

# Error estimates for a viscosity-splitting, finite element method for the incompressible Navier-Stokes equations

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**Summary** In this paper we provide an error analysis of a fractional-step, finite element method for the numerical solution of the incompressible Navier–Stokes equations. Under mild regularity assumptions on the continuous solution, we obtain first order error estimates in the time step size both for the intermediate and the end-of-step velocities of the method; we also give some error estimates for the pressure solution. We complete the analysis with some error estimates for a fully discrete, finite element version of the method.

**Key words** Incompressible viscous flow, Navier–Stokes equations, Fractional–step methods, Error analysis, Rate of convergence

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## 1 Introduction

The numerical solution of the unsteady, incompressible Navier–Stokes equations has received much attention in the last decades, and many numerical schemes are now available for that purpose. The difficulties encountered in this problem are mainly of three different kinds: the mixed type of the equations, which is due to the coupling of the momentum equation with the incompressibility condition, and, subsequently, the treatment of the pressure; the advective–diffusive

character of the equations, which have a viscous and a convective term; and finally, the nonlinearity of the problem.

Fractional step methods are becoming widely used in this context. By splitting the time advancement into a number of (generally two) substeps, they allow to separate the effects of the different operators appearing in the equations. They have been used together with different space discretizations, both finite difference ([1], [4], [17], [18]), finite element ([9], [13], [20]) and spectral element methods ([29]). However, semidiscrete presentations of these methods, in which the space variables are not discretized, seem more appropriate to study the time discretization itself.

The origin of this category of methods is generally credited to the work of Chorin (see [4]) and Temam (see [25]). They developed the well known *projection* method, which is a two step method in which the second step consists of the projection of an intermediate velocity field onto the space of solenoidal vector fields, thus enforcing incompressibility. The incompatibility of the projection boundary conditions with those of the original problem may introduce a numerical boundary layer of size  $O(\sqrt{\nu \delta t})$  in these methods (see [21] and [28]), where  $\nu$  is the kinematic viscosity and  $\delta t$  is the time step size. However, convergence of this method to a continuous solution as  $\delta t$  tends to zero was proved in [26], for the semidiscrete method, and [5], for a fully discrete method with periodic boundary conditions. The end-of-step velocities of the projection method do not converge in the space  $\mathbf{H}_0^1(\Omega)$ , since they do not satisfy the correct boundary conditions.

More recently, analytical studies of fractional step methods have turned into obtaining error estimates in the time step size, so as to establish their order of accuracy. Thus, J. Shen proved in [23] that the projection method, both with and without pressure correction, is first order accurate in a certain norm. Some imprecise steps in the proofs in [23] pointed out by J.L. Guermond in [14] were corrected in [24]. A more recent analysis given in [15] for a fully discrete, finite element version of the incremental fractional step projection method yielded error estimates of first order in the time step size and optimal order in the mesh size, assuming a finite element interpolation satisfying the discrete inf-sup condition. First order error estimates were also obtained by Long-an Ying (see [19] and the references therein) for another fractional step method, called *viscosity splitting* method, in which the viscosity is not fully uncoupled from incompressibility. In this sense, a fully discrete version of the so called  *$\theta$ -scheme* (see [12]), in which viscosity and incompressibility are also coupled, was

proved to converge to a continuous solution in [10] (see also [7] for a convergence analysis of a related parallel scheme). In [22] another fractional step method that keeps part of the viscous term in the second step is derived from an inexact factorization of the fully discrete original problem; this method is referred to as Yosida scheme in this reference.

In this paper we provide some error estimates for a *viscosity splitting*, fractional step method which was introduced and studied in [2]. It is a two-step scheme in which the nonlinearity and the incompressibility of the problem are split into different steps. It allows to enforce the original boundary conditions of the problem in all substeps of the scheme, which led to convergence of both the intermediate and end-of-step velocities of the method to a continuous solution in the spaces  $\mathbf{L}^2(\Omega)$  and  $\mathbf{H}_0^1(\Omega)$  (see [2]). Here we prove that these velocities are first order accurate in the time step size.

Moreover, the study of this method was originally motivated by the consideration of a well-known predictor–multicorrector algorithm (see [3]), as detailed in [2]; this fact provides a theoretical explanation of why the original boundary conditions of the problem can be prescribed in this algorithm, and in what sense it can be understood as a fractional step method.

The paper is organized as follows: in Section 2 we introduce the notation we use and some generalities about the incompressible Navier–Stokes equations, such as the regularity assumed for their solutions. In Section 3 we recall the fractional step method of [2] and introduce a finite element spatial approximation, while in Section 4 we give an error analysis for this method; we first obtain some error estimates for both the intermediate and the end-of-step velocities and then analyse the pressure solution. Finally, we also give some error estimates for the fully discrete, finite element solution which are of optimal order in the mesh size.

## 2 Preliminaries

The evolution of viscous, incompressible fluid flow in a bounded domain  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) is governed, in the primitive variable formulation, by the unsteady, incompressible Navier–Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p - \nu \Delta \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times (0, T) \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T) \quad (2)$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T) \quad (3)$$

$$\mathbf{u} = \mathbf{u}^0 \quad \text{in } \Omega \times \{0\} \quad (4)$$

where  $\mathbf{u}(\mathbf{x}, t) \in \mathbb{R}^d$  is the fluid velocity at position  $\mathbf{x} \in \Omega$  and time  $t \in (0, T)$  (with  $T > 0$  given),  $p(\mathbf{x}, t) \in \mathbb{R}$  is the fluid kinematic pressure,  $\nu > 0$  is the kinematic viscosity (which is assumed constant),  $\mathbf{f}(\mathbf{x}, t)$  is an external force term,  $\nabla$  is the gradient operator,  $\nabla \cdot$  is the divergence operator and  $\Delta$  is the Laplacian operator (here, and in what follows, boldface characters denote vector quantities). We consider only the homogeneous Dirichlet type boundary condition (3) for the sake of simplicity, and assume that the boundary of the domain  $\partial\Omega$  is at least of class  $\mathcal{C}^1$ .

In order to study some approximation schemes for this problem, we first introduce some notation. We denote by  $(\cdot, \cdot)$  the scalar product in  $L^2(\Omega)$ , and by  $\|u\|_0 = (u, u)^{1/2}$  its norm; the quotient space  $L_0^2(\Omega) = L^2(\Omega)/\mathbb{R}$  is needed in the case of Dirichlet type boundary conditions only, since the pressure is then determined only up to an additive constant; moreover, given  $m \in \mathbb{N}$ , the scalar product and norm in  $H^m(\Omega)$  are denoted by  $(u, v)_m$  and  $\|u\|_m$ , respectively. The space  $H^1(\Omega)$  contains a closed subspace  $H_0^1(\Omega)$  made up with functions which vanish at the boundary of  $\Omega$ ; the Poincaré–Friedrich inequality ensures that  $\|\nabla u\|_0 = (\nabla u, \nabla u)^{1/2}$  is a norm on  $H_0^1(\Omega)$ , equivalent to the norm induced by  $H^1(\Omega)$ . The dual space of  $H_0^1(\Omega)$  is denoted by  $H^{-1}(\Omega)$  with norm  $\|\cdot\|_{-1}$ , the duality pairing between these spaces being denoted by  $\langle \cdot, \cdot \rangle$ . All these definitions carry over to  $d$ -dimensional vector valued function spaces.

Due to the incompressibility condition (2), closed subspaces of solenoidal vector fields of these Hilbert spaces are also considered. Thus, we define:

$$\begin{aligned} H &= \{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \mathbf{u} = 0, \mathbf{n} \cdot \mathbf{u}|_{\partial\Omega} = 0\} \\ V &= \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \mid \nabla \cdot \mathbf{u} = 0\} \end{aligned}$$

Moreover, due to the unsteady character of the equations the following definitions are also needed: given  $p \in [1, \infty)$  and a Banach space  $W$ , the space  $L^p(0, T; W)$  is equipped with the norm:

$$\|u\|_{L^p(0, T; W)} = \left( \int_0^T \|u(t)\|_W^p dt \right)^{1/p}$$

and is also a Banach space with respect to this norm. The space of essentially bounded functions on  $(0, T)$  into  $W$  is denoted by  $L^\infty(0, T; W)$ . When  $W$  is a Hilbert space with scalar product  $(\cdot, \cdot)_W$ , the space  $L^2(0, T; W)$  is likewise with respect to:

$$(u, v) = \int_0^T (u(t), v(t))_W dt$$

In this notation, assuming  $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$  and  $\mathbf{u}_0 \in H$  problem (1)-(2)-(3)-(4) has at least one solution  $(\mathbf{u}, p)$  which satisfies  $\mathbf{u} \in L^\infty(0, T; H) \cap L^2(0, T; V)$  (see [27]). Uniqueness and more regularity of the solution can also be proved by assuming more regularity on the data  $\mathbf{f}$  and  $\mathbf{u}_0$  and the domain  $\Omega$ . In particular, we will assume that  $\mathbf{u}$  and  $p$  satisfy:

$$\mathbf{R1)} \quad \mathbf{u} \in \mathcal{C}^0(0, T; V) \cap L^\infty(0, T; \mathbf{H}^2(\Omega)), \quad \nabla p \in L^\infty(0, T; \mathbf{L}^2(\Omega))$$

$$\mathbf{R2a)} \quad \mathbf{u}_t \in L^2(0, T; \mathbf{L}^2(\Omega))$$

$$\mathbf{R2b)} \quad \mathbf{u}_t \in L^2(0, T; \mathbf{H}_0^1(\Omega))$$

$$\mathbf{R3)} \quad \int_0^T t \|\mathbf{u}_{tt}(t)\|_{-1}^2 dt \leq C$$

(the subindex  $t$  is employed hereafter for  $\frac{\partial}{\partial t}$ ). Here, and in what follows,  $C$  denotes a generic constant, possibly different at different occurrences, which may depend on the data  $\mathbf{f}$ ,  $\mathbf{u}_0$ ,  $T$  and  $\nu$ , the domain  $\Omega$  and the continuous solution  $\mathbf{u}$ , but is independent of the time step  $\delta t$  and the mesh size  $h$ . Conditions **R1**, **R2a**, **R2b** and **R3** can be proved, for instance, assuming that  $\Omega$  is of class  $\mathcal{C}^2$  (or is a convex polygon in  $\mathbb{R}^2$  or polyhedron in  $\mathbb{R}^3$ ) and that (see [16]):

$$\mathbf{u}_0 \in \mathbf{H}^2(\Omega) \cap V, \quad \mathbf{f}, \mathbf{f}_t \in L^\infty(0, T; \mathbf{L}^2(\Omega)), \quad \mathbf{u} \in L^\infty(0, T; \mathbf{H}_0^1(\Omega))$$

Under these assumptions, it was also shown in [23] that, according to the modifications introduced in [24]:

$$\mathbf{R4)} \quad \int_0^T \|\mathbf{u}_{tt}(t)\|_{V'}^2 dt \leq C$$

( $V'$  stands here for the dual space of  $V$ ). These regularity results will be used in what follows.

