \mathbb{R}^l \longrightarrow X$, $(\lambda_1, \dots, \lambda_l) \longrightarrow x + \lambda_1 v_1 + \dots + \lambda_l v_l$ is a miniversal deformation of x .

The following homogeneity property allows to choose a suitable element from each equivalent class, thus making easier the calculations.

Proposition 1. Let X be a differentiable manifold and $x_1, x_2 \in X$ be two equivalent elements (with respect to the equivalence relation defined in X derived from any action of a Lie group acting on X). Then there exists a diffeomorphism $f : X \longrightarrow X$ preserving the orbits such that $f(x_1) = x_2$.

PROOF. If $g \in \mathcal{G}$ is an element such that $x_2 = \alpha(g, x_1)$, it suffices to consider the mapping

$$\begin{aligned} f : X &\longrightarrow X \\ x &\longrightarrow \alpha(g, x) \end{aligned}$$

It is easy to check that this map is a diffeomorphism. \diamond

From now on, we will consider the manifold $X = \mathcal{S}_{n+1}$ and the following scalar product in \mathcal{S}_{n+1} : $\langle A_1, A_2 \rangle = \text{tr}(A_1 A_2^t)$.

Note that, in particular, we have $T_A \mathcal{S}_{n+1} = \mathcal{S}_{n+1}$ for all $A \in \mathcal{S}_{n+1}$.

Let $A = \begin{pmatrix} A_\infty & B \\ B^t & c \end{pmatrix} \in \mathcal{S}_{n+1}$ be a symmetric square matrix of order $n+1$, with $A_\infty \in \mathcal{M}_n(\mathbb{R})$, $B \in \mathcal{M}_{n \times 1}(\mathbb{R})$, $c \in \mathbb{R}$. If α is the action of a Lie group \mathcal{G} acting on \mathcal{S}_{n+1} so that the orbits are differentiable manifolds, it is well-known that $T_A \mathcal{O}(A) = d(\alpha_A)_I(T_I \mathcal{G})$ where α_A is the mapping

$$\begin{aligned} \alpha_A : \mathcal{G} &\longrightarrow \mathcal{S}_{n+1} \\ g &\longrightarrow \alpha_A(g) = \alpha(g, A) \end{aligned}$$

In the particular cases of the actions α_1 and α_2 defined at the beginning of this Section we obtain the following characterization of the tangent vector subspaces to the orbits.

Proposition 2. *Let us denote by $T_A \mathcal{O}_1(A)$, $T_A \mathcal{O}_2(A)$ the tangent spaces to the orbits $\mathcal{O}_1(A)$, $\mathcal{O}_2(A)$ of $A \in \mathcal{S}_{n+1}$ with respect to α_1 and α_2 , respectively. Then*

$$T_A \mathcal{O}_1(A) = \left\{ \begin{pmatrix} S^t A_\infty + A_\infty S + \lambda A_\infty & A_\infty P + \lambda B + S^t B \\ P^t A_\infty + B^t S + \lambda B^t & 2B^t P + \lambda c \end{pmatrix} \middle| \lambda \in \mathbb{R}, S \in \mathcal{M}_n(\mathbb{R}), P \in \mathbb{R}^n \right\}$$

$$T_A \mathcal{O}_2(A) = \left\{ \begin{pmatrix} S^t A_\infty + A_\infty S + \lambda A_\infty & A_\infty P + \lambda B + S^t B \\ P^t A_\infty + B^t S + \lambda B^t & 2B^t P + \lambda c \end{pmatrix} \middle| \lambda \in \mathbb{R}, S \in \mathcal{A}_n(\mathbb{R}), P \in \mathbb{R}^n \right\}$$

where $\mathcal{A}_n(\mathbb{R})$ is the manifold of skew-symmetric matrices ($S = -S^t$) of order n .

