Normal transversality and uniform bounds

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1 Introduction

Let A be a commutative ring. A graded A-algebra $U = \oplus_{n\geq 0} U_n$ is a *standard* A-algebra if $U_0 = A$ and $U = A[U_1]$ is generated as an A-algebra by the elements of U_1. A graded U-module $F = \oplus_{n\geq 0} F_n$ is a *standard* U-module if F is generated as an U-module by the elements of F_0, that is, $F_n = U_n F_0$ for all $n \geq 0$. In particular, $F_n = U_1 F_{n-1}$ for all $n \geq 1$. Given I, J, two ideals of A, we consider the following standard algebras: the *Rees algebra of* I, $\mathcal{R}(I) = \oplus_{n\geq 0} I^n t^n = A[It] \subset A[t]$, and the *multi-Rees algebra of* I and J, $\mathcal{R}(I,J) = \oplus_{n\geq 0}(\oplus_{p+q=n} I^p J^q u^p v^q) = A[Iu, Jv] \subset A[u,v]$. Consider the *associated graded ring of* I, $\mathcal{G}(I) = \mathcal{R}(I) \otimes A/I = \oplus_{n\geq 0} I^n/I^{n+1}$, and the *multi-associated graded ring of* I and J, $\mathcal{G}(I,J) = \mathcal{R}(I,J) \otimes A/(I+J) = \oplus_{n\geq 0}(\oplus_{p+q=n} I^p J^q/(I+J)I^p J^q)$. We can always consider the tensor product of two standard A-algebras $U = \oplus_{p\geq 0} U_p$ and $V = \oplus_{q\geq 0} V_q$ as a standard A-algebra with the natural grading $U \otimes V = \oplus_{n\geq 0}(\oplus_{p+q=n} U_p \otimes V_q)$. If M is an A-module, we have the standard modules: the *Rees module of* I *with respect to* M, $\mathcal{R}(I;M) = \oplus_{n\geq 0} I^n M t^n = M[It] \subset M[t]$ (a standard $\mathcal{R}(I)$-module), and the *multi-Rees module of* I and J *with respect to* M, $\mathcal{R}(I,J;M) = \oplus_{n\geq 0}(\oplus_{p+q=n} I^p J^q M u^p v^q) = M[Iu, Jv] \subset M[u,v]$ (a standard $\mathcal{R}(I,J)$-module). Consider the *associated graded module of* M *with respect to* I, $\mathcal{G}(I;M) = \mathcal{R}(I;M) \otimes A/I = \oplus_{n\geq 0} I^n M/I^{n+1} M$ (a standard $\mathcal{G}(I)$-module), and the *multi-associated graded module of* M *with respect to* I and J, $\mathcal{G}(I,J;M) = \mathcal{R}(I,J;M) \otimes A/(I+J) = \oplus_{n\geq 0}(\oplus_{p+q=n} I^p J^q M/(I+J)I^p J^q M)$ (a standard $\mathcal{R}(I,J)$-module). If U, V are two standard A-algebras and F is a standard U-module and G is a standard V-module, then $F \otimes G = \oplus_{n\geq 0}(\oplus_{p+q=n} F_p \otimes G_q)$ is a standard $U \otimes V$-module.

Denote by $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J;M) \to \mathcal{R}(I,J;M)$ and $\sigma : \mathcal{R}(I,J;M) \to \mathcal{R}(I+J;M)$ the natural surjective graded morphisms of standard $\mathcal{R}(I) \otimes \mathcal{R}(J)$-modules. Let $\varphi : \mathcal{R}(I) \otimes \mathcal{R}(J;M) \to \mathcal{R}(I+J;M)$ be $\sigma \circ \pi$. Denote by $\overline{\pi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I,J;M)$ and $\overline{\sigma} : \mathcal{G}(I,J;M) \to \mathcal{G}(I+J;M)$ the tensor product of π and σ by $A/(I+J)$; these are two natural surjective graded morphisms of standard $\mathcal{G}(I) \otimes \mathcal{G}(J)$-modules. Let $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I+J;M)$ be $\overline{\sigma} \circ \overline{\pi}$. The first purpose of this note is to prove the following theorem:

Theorem 1 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A-module. The following two conditions are equivalent:*

(i) $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I+J;M)$ *is an isomorphism.*

(ii) $\mathrm{Tor}_1(A/I^p, \mathcal{R}(J;M)) = 0$ *and* $\mathrm{Tor}_1(A/I^p, \mathcal{G}(J;M)) = 0$ *for all integers* $p \geq 1$.

In particular, $\mathcal{G}(I) \otimes \mathcal{G}(J) \simeq \mathcal{G}(I+J)$ *if and only if* $\mathrm{Tor}_1(A/I^p, A/J^q) = 0$ *and* $\mathrm{Tor}_2(A/I^p, A/J^q) = 0$ *for all integers* $p, q \geq 1$.