Error analysis of time integration schemes for time-dependent partial differential equations are usually given in terms of the following norms: given a Banach space  $W$  with norm  $\|\cdot\|_W$ , a continuous function  $u: [0, T] \rightarrow W$ , two real numbers  $p > 0$  and  $\alpha > 0$  and a time step size  $\delta t > 0$ , and taking  $t_n = n \delta t$  for  $n = 0, \dots, M = [T/\delta t]$ , a family of finite sequences  $\{u^n\}_{n=1, \dots, M}$  is said to be an order  $\alpha$  approximation of  $u$  in  $l^p(W)$  if there exists a constant  $C$  independent of  $\delta t$  such that, for all  $\delta t$ :

$$\left( \delta t \sum_{n=1}^M \|u(t_n) - u^n\|_W^p \right)^{1/p} < C \delta t^\alpha$$

Moreover,  $\{u^n\}_{n=1, \dots, M}$  is an order  $\alpha$  approximation of  $u$  in  $l^\infty(W)$  if:

$$\|u(t_n) - u^n\|_W < C \delta t^\alpha, \quad \forall n = 1, \dots, M$$

For the treatment of the convective term in the momentum equation (1), the following trilinear form is usually considered:

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \left( (\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w} \right), \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega), \mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{w} \in \mathbf{H}_0^1(\Omega)$$

This form is well defined and continuous on these spaces (see [27]), and it is skew-symmetric in its last two arguments if  $\mathbf{u} \in H$ , that is, if  $\nabla \cdot \mathbf{u} = 0$  and  $\mathbf{n} \cdot \mathbf{u} = 0$ :

$$c(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \quad \forall \mathbf{u} \in H, \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad (5)$$

Moreover,  $c$  possesses some continuity properties which hold when  $\Omega$  is of class  $\mathcal{C}^1$  (see [8]) and which we will use in our proofs, such as:

$$c(\mathbf{u}, \mathbf{v}, \mathbf{w}) \leq C \begin{cases} \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \\ \|\mathbf{u}\|_0 \|\mathbf{v}\|_2 \|\mathbf{w}\|_1 \\ \|\mathbf{u}\|_0 \|\mathbf{v}\|_1 \|\mathbf{w}\|_2 \\ \|\mathbf{u}\|_2 \|\mathbf{v}\|_1 \|\mathbf{w}\|_0 \\ \|\mathbf{u}\|_0^{1/2} \|\mathbf{u}\|_1^{1/2} \|\mathbf{v}\|_1 \|\mathbf{w}\|_1 \\ \|\mathbf{u}\|_1 \|\mathbf{v}\|_1 \|\mathbf{w}\|_0^{1/2} \|\mathbf{w}\|_1^{1/2} \end{cases}$$

Although this form is suitable for our analysis of the semidiscrete method, we will use the skew-symmetric part of  $c$  in the fully discrete problem, since incompressibility is only enforced weakly in the discrete setting; thus, we define:

$$\begin{aligned} \tilde{c}(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= (1/2) (c(\mathbf{u}, \mathbf{v}, \mathbf{w}) - c(\mathbf{u}, \mathbf{w}, \mathbf{v})), \\ &\forall \mathbf{u} \in \mathbf{H}^1(\Omega), \mathbf{v} \in \mathbf{H}_0^1(\Omega), \mathbf{w} \in \mathbf{H}_0^1(\Omega) \end{aligned}$$

Obviously, this form retains the continuity properties of the original form  $c$  (but for the last one), and is skew-symmetric in its last two arguments for any  $\mathbf{u} \in \mathbf{H}^1(\Omega)$ .

In some of our proofs we will also make use of the operator  $A^{-1}$ , defined as the inverse of the Stokes operator  $A = -P_H \Delta$ ,  $P_H$  being the projection onto  $H$ . The latter is defined for  $\mathbf{u} \in D(A) = V \cap \mathbf{H}^2(\Omega)$ , and is an unbounded, positive, self-adjoint closed operator onto  $H$ . Given  $\mathbf{u} \in H$ , by definition of  $A$ ,  $\mathbf{v} = A^{-1} \mathbf{u}$  is the solution of the following Stokes problem:

$$\begin{aligned} -\Delta \mathbf{v} + \nabla r &= \mathbf{u} && \text{in } \Omega \\ \nabla \cdot \mathbf{v} &= 0 && \text{in } \Omega \\ \mathbf{v} &= 0 && \text{on } \partial\Omega \end{aligned} \quad (6)$$

When  $\Omega$  is regular enough, there exists a constant  $C_1 > 0$  such that:

$$\|A^{-1} \mathbf{u}\|_s \leq C_1 \|\mathbf{u}\|_{s-2} \quad \text{for } s = 1, 2 \quad (7)$$

The following inequalities were given by J. Shen in [23] for  $(A^{-1}\mathbf{u}, \mathbf{u})$ , with  $\mathbf{u} \in H$ , and used there to deduce error estimates for the standard projection method:

$$C_2 \|\mathbf{u}\|_{-1}^2 \leq (A^{-1}\mathbf{u}, \mathbf{u}) \leq C_1 \|\mathbf{u}\|_{-1}^2$$

where  $C_1$  is the constant appearing in (7). But, as pointed out by J.L. Guermond in [14] and corrected in [24], the first inequality is not correct and has to be modified to:

$$C_2 \|\mathbf{u}\|_{V'}^2 \leq (A^{-1}\mathbf{u}, \mathbf{u}) \quad (8)$$

In our case, the following inequality (which can be easily proved) is also required:

$$\|A^{-1}\mathbf{u}\|_1 \leq C \|\mathbf{u}\|_{V'}$$

We will use this result in what follows.

### 3 Fractional-step, finite element method

#### 3.1 Fractional-step method

The fractional step method we analyse here was introduced in [2], where stability and convergence both in the spaces  $L^\infty(0, T; \mathbf{L}^2(\Omega))$  and  $L^2(0, T; \mathbf{H}_0^1(\Omega))$  and of both the intermediate and the end-of-step velocities to the continuous solution and where proved. Given  $\mathbf{u}^n \in V$ , approximation of  $\mathbf{u}$  at  $t = t_n$ , the time advancement to  $t_{n+1}$  is split into the following two steps:

*First step:* The first step of the method, which includes viscous and convective effects, consists of finding an intermediate velocity  $\mathbf{u}^{n+1/2}$  such that:

$$\frac{\mathbf{u}^{n+1/2} - \mathbf{u}^n}{\delta t} - \nu \Delta \mathbf{u}^{n+1/2} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1/2} = \mathbf{f}^n \quad (9)$$

$$\mathbf{u}^{n+1/2}|_{\partial\Omega} = 0 \quad (10)$$

*Second step:* Given  $\mathbf{u}^{n+1/2}$  from equation (9), find  $\mathbf{u}^{n+1}$  and  $p^{n+1}$  such that:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}}{\delta t} - \nu \Delta (\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}) + \nabla p^{n+1} = 0 \quad (11)$$

$$\nabla \cdot \mathbf{u}^{n+1} = 0 \quad (12)$$

$$\mathbf{u}^{n+1}|_{\partial\Omega} = 0 \quad (13)$$

As can be observed in (11), the main difference between this scheme and the standard projection method is the introduction of a viscous term in the incompressibility step, which allows the imposition of the original boundary condition (13) on the end-of-step velocity  $\mathbf{u}^{n+1}$ . Similar ideas can be found in the  $\theta$ -method of R. Glowinsky and others (see [12], for instance), in the first and third steps of the method of [20] and in several other methods such as [7], [19] or [29], all of which involve a last step with part of the viscous term. It can be observed in (9)–(10) and (11)–(12)–(13) how in this method convection is split from incompressibility, which are the two main difficulties of the problem, both of them still being coupled to viscosity. We have adopted here a first order linearized form of the convective term, although there are obviously other possibilities.

The motivations that led us to the study of this fractional step method are mainly twofold. First, it can be used to explain theoretically a class of predictor-multicorrector algorithms widely used in practice (see [2] for a more detailed explanation). These methods are based on an iterative scheme consisting of two steps per iteration with the same structure as the two steps above. Second, and this is the main concern of the present work, is the imposition of boundary conditions for the end-of-step velocity in fractional step methods. It is common practice among some users of the classical projection method to enforce all the boundary conditions for this field, although this is in principle not allowed if the viscous term in equation (11) is dropped. The present scheme, however, is not subject to this controversy; moreover, the fact that  $\mathbf{u}^{n+1}$  satisfies the correct boundary conditions led to improved convergence results in [2] with respect to those known for that variable in the standard projection method, and will allow us to obtain improved error estimates here too.

The computational efficiency of the scheme (9)–(13) was studied in [2]. The first step of the method, which is a linear, elliptic problem, can be seen as a linearized Burger’s problem; on the other hand, the second step has the structure of a Stokes (mixed) problem, the discretization of which leads to a symmetric system of linear equations. Based on ideas taken from the predictor–multicorrector algorithm used in [3], we developed in [2] an iterative technique for the solution of these two problems, in which each iteration consists of the solution of two linear systems with a diagonal matrix and a system with a symmetric, positive (semi)definite matrix which is the same for all iterations and time steps (and thus needs being computed and factorized only once at the beginning of the calculations); this iteration showed good convergence results in several test cases, which

makes the present fractional-step method feasible from a practical viewpoint. One drawback of this method is the need for the spatial discretization used to satisfy the discrete inf-sup compatibility condition, something which is nowadays known to apply to most versions of the standard projection method too (see [15]).