PROOF. For $i = 1, 2$, let us consider $\alpha_{iA} : \mathcal{G}_i \longrightarrow \mathcal{S}_{n+1}$ defined by

$$\alpha_{iA}(\lambda, S, P) = \lambda \begin{pmatrix} S^t A_\infty S & S^t A_\infty P + S^t B \\ P^t A_\infty S + B^t S & P^t A_\infty P + 2B^t P + c \end{pmatrix} \in \mathcal{S}_{n+1}$$

Then

$$\begin{aligned} \alpha_{iA}(1 + \varepsilon\lambda, I + \varepsilon S, \varepsilon P) &= \lambda \begin{pmatrix} I + \varepsilon S^t & 0 \\ \varepsilon P^t & 1 \end{pmatrix} A \begin{pmatrix} I + \varepsilon S & \varepsilon P \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} A_\infty & B \\ B^t & c \end{pmatrix} + \varepsilon \begin{pmatrix} \lambda A_\infty + S^t A_\infty + A_\infty S & A_\infty P + \lambda B + S^t B \\ P^t A_\infty + B^t S + \lambda B^t & 2B^t P + \lambda c \end{pmatrix} + O(\varepsilon^2) \end{aligned}$$

Then the result follows from the fact that \mathcal{G}_1 is an open submanifold of $\mathbb{R} \times \mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n$. It follows from elementary Differential Geometry that $T_I \mathcal{G}_1 = \mathbb{R} \times \mathcal{M}_n(\mathbb{R}) \times \mathbb{R}^n$. On the other hand, $T_I \mathcal{G}_2 = \mathbb{R} \times \mathcal{A}_n(\mathbb{R}) \times \mathbb{R}^n$. This also follows from elementary properties in Differential Geometry and from the fact that $T_I \mathcal{O}_n(\mathbb{R}) = \mathcal{A}_n(\mathbb{R})$. \diamond

In spite of the description above, it is not so easy to determine these tangent spaces or even only to calculate their dimension. It is easier to deal with their orthogonal vector

subspaces (with respect to any scalar product). Next Theorem shows that $T_A\mathcal{O}_1(A)^\perp$ and $T_A\mathcal{O}_2(A)^\perp$ may be identified with the sets of solutions of suitable linear equations systems.

Theorem 3. *Let us consider the following scalar product in \mathcal{S}_{n+1} : $\langle A_1, A_2 \rangle = \text{tr}(A_1 A_2^t)$. Let $M = \begin{pmatrix} M_1 & M_2 \\ M_2^t & M_4 \end{pmatrix}$ be a square matrix of order $n + 1$, with M_1 a symmetric matrix in $\mathcal{M}_n(\mathbb{R})$, $M_2 \in \mathcal{M}_{n \times 1}(\mathbb{R})$, $M_4 \in \mathbb{R}$. Then*

$$(a) \ M \in T_A\mathcal{O}_1(A)^\perp \iff \begin{cases} M_1 A_\infty + M_2 B^t = 0 \\ M_2^t A_\infty + M_4 B^t = 0 \\ M_2^t B + M_4 c = 0 \end{cases}$$

$$(b) \ M \in T_A\mathcal{O}_2(A)^\perp \iff \begin{cases} M_1 A_\infty - A_\infty M_1 + M_2 B^t - B M_2^t = 0 \\ M_2^t A_\infty + M_4 B^t = 0 \\ \text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c = 0 \end{cases}$$

PROOF. First, let us observe that

$$\begin{aligned} 0 &= \langle M, \begin{pmatrix} S^t A_\infty + A_\infty S + \lambda A_\infty & A_\infty P + \lambda B + S^t B \\ P^t A_\infty + B^t S + \lambda B^t & 2B^t P + \lambda c \end{pmatrix} \rangle \\ &= \text{tr} M \begin{pmatrix} S^t A_\infty + A_\infty S + \lambda A_\infty & A_\infty P + \lambda B + S^t B \\ P^t A_\infty + \lambda B^t + B^t S & 2B^t P + \lambda c \end{pmatrix} \\ &= 2\text{tr}(M_1 A_\infty S + M_2 B^t S) + 2[M_2^t A_\infty + M_4 B^t]P + \lambda[\text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c] \end{aligned}$$

(a) Let us assume $M \in T_A\mathcal{O}_1(A)^\perp$. This holds if, and only if, for all $\lambda \in \mathbb{R}$, $S \in \mathcal{M}_n(\mathbb{R})$ and $P \in \mathbb{R}^n$,

$$2\text{tr}(M_1 A_\infty S + M_2 B^t S) + 2[M_2^t A_\infty + M_4 B^t]P + \lambda[\text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c] = 0$$

