Bounding the Distance of a Controllable and Observable System to an Uncontrollable or Unobservable One

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ABSTRACT.- Let (A,B,C) be a triple of matrices representing a time-invariant linear system $\left.\begin{array}{l}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{array}\right\}$ under similarity equivalence, corresponding to a realization of a prescribed transfer function matrix.

In this paper we measure the distance between a irreducible realization, that is to say a controllable and observable triple of matrices (A,B,C) and the nearest reducible one that is to say uncontrollable or unobservable one.

Different upper bounds are obtained in terms of singular values of the controllability matrix $C(A,B,C)$, observability matrix $O(A,B,C)$ and controllability and observability matrix $CO(A,B,C)$ associated to the triple.

Introduction

We consider triples of matrices (A,B,C) with $A\in M_n(\mathbf{R})$, $B\in M_{n\times m}(\mathbf{R})$ and $C\in M_{p\times n}(\mathbf{R})$ corresponding to a time-invariant linear systems $\left.\begin{array}{r}\dot{x}(t)=Ax(t)+Bu(t)\\ y(t)=Cx(t)\end{array}\right\}$.
We consider the following action of the general linear group $Gl(n;\mathbf{R})$, according to the formula

$$(A_1,B_1,C_1)=(P^{-1}AP,P^{-1}B,CP)$$

The equivalence relation obtained from this action is such that two equivalent triples of matrices have the same transfer-function matrix.

We denote the space of these triples of matrices by $\mathcal{M}$ and the general linear group $Gl(n;\mathbf{R})$ by $\mathcal{G}$

We consider the set $\mathcal{M}^{co}=\{(A,B,C)\in\mathcal{M};(A,B,C)\,\text{controllable and observable}\}$. This is an open set in the space of all triples of matrices $\mathcal{M}$ and it is invariant with respect to the $\mathcal{G}$-action.

For each $(A,B,C)\in\mathcal{M}^{co}$ there exists an open neigbourhood of (A,B,C) relatively small, such that all triples of matrices in it are controllable and observable. Then it makes sense to consider the distances to the nearest uncontrollable, unobservable or uncontrollable and unobservable one, and to deduce safety neighbourhoods for controllable and observable triples of matrices.

The main goal of this paper is to show that different bounds of theese distances can be obtained. The method used for that as this one used in [**1**] for the case of pairs of matrices, is to explore the singular values of the controllability and observability matrices of the triple (A,B,C).

Several authors [**1**], [**2**], [**4**] analyze bounds on the distance from a given pair of matrices or a given pencil with qualitative different structure pair or pencil under different equivalent relation for pairs or strictly equivalence for pencils, as well as [**5**], [**6**], [**7**], [**9**] analyze the structural stability of a pair or a pencil and the hierarchic closure for pencils.

In this paper, the norm considered is the 2-norm, and given a triple $(A,B,C)\in\mathcal{M}$. We denote by A_c the companion matrix for A, that is to say

$$A_c=\begin{pmatrix}0&1&0&\dots&0\\0&0&1&\dots&0\\ \vdots&&&&\vdots\\0&0&0&\dots&1\\-\alpha_n&-\alpha_{n-1}&-\alpha_{n-2}&\dots&-\alpha_1\end{pmatrix}$$

where α_i are such that $\det(tI-A)=t^n+\alpha_1t^{n-1}+\ldots+\alpha_n$.

1. Preliminaries

We consider the following action of $\mathcal{G}$ on $\mathcal{M}$,

$$\alpha : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$$

defined by

$$\alpha(P, (A, B, C)) = (P^{-1}AP, P^{-1}B, CP)$$

The action defined by α induces the following equivalence relation between triples of matrices: (A_1, B_1, C_1) and (A_2, B_2, C_2) are called *equivalent* if and only if there exists $P \in \mathcal{G}$ such that $\alpha(P, (A_1, B_1, C_1)) = (A_2, B_2, C_2)$.

The controllability matrix of a triple $(A, B, C) \in \mathcal{M}$ is defined as

$$C(A, B, C) = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}.$$

The observability matrix of a triple $(A, B, C) \in \mathcal{M}$ is defined as

$$O(A, B, C) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

They are well known the following propositions (see [**3**] for more details).