### 3.2 Finite element approximation

We next consider a finite element approximation of the semidiscrete equations (9)-(10) and (11)-(12)-(13). For that purpose, we take a family of finite dimensional spaces  $V_h \subset \mathbf{H}_0^1(\Omega)$  and  $Q_h \subset L_0^2(\Omega)$  defined from standard finite element discretizations  $\{\Theta_h\}_{h>0}$  of the domain  $\Omega$  of mesh size  $h$ . The discrete problem then reads, in weak form:

*First step:* Given  $\mathbf{u}_h^n \in V_h$ , find  $\mathbf{u}_h^{n+1/2} \in V_h$  such that, for all  $\mathbf{v}_h \in V_h$ :

$$\begin{aligned} \frac{1}{\delta t}(\mathbf{u}_h^{n+1/2} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{n+1/2}, \nabla \mathbf{v}_h) + \tilde{c}(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) \\ = (\mathbf{f}^n, \mathbf{v}_h) \end{aligned} \quad (14)$$

*Second step:* Find  $\mathbf{u}_h^{n+1} \in V_h$  and  $p_h^{n+1} \in Q_h$  such that, for all  $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ :

$$\begin{aligned} \frac{1}{\delta t}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) + \nu(\nabla(\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n+1/2}), \nabla \mathbf{v}_h) \\ - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) = 0 \end{aligned} \quad (15)$$

$$(\nabla \cdot \mathbf{u}_h^{n+1}, q_h) = 0 \quad (16)$$

As was mentioned before, the second step of the method can be seen as a generalized Stokes problem; the approximating spaces  $V_h$  and  $Q_h$  are thus required to satisfy the standard discrete compatibility condition (see, for instance, [11]):

**H1)**  $\exists \beta > 0$  independent of  $h$  such that, for all  $h > 0$ :

$$\inf_{q_h \in Q_h - \text{Ker} B_h^t} \left( \sup_{\mathbf{v}_h \in V_h - \{\mathbf{0}\}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|_1 \|q_h\|_{Q_h / \text{Ker} B_h^t}} \right) \geq \beta > 0$$

Here, and in what follows, we use the linear continuous operators  $B_h: V_h \rightarrow Q_h'$  and  $B_h^t: Q_h \rightarrow V_h'$  defined by the relations:

$$B_h(\mathbf{v}_h)(q_h) = B_h^t(q_h)(\mathbf{v}_h) = (q_h, \nabla \cdot \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h, \forall q_h \in Q_h$$

Existence and uniqueness of solutions to problems (14) and (15)-(16) are easily established,  $p_h^{n+1}$  being determined up to an arbitrary element of  $\text{Ker}B_h^t$ . At this point, it is important to remark that it is not convenient to split the second step into a pressure Poisson equation and an update of the end-of-step velocity, as for the classical projection method. The latter is known to introduce some pressure stability (see [6] for the analysis of a method based on this stabilizing mechanism). In our case, (15)-(16) are the direct Galerkin approximation of (9)-(11)-(12), and thus the satisfaction of the discrete inf-sup condition is mandatory.

The family of finite element triangulations  $\{\Theta_h\}_{h>0}$  of the domain  $\Omega$  is assumed to be *regular*, and the finite element functions in  $V_h$  and  $Q_h$  are locally polynomials of degree at least  $k$  and  $k-1$ , respectively, in such a way that the following approximating properties hold:

**H2)**  $\exists \gamma > 0$  independent of  $h$  such that for every  $\mathbf{v} \in \mathbf{H}^r(\Omega)$  and  $q \in H^s(\Omega)$  and for all  $h > 0$ :

$$\inf_{\mathbf{v}_h \in V_h} \|\mathbf{v} - \mathbf{v}_h\|_m \leq \gamma h^{k_1-m} \|\mathbf{v}\|_r, \quad 0 \leq m \leq k_1, \quad k_1 = \min\{k+1, r\}$$

$$\inf_{q_h \in Q_h} \|q - q_h\|_m \leq \gamma h^{k_2-m} \|q\|_s, \quad 0 \leq m \leq k_2, \quad k_2 = \min\{k, s\}$$

Finally, due to the analysis technique employed here which deals with the temporal error first and then the spatial error, the following relationship between the time step size and the mesh size will also be assumed:

**H3)**  $\exists C > 0$  independent of  $\delta t$  and  $h$  such that:

$$\delta t \geq C h^2$$

This assumption does not impose an upper bound on the time step size, so that the semi-implicit scheme (9)-(11)-(12) remains unconditionally stable.

#### 4 Error analysis

We present here an error analysis of the fractional step method (9)-(11)-(12). We restrict to the first order scheme presented earlier with the linearized form of the convective term in (9), in which the *convective* velocity is approximated by its value at the previous time step. Similar error estimates to those presented here can be obtained for the fully nonlinear form  $(\mathbf{u}^{n+1/2} \cdot \nabla)\mathbf{u}^{n+1/2}$ , which is however computationally more costly.

#### 4.1 Error estimates for the semidiscrete velocities

Let us define the velocity error functions for this method as:

$$\begin{aligned} \mathbf{e}_c^{n+1} &= \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1} \\ \mathbf{e}_c^{n+1/2} &= \mathbf{u}(t_{n+1}) - \mathbf{u}^{n+1/2} \end{aligned}$$

where the subscript  $c$  refers to the fact that the space variables still remain ‘continuous’. We give a first estimate for  $\mathbf{e}_c^{n+1}$  and  $\mathbf{e}_c^{n+1/2}$  which shows that both  $\mathbf{u}^{n+1}$  and  $\mathbf{u}^{n+1/2}$  are order 1/2 approximations to  $\mathbf{u}$  in  $l^\infty(\mathbf{L}^2(\Omega))$  and in  $l^2(\mathbf{H}_0^1(\Omega))$ .

**Lemma 1** *Assume that **R1**, **R2a** and **R3** hold; then for  $N = 0, \dots, [T/\delta t] - 1$ , and for all  $\delta t > 0$ :*

$$\begin{aligned} \|\mathbf{e}_c^{N+1}\|_0^2 + \|\mathbf{e}_c^{N+1/2}\|_0^2 + \sum_{n=0}^N \{ \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_0^2 + \|\mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n\|_0^2 \} \\ + \delta t \nu \sum_{n=0}^N \{ \|\mathbf{e}_c^{n+1}\|_1^2 + \|\mathbf{e}_c^{n+1/2}\|_1^2 + \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_1^2 \} \leq C\delta t \end{aligned} \quad (17)$$

*Proof* The first part of the proof is similar to that of [23]. We call  $\mathbf{R}^n$  the truncation error defined by:

$$\begin{aligned} \frac{1}{\delta t}(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) - \nu \Delta(\mathbf{u}(t_{n+1})) + (\mathbf{u}(t_{n+1}) \cdot \nabla)\mathbf{u}(t_{n+1}) \\ + \nabla p(t_{n+1}) = \mathbf{f}(t_{n+1}) + \mathbf{R}^n \end{aligned} \quad (18)$$

so that:

$$\mathbf{R}^n = \frac{1}{\delta t} \int_{t_n}^{t_{n+1}} (t - t_n) \mathbf{u}_{tt}(t) dt$$

Subtracting (9) from (18), we get:

$$\begin{aligned} \frac{1}{\delta t}(\mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n) - \nu \Delta(\mathbf{e}_c^{n+1/2}) = \\ (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1/2} - (\mathbf{u}(t_{n+1}) \cdot \nabla)\mathbf{u}(t_{n+1}) + \mathbf{R}^n - \nabla p(t_{n+1}) \end{aligned} \quad (19)$$

We split the nonlinear terms on the right hand side of (19) into three terms, as in [23]:

$$\begin{aligned} (\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1/2} - (\mathbf{u}(t_{n+1}) \cdot \nabla)\mathbf{u}(t_{n+1}) = \\ - (\mathbf{e}_c^n \cdot \nabla)\mathbf{u}^{n+1/2} + ((\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})) \cdot \nabla)\mathbf{u}^{n+1/2} \\ - (\mathbf{u}(t_{n+1}) \cdot \nabla)\mathbf{e}_c^{n+1/2} \end{aligned} \quad (20)$$

and then take the inner product of (19) with  $2 \delta t \mathbf{e}_c^{n+1/2}$  and use the identity  $(a - b, 2a) = |a|^2 - |b|^2 + |a - b|^2$  to obtain:

$$\begin{aligned} & \| \mathbf{e}_c^{n+1/2} \|_0^2 - \| \mathbf{e}_c^n \|_0^2 + 2 \delta t \nu \| \mathbf{e}_c^{n+1/2} \|_1^2 + \| \mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n \|_0^2 \quad (21) \\ & = 2 \delta t \langle \mathbf{R}^n, \mathbf{e}_c^{n+1/2} \rangle - 2 \delta t (\nabla p(t_{n+1}), \mathbf{e}_c^{n+1/2}) \\ & \quad - 2 \delta t c(\mathbf{e}_c^n, \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1/2}) \\ & \quad + 2 \delta t c(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1/2}) \quad (22) \\ & \quad - 2 \delta t c(\mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}, \mathbf{e}_c^{n+1/2}) \end{aligned}$$

We bound each term in the RHS of (21) independently:

– Taylor residual term:

$$\begin{aligned} 2 \delta t \langle \mathbf{R}^n, \mathbf{e}_c^{n+1/2} \rangle & \leq 2 \delta t \| \mathbf{R}^n \|_{-1} \| \mathbf{e}_c^{n+1/2} \|_1 \\ & = \frac{\delta t \nu}{3} \| \mathbf{e}_c^{n+1/2} \|_1^2 + \frac{C}{\delta t} \left\| \int_{t_n}^{t_{n+1}} (t - t_n) \mathbf{u}_{tt} dt \right\|_{-1}^2 \\ & \leq \frac{\delta t \nu}{3} \| \mathbf{e}_c^{n+1/2} \|_1^2 + C \delta t \int_{t_n}^{t_{n+1}} t \| \mathbf{u}_{tt} \|_{-1}^2 dt \end{aligned}$$

– Pressure gradient term:

$$\begin{aligned} -2 \delta t (\nabla p(t_{n+1}), \mathbf{e}_c^{n+1/2}) & = -2 \delta t (\nabla p(t_{n+1}), \mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n) \\ & \leq \frac{1}{2} \| \mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n \|_0^2 + 2 \delta t^2 \| \nabla p(t_{n+1}) \|_0^2 \end{aligned}$$

since  $\nabla \cdot \mathbf{e}_c^n = 0$ .