In particular, if $P = 0$ and $\lambda = 0$, this equation is equivalent to

$$\text{tr}((M_1 A_\infty + M_2 B^t)S) = 0 \quad \forall S \in \mathcal{M}_n(\mathbb{R})$$

Let us denote by E_{ij} the matrices with the only non-zero entry being 1 at the position (i, j) (row i and column j). Varying $S = E_{ij}$ in the equation above it is not difficult to check that this equation is equivalent to

$$M_1 A_\infty + M_2 B^t = 0$$

Let us assume now that $S = 0$ and $\lambda = 0$. We denote by e_i the the $n \times 1$ -matrices with an only non-zero entry being 1 in row i . The equation

$$(M_2^t A_\infty + M_4 B^t)P = 0 \quad \forall P \in \mathbb{R}^n$$

is equivalent to

$$(M_2^t A_\infty + M_4 B^t) e_i = 0 \quad 1 \leq i \leq n$$

and these last equations hold if, and only if,

$$M_2^t A_\infty + M_4 B^t = 0$$

Finally, if $S = 0$ and $P = 0$, the equation above is equivalent to

$$\text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c = 0$$

We already know that $M_1 A_\infty + M_2 B^t = 0$ and thus

$$\text{tr}(M_1 A_\infty) + M_2^t B = 0$$

obtaining

$$M_2^t B + M_4 c = 0$$

The converse is straightforward true.

(b) Let us assume now $M \in T_A \mathcal{O}_2(A)^\perp$. This holds if, and only if, for all $\lambda \in \mathbb{R}$, $S \in \mathcal{A}_n(\mathbb{R})$ and $P \in \mathbb{R}^n$,

$$2\text{tr}(M_1 A_\infty S + M_2 B^t S) + 2[M_2^t A_\infty + M_4 B^t]P + \lambda[\text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c] = 0$$

In particular, if $P = 0$ and $\lambda = 0$, this equation is equivalent to

$$\text{tr}((M_1 A_\infty + M_2 B^t)S) = 0 \quad \forall S \in \mathcal{A}_n(\mathbb{R})$$

Let us denote by E_{ij} , $i \neq j$, the matrices with the only non-zero entries at the positions (i, j) and (j, i) (row i and column j and row j and column i , respectively), where the entries are 1 and -1 , respectively. The equation above is equivalent to the system of equations

$$\text{tr}((M_1 A_\infty + M_2 B^t)E_{ij}) = 0 \quad 1 \leq i, j \leq n; i \neq j$$

and these equalities hold if, and only if,

$$M_1 A_\infty - A_\infty M_1 + M_2 B^t - B M_2^t = 0$$

Let us assume now that $S = 0$ and $\lambda = 0$. We denote by e_i the $n \times 1$ -matrices with the only non-zero entry being 1 in row i . As in (a), the equation

$$(M_2^t A_\infty + M_4 B^t)P = 0 \quad \forall P \in \mathbb{R}^n$$

is equivalent to

$$M_2^t A_\infty + M_4 B^t = 0$$

Finally, if $S = 0$ and $P = 0$, the equation above is equivalent to

$$\text{tr}(M_1 A_\infty) + 2M_2^t B + M_4 c = 0$$

The converse is also straightforward true. \diamond

We observe that, after Theorem 3, a miniversal deformation can be explicitly described, for any $A \in \mathcal{S}_{n+1}$. Also the dimensions of the orbits with respect to α_1 and α_2 can be calculated.

We remember now the notion of structural stability as appears in [Wi80].

Definition 7. Let X be a topological space and \sim an equivalence relation defined on X . Then $x \in X$ is *structurally stable* if and only if there exists a neighbourhood \mathcal{U} of x such that $x \sim y$ for all $y \in \mathcal{U}$.

