An example of a non acyclic Koszul complex of a module

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In his paper [3], F. Sancho de Salas defines the *universal Koszul complex* of a module M over a sheaf of rings $\mathcal{O}$ as $\mathrm{Kos}(M) = \Lambda(M) \otimes_{\mathcal{O}} S(M)$, where $\Lambda(M)$ and $S(M)$ stand for the exterior and symmetric algebras of M, endowed with the usual differential, and he conjectures (Conjecture 2.3.) that $\mathrm{Kos}(M)$ is always acyclic. It is well known that for M flat or $\mathcal{O}$ an algebra over a field of characteristic zero, this is true (see [1] and [2] for definitions and proofs). We give now an example that it fails in characteristic 2. Recall that $\mathrm{Kos}^2(M)$, the homogeneous component of degree 2 of the Koszul complex $\mathrm{Kos}(M)$, is

$$0 \to \Lambda^2(M) \xrightarrow{\partial_{2,0}} M \otimes M \xrightarrow{\partial_{1,1}} S^2(M) \to 0\,,$$

where $\partial_{2,0}(u \wedge v) = v \otimes u - u \otimes v$ and $\partial_{1,1}(u \otimes v) = uv$. Let A be a local ring containing a field of characteristic 2 and let x, y, z be a system of parameters. Let $I = (x, y, z)$ be the ideal they generate and take $u = x(y \wedge z)$ in $\Lambda^2(I)$. To see $u \neq 0$, consider the bilinear surjective map $f : I \times I \to I^2/I^{[2]}$ defined by $f(a, b) = ab + I^{[2]}$, where $I^{[2]}$ is the ideal generated by the 2th powers of all elements of I. Since f vanishes over the elements (a, a), it extends to an epimorphism $f : \Lambda^2(I) \to I^2/I^{[2]}$. Remark that if 2 where invertible, $I^2 = I^{[2]}$ and $f = 0$. Since the characteristic is 2, $I^{[2]} = (x^2, y^2, z^2)$ and $f(u) = xyz + I^{[2]} \neq 0$ (by The Monomial Conjecture, see for instance, Theorem 9.2.1 in [2], or simply take A a regular ring and x, y, z a regular sequence). Hence $u \neq 0$. On the other hand,

$$x(y \otimes z) = (xy) \otimes z = y(x \otimes z) = x \otimes (yz) = z(x \otimes y) = (xz) \otimes y = x(z \otimes y)\,.$$

Therefore, $\partial_{2,0}(u) = x(z \otimes y) - x(y \otimes z) = 0$ and $H_2(\mathrm{Kos}^2(M)) = \mathrm{Ker}(\partial_{2,0}) \neq 0$. Remark that from general properties of the symmetic functor, it follows that $H_1(\mathrm{Kos}(M)) = 0$. Thus, $\mathrm{Kos}(M)$ is not a rigid complex.

References

[1] N. Bourbaki: Algèbre, Chapitre 10. Hermann, Masson, Paris, 1980.

[2] W. Bruns, J. Herzog: Cohen-Macaulay rings. Cambridge Studies in Advanced Math. 39. Cambridge University Press, 1993.

[3] F. Sancho de Salas: *Residues of a Pfaff system relative to an invariant subscheme.* Trans. Amer. Math. Soc. 352, 2000, 4019-4035.

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