

Singular separatrix splitting and the Poincaré-Melnikov method for area preserving maps

Amadeu Delshams and Rafael Ramírez-Ros
Departament de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Diagonal 647, 08028 Barcelona, Spain

Abstract

The splitting of separatrices of area preserving maps close to the identity is one of the most paradigmatic examples of an *exponentially small* or *singular* phenomenon. The intrinsic small parameter is the characteristic exponent $h > 0$ of the saddle fixed point. A standard technique to measure the splitting of separatrices is the so-called Poincaré-Melnikov method, which has several specific features in the case of analytic planar maps. The aim of this talk is to compare the predictions for the splitting of separatrices provided by the Poincaré-Melnikov method, with the analytic and numerical results in a simple example where computations in multiple-precision arithmetic are performed.

1 Introduction

In this talk, we review some results concerning the singular splitting of separatrices for the family of planar standard-like maps

$$F(x, y) = (y, -x + U'(y)) = \left(y, -x + \frac{2\mu_0 y}{1 + y^2} + \varepsilon V'(y) \right), \quad (1)$$

where $V(y) = \sum_{n \geq 1} V_n y^{2n}$ is an even entire function. These maps are area preserving.

If $\mu_0 + V_1 \varepsilon > 1$, the origin $O = (0, 0)$ is a saddle point with $\text{Spec}[DF_0(O)] = \{e^h, e^{-h}\}$, where the *characteristic exponent* $h > 0$ is given by $\cosh h = \mu_0 + V_1 \varepsilon$. In the weakly hyperbolic situation ($h \rightarrow 0^+$), the splitting size turns out to be exponentially small in h , so that its measure requires a very careful treatment, both from a numerical and an analytical point of view.

Our choice of the above maps, instead of more celebrated standard-like maps like the Hénon map or the standard map, is due to the fact that some explicit exponentially small estimates of the splitting size can be obtained for those maps. The estimates are computed using a discrete version of the Poincaré-Melnikov method.

This talk is organized as follows. In section 2, the discrete version of the Poincaré-Melnikov method for twist maps is described. Section 3 contains the analytical results about the splitting, both in the regular (Theorem 1) and singular (Theorem 2) cases. Finally, the numerical experiments are presented in section 4.

2 The Poincaré-Melnikov method

Let $F_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a *twist* diffeomorphism with twist generating function \mathcal{L}_0 , that is, $(X, Y) = F_0(x, y)$ if and only if $y = -\partial_1 \mathcal{L}_0(x, X)$ and $Y = \partial_2 \mathcal{L}_0(x, X)$. We will assume that F_0 has a *separatrix* Λ to a *saddle point* $z_0^\infty = (x_0^\infty, y_0^\infty) \notin \Lambda$; that is, the invariant curves of z_0^∞ coincide along the separatrix Λ .

Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a twist perturbation of F_0 , with generating function $\mathcal{L} = \mathcal{L}_0 + \varepsilon \mathcal{L}_1 + O(\varepsilon)$. For $|\varepsilon|$ small enough, F has a saddle point z^∞ “close” to z_0^∞ , whose local invariant curves are

“close” to the unperturbed separatrix, although, in general, the perturbed invariant curves no longer coincide.

To compute their distance, it is standard to invoke the Poincaré-Melnikov method which, in the case of area preserving maps, is based on the properties of the *Melnikov potential* (see [2, 6] and references therein)

$$L(z) = \sum_{k \in \mathbb{Z}} [\mathcal{L}_1(x_k, x_{k+1}) - \mathcal{L}_1(x_0^\infty, x_0^\infty)], \quad z_k = (x_k, y_k) = F_0^k(z), \quad z \in \Lambda. \quad (2)$$

If L has a non-degenerate critical point at $z = z_0$, then there exists a transverse homoclinic point $z_h = z_0 + O(\varepsilon)$ of the perturbed map. Moreover, if z_0 and z'_0 denote consecutive non-degenerate critical points of L , the pieces of the perturbed invariant curves between the associated homoclinic points z_h and z'_h enclose a region called *lobe*, whose area is invariant under F . This *lobe area* A has a nice expression in terms of the Melnikov potential:

$$A = \varepsilon [L(z'_0) - L(z_0)] + O(\varepsilon^2). \quad (3)$$

There exist similar theories for $2d$ -dimensional exact symplectic maps and Hamiltonian flows (see [3, 1] and references therein).

3 Analytical results

We are now ready to apply the Poincaré-Melnikov method to our maps.