– Nonlinear terms:

$$\begin{aligned} -2 \delta t c(\mathbf{e}_c^n, \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1/2}) & = -2 \delta t c(\mathbf{e}_c^n, \mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}) \\ & \leq C \delta t \| \mathbf{e}_c^n \|_0 \| \mathbf{u}(t_{n+1}) \|_2 \| \mathbf{e}_c^{n+1/2} \|_1 \\ & \leq \frac{\delta t \nu}{3} \| \mathbf{e}_c^{n+1/2} \|_1^2 + C \delta t \| \mathbf{e}_c^n \|_0^2 \end{aligned}$$

$$\begin{aligned} & 2 \delta t c(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1/2}) \\ & = 2 \delta t c(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}) \\ & \leq C \delta t \| \mathbf{u}(t_n) - \mathbf{u}(t_{n+1}) \|_0 \| \mathbf{u}(t_{n+1}) \|_2 \| \mathbf{e}_c^{n+1/2} \|_1 \\ & \leq \frac{\delta t \nu}{3} \| \mathbf{e}_c^{n+1/2} \|_1^2 + C \delta t \left\| \int_{t_n}^{t_{n+1}} \mathbf{u}_t dt \right\|_0^2 \\ & \leq \frac{\delta t \nu}{3} \| \mathbf{e}_c^{n+1/2} \|_1^2 + C \delta t^2 \int_{t_n}^{t_{n+1}} \| \mathbf{u}_t \|_0^2 dt \end{aligned}$$

$$-2 \delta t c(\mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}, \mathbf{e}_c^{n+1/2}) = 0$$

where we have used **R1** and the continuity and skew-symmetry properties of the trilinear form  $c$ . From all these inequalities we deduce:

$$\begin{aligned} & \|e_c^{n+1/2}\|_0^2 - \|e_c^n\|_0^2 + \delta t \nu \|e_c^{n+1/2}\|_1^2 + \frac{1}{2} \|e_c^{n+1/2} - e_c^n\|_0^2 \\ & \leq C \delta t \int_{t_n}^{t_{n+1}} t \|\mathbf{u}_{tt}\|_{-1}^2 dt + C \delta t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_0^2 dt \quad (23) \\ & + 2 \delta t^2 \|\nabla p(t_{n+1})\|_0^2 + C \delta t \|e_c^n\|_0^2 \end{aligned}$$

The proof is now different from that of [23]. We rewrite (11) as:

$$\frac{e_c^{n+1} - e_c^{n+1/2}}{\delta t} - \nu \Delta(e_c^{n+1} - e_c^{n+1/2}) - \nabla p^{n+1} = 0 \quad (24)$$

Taking the inner product of (24) with  $2 \delta t e_c^{n+1}$ , given that  $\nabla \cdot e_c^{n+1} = 0$  and that  $e_c^{n+1} = 0$  on  $\partial\Omega$ , we get:

$$\begin{aligned} & \|e_c^{n+1}\|_0^2 - \|e_c^{n+1/2}\|_0^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_0^2 \quad (25) \\ & + \delta t \nu \left( \|e_c^{n+1}\|_1^2 - \|e_c^{n+1/2}\|_1^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_1^2 \right) = 0 \end{aligned}$$

Adding up (23) and (25) for  $n = 0, \dots, N$ , we find:

$$\begin{aligned} & \|e_c^{N+1}\|_0^2 + \sum_{n=0}^N \left\{ \|e_c^{n+1} - e_c^{n+1/2}\|_0^2 + \frac{1}{2} \|e_c^{n+1/2} - e_c^n\|_0^2 \right\} \\ & + \delta t \nu \sum_{n=0}^N \left\{ \|e_c^{n+1}\|_1^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_1^2 \right\} \\ & \leq C \delta t \left( \int_0^T t \|\mathbf{u}_{tt}\|_{-1}^2 dt + \delta t \int_0^T \|\mathbf{u}_t\|_0^2 dt + \sup_{t \in [0, T]} \|\nabla p(t)\|_0^2 \right) \\ & + C \delta t \sum_{n=0}^N \|e_c^n\|_0^2 \end{aligned}$$

Applying the discrete Gronwall lemma to the last inequality and using the regularity properties of the solution  $(\mathbf{u}, p)$ , we obtain:

$$\begin{aligned} & \|e_c^{N+1}\|_0^2 + \sum_{n=0}^N \left\{ \|e_c^{n+1} - e_c^{n+1/2}\|_0^2 + \|e_c^{n+1/2} - e_c^n\|_0^2 \right\} \\ & + \delta t \nu \sum_{n=0}^N \left\{ \|e_c^{n+1}\|_1^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_1^2 \right\} \quad (26) \\ & \leq C \delta t \end{aligned}$$

Finally, the bounds for  $\mathbf{u}^{n+1/2}$  follow from (26) and the triangle inequality, so that (17) is proved.

*Remark 1* Lemma 1 shows, in particular, that the method provides uniformly stable velocities in  $\mathbf{H}_0^1(\Omega)$ , that is to say, that there exists a constant  $C > 0$  independent of the time step  $\delta t$  such that for all  $n = 0, \dots, [T/\delta t] - 1$ :

$$\|\mathbf{u}^{n+1}\|_1 \leq C, \quad \|\mathbf{u}^{n+1/2}\|_1 \leq C \quad (27)$$

since  $\|\mathbf{e}_c^{n+1}\|_1 \leq C$ ,  $\|\mathbf{e}_c^{n+1/2}\|_1 \leq C$  and  $\mathbf{u} \in L^\infty(0, T; \mathbf{H}_0^1(\Omega))$ . Moreover, we also have:

$$\|\mathbf{e}_c^{n+1}\|_0 \leq C \delta t^{1/2}, \quad \|\mathbf{e}_c^{n+1/2}\|_0 \leq C \delta t^{1/2} \quad (28)$$

We will use these bounds later on.

Next we give a first order error estimate for both  $\mathbf{u}^{n+1/2}$  and  $\mathbf{u}^{n+1}$  in the norm of  $l^2(\mathbf{L}^2(\Omega))$ , which is what was proven for the standard projection method in [23] when applied to the (linear) Stokes problem, that is, when dropping the convective term in (1), according to the amendments of [24]:

**Theorem 1** *Assume R1, R2a, R3 and R4 hold; then, for  $N = 0, \dots, [T/\delta t] - 1$  and for small enough  $\delta t$ :*

$$\|\mathbf{e}_c^{N+1}\|_{V'}^2 + \delta t \sum_{n=0}^N \left( \|\mathbf{e}_c^{n+1}\|_0^2 + \|\mathbf{e}_c^{n+1/2}\|_0^2 \right) \leq C \delta t^2 \quad (29)$$

that is,  $\mathbf{u}^{n+1}$  converges to  $\mathbf{u}(t_{n+1})$  in  $l^2(\mathbf{L}^2(\Omega)) \cap l^\infty(V')$  with order  $\delta t$ .

*Proof* By adding (9) and (11), we get:

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\delta t} - \nu \Delta \mathbf{u}^{n+1} + (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1/2} + \nabla p^{n+1} = \mathbf{f}(t_{n+1}) \quad (30)$$

Calling  $r_c^{n+1} = p(t_{n+1}) - p^{n+1}$  the pressure error and subtracting (30) from (18), we have:

$$\begin{aligned} \frac{1}{\delta t} (\mathbf{e}_c^{n+1} - \mathbf{e}_c^n) - \nu \Delta (\mathbf{e}_c^{n+1}) + \nabla r_c^{n+1} \\ = (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1/2} - (\mathbf{u}(t_{n+1}) \cdot \nabla) \mathbf{u}(t_{n+1}) + \mathbf{R}^n \end{aligned} \quad (31)$$

We take the inner product of (31) with  $2 \delta t A^{-1} \mathbf{e}_c^{n+1}$ , as in [23], and use the self-adjointness of  $A^{-1}$  to get:

$$\begin{aligned} (\mathbf{e}_c^{n+1}, A^{-1} \mathbf{e}_c^{n+1}) - (\mathbf{e}_c^n, A^{-1} \mathbf{e}_c^n) + (\mathbf{e}_c^{n+1} - \mathbf{e}_c^n, A^{-1} (\mathbf{e}_c^{n+1} - \mathbf{e}_c^n)) \\ - 2 \delta t \nu (\Delta \mathbf{e}_c^{n+1}, A^{-1} \mathbf{e}_c^{n+1}) \\ = 2 \delta t c(\mathbf{u}^n, \mathbf{u}^{n+1/2}, A^{-1} \mathbf{e}_c^{n+1}) \\ - 2 \delta t c(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), A^{-1} \mathbf{e}_c^{n+1}) \end{aligned} \quad (32)$$

$$+ 2 \delta t \langle \mathbf{R}^n, A^{-1} \mathbf{e}_c^{n+1} \rangle \quad (33)$$

The treatment of the term  $-2\delta t\nu(\Delta\mathbf{e}_c^{n+1}, A^{-1}\mathbf{e}^{n+1})$  is simpler in our case than for the standard projection method. In fact, if we take  $\mathbf{u} = \mathbf{e}_c^{n+1}$  in (6), we have:

$$\begin{aligned} -2\delta t\nu(\Delta\mathbf{e}_c^{n+1}, A^{-1}\mathbf{e}_c^{n+1}) &= 2\delta t\nu(\mathbf{e}_c^{n+1}, -\Delta(A^{-1}\mathbf{e}_c^{n+1})) \\ &= 2\delta t\nu(\mathbf{e}_c^{n+1}, \mathbf{e}_c^{n+1} - \nabla r) \\ &= 2\delta t\nu\|\mathbf{e}_c^{n+1}\|_0^2 \end{aligned}$$

since  $\nabla \cdot \mathbf{e}_c^{n+1} = 0$ . The RHS terms in (33) are bounded as follows. For the Taylor residual term we have:

$$\begin{aligned} 2\delta t \langle \mathbf{R}^n, A^{-1}\mathbf{e}_c^{n+1} \rangle &\leq 2\delta t \|\mathbf{R}^n\|_{V'} \|\mathbf{e}_c^{n+1}\|_1 \\ &\leq C\delta t \|\mathbf{R}^n\|_{V'} \|\mathbf{e}_c^{n+1}\|_{V'} \\ &\leq \delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2 + C\delta t \|\mathbf{R}^n\|_{V'}^2 \\ &\leq \delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2 \\ &\quad + C\delta t^{-1} \int_{t_n}^{t_{n+1}} (t - t_n)^2 dt \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|_{V'}^2 dt \\ &\leq \delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2 + C\delta t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|_{V'}^2 dt \end{aligned}$$