PROPOSITION (1.1). *a) The rank of the controllability matrix is invariant under the equivalence relation considered.*

b) A triple of matrices $(A, B, C) \in \mathcal{M}$ is controllable if and only if the controllability matrix has full rank, i.e.

$$\operatorname{rank} C(A, B, C) = n.$$

PROPOSITION (1.2). *a) The rank of the observability matrix is invariant under the equivalence relation considered.*

b) A triple of matrices $(A, B, C) \in \mathcal{M}$ is observable if and only if the observability matrix has full rank, i.e.

$$\operatorname{rank} O(A, B, C) = n.$$

The controllability and observability matrix of a triple $(A, B, C) \in \mathcal{M}$ is defined as

$$CO(A, B, C) = O(A, B, C) \cdot C(A, B, C) = \begin{pmatrix} CB & CAB & \dots & CA^{n-1}B \\ \vdots & & & \vdots \\ CA^{n-1}B & CA^{n}B & \dots & CA^{2n-2}B \end{pmatrix}$$

PROPOSITION (1.3). *The rank of the controllability and observability matrix is invariant under the equivalence relation considered.*

PROOF: Let (A_1, B_1, C_1) and (A_2, B_2, C_2) equivalent triples. Then there exist invertible matrix P such that $(A_2, B_2, C_2) = (P^{-1}A_1P, P^{-1}B_1, C_1P)$. So

$$C_2A_2^kB_2 = C_1PP^{-1}A_1^kPP^{-1}B_1 = C_1A_1^kB_1.$$

PROPOSITION (1.4). *A triple of matrices (A, B, C) is controllable and observable if and only if*

$$\operatorname{rank} CO(A, B, C) = n.$$

PROOF: It follows from Sylvester's inequality (see [**8**] for details),

$$\operatorname{rank} O(A, B, C) + \operatorname{rank} C(A, B, C) - n \leq \operatorname{rank} CO(A, B, C) \leq \\ \leq \min(\operatorname{rank} O(A, B, C), \operatorname{rank} C(A, B, C)).$$

2. The μ-Distance.

The open character of $\mathcal{M}^{co}$, allows us to ensure that if $(A, B, C) \in \mathcal{M}$ is a controllable and observable triple of matrices there exists a neigborhood $\mathcal{U}$ in $\mathcal{M}$ such that for all $(A_1, B_1, C_1) \in \mathcal{U}$ then (A_1, B_1, C_1) is also a controllable and observable. Therefore it makes sense to consider the distance to the nearest uncontrollable or unobservable or uncontrollable and unobservable triple.

DEFINITION (2.1): We define a norm in the space $\mathcal{M}$ in the following manner

$$\text{for all } (A, B, C) \in \mathcal{M}, \qquad \|(A, B, C)\| = \left\| \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \right\|,$$

where $\left\| \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \right\|$ is any matrix norm.

DEFINITION (2.2): For a given controllable and observable triple of matrices $(A, B, C) \in \mathcal{M}^{co}$ we define the distance between (A, B, C) and a nearest uncontrollable triple by

$$\mu^c(A, B, C) = \min_{(\delta A, \delta B, \delta C)} \|(\delta A, \delta B, \delta C)\|$$

where $(\delta A, \delta B, \delta C) \in \mathcal{M}$ such that $(A + \delta A, B + \delta B, C + \delta C)$ is uncontrollable.

DEFINITION (2.3): For a given controllable and observable triple of matrices $(A, B, C) \in \mathcal{M}^{co}$ we define the distance between (A, B, C) and a nearest unobservable triple by

$$\mu^o(A, B, C) = \min_{(\delta A, \delta B, \delta C)} \|(\delta A, \delta B, \delta C)\|$$

where $(\delta A, \delta B, \delta C) \in \mathcal{M}$ such that $(A+\delta A, B+\delta B, C+\delta C)$ is unobservable.

