Orbits of Controllable and Observable Systems

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ABSTRACT.- Let a time-invariant linear system $\left.\begin{array}{l}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{array}\right\}$ corresponding to a realization of a prescribed transfer function matrix can be represented by triples of matrices (A,B,C). The permitted transformations of basis changes in the space state on the systems can be seen in the space of triples of matrices as similarity equivalence. In this paper we give a geometric characteriaztion of controllable and observable systems as orbits under a Lie group action. As a corollary we obtain a lower bound of the distance between a controllable and observable triple and the nearest uncontrollable one.

INTRODUCTION

In the space of triples of matrices $(A,B,C)\in M_n(\mathbf{C})\times M_{p\times n}(\mathbf{C})\times M_{n\times m}(\mathbf{C})$ corresponding to a time-invariant linear systems $\left.\begin{array}{l}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{array}\right\}$. We consider the following action of the general linear group $Gl(n;\mathbf{C})$, according to the formula

$$(A_1,B_1,C_1)=(P^{-1}AP,P^{-1}B,CP)$$

We denote the space of triples of matrices $M_n(\mathbf{C})\times M_{n\times m}(\mathbf{C})\times M_{p\times n}(\mathbf{C})$ by $\mathcal{M}$ and the general linear group $Gl(n;\mathbf{C})$ by $\mathcal{G}$

The sets of equivalent triples under the group action are differentiable manifolds called orbits.

The controllability and observability character of a triple is invariant by the group Lie action, so given a controllable and observable triple of matrices the nearest non-controllable or non observable one remains obviously, in another orbit. Then the problem can be reduced to compute the distance from (A,B,C) to the orbits of uncontrollable or unobservable triples. For that we explore the rank of a matrix representing the tangent space to the orbit of the triple (A,B,C).

The norm considered in this paper is the Frobenius norm.

1. Preliminaries

We will denote the general linear group by $\mathcal{G}$ and its unit element by I. This is a complex manifold and its tangent space at the identity is $T_I\mathcal{G}=M_n(\mathbf{C})$.

Definition (1.1): We consider the following action of $\mathcal{G}$ on $\mathcal{M}$,

$$\alpha:\mathcal{G}\times\mathcal{M}\longrightarrow\mathcal{M}$$

defined by

$$\alpha(P,(A,B,C))=(P^{-1}AP,P^{-1}B,CP)$$

The action defined by α induces the following equivalence relation between triples of matrices: (A_1,B_1,C_1) and (A_2,B_2,C_2) are called *equivalent* if and only if there exists $P\in\mathcal{G}$ such that $\alpha(P,(A_1,B_1,C_1))=(A_2,B_2,C_2)$. This equivalence relation corresponds with the permitted operations of basis change in the state space used in Linear System theory

The differentiable manifold of triples of matrices in $\mathcal{M}$ which are equivalent to (A,B,C) is its orbit under the action of α and we will denote it by $\mathcal{O}(A,B,C)$. The orbits verify the following condition.

Proposition (1.1). *Let* $(A_0,B_0,C_0)\in\overline{\mathcal{O}(A,B,C)}$. *Then* $\mathcal{O}(A_0,B_0,C_0)\subset\overline{\mathcal{O}(A,B,C)}$.

Proof: Let $(A_0,B_0,C_0)\in\overline{\mathcal{O}(A,B,C)}$, then

$$(A_0,B_0,C_0)=\lim_{n\to\infty}(A_n,B_n,C_n)$$

with $(A_n,B_n,C_n)\in\mathcal{O}(A,B,C)$, Then for all $(S^{-1}A_0S,S^{-1}B_0,C_0S)$ we consider $(S^{-1}A_nS,S^{-1}B_n,C_nS)\in\mathcal{O}(A,B,C)$, and

$$(S^{-1}A_0S,S^{-1}B_0,C_0S)=\lim_{n\to\infty}(S^{-1}A_nS,S^{-1}B_n,C_nS)$$

The orbits verify the homogenity property:

Proposition (1.2). *Let* (A_1,B_1,C_1), $(A_2,B_2,C_2)\in\mathcal{O}(A,B,C)$, *then there exists a diffeomorphism* $h:\mathcal{M}\longrightarrow\mathcal{M}$ *preserving orbits and such that* $h(A_1,B_1,C_1)=(A_2,B_2,C_2)$.