To begin with, the maps (1) are twist maps with generating function $\mathcal{L}(x, X) = -xX + U(X)$. Moreover, the origin is a saddle point with characteristic exponent $h > 0$, $\cosh h = \mu$, provided that $\mu := \mu_0 + V_1\varepsilon > 1$.

It is convenient to split the function $U(y) = \mu_0 \log(1 + y^2) + \varepsilon V(y)$ as $U(y) = U_0(y) + \varepsilon U_1(y)$, with $U_0(y) = \mu \log(1 + y^2)$ and $U_1(y) = V(y) - V_1 \log(1 + y^2)$. Then the maps (1) take the form

$$F(x, y) = F_0(x, y) + \varepsilon F_1(x, y), \quad F_0(x, y) = (y, -x + U_0(y)), \quad F_1(x, y) = (0, U_1(y)),$$

and so $\mathcal{L}(x, X) = \mathcal{L}_0(x, X) + \varepsilon \mathcal{L}_1(x, X)$, with $\mathcal{L}_0(x, X) = -xX + U_0(X)$ and $\mathcal{L}_1(X) = U_1(X)$.

When $\varepsilon = 0$, the unperturbed map

$$F_0(x, y) = (y, -x + U'_0(y)) = \left(y, -x + \frac{2\mu y}{1 + y^2} \right)$$

is called *McMillan map* [8] and is *integrable*, with a polynomial first integral given by

$$I_0(x, y) = x^2 - 2\mu xy + y^2 + x^2 y^2.$$

Since $\mu > 1$, the zero level of I_0 is a lemniscate, whose loops are *separatrices* to the origin. The separatrix Λ in the quadrant $\{x, y > 0\}$ can be parameterized by

$$z_0(t) = (x_0(t), y_0(t)) = (\xi_0(t - h/2), \xi_0(t + h/2)), \quad \xi_0(t) = \gamma \operatorname{sech} t, \quad \gamma = \sinh h. \quad (4)$$

which is a *natural* parameterization; that is, $F_0(z_0(t)) = z_0(t + h)$.

When $\varepsilon \neq 0$, the map F can be thought as a perturbed McMillan map, although the characteristic exponent of the origin does *not* change, since the perturbation $\varepsilon U'_1(y)$ begins with cubic terms. These two parameters, the *characteristic exponent* $h > 0$ and the *perturbation strength* ε , will be considered the intrinsic parameters of our family of maps.

Since $U(y)$ is an even function, F is an odd map, and thus the perturbed invariant curves of the origin are symmetric with respect to the origin, so that we concentrate only on the positive quadrant $\{x, y > 0\}$.

By the form of the perturbation, $R^+(x, y) = (y, x)$ is a reversor of F , as well as the involution $R^- := F \circ R^+$, which is given by $R^-(x, y) = (x, -y + U'(x))$: $F_0^{-1} = R^\pm \circ F_0 \circ R^\pm$. Consequently, the perturbed invariant curves intersect at points $z^\pm \in C^\pm = \{z : R^\pm z = z\}$, which give rise to two *primary* symmetric homoclinic orbits \mathcal{O}^\pm on the quadrant $\{x, y > 0\}$.

Using the parameterization (4), the Melnikov potential (2) of our problem is given by

$$L(t) := L(z_0(t)) = \sum_{n \in \mathbb{Z}} U_1(y_0(t + hn)) = \sum_{n \in \mathbb{Z}} V(\xi_0(t + h/2 + hn)),$$

and is a *doubly-periodic* function: $L(t) = L(t + h) = L(t + \pi i)$. As a consequence, the series of $L(t)$ can be computed explicitly through the study of the complex singularities of $L(t)$ [2], leading to the expression (modulo an additive constant):

$$L(t) = e^{-\pi^2/h} \cos(2\pi t/h) \left[-\Theta^0(h)/2 + \mathcal{O}(e^{-2\pi^2/h}) \right], \quad (5)$$

in terms of an even entire function $\Theta^0(h)$ that satisfies

$$\Theta^0(h) = 8\pi \sum_{n \geq 1} \frac{(2\pi)^{2n-1}}{(2n-1)!} V_n + \mathcal{O}(h^2) = 8\pi \widehat{V}(2\pi) + \mathcal{O}(h^2), \quad (6)$$

where $\widehat{V}(\xi) := \sum_{n \geq 1} V_n \xi^{2n-1} / (2n-1)!$ is the so-called *Borel transform* of $V(y)$. We refer to [4] for the details.