DEFINITION (2.4): For a given controllable and observable triple of matrices $(A,B,C) \in \mathcal{M}^{co}$ we define the distance between (A,B,C) and a nearest uncontrollable and unobservable triple by

$$\mu^{co}(A,B,C) = \min_{(\delta A, \delta B, \delta C)} \|(\delta A, \delta B, \delta C)\|$$

where $(\delta A, \delta B, \delta C) \in \mathcal{M}$ such that $(A+\delta A, B+\delta B, C+\delta C)$ is uncontrollable and unobservable.

We remark that $\mu^{co} \geq \max\{\mu^c, \mu^o\}$ as we can see in the following example:

Let (A,B,C) with $A \in M_1(\mathbf{R})$, $A = (a)$, $B \in M_1(\mathbf{R})$, $B = (1)$, $C \in M_1(\mathbf{R})$, $C = (1)$, $\mu^c = 1$, $\mu^o = 1$, $\mu^{co} = \sqrt{2}$. Then make sense to consider μ^{co}.

The matrix norm considered in the follows is the 2-norm: $\|A\|_2 = \sigma_1$ where σ_1 is the largest singular value of A.

It is evident that if P is an orthogonal matrix and we consider $(A_1, B_1, C_1) = (P^{-1}AP, P^{-1}B, CP)$ we have

$$\mu^*(A_1, B_1, C_1) = \mu^*(A,B,C).$$

for $* = c$, o, or co.

3. μ^*-distance and relationship with $C(A,B,C)$, $O(A,B,C)$ and $CO(A,B,C)$ matrices

Now we analyze as a bound of $\|(\delta A, \delta B, \delta C)\|_2$ can be deduced from the controllability, observability and controllability and observability matrices of a given triple of matrices (A,B,C). In this case we obtain bounds for μ^*, $* = c$, o, or co.

Given a triple $(A,B,C) \in \mathcal{M}^{co}$, the controllability matrix of (A,B,C), is independent of the matrix C. Then we can reduce to the pair (A,B) and consider the bound of $\|(\delta A, \delta B)\|_2$ where $(\delta A, \delta B)$ is in such a way that $(A+\delta A, B+\delta B)$ is uncontrollable, obtained by D.L. Boley and W-S Lu in [**2**], and we deduce the following Theorem.

THEOREM (3.1). *For a given triple* $(A,B,C) \in \mathcal{M}^{co}$ *we have*

$$\mu^c(A,B,C) \leq \min\left(\left(1 + \frac{\|A_c\|_2}{\sigma_1^c}\right)\sigma_2^c, \ldots, \left(1 + \frac{\|A_c\|_2}{\sigma_{n-1}^c}\right)\sigma_n^c\right).$$

where σ_i^c, $i = 1, \ldots, n$ *are the singular values of the controllability matrix* $C(A,B,C)$.

Now, taking into account that the observability matrix $O(A,B,C)$ is independent of the matrix B and $O(A,B,C)^t = C(A^t, C^t, B^t)$, we have

THEOREM (3.2). *For a given triple* $(A,B,C)\in\mathcal{M}^{co}$ *we have*

$$\mu^o(A,B,C)\le \min\left(\left(1+\frac{\|A_c\|_2}{\sigma_1^o}\right)\sigma_2^o,\dots,\left(1+\frac{\|A_c\|_2}{\sigma_{n-1}^o}\right)\sigma_n^o\right)$$

where σ_i^o, $i=1,\dots,n$ *are the singular values of the observability matrix* $O(A,B,C)$.

Now we are interested to obtain a bound related to the $CO(A,B,C)$ matrix. Firstly, we obtain a bound relating the $O(A,B,C)$ and $CO(A,B,C)$

Calling $\begin{bmatrix}\Sigma^{co} & 0\\ 0 & 0\end{bmatrix}$ the s.v.d. of $CO(A,B,C)$ and σ_i^{co} the singular values we have

$$CO(A,B,C)=X^t\begin{bmatrix}\Sigma^{co} & 0\\ 0 & 0\end{bmatrix}Y$$

where X and Y are orthogonal matrices.