Proof: If (A_1,B_1,C_1), $(A_2,B_2,C_2)\in\mathcal{O}(A,B,C)$ there exists $P\in\mathcal{G}$ such that $(A_2,B_2,C_2)=(P^{-1}A_1P,P^{-1}B_1,C_1P)$. Then it suffices to consider

$$\begin{gathered}h:\mathcal{M}\longrightarrow\mathcal{M}\\(A,B,C)\longrightarrow(P^{-1}AP,P^{-1}B,CP)\end{gathered}$$

This proposition permit us to consider a selected triple in the orbit called canonical reduced form and denoted by (A_c,B_c,C_c).

Remark (1.1): If $m=1$ and the triple (A,B,C) is controllable it is easy to obtain a canonical reduced form it suffices to take $P=(\,B\quad AB\quad\dots\quad A^{n-1}B\,)$ and $A_c=P^{-1}AP$, $B_c=P^{-1}B$ and $C_c=CP$. This method can be generalized (see [**3**]).

We are interested in to know if it is possible to define an homomorphism assigning each orbit its canonical reduced form, that is to say we are interested in the existence or non, of continuous canonical forms. Hazewinkel in [**3**], prove that it is only possible if $m=1$ or $p=1$ and if we define the map on the space of controllable and observable triples.

Let $\mathcal{M}^{co}\subset\mathcal{M}$ be the set of controllable and observable triples. Obviously $\mathcal{M}^{co}$ is a $\mathcal{G}$-invariant space.

Definition (1.2): A canonical form for $\mathcal{G}$ acting on $\mathcal{M}^{co}$ is a mapping

$$c:\mathcal{M}^{co}\longrightarrow\mathcal{M}^{co}$$

such that the following properties hold

1) $c(A,B,C)=(A_c,B_c,C_c)$
2) $\forall(A',B',C')\in\mathcal{O}(A,B,C)$, $c(A',B',C')=c(A,B,C)$
3) If $c(A',B',C')=c(A,B,C)$ then $(A',B',C')\in\mathcal{O}(A,B,C)$.

Proposition (1.3) [**3**]. *The map c is continuous if and only if $m=1$ or $p=1$.*

Proposition (1.4). *Let $(A_n,B_n,C_n)\in\mathcal{O}(A,B,C)\subset\mathcal{M}^{co}$ with $m=1$ or $p=1$, such that $\lim_{n\to\infty}(A_n,B_n,C_n)=(A_0,B_0,C_0)\in\mathcal{M}$. Then $(A_0,B_0,C_0)\in\mathcal{M}^{co}$ if and only if $(A_0,B_0,C_0)\in\mathcal{O}(A,B,C)$.*

Proof: For all n, we have $c(A_n,B_n,C_n)=(A_c,B_c,C_c)$ and c is continuous in $\mathcal{M}^{co}$ then $c(A_0,B_0,C_0)=(A_c,B_c,C_c)$ if and only if $(A_0,B_0,C_0)\in\mathcal{O}(A,B,C)$.

Corollary (1.1). *If $(A_0,B_0,C_0)\in\overline{\mathcal{O}(A,B,C)}-\mathcal{O}(A,B,C)$, then $(A_0,B_0,C_0)\notin\mathcal{M}^{co}$.*

2. The tangent space to the orbit

To compute the dimension of orbits may be very tedious if one use the definition of orbits, but taking into accouint the differentiable character of the manifold defining orbits it is easier to compute the dimension troughout the tangent space.