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The morphism $\overline{\varphi}$ has been studied by Hironaka [**5**], Grothendieck [**3**] and Hermann, Ikeda and Orbanz [**4**], among others, but assuming always A is normally flat along I (see 21.11 in [**4**]). We will see how Theorem 1 generalizes all this former work.

Let us now recall some definitions in order to state the second purpose of this note. If U is a standard A-algebra and F is a graded U-module, put $s(F) = \min\{r \geq 1 \mid F_n = 0 \text{ for all } n \geq r+1\}$, where $s(F)$ may possibly be infinite. If $U_+ = \oplus_{n>0} U_n$ and $r \geq 1$, the following three conditions are equivalent: F can be generated by elements of degree at most r; $s(F/U_+F) \leq r$; and $F_n = U_1 F_{n-1}$ for all $n \geq r+1$. If $\varphi : G \to F$ is a surjective graded morphism of graded U-modules, we denote by $E(\varphi)$ the graded A-module $E(\varphi) = \ker\varphi / U_+ \ker\varphi = \ker\varphi_0 \oplus (\oplus_{n\geq 1} \ker\varphi_n / U_1 \ker\varphi_{n-1}) = \oplus_{n\geq 0} E(\varphi)_n$. If F is a standard U-module, take $\mathbf{S}(U_1)$ the symmetric algebra of U_1, $\alpha : \mathbf{S}(U_1) \to U$ the surjective graded morphism of standard A-algebras induced by the identity on U_1 and $\gamma : \mathbf{S}(U_1) \otimes F_0 \stackrel{\alpha\otimes 1}{\to} U \otimes F_0 \to F$ the composition of $\alpha \otimes 1$ with the structural morphism. Since F is a standard U-module, γ is a surjective graded morphism of graded $\mathbf{S}(U_1)$-modules. The *module of effective n-relations* of F is defined to be $E(F)_n = E(\gamma)_n = \ker\gamma_n / U_1 \ker\gamma_{n-1}$ (for $n = 0$, $E(F)_n = 0$). Put $E(F) = \oplus_{n\geq 1} E(F)_n = \oplus_{n\geq 1} E(\gamma)_n = E(\gamma) = \ker\gamma / \mathbf{S}_+(U_1)\ker\gamma$. The *relation type* of F is defined to be $\mathrm{rt}(F) = s(E(F))$, that is, $\mathrm{rt}(F)$ is the minimum positive integer $r \geq 1$ such that the effective n-relations are zero for all $n \geq r+1$. A *symmetric presentation* of a standard U-module F is a surjective graded morphism of standard V-modules $\varphi : G \to F$, with $\varphi : G = V \otimes M \stackrel{f\otimes h}{\to} U \otimes F_0 \to F$, where V is a symmetric A-algebra, $f : V \to U$ is a surjective graded morphism of standard A-algebras, $h : M \to F_0$ is an epimorphism of A-modules and $U \otimes F_0 \to F$ is the structural morphism. One can show (see [**8**]) that $E(F)_n = E(\varphi)_n$ for all $n \geq 2$ and $s(E(F)) = s(E(\varphi))$. Thus the module of effective n-relations and the relation type of a standard U-module are independent of the chosen symmetric presentation. Roughly speaking, the relation type of F is the largest degree of any minimal homogeneous system of generators of the submodule defining F as a quotient of a polynomial ring with coefficients in F_0. For an ideal I of A and an A-module M, the module of effective n-relations and the relation type of I with repect to M are defined to be $E(I;M)_n = E(\mathcal{R}(I;M))_n$ and $\mathrm{rt}(I;M) = \mathrm{rt}(\mathcal{R}(I;M))$, respectively. Then:

Theorem 2 *Let A be a commutative ring, U and V two standard A-algebras, F a standard U-module and G a standard V-module. Then $U \otimes V$ is a standard A-algebra, $F \otimes G$ is a standard $U \otimes V$-module and* $\mathrm{rt}(F \otimes G) \leq \max(\mathrm{rt}(F), \mathrm{rt}(G))$.

As a consequence of Theorems 1 and 2, one deduces the existence of an uniform bound for the relation type of all maximal ideals of an excellent ring.

Theorem 3 *Let A be an excellent (or $J-2$) ring and let M be a finitely generated A-module. Then there exists an integer $s \geq 1$ such that, for all maximal ideals $\mathfrak{m}$ of A, the relation type of $\mathfrak{m}$ with respect to M satisfies* $\mathrm{rt}(\mathfrak{m}; M) \leq s$.

In fact, Theorem 3 could also been deduced from the proof of Theorem 4 of Trivedi in [**9**]. Finally, and using Theorem 2 of [**8**], one can recover the following result of Duncan and O'Carroll.