REMARK (3.1): If $(A_1,B_1,C_1)=(XAX^{-1},XB,CX^{-1})$ with $X^t=X^{-1}$, then

$$\begin{aligned}CO(A_1,B_1,C_1)&=O(A_1,B_1,C_1)C(A_1,B_1,C_1)=\\&=O(A,B,C)X^tXC(A,B,C)=\\&=O(A,B,C)C(A,B,C)=CO(A,B,C)\end{aligned}$$

LEMMA (3.1). *For a given triple* $(A,B,C)\in\mathcal{M}^{co}$ *there exists an orthogonal matrix* P *such that*

$$A_1=P^{-1}AP=\begin{pmatrix}\overline{A}_1 & \overline{A}_2\\ \overline{A}_3 & \overline{A}_4\end{pmatrix},\quad B_1=P^{-1}B=\begin{pmatrix}\overline{B}_1\\ \overline{B}_2\end{pmatrix},\quad C_1=CP=(\,\overline{C}_1\quad \overline{C}_2\,)$$

where $\overline{A}_1\in M_r(\mathbf{R})$, $\overline{B}_1\in M_{r\times m}(\mathbf{R})$, $\overline{C}_1\in M_{p\times r}(\mathbf{R})$ $1\le r\le n-1$, *with*

$$\|\overline{A}_2\|_2\le\|A_c\|_2\frac{\sigma_{r+1}^o}{\sigma_r^o},\quad \|\overline{B}_1\|_2\le\frac{\sigma_1^{co}}{\sigma_r^o}\quad\text{and}\quad \|\overline{C}_2\|_2\le\sigma_{r+1}^o$$

PROOF: Let (A,B,C) be a triple in $\mathcal{M}^{co}$, $\begin{bmatrix}\Sigma^{co} & 0\\ 0 & 0\end{bmatrix}$ the s.v.d of $CO(A,B,C)$:

$$CO(A,B,C)=X^t\begin{bmatrix}\Sigma^{co} & 0\\ 0 & 0\end{bmatrix}Y$$

where X and Y are orthogonal matrices.

$$X^t\begin{bmatrix}\Sigma^{co} & 0\\ 0 & 0\end{bmatrix}Y=O(A,B,C)C(A,B,C)=O(A,B,C)PP^{-1}C(A,B,C)$$

where P is such that $O(A,B,C)=Q\begin{bmatrix}\Sigma^o\\0\end{bmatrix}P^{-1}$, P, Q being orthogonals and $\begin{bmatrix}\Sigma^o\\0\end{bmatrix}$ the s.v.d of $O(A,B,C)$.

We consider $(A_1,B_1,C_1)=(P^{-1}AP,P^{-1}B,CP)$, then

$$O(A_1,B_1,C_1)=Q\begin{bmatrix}\Sigma^o\\0\end{bmatrix}$$

$$X^t\begin{bmatrix}\Sigma^{co}&0\\0&0\end{bmatrix}Y\begin{pmatrix}I_m\\0\end{pmatrix}=Q\begin{bmatrix}\Sigma^o\\0\end{bmatrix}B_1$$

$$\begin{aligned}B_1&=\left(Q\begin{bmatrix}\Sigma^o\\0\end{bmatrix}\right)^+X^t\begin{bmatrix}\Sigma^{co}&0\\0&0\end{bmatrix}Y\begin{pmatrix}I_m\\0\end{pmatrix}=\\&=\begin{bmatrix}\Sigma^o\\0\end{bmatrix}^+Q^tX^t\begin{bmatrix}\Sigma^{co}&0\\0&0\end{bmatrix}Y\begin{pmatrix}I_m\\0\end{pmatrix}\end{aligned}$$

$$\begin{bmatrix}\Sigma^o\\0\end{bmatrix}=\begin{pmatrix}\sigma_1^0&&&&&\\&\ddots&&&&\\&&\sigma_r^o&&&\\&&&\sigma_{r+1}^0&&\\&&&&\ddots&\\&&&&&\sigma_n^o\\0&&&&&0\\\vdots&&&&&\vdots\\0&&&&&0\end{pmatrix}=\begin{pmatrix}\Sigma_r^o&0\\0&\Sigma_{\overline{r}}^o\\0&0\end{pmatrix}$$

$$\begin{bmatrix}\Sigma^o\\0\end{bmatrix}^+=\begin{pmatrix}\Sigma_r^{o-1}&0&0\\0&\Sigma_{\overline{r}}^{o-1}&0\end{pmatrix}$$