Let (A,B,C) be a triple of matrices in $\mathcal{M}$. It is not dificult to check that the tangent space of its orbit $T_{(A,B,C)}\mathcal{O}(A,B,C)$ is given in the following manner

$$T_{(A,B,C)}\mathcal{O}(A,B,C)=\{(X,Y,Z)=([A,P],-PB,CP);\ P\in T_I\mathcal{G}=M_n(\mathbf{C})\}.$$

Using the Kronecker products and vec-operator (see [**4**] for their definition and properties), we can represent the $n^2+nm+pn$ vectors $(X,Y,Z)\in T_{(A,B,C)}\mathcal{O}(A,B,C)$ in the form

$$\begin{pmatrix}\operatorname{vec}(X)\\ \operatorname{vec}(Y)\\ \operatorname{vec}(Z)\end{pmatrix}=\begin{pmatrix}A\otimes I_n-I_n\otimes A^t\\ -I_n\otimes B^t\\ C\otimes I_n\end{pmatrix}(\operatorname{vec}(P)).$$

In this notation, we may say that the tangent space is the range of the $(n^2+nm+pn)\times n^2$-matrix

$$\mathbf{T}=\begin{pmatrix}A\otimes I_n-I_n\otimes A^t\\ -I_n\otimes B^t\\ C\otimes I_n\end{pmatrix}.$$

Then we have the following result.

Theorem (2.1).

$$\dim T_{(A,B,C)}\mathcal{O}(A,B,C)=\operatorname{rank}\mathbf{T}.$$

Remark (2.1): $\operatorname{rank}\mathbf{T}\le n^2<\dim\mathcal{M}=n^2+nm+np$, then there are not open orbits, but there are orbits of dimension n^2.

Example (2.1): Let $(A,B,C)\in\mathcal{M}$ with $m=p=1$ be a triple of matrices with $A=\operatorname{diag}(a_1,\ldots,a_n)$ $a_i\neq a_j$ for all $i\neq j$, $B=\begin{pmatrix} b_1 & \ldots & b_n\end{pmatrix}^t$, $C=\begin{pmatrix} c_1 & \ldots & c_n\end{pmatrix}$ with $b_i\neq 0$ and $c_i\neq 0$ for all $i=1,\ldots,n$. Then $\dim\mathcal{O}(A,B,C)=n^2$.

For that, it is sufficient to compute $\operatorname{rank}\mathbf{T}$.

$$
\operatorname{rank}\begin{pmatrix}
0 &&&&&&&& \\
& a_1-a_2 &&&&&&& \\
&& \ddots &&&&&& \\
&&& a_1-a_n &&&&& \\
&&&& \ddots &&&& \\
&&&&& a_n-a_1 &&& \\
&&&&&& \ddots && \\
&&&&&&& a_n-a_{n-1} & \\
&&&&&&&& 0 \\
-b_1 & -b_2 & \ldots & -b_n & 0 & \ldots &&& 0 \\
&&&&& \ddots &&& \\
&&&&&& -b_1 & \ldots & -b_n \\
c_1 &&&&&& c_n && \\
& \ddots &&&&&& \ddots & \\
&& \ddots &&&&&& \ddots \\
&& c_1 &&&&&& c_n
\end{pmatrix} =
$$

$$
\operatorname{rank}\begin{pmatrix}
b_1 &&&&&&&&& \\
& a_1-a_2 &&&&&&&& \\
&& \ddots &&&&&&& \\
&&& a_1-a_n &&&&&& \\
&&&& \ddots &&&&& \\
&&&&& a_n-a_1 &&&& \\
&&&&&& \ddots &&& \\
&&&&&&& a_n-a_{n-1} && \\
&&&&&&&& b_n & \\
c_1 &&&&&&& c_n && \\
& \ddots &&&&&& \ddots && \\
&& \ddots &&&&&&& \ddots \\
&&& c_1 &&&&&& c_n
\end{pmatrix} = n^2.
$$

Notice that the triple (A,B,C) is controllable and observable but this is not a necessary condition

Example: $A=\begin{pmatrix} a & \\ & b\end{pmatrix}$, $B=\begin{pmatrix} b_1\\ 0\end{pmatrix}$, $C=\begin{pmatrix} c_1 & c_2\end{pmatrix}$ with $a\neq b$, $b_1\neq 0$, $c_2\neq 0$.