Corollary 4 [**2**] *Let A be an excellent (or $J-2$) ring and let $N \subseteq M$ be two finitely generated A-modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals $\mathfrak{m}$ of A, $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

2 Normal transversality

Lemma 2.1 *Let A be a commutative ring, I an ideal of A, U a standard A-algebra, F and G two standard U-modules and $\varphi : G \to F$ a surjective graded morphism of standard A-alegbras. If $\overline{A} = A/I$, then $\overline{U} = U \otimes \overline{A}$ is a standard $\overline{A}$-algebra, $\overline{F} = F \otimes \overline{A}$ and $\overline{G} = G \otimes \overline{A}$ are two standard $\overline{U}$-modules and $\overline{\varphi} = \varphi \otimes 1_{\overline{A}} : \overline{G} \to \overline{F}$ is a surjective graded morphism of standard $\overline{U}$-modules. Moreover, $s(E(\overline{\varphi})) \leq s(E(\varphi))$.*

Proof. If we tensor $0 \to \ker\varphi_i \to G_i \stackrel{\varphi_i}{\to} F_i \to 0$ by $\overline{A}$ we get $\ker\varphi_i \otimes \overline{A} \to \overline{G}_i \stackrel{\overline{\varphi}_i}{\to} \overline{F}_i \to 0$, exact sequences which induce epimorphisms $\ker\varphi_i \otimes \overline{A} \to \ker\overline{\varphi}_i$. On the other hand, if we tensor $U_1 \otimes \ker\varphi_{n-1} \to \ker\varphi_n \to E(\varphi)_n \to 0$ by $\overline{A}$ we get the exact sequence

$$\overline{U}_1 \otimes \ker\varphi_{n-1} \otimes \overline{A} \to \ker\varphi_n \otimes \overline{A} \to E(\varphi)_n \otimes \overline{A} \to 0\,.$$

We thus have the following commutative diagram of exact rows

$$\begin{array}{ccccccc}
\overline{U}_1 \otimes \ker\varphi_{n-1} \otimes \overline{A} & \longrightarrow & \ker\varphi_n \otimes \overline{A} & \longrightarrow & E(\varphi)_n \otimes \overline{A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\overline{U}_1 \otimes \ker\overline{\varphi}_{n-1} & \longrightarrow & \ker\overline{\varphi}_n & \longrightarrow & E(\overline{\varphi})_n & \longrightarrow & 0,
\end{array}$$

from where we deduce an epimorphism $E(\varphi)_n \otimes \overline{A} \to E(\overline{\varphi})_n$. In particular, $s(E(\overline{\varphi})) \leq s(E(\varphi))$. ∎

Lemma 2.2 *Let A be a commutative ring, I, J two ideals of A and M an A-module. Consider $\sigma : \mathcal{R}(I,J;M) \to \mathcal{R}(I+J;M)$ and $\overline{\sigma} = \mathcal{G}(I,J;M) \to \mathcal{G}(I+J;M)$. Then*

(*a*) $\ker(\sigma_1) \simeq IM \cap JM$.

(*b*) $\ker(\overline{\sigma}_1) = 0$ *if and only if* $IM \cap JM \subset I(I+J)M \cap (I+J)JM$.

(*c*) *If $I^pM \cap J^qM = I^pJ^qM$ for all integers $p, q \geq 1$, then $s(E(\sigma)) = 1$ and $\overline{\sigma}$ is an isomorphism.*

Proof. Consider $0 \to IM \cap JM \stackrel{\rho}{\to} IM \oplus JM \stackrel{\sigma_1}{\to} (I+J)M \to 0$ where $\rho(a) = (a, -a)$ and $\sigma_1(a,b) = a+b$. Clearly it is an exact sequence of A-modules. Thus $\ker(\sigma_1) = \rho(IM \cap JM) \simeq IM \cap JM$. If we tensor this exact sequence by $A/(I+J)$ we get $(IM \cap JM) \otimes A/(I+J) \stackrel{\overline{\rho}}{\to} (IM \oplus JM) \otimes A/(I+J) \stackrel{\overline{\sigma}_1}{\to} (I+J)M/(I+J)^2M \to 0$. Then

$$\ker(\overline{\sigma}_1) = \mathrm{im}\overline{\rho} = \{(\overline{a}, -\overline{a}) \in IM/I(I+J)M \oplus JM/(I+J)JM \mid a \in IM \cap JM\}\,.$$