Remark. In the case where X is a differentiable manifold and \sim is the equivalence relation defined by the action of a Lie group acting on X , giving rise to orbits which are also differentiable manifolds, we have that the following statements are equivalent:

- (a) x is structurally stable.
- (b) $\mathcal{O}(x)$ is an open submanifold.
- (c) $\dim \mathcal{O}(x) = \dim X$.
- (d) $\dim T_x \mathcal{O}(x)^\perp = 0$.

This characterization allows to know, from the study of the systems in the statement of Theorem 3, which matrices $A \in \mathcal{S}_{n+1}$ are structurally stable under the equivalence relations deduced from the actions of the Lie groups \mathcal{G}_1 and \mathcal{G}_2 .

Note that when dealing with the equivalence relation derived from the action α_1 the structurally stable matrices can be characterized as follows.

Proposition 3. Structurally stable matrices in \mathcal{S}_{n+1} under the affine equivalence relation are those symmetric matrices $A = \begin{pmatrix} A_\infty & B \\ B^t & c \end{pmatrix}$ such that A_∞ and A have full rank.

PROOF. This follows from the lower semicontinuity of the rank of the matrices or the direct inspection of the linear equations system in the statement of Theorem 3 (a). \diamond .

Similarly, we note that there are no structurally stable symmetric matrices in \mathcal{S}_{n+1} with respect to the equivalence relation derived from the action α_2 .

To finish this Section, we study whether the partitions of \mathcal{S}_{n+1} in orbits, according to both lie group actions, are stratifications of \mathcal{S}_{n+1} and, if so, the regularity properties they satisfy, recalling first the concept of stratification.

Definition 8. If X is a differentiable manifold, a partition $X = \cup_{i \in I} X_i$ is called a *stratification* of X when it is locally finite (that is to say, for all $x \in X$, there exists a neighbourhood U of x meeting only a finite number of subsets X_i) and for all $i \in I$, X_i is a differentiable submanifold of X . Then each X_i is called a *stratum*.

Note that the partition in \mathcal{S}_{n+1} according to the α_1 -action is finite. Therefore this partition is a stratification: the strata being the orbits under this action. Let us denote by $\Sigma_1 = \cup_{A \in \mathcal{S}_{n+1}} \mathcal{O}_1(A)$ this stratification. Nevertheless, we can observe that the partition deduced from the α_2 -action is not locally finite, therefore the partition of \mathcal{S}_{n+1} corresponding to this action is not a stratification.

Let us remember the *Whitney regularity condition* (see [Gi76], for instance). Let U, V be two disjoint submanifolds of \mathbb{R}^p , and let $x \in U \cap \overline{V}$. V is said to be Whitney regular over U at x when the following condition holds: let $(u_i), (v_i)$ be two sequences, in U and V , respectively, converging to x and such that $u_i \neq v_i \forall i$. Let us denote by L_i the line spanned by $v_i - u_i$ and T_i be the vector subspace $T_{v_i}V$. If $(L_i)_i$ converges to L and $(T_i)_i$ converges to T , in the corresponding Grassmannians of subspaces of \mathbb{R}^p , then $L \subseteq T$. Since given a diffeomorphism of an open set M of \mathbb{R}^p onto another open set M' of \mathbb{R}^p mapping U, V and x to U', V' and x' , respectively, V is Whitney regular over U at x if, and only if, V' is Whitney regular over U' at x' , we can define the Whitney regularity condition in the case of an arbitrary differentiable manifold X .

A stratification Σ of a differentiable manifold X is said to be a *Whitney stratification* if for any pair of strata $U, V \in \Sigma$, V is Whitney regular over U at x whenever $U \cap \overline{V} \neq \emptyset$.

To finish this geometrical approach, we can state the following result, which is a straightforward consequence of the combination of several facts mentioned previously.

Proposition 4. The stratification in \mathcal{S}_{n+1} defined by the α_1 -action is Whitney regular.

PROOF. This follows from the fact that the strata are in this case the (finitely many) orbits under this action.

4. Perturbation of quadrics

Let Q be a (non-empty) set of points (x_1, \dots, x_n) in \mathbb{R}^n given by an equation of the form $X^t A X = 0$, $A \in \mathcal{S}_{n+1}$, $X = (x_1 \dots x_n 1)^t$, that is to say, a quadric.