$$
\operatorname{rank}\mathbf{T}=\operatorname{rank}\begin{pmatrix}
0 & 0 & 0 & 0\\
0 & a-b & 0 & 0\\
0 & 0 & b-a & 0\\
0 & 0 & 0 & 0\\
-b_1 & 0 & 0 & 0\\
0 & 0 & -b_1 & 0\\
c_1 & 0 & c_2 & 0\\
0 & c_1 & 0 & c_2
\end{pmatrix}=\operatorname{rank}\begin{pmatrix}
b_1 & 0 & 0 & 0\\
0 & a-b & 0 & 0\\
0 & 0 & b-a & 0\\
0 & 0 & 0 & c_2\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{pmatrix}=4=n^2.
$$

For $m = p = 1$ a necessary and sufficient condition to $\dim \mathcal{O}(A,B,C) = n^2$ is given in the following proposition.

PROPOSITION (2.2). *Let $(A,B,C) \in \mathcal{M}$ with $m = p = 1$. $\dim \mathcal{O}(A,B,C) = n^2$ if and only if the triple (A,B,C) be controllable or observable.*

PROOF: Taking into account that $\forall (A_1,B_1,C_1) \in \mathcal{O}(A,B,C)$, then $\mathcal{O}(A_1,B_1,C_1) = \mathcal{O}(A,B,C)$ we can restreint to the canonical reduced form.

Then, let $(A,B,C) \in \mathcal{M}^{co}$ with $m = p = 1$, such that

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & \alpha_1 \\ 1 & 0 & & 0 & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_n \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & \dots & c_n \end{pmatrix}$$

$$\operatorname{rank} \mathbf{T} = \operatorname{rank} \begin{pmatrix} -A^t & 0 & 0 & \dots & 0 & \alpha_1 I_n \\ I_n & -A^t & 0 & \dots & 0 & \alpha_2 I_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I_n & \alpha_n I_n - A^t \\ -B^t & 0 & 0 & \dots & 0 & 0 \\ 0 & -B^t & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -B^t \\ c_1 I_n & c_2 I_n & c_3 I_n & \dots & c_{n-1} I_n & c_n I_n \end{pmatrix}.$$

Making block elementary transformations we obtain

$$\operatorname{rank} \mathbf{T} = \operatorname{rank} \begin{pmatrix} I_n & & & & \\ & I_n & & & \\ & & \ddots & & \\ & & & I_n & \\ & & & & C(A,B,C) \\ & & & & O(A,B,C)^t \end{pmatrix} = n^2$$

where $C(A,B,C) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ (the controllability matrix) and $O(A,B,C)^t = \begin{pmatrix} C^t & A^tC^t & \dots & {A^t}^{n-1}C^t \end{pmatrix}$ (the transpose observability matrix).

Then, if $\operatorname{rank} \mathbf{T} < n^2$, the triple (A,B,C) is neither controllable or observable.

We observe that we can obtain the dimension of $T_{(A,B,C)}\mathcal{O}(A,B,C)$ from the singular value decomposition (s.v.d.) of the matrix $\mathbf{T}$.

PROPOSITION (2.3).

$$\dim T_{(A,B,C)}\mathcal{O}(A,B,C) = \text{number of non-zero singular values of } \mathbf{T}.$$

3. The normal space to the orbit

We may define the normal space $T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp}$ as the orthogonal to the tangent space $T_{(A,B,C)}\mathcal{O}(A,B,C)$. The orthogonality is defined with respect to the following usual inner product.

DEFINITION (3.1):

$$< (A_1,B_1,C_1),(A_2,B_2,C_2) >= \operatorname{trace} A_1A_2^* + \operatorname{trace} B_1B_2^* + \operatorname{trace} C_1C_2^*.$$

Obviously, we have the following.

Corollary (3.1).

$$\dim T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp} = n^2 + nm + np - \operatorname{rank}\mathbf{T} = \dim \operatorname{Ker}\mathbf{T} + nm + np.$$

We can compute $T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp}$ solving a linear matricial system:

$$T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp} = \{(X,Y,Z) \mid [X^*,A] - BY^* + Z^*C = 0\}$$

Example (3.1): Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$, then

$$T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp} = \left\{\left(\begin{pmatrix} x_{11} & x_{21} \\ x_{21}+z_2 & x_{21}+x_{11} \end{pmatrix}, \begin{pmatrix} z_2 \\ z_1+z_2 \end{pmatrix}, \begin{pmatrix} z_1 & z_2 \end{pmatrix}\right) \forall x_{11}, x_{21}, z_1, z_2 \in \mathbf{C}\right\}$$

$\dim T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp} = 4$ and $\dim \mathcal{O}(A,B,C) = \dim \mathcal{M} - \dim T_{(A,B,C)}\mathcal{O}(A,B,C)^{\perp} = 8 - 4 = 4$.