For the nonlinear terms, we use the splitting (20) to express them as:

$$\begin{aligned} &2\delta t \left( c(\mathbf{u}^n, \mathbf{u}^{n+1/2}, A^{-1}\mathbf{e}_c^{n+1}) - c(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), A^{-1}\mathbf{e}_c^{n+1}) \right) \\ &= 2\delta t \left( -c(\mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}, A^{-1}\mathbf{e}_c^{n+1}) \right. \\ &\quad \left. + c(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \mathbf{u}^{n+1/2}, A^{-1}\mathbf{e}_c^{n+1}) \right. \\ &\quad \left. - c(\mathbf{e}_c^n, \mathbf{u}^{n+1/2}, A^{-1}\mathbf{e}_c^{n+1}) \right) \end{aligned}$$

which we call I, II and III, respectively. Then:

$$\begin{aligned} \text{I} &\leq C\delta t \|\mathbf{u}(t_{n+1})\|_2 \|\mathbf{e}_c^{n+1}\|_1 \|\mathbf{e}_c^{n+1/2}\|_0 \\ &\leq C\delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2 + \frac{\delta t\nu}{4} \|\mathbf{e}_c^{n+1/2}\|_0^2 \\ &= C\delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2 + \frac{\delta t\nu}{4} \left\{ \|\mathbf{e}_c^{n+1}\|_0^2 + \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_0^2 \right. \\ &\quad \left. + \delta t\nu \|\mathbf{e}_c^{n+1}\|_1^2 + \delta t\nu \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_1^2 - \delta t\nu \|\mathbf{e}_c^{n+1/2}\|_1^2 \right\} \end{aligned}$$

where we have used (5) and (25);

$$\begin{aligned} \text{II} &\leq C\delta t \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_0 \|\mathbf{u}^{n+1/2}\|_1 \|\mathbf{e}_c^{n+1}\|_2 \\ &\leq C\delta t \left\| \int_{t_n}^{t_{n+1}} \mathbf{u}_t dt \right\|_0 \|\mathbf{e}_c^{n+1}\|_0 \\ &\leq C\delta t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_0^2 dt + \frac{\delta t\nu}{4} \|\mathbf{e}_c^{n+1}\|_0^2 \end{aligned}$$

where we have used the bound (27);

$$\begin{aligned} \text{III} &= 2 \delta t c(\mathbf{e}_c^n, A^{-1} \mathbf{e}_c^{n+1}, \mathbf{u}(t_{n+1})) - 2 \delta t c(\mathbf{e}_c^n, A^{-1} \mathbf{e}_c^{n+1}, \mathbf{e}_c^{n+1/2}) \\ &= \text{III}_a + \text{III}_b \end{aligned}$$

so that:

$$\begin{aligned} \text{III}_a &\leq C \delta t \|\mathbf{e}_c^n\|_0 \|A^{-1} \mathbf{e}_c^{n+1}\|_1 \|\mathbf{u}(t_{n+1})\|_2 \\ &\leq C \delta t \|\mathbf{e}_c^n\|_0 \|\mathbf{e}_c^{n+1}\|_{V'} \\ &\leq C \delta t \left( \|\mathbf{e}_c^{n+1}\|_0 + \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_0 + \|\mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n\|_0 \right) \|\mathbf{e}_c^{n+1}\|_{V'} \\ &\leq \frac{\delta t \nu}{4} \|\mathbf{e}_c^{n+1}\|_0^2 + C \delta t \left( \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_0^2 + \|\mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n\|_0^2 \right) \\ &\quad + C \delta t \|\mathbf{e}_c^{n+1}\|_{V'}^2, \end{aligned}$$

due to **R1** and the triangle inequality; finally:

$$\begin{aligned} \text{III}_b &\leq C \delta t \|\mathbf{e}_c^n\|_0 \|A^{-1} \mathbf{e}_c^{n+1}\|_2 \|\mathbf{e}_c^{n+1/2}\|_1 \\ &\leq C \delta t \|\mathbf{e}_c^n\|_0 \|\mathbf{e}_c^{n+1}\|_0 \|\mathbf{e}_c^{n+1/2}\|_1 \\ &\leq C \delta t^{3/2} \|\mathbf{e}_c^{n+1}\|_0 \|\mathbf{e}_c^{n+1/2}\|_1 \\ &\leq \frac{\delta t \nu}{4} \|\mathbf{e}_c^{n+1}\|_0^2 + C \delta t^2 \|\mathbf{e}_c^{n+1/2}\|_1^2 \end{aligned}$$

where we have used (28). Adding up (33) for  $n = 0, \dots, N$ , and using all these inequalities, we get:

$$\begin{aligned} (\mathbf{e}_c^{N+1}, A^{-1} \mathbf{e}_c^{N+1}) &+ \sum_{n=0}^N (\mathbf{e}_c^{n+1} - \mathbf{e}_c^n, A^{-1} (\mathbf{e}_c^{n+1} - \mathbf{e}_c^n)) + \delta t \nu \sum_{n=0}^N \|\mathbf{e}_c^{n+1}\|_0^2 \\ &\leq C \delta t^2 \int_0^T \|\mathbf{u}_{tt}\|_{V'}^2 dt + C \delta t^2 \int_0^T \|\mathbf{u}_t\|_0^2 dt \\ &\quad + C \delta t \sum_{n=0}^N \|\mathbf{e}_c^{n+1}\|_{V'}^2 + C \delta t^2 \sum_{n=0}^N \|\mathbf{e}_c^{n+1}\|_1^2 \\ &\quad + C \delta t \sum_{n=0}^N \{ \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_0^2 + \|\mathbf{e}_c^{n+1/2} - \mathbf{e}_c^n\|_0^2 \} \\ &\quad + C \delta t^2 \sum_{n=0}^N \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^{n+1/2}\|_1^2 + C \delta t^2 \sum_{n=0}^N \|\mathbf{e}_c^{n+1/2}\|_1^2 \end{aligned}$$

Using now (8), the regularity properties **R2a** and **R4** of the continuous solution and the estimates of Lemma 1, we get:

$$\|\mathbf{e}_c^{N+1}\|_{V'}^2 + \sum_{n=0}^N \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^n\|_{V'}^2 + \delta t \nu \sum_{n=0}^N \|\mathbf{e}_c^{n+1}\|_0^2$$

$$\leq C \delta t^2 + C \delta t \sum_{n=0}^N \|e_c^{n+1}\|_V^2,$$

For sufficiently small  $\delta t$ , we can apply the discrete Gronwall lemma to the last inequality, and we get:

$$\begin{aligned} \|e_c^{N+1}\|_V^2 + \sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_V^2 + \delta t \nu \sum_{n=0}^N \|e_c^{n+1}\|_0^2 \\ \leq C \delta t^2 \end{aligned} \quad (34)$$

and the estimate for  $\mathbf{u}^{n+1}$  is proved. For  $\mathbf{u}^{n+1/2}$ , we have:

$$\begin{aligned} \delta t \nu \sum_{n=0}^N \|e_c^{n+1/2}\|_0^2 &\leq 2 \delta t \nu \sum_{n=0}^N \left( \|e_c^{n+1}\|_0^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_0^2 \right) \\ &\leq C \delta t^2 \end{aligned}$$

due to (34) and Lemma 1, so that (29) is proved.

The error estimates of Theorem 1 can be improved to first order in the norms of  $l^\infty(\mathbf{L}^2(\Omega))$  and  $l^2(\mathbf{H}_0^1(\Omega))$  for the end-of-step velocities  $\mathbf{u}^{n+1}$  assuming some slightly stronger regularity on the continuous solution, namely, **R2b** rather than **R2a**. Estimates in these norms were also obtained in [15] for the intermediate velocities of a fully discrete, incremental version of the fractional step projection method, assuming a finite element spatial discretization satisfying the discrete inf-sup condition and under much stronger regularity assumptions on the continuous solution:

**Theorem 2** *Assume that **R1**, **R2b**, **R3** and **R4** hold; then, for  $N = 0, \dots, [T/\delta t] - 1$ , and for small enough  $\delta t$ :*

$$\|e_c^{N+1}\|_0^2 + \delta t \nu \sum_{n=0}^N \|e_c^{n+1}\|_1^2 \leq C \delta t^2 \quad (35)$$

that is,  $\mathbf{u}^{n+1}$  converges to  $\mathbf{u}(t_{n+1})$  in  $l^2(\mathbf{H}_0^1(\Omega)) \cap l^\infty(\mathbf{L}^2(\Omega))$  with order  $\delta t$ .

*Proof* Unlike for the standard projection method, we can take the inner product of (31) with  $2\delta t e_c^{n+1}$ , since in our case  $e_c^{n+1} \in V$ , to get:

$$\begin{aligned} \|e_c^{n+1}\|_0^2 - \|e_c^n\|_0^2 + \|e_c^{n+1} - e_c^n\|_0^2 + 2 \delta t \nu \|e_c^{n+1}\|_1^2 \\ = 2 \delta t c(\mathbf{u}^n, \mathbf{u}^{n+1/2}, e_c^{n+1}) - 2 \delta t c(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), e_c^{n+1}) \\ + 2 \delta t \langle \mathbf{R}^n, e_c^{n+1} \rangle \end{aligned} \quad (36)$$

The RHS terms in (36) are bounded as follows. For the Taylor residual term we have:

$$\begin{aligned}
2\delta t \langle \mathbf{R}^n, \mathbf{e}_c^{n+1} \rangle &\leq 2\delta t \|\mathbf{R}^n\|_{V'} \|\mathbf{e}_c^{n+1}\|_1 \\
&\leq \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2 + C\delta t \|\mathbf{R}^n\|_{V'}^2 \\
&\leq \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2 + C\delta t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_{tt}\|_{V'}^2 dt
\end{aligned}$$