From formula (5), one sees that the Melnikov potential $L(t)$ is $\mathcal{O}(\exp(-\pi^2/h))$, and that for $\widehat{V}(2\pi) \neq 0$ and h small enough, all the critical points of $L(t)$ are non-degenerate and contained in $h\mathbb{Z}/2$. Therefore, the Poincaré-Melnikov formula (3) gives the following result.

Theorem 1 *Assume that $\widehat{V}(2\pi) \neq 0$. Then, for $0 < |\varepsilon| < \varepsilon^*(h) = \mathcal{O}(\exp(-\pi^2/h))$, the map (1) has exactly two transverse, symmetric, primary homoclinic orbits in the quadrant $\{x, y > 0\}$. These orbits determine a lobe with area $A = \varepsilon A_{\text{Mel}} + \mathcal{O}(\varepsilon^2)$, where the first order in ε approximation A_{Mel} is given by*

$$A_{\text{Mel}} = L(h/2) - L(0) = e^{-\pi^2/h} \left[\Theta^0(h) + \mathcal{O}(e^{-2\pi^2/h}) \right]. \quad (7)$$

In the singular case ($h \rightarrow 0^+$) and $\varepsilon = \mathcal{O}(h^p)$, $p > 0$, Theorem 1 cannot be applied, since it requires ε to be exponentially small in h . For $p > 6$, the following theorem [4, 5] gives an asymptotic expression for the lobe area in terms of the Melnikov potential, and establishes transversal splitting of separatrices.

Theorem 2 *Assume that $\varepsilon = \mathcal{O}(h^p)$, $p > 6$. Then, if $\widehat{V}(2\pi) \neq 0$, there exists $h^* > 0$ such that the map (1) has exactly two transverse, symmetric, primary homoclinic orbits in the first quadrant, for all $0 < h < h^*$. Moreover, they enclose a lobe with area*

$$A = \varepsilon e^{-\pi^2/h} \left[8\pi \widehat{V}(2\pi) + \mathcal{O}(h^2) \right] \quad (h \rightarrow 0^+).$$

As a consequence, for $\varepsilon = o(h^p)$, $p > 6$, the lobe area A is given, in first order, by the Poincaré-Melnikov method. At this very moment, one is confronted with several questions about the expression of the lobe area $A = A(h, \varepsilon)$ in the singular case $h \rightarrow 0^+$:

1. Is the lobe area A also given by the Poincaré-Melnikov for $\varepsilon = o(h^p)$, $0 < p \leq 6$?
2. Is A also given by the Poincaré-Melnikov for $\varepsilon \rightarrow 0$ independently of $h \rightarrow 0^+$?
3. What is the expression for A for ε fixed and $h \rightarrow 0^+$?

The singular cases **1** and **2** are *perturbative*, whereas the singular case **3** is *non-perturbative*. To deal with them, we performed some numerical experiments to compute the lobe area A , as explained in the next section.

4 Numerical results

We now study numerically the cases not covered by the analytical results in the singular limit.

The computation of the lobe area A near the singular limit is a delicate question, since several numerical problems are intrinsic to it. To mention only a few: (i) the dynamics near the saddle point is extremely slow, due to the weakly hyperbolic character of the fixed point; (ii) the lobe area is computed as a difference of exponentially close actions, giving rise to a massive cancellation of significant digits; and (iii) the general algorithms for computing homoclinic points have condition numbers inversely proportional to the splitting size, being exponentially ill-conditioned for our singular maps.

To overcome these difficulties, we are forced to: (i') expand the local invariant curves up to very high (but optimal) order; (ii') use an expensive multiple-precision arithmetic; and (iii') take the greatest advantage of symmetries and/or reversors. For instance, the computations in the most extreme cases (in which $h = 0.001$ and the lobe area A was less than 10^{-4200}) have required more than 5200 digits in the arithmetic expanding the invariant curves up to order 1300. We refer to [5] for the details and the implementation of the programs.

All these items encourage us to restrict the numerical experiments to the simplest perturbations; that is, for the linear perturbation $\varepsilon V'(y) = \varepsilon y$, and the cubic one $\varepsilon V'(y) = \varepsilon y^3$.