Obviously, two quadrics Q_1 and Q_2 defined by equations $X^t A_1 X = 0$, $X^t A_2 X = 0$ with $A_1, A_2 \in \mathcal{S}_{n+1}$ are equivalent with respect to the affine equivalence (respectively, with respect to the metric equivalence) if, and only if, so are the matrices A_1 and A_2 .

The concept of structurally stable quadrics will be translated to the concept of structurally stable matrices defining the quadric. In general, we will identify the quadric with a matrix defining it and make use of the geometric study made in §3 in \mathcal{S}_{n+1} . Further details about the relationship between symmetric square matrices and quadrics with respect to the affine and metric equivalence can be found in [Pu95], Chapter 14, §6 and §11. Concretely, we can state the following definition.

Definition 9. We will say that a quadric is *structurally stable* when it is defined by a structurally stable matrix $A \in \mathcal{S}_{n+1}$.

The Remark at the end of the preceding Section shows that there are no structurally stable quadrics with respect to the metric equivalence. When considering the affine equivalence, structurally stable quadrics are those which are both non-degenerate and with center (note that the center is unique in this case).

In particular, in the case where $n = 2$, the only structurally stable conics are ellipses and hyperbolas (and imaginary ellipses, if we wished to take in consideration imaginary conics). Obviously, in all these cases the dimension of the orbits is 6.

Let us consider the case of the parabola and the affine equivalence. A reduced affine equation is: $x^2 + 2y = 0$. It is obvious that we can always restrict ourselves to consider the reduced equation, as deduced from the homogeneity property in Proposition 1.

A matrix $A \in \mathcal{S}_{n+1}$ defining this conic is then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The set of solutions of the system in Theorem 3 (a) is:

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| m \in \mathbb{R} \right\}$$

We deduce that

$$T_A \mathcal{O}_1(A)^\perp = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

and $\dim T_A \mathcal{O}_1(A) = 5$.

When considering the metric equivalence, a metric reduced equation is $x^2 + \mu y = 0$, a matrix A defining this conic is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\mu}{2} \\ 0 & \frac{\mu}{2} & 0 \end{pmatrix}$$

and the set of solutions of the system in Theorem 3 (b) is:

$$\left\{ \left(\begin{pmatrix} -\mu m_2 & 0 & 0 \\ 0 & m_1 & m_2 \\ 0 & m_2 & 0 \end{pmatrix} \right) \middle| m_1, m_2 \in \mathbb{R} \right\}$$

We deduce that

$$T_A \mathcal{O}_2(A)^\perp = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\mu & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right]$$

and $\dim T_A \mathcal{O}_2(A) = 4$.

The other cases can be handled in a similar way.

Next Tables show the dimension of the orbits with respect to both equivalence relations in the cases of conics in \mathbb{R}^2 and quadrics in \mathbb{R}^3 .

Conic	Dimension of the orbit with respect to α_1	Dimension of the orbit with respect to α_2
Ellipse	6	3 or 4 (*)
Hyperbola	6	4
Parabola	5	4
Non-parallel lines	5	4
Parallel lines	4	3
Double line	3	3
Line	3	3

(*) depending on the eigenvalues of A_∞ being equal or different, respectively.

Table 1: Conics in \mathbb{R}^2 and dimension of orbits

Quadric	Dimension of the orbit with respect to α_1	Dimension of the orbit with respect to α_2
Ellipsoid	10	4, 6 or 7 (*)
Hyperboloid of one sheet	10	6 or 7 (**)
Hyperboloid of two sheets	10	6 or 7 (**)
Elliptic paraboloid	9	6 or 7 (**)
Hyperbolic paraboloid	9	7
Real con	9	6 or 7 (**)
Real elliptic cylinder	8	5 or 6 (**)
Hyperbolic cylinder	8	6
Parabolic cylinder	7	6
Non-parallel planes	7	6
Parallel planes	5	4
Double plane	4	4
Plane	4	4

(*) depending on the eigenvalues of A_∞ being all equal, two of them equal or different, respectively.

(**) depending on the eigenvalues of A_∞ being two of them equal or different, respectively.