For the nonlinear terms, we use again the splitting (20) to express them as:

$$\begin{aligned}
&2\delta t \left( c(\mathbf{u}^n, \overline{\mathbf{u}^{n+1/2}}, \mathbf{e}_c^{n+1}) - c(\mathbf{u}(t_{n+1}), \mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1}) \right) \\
&= 2\delta t \left\{ -c(\mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1/2}, \mathbf{e}_c^{n+1}) + c(\mathbf{u}(t_n) - \mathbf{u}(t_{n+1}), \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1}) \right. \\
&\quad \left. - c(\mathbf{e}_c^n, \mathbf{u}^{n+1/2}, \mathbf{e}_c^{n+1}) \right\}
\end{aligned}$$

which we call again I, II and III, respectively; then:

$$\begin{aligned}
\text{I} &\leq C\delta t \|\mathbf{u}(t_{n+1})\|_2 \|\mathbf{e}_c^{n+1}\|_1 \|\mathbf{e}_c^{n+1/2}\|_0 \\
&\leq \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2 + C\delta t \|\mathbf{e}_c^{n+1/2}\|_0^2 \\
\text{II} &\leq C\delta t \|\mathbf{u}(t_n) - \mathbf{u}(t_{n+1})\|_1 \|\mathbf{u}^{n+1/2}\|_1 \|\mathbf{e}_c^{n+1}\|_1 \\
&= C\delta t \left\| \int_{t_n}^{t_{n+1}} \mathbf{u}_t dt \right\|_1 \|\mathbf{e}_c^{n+1}\|_1 \\
&\leq C\delta t^2 \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_1^2 dt + \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2 \\
\text{III} &= 2\delta t c(\mathbf{e}_c^n, \mathbf{e}_c^{n+1/2}, \mathbf{e}_c^{n+1}) - 2\delta t c(\mathbf{e}_c^n, \mathbf{u}(t_{n+1}), \mathbf{e}_c^{n+1}) \\
&= \text{IIIa} + \text{IIIb}
\end{aligned}$$

so that:

$$\begin{aligned}
\text{IIIa} &\leq C\delta t \|\mathbf{e}_c^n\|_1 \|\mathbf{e}_c^{n+1}\|_1 \|\mathbf{e}_c^{n+1/2}\|_0^{1/2} \|\mathbf{e}_c^{n+1/2}\|_1^{1/2} \\
&\leq C\delta t \|\mathbf{e}_c^n\|_1 \|\mathbf{e}_c^{n+1}\|_1 \|\mathbf{e}_c^{n+1/2}\|_0^{1/2} \\
&\leq C\delta t^{5/4} \|\mathbf{e}_c^n\|_1 \|\mathbf{e}_c^{n+1}\|_1 \\
&\leq C\delta t^{3/2} \nu \|\mathbf{e}_c^n\|_1^2 + \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2 \\
\text{IIIb} &\leq C\delta t \|\mathbf{e}_c^n\|_0 \|\mathbf{u}(t_{n+1})\|_2 \|\mathbf{e}_c^{n+1}\|_1 \\
&\leq C\delta t \|\mathbf{e}_c^n\|_0^2 + \frac{\delta t \nu}{5} \|\mathbf{e}_c^{n+1}\|_1^2
\end{aligned}$$

where we have used (28) and the continuity properties of the trilinear form  $c$ . Adding up (36) for  $n = 0, \dots, N$ , taking into account (25) for the term I, and the previous inequalities, we get:

$$\begin{aligned} & \|e_c^{N+1}\|_0^2 + \sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_0^2 + \delta t \nu \sum_{n=0}^N \|e_c^{n+1}\|_1^2 + C \delta t^2 \nu \sum_{n=0}^N \|e_c^{n+1/2}\|_1^2 \\ & \leq C \delta t^2 \int_0^T \|\mathbf{u}_{tt}\|_{V'}^2 dt + C \delta t^2 \int_0^T \|\mathbf{u}_t\|_1^2 dt \\ & + C \delta t \sum_{n=0}^N \|e_c^{n+1}\|_0^2 + C \delta t \sum_{n=0}^N \|e_c^{n+1} - e_c^{n+1/2}\|_0^2 \\ & + C \delta t^2 \sum_{n=0}^N \{\|e_c^{n+1}\|_1^2 + \|e_c^{n+1} - e_c^{n+1/2}\|_1^2\} + C \delta t^{3/2} \nu \sum_{n=0}^N \|e_c^n\|_1^2 \end{aligned}$$

Using the regularity properties of the solution **R2b** and **R4** and the estimates of Lemma 1, we get:

$$\begin{aligned} & \|e_c^{N+1}\|_0^2 + \sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_0^2 + \delta t \nu \sum_{n=0}^N \|e_c^{n+1}\|_1^2 + C \delta t^2 \nu \sum_{n=0}^N \|e_c^{n+1/2}\|_1^2 \\ & \leq C \delta t^2 + C \delta t \sum_{n=0}^N \|e_c^{n+1}\|_0^2 + C \delta t^{3/2} \nu \sum_{n=0}^N \|e_c^n\|_1^2 \end{aligned}$$

For sufficiently small  $\delta t$ , we can apply the discrete Gronwall lemma to the last inequality and take the last term to the left-hand-side, to get:

$$\|e_c^{N+1}\|_0^2 + \sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_0^2 + \delta t \nu \sum_{n=0}^N \|e_c^{n+1}\|_1^2 \leq C \delta t^2$$

and (29) is proved.

#### 4.2 Error estimates for the semidiscrete pressure

As a side product of the estimates of Theorem 2, we obtain order 1/2 error estimates for the pressure approximation in  $l^2(L_0^2(\Omega))$ , which is what one can expect for the present scheme. We first recall a technical result, similar to that of Lemma A1 in [24]. In Theorem 2 we have proved, in particular, that:

$$\sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_0^2 \leq C \delta t^2$$

This implies that:

$$\sum_{n=0}^N \|e_c^{n+1} - e_c^n\|_{-1}^2 \leq C \delta t^2 \quad (37)$$

since for all  $\mathbf{v} \in \mathbf{L}^2(\Omega)$ ,  $\|\mathbf{v}\|_{-1} \leq \|\mathbf{v}\|_0$ . This is what we actually use to prove the following error estimate for the pressure:

**Theorem 3** *Assume that **R1**, **R2b**, **R3** and **R4** hold; then, for  $N = 0, \dots, [T/\delta t] - 1$  and for small enough  $\delta t$ :*

$$\delta t \sum_{n=0}^N \|p(t_{n+1}) - p^{n+1}\|_{L_0^2(\Omega)}^2 \leq C \delta t \quad (38)$$

that is,  $p^{n+1}$  converges to  $p(t_{n+1})$  in  $l^2(L_0^2(\Omega))$  with order  $\delta t^{1/2}$ .

*Proof* We rewrite (31) as:

$$\begin{aligned} -\nabla r_c^{n+1} &= \frac{1}{\delta t} (e_c^{n+1} - e_c^n) - \nu \Delta(e_c^{n+1}) - \mathbf{R}^n \\ &\quad - (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1/2} + (\mathbf{u}(t_{n+1}) \cdot \nabla) \mathbf{u}(t_{n+1}) \end{aligned} \quad (39)$$

Using the continuous LBB condition:

$$\|r_c^{n+1}\|_{L_0^2(\Omega)} \leq C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\nabla r_c^{n+1}, \mathbf{v})}{\|\mathbf{v}\|_1} \quad (40)$$

we need to bound the products of the RHS of (39) with an arbitrary  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ . We have:

$$\begin{aligned} \frac{1}{\delta t} (e_c^{n+1} - e_c^n, \mathbf{v}) &\leq \frac{1}{\delta t} \|e_c^{n+1} - e_c^n\|_{-1} \|\mathbf{v}\|_1 \\ \langle -\nu \Delta(e_c^{n+1}), \mathbf{v} \rangle &= \nu \|e_c^{n+1}\|_1 \|\mathbf{v}\|_1 \\ \langle -\mathbf{R}^n, \mathbf{v} \rangle &\leq \|\mathbf{R}^n\|_{-1} \|\mathbf{v}\|_1 \leq C \left( \int_{t_n}^{t_{n+1}} t \|\mathbf{u}_{tt}\|_{-1}^2 dt \right)^{1/2} \|\mathbf{v}\|_1 \end{aligned}$$

For the nonlinear terms, we use the following splitting:

$$\begin{aligned} & - (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1/2} + (\mathbf{u}(t_{n+1}) \cdot \nabla) \mathbf{u}(t_{n+1}) \\ &= \left( (\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) \cdot \nabla \right) \mathbf{u}(t_{n+1}) + (e_c^n \cdot \nabla) \mathbf{u}(t_{n+1}) \\ &+ (\mathbf{u}^n \cdot \nabla) e_c^{n+1/2} \end{aligned} \quad (41)$$

Calling I, II and III the three terms obtained after testing (41) with  $\mathbf{v}$ , we have:

$$\begin{aligned} \text{I} &\leq C \|\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)\|_0 \|\mathbf{u}(t_{n+1})\|_2 \|\mathbf{v}\|_1 \\ &\leq C (\delta t \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_0^2 dt)^{1/2} \|\mathbf{v}\|_1 \end{aligned}$$

$$\text{II} \leq C \|\mathbf{e}_c^n\|_1 \|\mathbf{u}(t_{n+1})\|_1 \|\mathbf{v}\|_1 \leq C \|\mathbf{e}_c^n\|_1 \|\mathbf{v}\|_1$$

$$\text{III} \leq C \|\mathbf{u}^n\|_1 \|\mathbf{e}_c^{n+1/2}\|_1 \|\mathbf{v}\|_1 \leq C \|\mathbf{e}_c^{n+1/2}\|_1 \|\mathbf{v}\|_1$$

where we have used **R1** and (27). Thus, we obtain:

$$\begin{aligned} \|r_c^{n+1}\|_{L_0^2(\Omega)} &\leq \frac{C}{\delta t} \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^n\|_{-1} \\ &\quad + C \left\{ \|\mathbf{e}_c^{n+1}\|_1 + \|\mathbf{e}_c^n\|_1 + \|\mathbf{e}_c^{n+1/2}\|_1 \right. \\ &\quad \left. + \left( \int_{t_n}^{t_{n+1}} t \|\mathbf{u}_{tt}\|_{-1}^2 dt \right)^{1/2} + \left( \delta t \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_0^2 dt \right)^{1/2} \right\} \end{aligned}$$

which yields:

$$\begin{aligned} \|r_c^{n+1}\|_{L_0^2(\Omega)}^2 &\leq \frac{C}{\delta t^2} \|\mathbf{e}_c^{n+1} - \mathbf{e}_c^n\|_{-1}^2 \\ &\quad + C \left\{ \|\mathbf{e}_c^{n+1}\|_1^2 + \|\mathbf{e}_c^n\|_1^2 + \|\mathbf{e}_c^{n+1/2}\|_1^2 \right. \\ &\quad \left. + \int_{t_n}^{t_{n+1}} t \|\mathbf{u}_{tt}\|_{-1}^2 dt + \delta t \int_{t_n}^{t_{n+1}} \|\mathbf{u}_t\|_0^2 dt \right\} \end{aligned}$$

and (38) results from (37), the regularity properties **R3** and **R2a** (which is implied by **R2b**) of the continuous solution  $\mathbf{u}$ , and the estimates of Lemma 1.