When $V(y)$ is a polynomial the Melnikov potential $L(t)$ is an elliptic function, and $\Theta^0(h)$ can be explicitly computed in a finite number of steps [2]:

$$\Theta^0(h) = \begin{cases} 8\pi^2\gamma^2h^{-2} & \text{for } V'(y) = y \\ \frac{8}{3}\pi^2\gamma^4h^{-2}[1 + \pi^2h^{-2}] & \text{for } V'(y) = y^3 \end{cases} . \quad (8)$$

Taking into account the expression (7) of Theorem 1, in the singular case we look, for every fixed ε , for a formula for the lobe area of the form

$$A = \varepsilon e^{-\pi^2/h} \left[\Theta^\varepsilon(h) + O(e^{-2\pi^2/h}) \right], \quad (\text{fixed } \varepsilon, h \rightarrow 0^+), \quad (9)$$

for a function $\Theta^\varepsilon(h)$ given by an asymptotic series of the form

$$\Theta^\varepsilon(h) \sim \sum_{n \geq 0} \Theta_n^\varepsilon h^{2n}, \quad (\text{fixed } \varepsilon, h \rightarrow 0^+). \quad (10)$$

The sign \sim means that the series above needs not to be convergent, but only asymptotic: for every $N \geq 0$,

$$\left| \Theta^\varepsilon(h) - \sum_{n=0}^N \Theta_n^\varepsilon h^{2n} \right| = O(h^{2N+2}).$$

After computing and analyzing a relevant number of coefficients Θ_n^ε (up to $n = 100$, say), the numerical results obtained for ε fixed and $h \rightarrow 0^+$ (the non-perturbative case) can be summarized as follows.

Numerical result 1 *For the linear and cubic perturbations, the following asymptotic expansion for the lobe area A holds*

$$A \sim \varepsilon e^{-\pi^2/h} \sum_{n \geq 0} \Theta_n^\varepsilon h^{2n} \quad (\varepsilon \text{ fixed, } h \rightarrow 0^+),$$

where the coefficients Θ_n^ε verify

$$\Theta_n^\varepsilon = (2n)!(2\pi^2)^{-2n}(2n)^4 \left[\Xi_\infty^\varepsilon + O(n^{-1}) \right], \quad (11)$$

as $n \rightarrow +\infty$, for some constant $\Xi_\infty^\varepsilon \neq 0$. ($\Xi_\infty^\varepsilon < 0$ for $\varepsilon > 0$.)

In particular, the series $\Theta^\varepsilon(h) = \sum_{n \geq 0} \Theta_n^\varepsilon h^{2n}$ is divergent but it is Gevrey-1, i.e., its Borel transform $\widehat{\Theta}^\varepsilon(h) = \sum_n \Theta_n^\varepsilon \xi^{2n-1} / (2n-1)!$ is convergent, for $|\xi| < 2\pi^2$.

The next step is to consider the perturbative case, and therefore to check whether all the previous objects $\Theta^\varepsilon(h)$, Θ_n^ε , Ξ_∞^ε , tend to well-defined limits, as $\varepsilon \rightarrow 0$. This was confirmed by the numerical experiments.

Numerical result 2 *For the linear and cubic perturbations, the objects $\Theta^\varepsilon(h)$, Θ_n^ε , Ξ_∞^ε , introduced in the Numerical Result 1, tend to well-defined limits, as $\varepsilon \rightarrow 0$. More precisely,*

1. $\Theta^\varepsilon(h) = \Theta^0(h) + O(\varepsilon)$, uniformly in $h \in (0, 1]$.
2. $\Theta_n^\varepsilon = \Theta_n^0 + O(\varepsilon)$, non-uniformly in $n \geq 0$.
3. $\Xi_\infty^\varepsilon = \varepsilon \Xi_\infty^0 + O(\varepsilon^2)$, where $\Xi_\infty^0 = \begin{cases} -12\pi^{-4} & \text{if } V'(y) = y \\ -16/3 & \text{if } V'(y) = y^3 \end{cases}$.

5 Closing

From the numerical results computed for the map (1), one sees that the Melnikov potential provides the right answer even in the singular case $\varepsilon \rightarrow 0$ independently of $h \rightarrow 0^+$. Therefore, one should improve analytically the hypothesis $\varepsilon = o(h^p)$, $p > 6$ in Theorem 2.

As a matter of fact, the Numerical result 1 suggests that resurgence tools [7] may be successful to fill this gap between analytical and numerical results. Asymptotic expressions similar to (11) seem to be true [9] for some celebrated entire maps, like the Hénon map or the standard map.

Acknowledgments

This work was partially supported by the Catalan grant CIRIT 1998SGR-00041. Research by A. Delshams is also supported by the Spanish grant DGICYT PB94-0215.

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E-mail addresses of the authors: amadeu@ma1.upc.es, rafael@tere.upc.es.