We denote by Y_m the upper left $m\times m$ submatrix of Y, then

$$Y\begin{pmatrix}I_m\\0\end{pmatrix}=\begin{pmatrix}Y_m\\Y_{nm-m}\end{pmatrix}$$

and

$$\begin{bmatrix}\Sigma^{co}&0\\0&0\end{bmatrix}Y\begin{pmatrix}I_m\\0\end{pmatrix}=\begin{bmatrix}\Sigma^{co}&0\\0&0\end{bmatrix}\begin{pmatrix}Y_m\\Y_{nm-m}\end{pmatrix}=\begin{pmatrix}\Sigma^{co}Y_m^p\\0\end{pmatrix}$$

where Y_m^p denote the upper $n\times m$ submatrix of $\begin{pmatrix}Y_m\\Y_{nm-m}\end{pmatrix}$

In the other hand, partitioning the matrix $Q^tX^t=\begin{pmatrix}S_1\\S_2\\S_3\end{pmatrix}$ with $S_1\in M_{r\times np}(\mathbf{R})$, $S_2\in M_{(n-r)\times np}(\mathbf{R})$, $S_3\in M_{(np-n)\times np}(\mathbf{R})$

$$\begin{pmatrix}\Sigma^o\\0\end{pmatrix}^+Q^tX^t=\begin{pmatrix}\Sigma_r^{o-1}S_1\\\Sigma_{\overline{r}}^{o-1}S_2\end{pmatrix}$$

so

$$B_1 = \begin{pmatrix} \Sigma_r^{o-1} S_1 \\ \Sigma_{\overline{r}}^{o-1} S_2 \end{pmatrix} \begin{pmatrix} \Sigma^{co} Y_m^p \\ 0 \end{pmatrix}$$

Now partitioning the matrix $S_1 = (\, S_1^p \quad S_1^s \,)$ with $S_1^p \in M_{r\times n}(\mathbf{R})$, $S_2 = (\, S_2^p \quad S_2^s \,)$ with $S_2^p \in M_{(n-r)\times n}(\mathbf{R})$, the matrix B_1 can be witten as

$$B_1 = \begin{pmatrix} \Sigma_r^{o-1} S_1^p \Sigma^{co} Y_m^p \\ \Sigma_{\overline{r}}^{o-1} S_2^p \Sigma^{co} Y_m^p \end{pmatrix} = \begin{pmatrix} \overline{B}_1 \\ \overline{B}_2 \end{pmatrix}$$

with $\overline{B}_1 \in M_{r\times m}(\mathbf{R})$

$$\|\overline{B}_1\|_2 = \|\Sigma_r^{o-1} S_1^p \Sigma^{co} Y_m^p\|_2 \leq \|\Sigma_r^{o-1}\|_2 \|S_1^p\|_2 \|\Sigma^{co}\|_2 \|Y_m^p\|_2$$

Taking into account that

$$\|\Sigma_r^{o-1}\|_2 = \sigma_r^{o-1}, \quad \|\Sigma^{co}\|_2 = \sigma_1^{co}$$

$$\|S_1^p\|_2 \leq \left\| \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \right\|_2 = 1$$

$$\|Y_m^p\|_2 \leq \|Y\|_2 = 1$$

and

$$\|\overline{B}_1\|_2 \leq \frac{\sigma_1^{co}}{\sigma_r^o}$$

Theorem (3.3). *For a given triple $(A, B, C) \in \mathcal{M}^{co}$ we have*

$$\mu^{co}(A, B, C) \leq (\|A_c\|_2 \sigma_{r+1}^o + \sigma_1^{co}) \frac{1}{\sigma_r^o} + \sigma_{r+1}^o, \quad 1 \leq r \leq n-1$$

Proof: We consider $(A_1 + \delta A_1, B_1 + \delta B_1, C_1 + \delta C_1)$ with

$$\delta A_1 = \begin{pmatrix} 0 & -\overline{A}_2 \\ 0 & 0 \end{pmatrix}, \quad \delta B_1 = \begin{pmatrix} -\overline{B}_1 \\ 0 \end{pmatrix}, \quad \delta C_1 = (\, 0 \quad -\overline{C}_2 \,)$$

The triple $(A_1 + \delta A_1, B_1 + \delta B_1, C_1 + \delta C_1)$ is an uncontrollable and unobservable triple of matrices for all $1 \leq r \leq n-1$.