#### 4.3 Error estimates for the fully discrete solution

We finally present an error analysis for the fully discrete, finite element solution  $(\mathbf{u}_h^{n+1/2}, \mathbf{u}_h^{n+1}, p_h^{n+1})$  as an approximation of the semidiscrete, fractional-step solution  $(\mathbf{u}^{n+1/2}, \mathbf{u}^{n+1}, p^{n+1})$ . We define the ‘discrete’ errors as:

$$\begin{aligned} \mathbf{e}_d^{n+1} &= \mathbf{u}^{n+1} - \mathbf{u}_h^{n+1} \\ \mathbf{e}_d^{n+1/2} &= \mathbf{u}^{n+1/2} - \mathbf{u}_h^{n+1/2} \\ r_d^{n+1} &= p^{n+1} - p_h^{n+1} \end{aligned}$$

We also use the following notation for the error functions associated to the finite element spaces used:

$$\begin{aligned}
E_n(h) &= \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1 + \frac{1}{h} \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_0 \\
&\quad + \inf_{q_h \in Q_h} \|p^{n+1} - q_h\|_0 \\
&\quad + \inf_{\mathbf{w}_h \in \text{Ker}B_h} \|\mathbf{u}^{n+1} - \mathbf{w}_h\|_1 + \frac{1}{h} \inf_{\mathbf{w}_h \in \text{Ker}B_h} \|\mathbf{u}^{n+1} - \mathbf{w}_h\|_0 \\
E(h) &= \max_{n=0, \dots, N} E_n(h)
\end{aligned}$$

It is well known that under the discrete inf-sup condition **H1**, optimal order approximation both in  $H_0^1(\Omega)$  and in  $L^2(\Omega)$  of solenoidal vector fields can be achieved by means of discretely divergence free finite element functions, that is, functions  $\mathbf{w}_h$  in  $\text{Ker}B_h$ . In fact, one has the following result:

**Lemma 2** *Let  $\mathbf{u} \in V$  and assume that the discrete spaces  $V_h$  and  $Q_h$  satisfy the inf-sup condition **H1**; then:*

$$\inf_{\mathbf{w}_h \in \text{Ker}B_h} \|\mathbf{u} - \mathbf{w}_h\|_1 \leq C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1$$

Moreover, if  $\Omega$  is of class  $C^2$ , so that the inverse of the Stokes operator  $A^{-1}$  verifies the shift (7) with  $s = 2$ , then:

$$\inf_{\mathbf{w}_h \in \text{Ker}B_h} \|\mathbf{u} - \mathbf{w}_h\|_0 \leq C h \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1$$

*Proof* Let us consider the following Stokes problem: find  $(\mathbf{y}, a) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that, for all  $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ :

$$\begin{aligned}
(\nabla \mathbf{y}, \nabla \mathbf{v}) - (a, \nabla \cdot \mathbf{v}) &= (\nabla \mathbf{u}, \nabla \mathbf{v}) \\
(q, \nabla \cdot \mathbf{y}) &= 0
\end{aligned}$$

Given  $\mathbf{u} \in V$ ,  $(\nabla \mathbf{u}, \nabla \mathbf{v})$  is a linear, continuous functional on  $\mathbf{H}_0^1(\Omega)$ , and thus this problem admits a unique solution  $(\mathbf{y}, a) = (\mathbf{u}, 0)$ . We next consider the finite element discrete problem of finding  $(\mathbf{y}_h, a_h) \in V_h \times Q_h$  such that, for all  $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ :

$$\begin{aligned}
(\nabla \mathbf{y}_h, \nabla \mathbf{v}_h) - (a_h, \nabla \cdot \mathbf{v}_h) &= (\nabla \mathbf{u}, \nabla \mathbf{v}_h) \\
(q_h, \nabla \cdot \mathbf{y}_h) &= 0
\end{aligned}$$

Standard approximation results for this problem (see, for instance, [11]) allow us to conclude that:

$$\inf_{\mathbf{w}_h \in \text{Ker}B_h} \|\mathbf{u} - \mathbf{w}_h\|_1 \leq \|\mathbf{u} - \mathbf{y}_h\|_1 \leq C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1$$

and, if  $\Omega$  is regular enough, that:

$$\inf_{\mathbf{w}_h \in \text{Ker} B_h} \|\mathbf{u} - \mathbf{w}_h\|_0 \leq \|\mathbf{u} - \mathbf{y}_h\|_0 \leq Ch \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_1$$

These results ensure that the error functions  $E_n(h)$  have an optimal order behaviour with respect to the mesh size  $h$ . Our error estimates for the fully discrete solution are given next:

**Theorem 4** *Assume that **R1**, **R2b**, **R3**, **H1** and **H3** hold; then for  $N = 0, \dots, [T/\delta t] - 1$ , and for small enough  $\delta t$  and  $h$ :*

$$\begin{aligned} \|\mathbf{e}_d^{N+1}\|_0^2 + \|\mathbf{e}_d^{N+1/2}\|_0^2 + \delta t \nu \sum_{n=0}^N \left\{ \|\mathbf{e}_d^{n+1}\|_1^2 + \|\mathbf{e}_d^{n+1/2}\|_1^2 \right\} \\ \leq C (E(h))^2 \end{aligned} \quad (42)$$

*Proof* From (9)-(11)-(12) and (14)-(15)-(16), we get, for all  $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ :

$$\frac{1}{\delta t} (\mathbf{e}_d^{n+1/2} - \mathbf{e}_d^n, \mathbf{v}_h) + \nu (\nabla \mathbf{e}_d^{n+1/2}, \nabla \mathbf{v}_h) = \quad (43)$$

$$\tilde{c}(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{v}_h) - \tilde{c}(\mathbf{u}^n, \mathbf{u}^{n+1/2}, \mathbf{v}_h)$$

$$\frac{1}{\delta t} (\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}, \mathbf{v}_h) + \nu (\nabla (\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}), \nabla \mathbf{v}_h) \quad (44)$$

$$- (r_d^{n+1}, \nabla \cdot \mathbf{v}_h) = 0$$

$$(\nabla \cdot \mathbf{e}_d^{n+1}, q_h) = 0 \quad (45)$$

Given  $(\mathbf{v}_h, \mathbf{w}_h, q_h) \in V_h \times \text{Ker}(B_h) \times Q_h$  arbitrary, from (43)-(44)-(45) we have:

$$\begin{aligned} \|\mathbf{e}_d^{n+1/2}\|_0^2 - \|\mathbf{e}_d^n\|_0^2 + \|\mathbf{e}_d^{n+1/2} - \mathbf{e}_d^n\|_0^2 + 2\nu \delta t \|\nabla \mathbf{e}_d^{n+1/2}\|_0^2 \\ + \|\mathbf{e}_d^{n+1}\|_0^2 - \|\mathbf{e}_d^{n+1/2}\|_0^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_0^2 \\ + \nu \delta t \left\{ \|\nabla \mathbf{e}_d^{n+1}\|_0^2 - \|\nabla \mathbf{e}_d^{n+1/2}\|_0^2 + \|\nabla (\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2})\|_0^2 \right\} \\ = 2 (\mathbf{e}_d^{n+1/2} - \mathbf{e}_d^n, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \\ + 2\nu \delta t (\nabla \mathbf{e}_d^{n+1/2}, \nabla (\mathbf{u}^{n+1/2} - \mathbf{v}_h)) \\ - 2\delta t \tilde{c}(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{u}_h^{n+1/2} - \mathbf{v}_h) \\ + 2\delta t \tilde{c}(\mathbf{u}^n, \mathbf{u}^{n+1/2}, \mathbf{u}_h^{n+1/2} - \mathbf{v}_h) \\ + 2(\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}, \mathbf{u}^{n+1} - \mathbf{w}_h) \\ + 2\nu \delta t (\nabla (\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}), \nabla (\mathbf{u}^{n+1} - \mathbf{w}_h)) \\ - 2\delta t (r_d^{n+1}, \nabla \cdot (\mathbf{u}^{n+1} - \mathbf{w}_h)) \\ + 2\delta t (\nabla \cdot \mathbf{e}_d^{n+1}, p^{n+1} - q_h) \end{aligned}$$

We bound each term in the RHS as follows:

$$\begin{aligned}
2(e_d^{n+1/2} - e_d^n, \mathbf{u}^{n+1/2} - \mathbf{v}_h) &\leq \frac{1}{2} \|e_d^{n+1/2} - e_d^n\|_0^2 + C \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_0^2 \\
2\nu \delta t (\nabla e_d^{n+1/2}, \nabla(\mathbf{u}^{n+1/2} - \mathbf{v}_h)) &\leq 2\nu \delta t \|\nabla e_d^{n+1/2}\|_0 \|\nabla(\mathbf{u}^{n+1/2} - \mathbf{v}_h)\|_0 \\
&\leq \frac{\nu \delta t}{7} (\|\nabla e_d^{n+1}\|_0^2 + \|\nabla(e_d^{n+1} - e_d^{n+1/2})\|_0^2) \\
&\quad + C \delta t \|\nabla(\mathbf{u}^{n+1/2} - \mathbf{v}_h)\|_0^2 \\
2(e_d^{n+1} - e_d^{n+1/2}, \mathbf{u}^{n+1} - \mathbf{w}_h) &\leq \frac{1}{2} \|e_d^{n+1} - e_d^{n+1/2}\|_0^2 + C \|\mathbf{u}^{n+1} - \mathbf{w}_h\|_0^2 \\
2\nu \delta t (\nabla(e_d^{n+1} - e_d^{n+1/2}), \nabla(\mathbf{u}^{n+1} - \mathbf{w}_h)) &\leq \frac{\nu \delta t}{7} \|\nabla(e_d^{n+1} - e_d^{n+1/2})\|_0^2 \\
&\quad + C \delta t \|\nabla(\mathbf{u}^{n+1} - \mathbf{w}_h)\|_0^2 \\
-2\delta t (r_d^{n+1}, \nabla \cdot (\mathbf{u}^{n+1} - \mathbf{w}_h)) &= -2\delta t (p^{n+1} - q_h, \nabla \cdot (\mathbf{u}^{n+1} - \mathbf{w}_h)) \\
&\quad + 2\delta t (p_h^{n+1} - q_h, \nabla \cdot (\mathbf{u}^{n+1} - \mathbf{w}_h)) \\
&\leq \delta t \|p^{n+1} - q_h\|_0^2 + C \delta t \|\mathbf{u}^{n+1} - \mathbf{w}_h\|_1^2
\end{aligned}$$

since  $\nabla \cdot \mathbf{u}^{n+1} = 0$  and we have taken  $\mathbf{w}_h$  in  $\text{Ker}(B_h)$ . Moreover:

$$2\delta t (\nabla \cdot e_d^{n+1}, p^{n+1} - q_h) \leq C \delta t \|p^{n+1} - q_h\|_0^2 + \frac{\nu \delta t}{7} \|\nabla e_d^{n+1}\|_0^2$$