Hence $\ker(\overline{\sigma}_1) = 0$ if and only if $IM \cap JM \subset I(I+J)M \cap (I+J)JM$. Now, let us prove (*c*). Let $z \in \ker\sigma_n \subset \mathcal{R}(I,J;M)_n = \oplus_{p+q=n} I^pJ^qMu^pv^q \subset M[u,v]$. Thus, $z = a_0u^n + a_1u^{n-1}v + \ldots + a_{n-1}uv^{n-1} + a_nv^n$, $a_i \in I^{n-i}J^iM$, and $0 = \sigma_n(z) = (a_0 + a_1 + \ldots + a_{n-1} + a_n)t^n \in \mathcal{R}(I+J;M)_n = (I+J)^nMt^n$. So $a_0 + a_1 + \ldots + a_{n-1} + a_n = 0$. Let us denote (see [**10**], page 134):

$$\begin{cases}
b_0 = a_0 \in I^nM \cap JM = I^nJM \\
b_1 = a_0 + a_1 \in I^{n-1}M \cap J^2M = I^{n-1}J^2M \text{ and } a_1 = b_1 - b_0 \\
b_2 = a_0 + a_1 + a_2 \in I^{n-2}M \cap J^3M = I^{n-2}J^3M \text{ and } a_2 = b_2 - b_1 \\
\ldots \\
b_{n-2} = a_0 + \ldots + a_{n-2} \in I^2M \cap J^{n-1}M = I^2J^{n-1}M \text{ and } a_{n-2} = b_{n-2} - b_{n-3} \\
b_{n-1} = a_0 + \ldots + a_{n-1} \in IM \cap J^nM = IJ^nM \text{ and } a_{n-1} = b_{n-1} - b_{n-2} \\
a_n = -b_{n-1} \in IJ^nM\,.
\end{cases}$$

We can rewrite z in $M[u,v]$ in the following manner:

$$\begin{aligned}
z &= a_0u^n + a_1u^{n-1}v + \ldots + a_{n-1}uv^{n-1} + a_nv^n = \\
&= b_0u^n + (b_1-b_0)u^{n-1}v + (b_2-b_1)u^{n-2}v^2 + \ldots + \\
&+ (b_{n-2}-b_{n-3})u^2v^{n-2} + (b_{n-1}-b_{n-2})uv^{n-1} + (-b_{n-1})v^n = \\
&= \underbrace{(b_0u^{n-1} + b_1u^{n-2}v + b_2u^{n-3}v^2 + \ldots + b_{n-2}uv^{n-2} + b_{n-1}v^{n-1})}_{p(u,v)}(u-v) := p(u,v)(u-v)\,,
\end{aligned}$$

where $p(u,v) \in A[Iu,Jv]_{n-1} \cdot (IJM) = \mathcal{R}(I,J)_{n-1} \cdot (IJM)$. Since by hypothesis $IM \cap JM = IJM$, then $\ker(\sigma_1) = (IJM)(u-v)$, $\ker(\overline{\sigma}_1) = 0$ and $z = p(u,v)(u-v) \in \mathcal{R}(I,J)_{n-1} \cdot (IJM)(u-v) = \mathcal{R}(I,J)_{n-1} \cdot \ker\sigma_1$. Thus $\ker\sigma_n = \mathcal{R}(I,J)_{n-1} \cdot \ker\sigma_1$ for all $n \geq 2$ and $s(E(\sigma)) = 1$. By Lemma 2.1, $s(E(\overline{\sigma})) \leq s(E(\sigma)) = 1$. Therefore $\ker(\overline{\sigma}_n) = \mathcal{G}(I,J)_{n-1} \cdot \ker(\overline{\sigma}_1) = 0$ for all $n \geq 2$ and $\overline{\sigma}$ is an isomorphism. ∎

Proposition 2.3 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A-module. The following two conditions are equivalent:*

(i) $\overline{\sigma} : \mathcal{G}(I,J;M) \to \mathcal{G}(I+J;M)$ *is an isomorphism.*

(ii) $I^pM \cap J^qM = I^pJ^qM$ *for all integers* $p,q \geq 1$.

Proof. Remark that we can suppose A is local. By Lemma 2.2, $(ii) \Rightarrow (i)$. Let us see $(i) \Rightarrow (ii)$, proving by double induction in $p,q \geq 1$ that

$$I^pM \cap J^qM \subset I^p(I+J)J^{q-1}M \cap (I+J)^pJ^qM\,.$$

Remark that if $I^pM \cap J^qM \subset I^p(I+J)J^{q-1}M$ for all $p,q \geq 1$, then $I^pM \cap J^qM \subset I^{p+1}M + I^pJ^qM$ and $I^pM \cap J^qM \subset I^{p+1}M \cap J^qM + I^pJ^qM$. Recursively, and using A is noetherian local and M is finitely generated, $I^pM \cap J^qM \subset (\cap_{r\geq 1} I^{p+r}M \cap J^qM) + I^pJ^qM \subset (\cap_{n\geq 1} I^nM) + I^pJ^qM = I^pJ^qM$, concluding (ii). Take $q = 1$. Let us prove by induction in $p \geq 1$ that