Then

$$\|(\delta A_1, \delta B_1, \delta C_1)\|_2 \geq \mu^{co}(A_1, B_1, C_1) = \mu^{co}(A, B, C)$$

Finally, in this case we have

$$\|(\delta A_1, \delta B_1, \delta C_1)\|_2 \leq \|\delta A_1\|_2 + \|\delta B_1\|_2 + \|\delta C_1\|_2 =$$
$$= \|\overline{A}_2\|_2 + \|\overline{B}_1\|_2 + \|\overline{C}_2\|_2 \leq \|A_c\|_2 \frac{\sigma_{r+1}^o}{\sigma_r^o} + \frac{\sigma_1^{co}}{\sigma_r^o} + \sigma_{r+1}^o.$$

Now we deduce a bound relating the $C(A, B, C)$ and $CO(A, B, C)$ matrices using the duality relation that there exist into $C(A, B, C)$ and $O(A, B, C)$.

THEOREM (3.4). *Let* $(A,B,C)\in\mathcal{M}^{co}$, *then*

$$\mu^{co}(A,B,C)\leq\left(\|A_c\|_2\sigma^c_{r+1}+\sigma^{co}_1\right)\frac{1}{\sigma^c_r}+\sigma^c_{r+1},\ 1\leq r\leq n-1$$

PROOF: For that it suffices to observe that if (A,B,C), $A\in M_n(\mathbf{R})$, $B\in M_{n\times m}(\mathbf{R})$, $C\in M_{p\times n}(\mathbf{R})$ is a controllable and observable triple then the triple (A^t,C^t,B^t) is also a controllable and observable triple. And in the other hand, we observe that given a matrix $M\in M_{r\times s}(\mathbf{R})$, M and M^t have the same non-zero singular values

Theorems (3.3) and (3.4) permit us to deduce the following bound for μ^{co}.

COROLLARY (3.4). *Let* $(A,B,C)\in\mathcal{M}^{co}$. *Then*

$$\mu^{co}(A,B,C)\leq$$
$$\min\left(\left(\|A_c\|_2\sigma^o_{r+1}+\sigma^{co}_1\right)\frac{1}{\sigma^o_r}+\sigma^o_{r+1},\left(\|A_c\|_2\sigma^c_{r+1}+\sigma^{co}_1\right)\frac{1}{\sigma^c_r}+\sigma^c_{r+1}\right),$$

for $1\leq r\leq n-1$.

EXAMPLE (3.1): Let (A,B,C) the triple defined as follows

$$A=\begin{pmatrix}0.1&0.1&0\\0&0.01&0.01\\0&0&0.01\end{pmatrix},\ B=\begin{pmatrix}0\\0\\0.1\end{pmatrix},\ C=\begin{pmatrix}0.1&0&0\end{pmatrix}$$

$$C(A,B,C)=\begin{pmatrix}0&0&0.0001\\0&0.001&0.00002\\0.1&0.001&0.00001\end{pmatrix}$$

$$O(A,B,C)=\begin{pmatrix}0.1&0&0\\0.01&0.01&0\\0.001&0.0011&0.0001\end{pmatrix}$$

$$CO(A,B,C)=\begin{pmatrix}0&0&10^{-5}\\0&10^{-5}&0.12\cdot10^{-5}\\10^{-5}&0.12\cdot10^{-5}&0.123\cdot10^{-6}\end{pmatrix}$$

$$\mu^c=0.01107294359$$

$$\mu^o=0.01010136582$$

$$\mu^{co}=0.01118193072$$

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