Table 2: Quadrics in \mathbb{R}^3 and dimension of the orbits

Since the stratification Σ_1 of \mathcal{S}_{n+1} is Whitney regular, a miniversal deformation is a parametrized family which is generically transverse to the stratification. That is to say, the set of families which are transverse to the stratification is an open and dense set in \mathcal{S}_{n+1} . According to Thom theorem, the induced partition in the space of parameters is also a (Whitney) stratification and the codimensions are the same.

Thus we can show the bifurcation diagrams, obtained from the miniversal deformations. From them we can deduce the quadrics meeting all neighbourhood of a given one, and which are the most probable. Concretely, we show the case of the parabola, the other cases being handled similarly.

When dealing with the affine equivalence, we have to study which conics are defined by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $m \in \mathbb{R}$.

It is easy to check that there are only the following possibilities:

1. $m = 0$, in which case the conic is a parabola

2. $m > 0$, in which case the conic is a (real) ellipse
3. $m < 0$, in which case the conic is a hyperbola

We obtain the following bifurcation diagram:

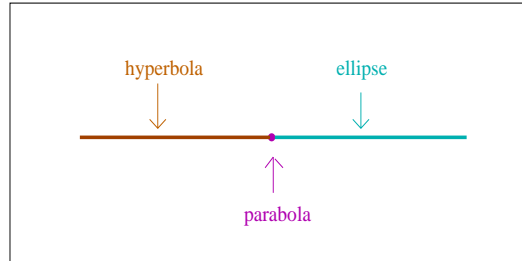


Figure 1: Bifurcation diagram of the parabola

In the other cases we obtain the following bifurcation diagrams.

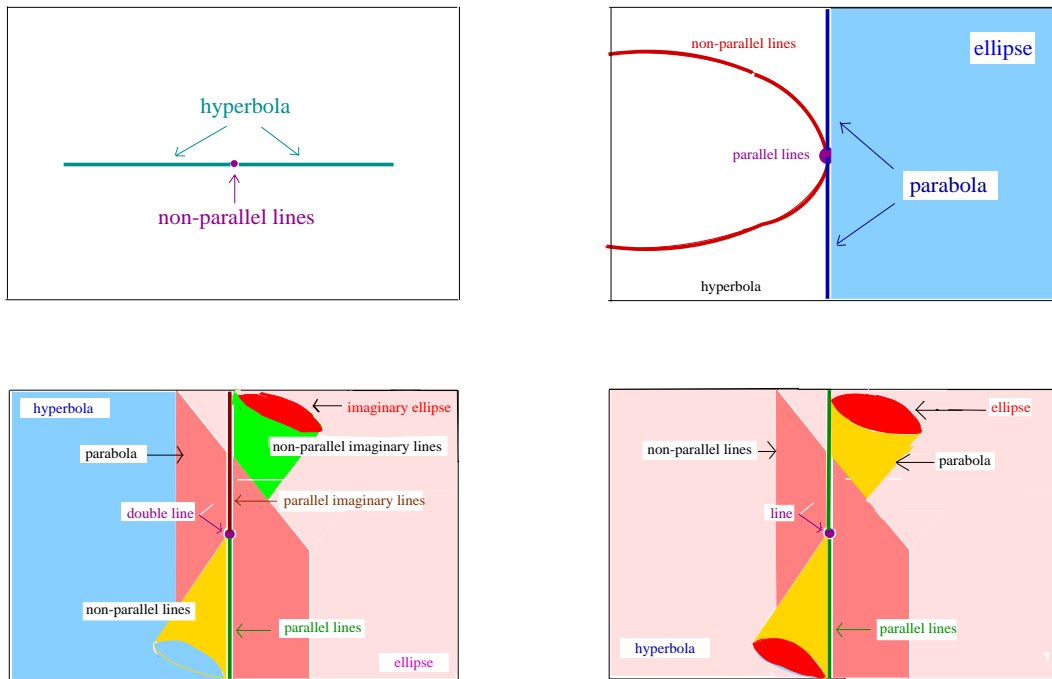


Figure 2: Bifurcation diagrams with respect to the affine equivalence of conics in \mathbb{R}^2

Remark. Note that a perturbation of an “imaginary” conics and quadrics may give rise to a real conic or quadric.