$$I^pM \cap JM \subset I^p(I+J)M \cap (I+J)^pJM\,.$$

For $p = 1$, we apply Lemma 2.2, (b), using the hypothesis $\overline{\sigma}_1$ is an isomorphism. Suppose

$$I^pM \cap JM \subset I^p(I+J)M \cap (I+J)^pJM$$

is true and let us prove

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)M \cap (I+J)^{p+1}JM\,.$$

Then $I^{p+1}M \cap JM \subset I^pM \cap JM \subset (I+J)^pJM$. Consider the short complex of A-modules:

$$I^{p+1}M \cap JM \xrightarrow{\alpha} I^{p+1}M \oplus (I+J)^pJM \xrightarrow{\beta} (I+J)^{p+1}M\,,$$

where $\alpha(a) = (a,-a)$ and $\beta(a,b) = a+b$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A-modules such that $\beta \circ \gamma = \sigma_{p+1}$. If we tensor this short complex by $A/(I+J)$ we obtain:

$$(I^{p+1}M \cap JM) \otimes A/(I+J) \xrightarrow{\overline{\alpha}} I^{p+1}M/I^{p+1}(I+J)M \oplus (I+J)^pJM/(I+J)^{p+1}JM$$
$$I^{p+1}M/I^{p+1}(I+J)M \oplus (I+J)^pJM/(I+J)^{p+1}JM \xrightarrow{\overline{\beta}} (I+J)^{p+1}M/(I+J)^{p+2}M\,,$$

with $\overline{\beta} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{p+1} = \overline{\beta} \circ \overline{\gamma}$ is an isomorphism and $\overline{\gamma}$ is an epimorphism, then $\overline{\gamma}$ is an isomorphism, $\overline{\beta}$ is an isomorphism, $\overline{\alpha} = 0$ and

$$I^{p+1}M \cap JM \subset I^{p+1}(I+J)M \cap (I+J)^{p+1}JM\,.$$

In particular, $I^pM \cap JM \subset I^p(I+J)M$ for all $p \geq 1$, so $I^pM \cap JM \subset I^{p+1}M + I^pJM$ and therefore $I^pM \cap JM \subset I^{p+1}M \cap JM + I^pJM$. Recursively, and using A is noetherian local and M is finitely generated, $I^pM \cap JM \subset (\cap_{r\geq 1} I^{p+r}M \cap JM) + I^pJM \subset (\cap_{n\geq 1} I^nM) + I^pJM = I^pJM$ concluding $I^pM \cap JM = I^pJM$ for all $p \geq 1$. Remark that, by the symmetry of the problem, $IM \cap J^qM = IJ^qM$ for all $q \geq 1$. Now, fix $q \geq 1$ and suppose

$$I^pM \cap J^qM \subset I^p(I+J)J^{q-1}M \cap (I+J)^pJ^qM$$

holds for all $p \geq 1$ (in particular, $I^pM \cap J^qM = I^pJ^qM$ for all $p \geq 1$). Let us prove, by induction in $p \geq 1$, that

$$I^pM \cap J^{q+1}M \subset I^p(I+J)J^qM \cap (I+J)^pJ^{q+1}M\,.$$

For $p = 1$, we have $IM \cap J^{q+1}M = IJ^{q+1}M \subset I(I+J)J^qM \cap (I+J)J^{q+1}M$. Suppose we have

$$I^pM \cap J^{q+1}M \subset I^p(I+J)J^qM \cap (I+J)^pJ^{q+1}M$$

and let us prove

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)J^qM \cap (I+J)^{p+1}J^{q+1}M\,.$$

Then $I^{p+1}M \cap J^{q+1}M \subset I^pM \cap J^{q+1}M \subset (I+J)^pJ^{q+1}M$ and $I^{p+1}M \cap J^{q+1}M \subset I^{p+1}M \cap J^qM = I^{p+1}J^qM$. Consider the short complex of A-modules:

$$I^{p+1}M \cap J^{q+1}M \stackrel{\alpha}{\longrightarrow} I^{p+q+1}M \oplus \ldots \oplus I^{p+1}J^qM \oplus (I+J)^pJ^{q+1}M \stackrel{\beta}{\longrightarrow} (I+J)^{p+q+1}M\,,$$

where $\alpha(a) = (0,\ldots,0,a,-a)$ and $\beta(a_1,\ldots,a_{q+2}) = a_1 + \ldots + a_{q+2}$. Remark that $\beta \circ \alpha = 0$, β is surjective and that there exists a natural epimorphism γ of A-modules such that $\beta \circ \gamma = \sigma_{p+q+1}$. If we tensor this complex by $A/(I+J)$ we obtain $\overline{\beta} \circ \overline{\alpha} = 0$. Since $\overline{\sigma}_{p+q+1} = \overline{\beta} \circ \overline{\gamma}$ is an isomorphism and $\overline{\gamma}$ is an epimorphism, then $\overline{\gamma}$ is an isomorphism, $\overline{\beta}$ is an isomorphism, $\overline{\alpha} = 0$ and

$$I^{p+1}M \cap J^{q+1}M \subset I^{p+1}(I+J)J^qM \cap (I+J)^{p+1}J^{q+1}JM\,. \blacksquare$$