4. Application.

The open character of $\mathcal{M}^{co}$, allows us to ensure that if $(A,B,C) \in \mathcal{M}$ is a controllable and observable triple of matrices there exists a neigborhood $\mathcal{U}$ in $\mathcal{M}$ such that for all $(A_1, B_1, C_1) \in \mathcal{U}$ then (A_1, B_1, C_1) is also a controllable and observable. Therefore it makes sense to consider the distance to the nearest uncontrollable or unobservable triple.

Definition (4.1): We define a norm in the space $\mathcal{M}$ in the following manner

$$\text{for all } (A,B,C) \in \mathcal{M}, \qquad \|(A,B,C)\| = \left\|\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\right\|,$$

where $\left\|\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\right\|$ is any matrix norm.

Definition (4.2): For a given controllable and observable triple of matrices $(A,B,C) \in \mathcal{M}$ we define the distance between (A,B,C) and a nearest uncontrollable or unobservable one by

$$d = \min_{(\delta A, \delta B, \delta C)} \|(\delta A, \delta B, \delta C)\|,$$

where $(\delta A, \delta B, \delta C) \in \mathcal{M}$ such that $(A + \delta A, B + \delta B, C + \delta C)$ is uncontrollable or unobservable.

The s.v.d. characterization of the dimension of $\mathcal{O}(A,B,C)$ leads to the following Theorem.

Theorem (4.1). *For a given controllable and observable triple of matrices* $(A,B,C) \in \mathcal{M}^{co}$ *with* $m = p = 1$ *a lower bound on the distance to the closest triple* $(A+\delta A, B+\delta B, C+\delta C)$ *with* $\dim \mathcal{O}(A+\delta A, B+\delta B, C+\delta C) = n^2 - \ell$ *and* $\ell \geq 1$ *is given by*

$$\|(\delta A, \delta B, \delta C)\| \geq \frac{1}{\sqrt{2n}} \left(\sum_{i=n-\ell+1}^{a} \sigma_i^2(\mathbf{T})\right)^{1/2} \geq \frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n}}.$$

PROOF: Let $(A+\delta A, B+\delta B, C+\delta C)$ be a perturbed triple of matrices with

$$\dim T\mathcal{O}(A+\delta A, B+\delta B, C+\delta C) = n^2 - \ell.$$

Then

$$\operatorname{rank}(\mathbf{T}+\delta\mathbf{T}) < n^2,$$

where

$$\delta\mathbf{T} = \begin{pmatrix} \delta A \otimes I_n - I_n \otimes \delta A^t \\ -I_n \otimes \delta B^t \\ \delta C \otimes I_n \end{pmatrix},$$

$$\|(\delta A, \delta B, \delta C)\|_F \leq \|\delta\mathbf{T}\|_F \leq \sqrt{2n}\|(\delta A, \delta B, \delta C)\|_F.$$

The Eckart-Young and Mirsky Theorem for finding the closest matrix of a given rank (see [**2**]), gives that the size of the smallest perturbation in Frobenius norm that reduces the rank in $\mathbf{T}$ from n^2 to $n^2-\ell$ with $\ell \geq 1$, is

$$\left(\sum_{i=n^2-\ell+1}^{n^2} \sigma_i^2(\mathbf{T}) \right)^{1/2}.$$

Then,

$$\|(\delta A, \delta B, \delta C)\|_F \geq \frac{1}{\sqrt{2n}} \left(\sum_{i=n^2-\ell+1}^{n^2} \sigma_i^2(\mathbf{T}) \right)^{1/2} \geq \frac{\sigma_{\min}(\mathbf{T})}{\sqrt{2n}}.$$

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