A similar study can be done in \mathbb{R}^n , $n \geq 3$. Here we present some examples in the case where $n = 3$.

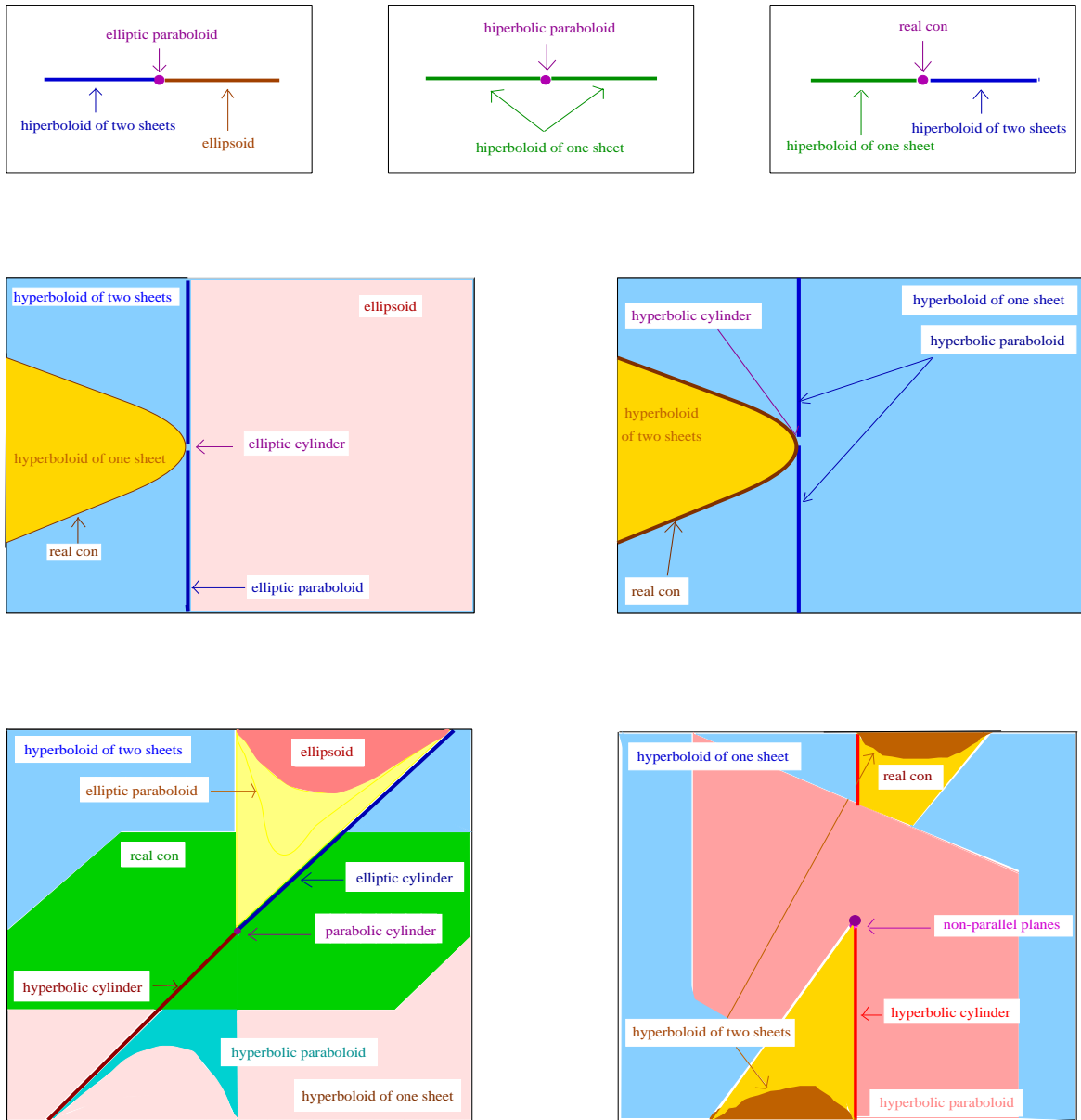


Figure 3: Bifurcation diagrams with respect to the affine equivalence of quadrics in \mathbb{R}^3

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