Proposition 2.4 *Let A be a commutative ring, I an ideal of A and $\lambda : M \otimes N \to P$ an epimorphism of A-modules. Consider $f : \mathcal{R}(I;M) \otimes N \to \mathcal{R}(I;P)$ and $\overline{f} = f \otimes 1_{A/I} : \mathcal{G}(I;M) \otimes N \to \mathcal{G}(I;P)$ the natural surjective graded morphisms of standard modules. Then, for each integer $n \geq 2$, there exists an exact sequence of A-modules $E(f)_{n+1} \to E(f)_n \to E(\overline{f})_n \to 0$. In particular, if A is noetherian, M, N, P are finitely generated and $\overline{f}$ is an isomorphism, then f is an isomorphism.*

Proof. For each integer $n \geq 1$, tensor $0 \to I^nM \to M \to M/I^nM \to 0$ by N and get the exact sequence $I^nM \otimes N \to M \otimes N \to M \otimes N/I^n(M \otimes N) \to 0$. In particular, there exist epimorphisms $I^nM \otimes N \to I^n(M \otimes N)$ which induce a surjective graded morphism of standard $\mathcal{R}(I)$-modules $g : \mathcal{R}(I;M) \otimes N \to \mathcal{R}(I;M \otimes N)$. Clearly λ induces a surjective graded morphism of standard $\mathcal{R}(I)$-modules $h : \mathcal{R}(I;M \otimes N) \to \mathcal{R}(I;P)$. The composition defines the surjective graded morphism of standard $\mathcal{R}(I)$-modules $f = h \circ g : \mathcal{R}(I;M) \otimes N \to \mathcal{R}(I;P)$. If we tensor f by A/I, we get $\overline{f} : \mathcal{G}(I;M) \otimes N \to \mathcal{G}(I;P)$ a surjective graded morphism of standard $\mathcal{G}(I)$-modules.

Let X be an A-module. The following is a commutative diagram of exact columns with rows the last three nonzero terms of the complexes $\mathcal{K}(\mathcal{R}(I;X))_{n+1}$, $\mathcal{K}(\mathcal{R}(I;X))_n$ and $\mathcal{K}(\mathcal{G}(I;X))_n$ (see Proposition 2.6 in [**8**] for more details):

$$\begin{array}{ccccccccc}
\mathcal{K}(\mathcal{R}(I;X))_{n+1} & \cdots & \mathbf{\Lambda}_2 I\otimes I^{n-1}X & \xrightarrow{\partial_{2,n+1}} & I\otimes I^n X & \xrightarrow{\partial_{1,n+1}} & I^{n+1}X & \longrightarrow & 0 \\
\downarrow u. & & \downarrow u_2 & & \downarrow u_1 & & \downarrow u_0 & & \\
\mathcal{K}(\mathcal{R}(I;X))_{n} & \cdots & \mathbf{\Lambda}_2 I\otimes I^{n-2}X & \xrightarrow{\partial_{2,n}} & I\otimes I^{n-1} X & \xrightarrow{\partial_{1,n}} & I^{n}X & \longrightarrow & 0 \\
\downarrow v. & & \downarrow v_2 & & \downarrow v_1 & & \downarrow v_0 & & \\
\mathcal{K}(\mathcal{G}(I;X))_{n} & \cdots & \mathbf{\Lambda}_2 I/I^2\otimes I^{n-2}X/I^{n-1}X & \xrightarrow[\partial_{2,n}]{} & I/I^2\otimes I^{n-1} X/I^nX & \xrightarrow[\partial_{1,n}]{} & I^{n}X/I^{n+1}X & \longrightarrow & 0
\end{array}$$