The nonlinear terms are treated as follows:

$$\begin{aligned}
&2\delta t \left( -\tilde{c}(\mathbf{u}_h^n, \mathbf{u}_h^{n+1/2}, \mathbf{u}_h^{n+1/2} - \mathbf{v}_h) + \tilde{c}(\mathbf{u}^n, \mathbf{u}^{n+1/2}, \mathbf{u}_h^{n+1/2} - \mathbf{v}_h) \right) \\
&= 2\delta t \left( -\tilde{c}(\mathbf{u}_h^n, e_d^{n+1/2}, e_d^{n+1/2}) + \tilde{c}(\mathbf{u}_h^n, e_d^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \right. \\
&\quad \left. - \tilde{c}(e_d^n, \mathbf{u}^{n+1/2}, e_d^{n+1/2}) + \tilde{c}(e_d^n, \mathbf{u}^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \right)
\end{aligned}$$

The first term in the RHS is zero, due to the skew-symmetry of the trilinear form  $\tilde{c}$ . For the second one, we have:

$$\begin{aligned}
&2\delta t \tilde{c}(\mathbf{u}_h^n, e_d^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \\
&= -2\delta t \tilde{c}(e_d^n, e_d^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) + 2\delta t \tilde{c}(\mathbf{u}^n, e_d^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \\
&\leq C \nu \delta t \left( \|e_d^n\|_1 \|e_d^{n+1/2}\|_1 \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1 \right. \\
&\quad \left. + \|\mathbf{u}^n\|_1 \|e_d^{n+1/2}\|_1 \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1 \right) \\
&\leq C \nu \delta t \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1 \left( \|e_d^n\|_1^2 + \|e_d^{n+1}\|_1^2 + \|e_d^{n+1} - e_d^{n+1/2}\|_1^2 \right) \\
&\quad + \frac{\nu \delta t}{7} \left( \|\nabla e_d^{n+1}\|_0^2 + \|\nabla(e_d^{n+1} - e_d^{n+1/2})\|_0^2 \right) + C \delta t \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1^2
\end{aligned}$$

since, according to Remark 1,  $\|\mathbf{u}^n\|_1 \leq C$ ; furthermore:

$$\begin{aligned}
& -2\delta t \tilde{c}(\mathbf{e}_d^n, \mathbf{u}^{n+1/2}, \mathbf{e}_d^{n+1/2}) \\
& = 2\delta t \left( \tilde{c}(\mathbf{e}_d^n, \mathbf{e}_c^{n+1/2}, \mathbf{e}_d^{n+1/2}) - \tilde{c}(\mathbf{e}_d^n, \mathbf{u}(t_{n+1}), \mathbf{e}_d^{n+1/2}) \right) \\
& \leq C\delta t \|\mathbf{e}_d^n\|_0^{1/2} \|\mathbf{e}_d^n\|_1^{1/2} \|\mathbf{e}_c^{n+1/2}\|_1 \|\mathbf{e}_d^{n+1/2}\|_1 \\
& + C\delta t \|\mathbf{e}_d^n\|_0 \|\mathbf{u}(t_{n+1})\|_2 \|\mathbf{e}_d^{n+1/2}\|_1 \\
& \leq C\delta t^{3/2} \|\mathbf{e}_d^n\|_0^{1/2} \|\mathbf{e}_d^n\|_1^{1/2} \|\mathbf{e}_d^{n+1/2}\|_1 + C\delta t \|\mathbf{e}_d^n\|_0 \|\mathbf{e}_d^{n+1/2}\|_1 \\
& \leq C\delta t^2 \|\mathbf{e}_d^n\|_0 \|\mathbf{e}_d^n\|_1 + \frac{\nu\delta t}{7} \left( \|\nabla \mathbf{e}_d^{n+1}\|_0^2 + \|\nabla(\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2})\|_0^2 \right) \\
& + C\delta t \|\mathbf{e}_d^n\|_0^2 + \frac{\nu\delta t}{7} \left( \|\nabla \mathbf{e}_d^{n+1}\|_0^2 + \|\nabla(\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2})\|_0^2 \right) \\
& \leq C\delta t \|\mathbf{e}_d^n\|_0^2 + \nu\delta t^3 \|\mathbf{e}_d^n\|_1^2 \\
& + \frac{\nu\delta t}{7} \left( \|\nabla \mathbf{e}_d^{n+1}\|_0^2 + \|\nabla(\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2})\|_0^2 \right)
\end{aligned}$$

where we have used the continuity properties of the trilinear form  $\tilde{c}$ , the bound  $\|\mathbf{e}_c^{n+1/2}\|_1 \leq C\delta t^{1/2}$ , which follows from Theorem 2, and the regularity property **R1** of the continuous solution. In a similar way, it can be shown that:

$$\begin{aligned}
& 2\delta t \tilde{c}(\mathbf{e}_d^n, \mathbf{u}^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) \\
& = -2\delta t \left( \tilde{c}(\mathbf{e}_d^n, \mathbf{e}_c^{n+1/2}, \mathbf{u}^{n+1/2} - \mathbf{v}_h) + \tilde{c}(\mathbf{e}_d^n, \mathbf{u}(t_{n+1}), \mathbf{u}^{n+1/2} - \mathbf{v}_h) \right) \\
& \leq C\delta t \|\mathbf{e}_d^n\|_0^2 + \nu\delta t^3 \|\mathbf{e}_d^n\|_1^2 + C\delta t \|\mathbf{u}^{n+1/2} - \mathbf{v}_h\|_1^2
\end{aligned}$$

Combining all the above inequalities and taking the infimum with respect to  $(\mathbf{v}_h, \mathbf{w}_h, q_h) \in V_h \times \text{Ker}B_h \times Q_h$ , we get:

$$\begin{aligned}
& \|\mathbf{e}_d^{n+1}\|_0^2 - \|\mathbf{e}_d^n\|_0^2 + \|\mathbf{e}_d^{n+1/2} - \mathbf{e}_d^n\|_0^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_0^2 \\
& + \nu\delta t \left( \|\mathbf{e}_d^{n+1}\|_1^2 + \|\mathbf{e}_d^{n+1/2}\|_1^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_1^2 \right) \\
& \leq Ch^2 (E_n(h))^2 + C\delta t (E_n(h))^2 + C\delta t \|\mathbf{e}_d^n\|_0^2 + \nu\delta t^3 \|\mathbf{e}_d^n\|_1^2 \\
& + C\nu\delta t (E_n(h)) \left( \|\mathbf{e}_d^n\|_1^2 + \|\mathbf{e}_d^{n+1}\|_1^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_1^2 \right)
\end{aligned}$$

Adding up this inequality for  $n = 0, \dots, N$ , we get:

$$\begin{aligned}
& \|\mathbf{e}_d^{N+1}\|_0^2 + \sum_{n=0}^N \left( \|\mathbf{e}_d^{n+1/2} - \mathbf{e}_d^n\|_0^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_0^2 \right) \\
& + \nu\delta t \sum_{n=0}^N \left( \|\mathbf{e}_d^{n+1}\|_1^2 + \|\mathbf{e}_d^{n+1/2}\|_1^2 + \|\mathbf{e}_d^{n+1} - \mathbf{e}_d^{n+1/2}\|_1^2 \right)
\end{aligned}$$

$$\begin{aligned} &\leq C \left(1 + \frac{h^2}{\delta t}\right) (E(h))^2 + C \delta t \sum_{n=0}^N \|e_d^n\|_0^2 + \nu \delta t^3 \sum_{n=0}^N \|e_d^n\|_1^2 \\ &+ C (E(h)) \nu \delta t \sum_{n=0}^N \left( \|e_d^{n+1}\|_1^2 + \|e_d^{n+1} - e_d^{n+1/2}\|_1^2 \right) \end{aligned}$$

and (42) follows for small enough  $\delta t$  and  $h$  (since the last two terms can then be passed over to the LHS), due to the discrete Gronwall inequality, condition **H3** and the triangle inequality to bound  $\|e_d^{N+1/2}\|_0^2$ .

Combining Theorems 2 and 4, we have an estimate for the overall error of the method,  $e^{n+1} = \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1} = e_c^{n+1} + e_d^{n+1}$ :

**Corollary 1** *Assume that **R1**, **R2b**, **R3**, **H1**, **H2** and **H3** hold; assume also that for all  $n = 0, \dots, [T/\delta t] - 1$ ,  $\mathbf{u}^{n+1}, \mathbf{u}^{n+1/2} \in \mathbf{H}^k(\Omega)$  and  $p^{n+1} \in H^{k-1}(\Omega)$ , and they are uniformly bounded in these spaces; then for  $N = 0, \dots, [T/\delta t] - 1$ , and for small enough  $\delta t > 0$  and  $h$ :*

$$\|e^{N+1}\|_0^2 + \delta t \nu \sum_{n=0}^N \|e^{n+1}\|_1^2 \leq C (\delta t^2 + h^{2k}) \quad (46)$$

Estimate (46) says that the present scheme is first order accurate in the time step size and provides optimal order accuracy in the mesh size in the norms of  $l^\infty(\mathbf{L}^2(\Omega))$  and  $l^2(\mathbf{H}_0^1(\Omega))$ .

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