In other words, $\mathcal{K}(\mathcal{R}(I;X))_{n+1} \stackrel{u.}{\to} \mathcal{K}(\mathcal{R}(I;X))_n \stackrel{v.}{\to} \mathcal{K}(\mathcal{G}(I;X))_n \to 0$ is an exact sequence of complexes. It induces the morphisms in homology: $H_1(\mathcal{K}(\mathcal{R}(I;X))_{n+1}) \stackrel{u}{\to} H_1(\mathcal{K}(\mathcal{R}(I;X))_n)$ and $H_1(\mathcal{K}(\mathcal{R}(I;X))_n) \stackrel{v}{\to} H_1(\mathcal{K}(\mathcal{G}(I;X))_n)$. By Proposition 2.6 in [**8**], $H_1(\mathcal{K}(\mathcal{R}(I;X))_n) = E(I;X)_n$ and $H_1(\mathcal{K}(\mathcal{G}(I;X))_n) = E(\mathcal{G}(I;X))_n$. Thus we have $E(I;X)_{n+1} \stackrel{u}{\to} E(I;X)_n \stackrel{v}{\to} E(\mathcal{G}(I;X))_n$. Since $v.\circ u. = 0$, then $v\circ u = 0$. Since u_0 is injective, then $\ker v \subset \operatorname{im} u$. Since $H_0(\mathcal{K}(\mathcal{R}(I;X))_{n+1}) = 0$, then v is surjective. So $E(I;X)_{n+1} \stackrel{u}{\to} E(I;X)_n \stackrel{v}{\to} E(\mathcal{G}(I;X))_n \to 0$ is an exact sequence of A-modules. For $X = P$ we get the exact sequence of A-modules: $E(I;P)_{n+1} \stackrel{u}{\to} E(I;P)_n \stackrel{v}{\to} E(\mathcal{G}(I;P))_n \to 0$. Take $X = M$ in $\mathcal{K}(\mathcal{R}(I;X))_{n+1} \stackrel{u.}{\to} \mathcal{K}(\mathcal{R}(I;X))_n \stackrel{v.}{\to} \mathcal{K}(\mathcal{G}(I;X))_n \to 0$ and tensor it by N. Then we get the exact sequence of complexes

$$\mathcal{K}(\mathcal{R}(I;M))_{n+1}\otimes N \stackrel{\alpha.=u.\otimes 1}{\longrightarrow} \mathcal{K}(\mathcal{R}(I;M))_n\otimes N \stackrel{\beta.=v.\otimes 1}{\longrightarrow} \mathcal{K}(\mathcal{G}(I;M))_n\otimes N \longrightarrow 0\,.$$

That is, we obtain the exact sequence:

$$\mathcal{K}(\mathcal{R}(I;M)\otimes N)_{n+1} \xrightarrow{\alpha.} \mathcal{K}(\mathcal{R}(I;M)\otimes N)_n \xrightarrow{\beta.} \mathcal{K}(\mathcal{G}(I;M)\otimes N)_n \longrightarrow 0\,,$$

which induces the morphisms in homology

$$H_1(\mathcal{K}(\mathcal{R}(I;M)\otimes N)_{n+1}) \stackrel{\alpha}{\to} H_1(\mathcal{K}(\mathcal{R}(I;M)\otimes N)_n) \stackrel{\beta}{\to} H_1(\mathcal{K}(\mathcal{G}(I;M)\otimes N)_n)\,.$$

Again, by Proposition 2.6 in [**8**], $H_1(\mathcal{K}(\mathcal{R}(I;M)\otimes N)_n) = E(\mathcal{R}(I;M)\otimes N)_n$ and $H_1(\mathcal{K}(\mathcal{G}(I;M)\otimes N)_n) = E(\mathcal{G}(I)\otimes M)_n$. Moreover, since $\beta.\circ\alpha. = 0$, then $\beta\circ\alpha = 0$, and since $H_0(\mathcal{K}(\mathcal{R}(I;M)\otimes N)_{n+1}) = 0$, then β is an epimorphism. Thus we have

$$E(\mathcal{R}(I;M)\otimes N)_{n+1} \xrightarrow{\alpha} E(\mathcal{R}(I;M)\otimes N)_n \xrightarrow{\beta} E(\mathcal{G}(I;M)\otimes N)_n \longrightarrow 0$$

with $\beta\circ\alpha = 0$ and β surjective. Remark that since we do not know if $\alpha_0 = u_0\otimes 1$ is injective, we can not deduce $\ker\beta \subset \operatorname{im}\alpha$. On the other hand, consider $g : \mathbf{S}(I)\otimes M\otimes N \to \mathcal{R}(I;M)\otimes N$ and $\overline{g} : \mathbf{S}(I/I^2)\otimes M\otimes N \to \mathcal{G}(I;M)\otimes N$ the natural surjective graded morphisms of standard modules, where $\mathbf{S}(I)$, $\mathbf{S}(I/I^2)$ stands for the symmetric algebras of I and I/I^2, respectively. By Lemma 2.3 in [**8**], for each $n\geq 2$, there exists exact sequences of A-modules $E(g)_n \to E(f\circ g)_n \to E(f)_n \to 0$ and $E(\overline{g})_n \to E(\overline{f}\circ\overline{g})_n \to E(\overline{f})_n \to 0$. In other words, we have exact sequences

$$E(\mathcal{R}(I;M)\otimes N)_n \to E(\mathcal{R}(I;P))_n \to E(f)_n \to 0 \text{ and}$$
$$E(\mathcal{G}(I;M)\otimes N)_n \to E(\mathcal{G}(I;P))_n \to E(\overline{f})_n \to 0\,.$$

Consider the following commutative diagram of exact columns:

$$
\begin{array}{ccccccc}
E(\mathcal{R}(I;M)\otimes N)_{n+1} & \xrightarrow{\alpha} & E(\mathcal{R}(I;M)\otimes N)_n & \xrightarrow{\beta} & E(\mathcal{G}(I;M)\otimes N)_n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
E(\mathcal{R}(I;P))_{n+1} & \xrightarrow{u} & E(\mathcal{R}(I;P))_n & \xrightarrow{v} & E(\mathcal{G}(I;P))_n & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
E(f)_{n+1} & & E(f)_n & & E(\overline{f})_n & &
\end{array}
$$

The commutativity induces two morphisms $\xi : E(f)_{n+1} \to E(f)_n$ and $\mu : E(f)_n \to E(\overline{f})_n$. Since $v \circ u = 0$, then $\mu \circ \xi = 0$. Since v is surjective, then μ is surjective too. Since β is surjective and the middle row is exact, then $\ker\mu \subset \operatorname{im}\xi$. Therefore,

$$E(f)_{n+1} \xrightarrow{\xi} E(f)_n \xrightarrow{\mu} E(\overline{f})_n \longrightarrow 0$$

is an exact sequence of A modules. Finally, if A is noetherian and M, N and P are finitely generated, then $E(f)_n = 0$ for $n \gg 0$ big enough. ∎

Theorem 2.5 *Let A be a noetherian ring, I, J two ideals of A and M a finitely generated A-module. The following two conditions are equivalent:*

(i) $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I+J;M)$ *is an isomorphism.*

(ii) $\operatorname{Tor}_1(A/I^p, \mathcal{R}(J;M)) = 0$ *and* $\operatorname{Tor}_1(A/I^p, \mathcal{G}(J;M)) = 0$ *for all integers* $p \geq 1$.

In particular, $\mathcal{G}(I) \otimes \mathcal{G}(J) \simeq \mathcal{G}(I+J)$ *if and only if* $\operatorname{Tor}_1(A/I^p, A/J^q) = 0$ *and* $\operatorname{Tor}_2(A/I^p, A/J^q) = 0$ *for all integers* $p, q \geq 1$.

Proof. Remark that $\operatorname{Tor}_1(A/I^p, J^qM) = \ker(\pi_{p,q} : I^p \otimes J^qM \to I^pJ^qM)$. Moreover, under the hypothesis $\operatorname{Tor}_1(A/I^p, \mathcal{R}(J;M)) = 0$ for all $p \geq 1$, then the following two conditions are equivalent:

- $\operatorname{Tor}_1(A/I^p, \mathcal{G}(J;M)) = 0$ for all $p \geq 1$.
- $I^pM \cap J^qM = I^pJ^qM$ for all $p, q \geq 1$.

Suppose (ii) holds, i.e., $\operatorname{Tor}_1(A/I^p, J^qM) = 0$ and $I^pM \cap J^qM = I^pJ^qM$ for all $p, q \geq 1$. Then, $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J;M) \to \mathcal{R}(I,J;M)$ is an isomorphism and, by Lemma 2.2, $\overline{\sigma} : \mathcal{G}(I,J;M) \to \mathcal{G}(I+J;M)$ is an isomorphism. Thus $\overline{\varphi} = \overline{\sigma} \circ \overline{\pi}$ is an isomorphism and (i) holds. Let us now prove $(i) \Rightarrow (ii)$. If $\overline{\varphi} = \overline{\sigma} \circ \overline{\pi}$ is an isomorphism , then $\overline{\sigma}$ and $\overline{\pi}$ are two isomorphisms. By Proposition 2.3, $\overline{\sigma}$ an isomorphism implies $I^pM \cap J^qM = I^pJ^qM$ for all $p, q \geq 1$. In particular,

$$\mathcal{R}(I; J^qM/J^{q+1}M)_p = \frac{I^pJ^qM + J^{q+1}M}{J^{q+1}M} = \frac{I^pJ^qM}{I^pJ^qM \cap J^{q+1}M} = \frac{I^pJ^qM}{I^pJ^{q+1}M} = \mathcal{G}(J; I^pM)_q \text{ and}$$
$$\mathcal{G}(I; J^qM/J^{q+1}M)_p = \frac{I^pJ^qM + J^{q+1}M}{I^{p+1}J^qM + J^{q+1}M} = \frac{I^pJ^qM}{(I+J)I^pJ^qM} = \mathcal{G}(I,J;M)_{p,q}.$$

Fix $q \geq 1$. Since $\overline{\pi}_{p,q} : \mathcal{G}(I)_p \otimes \mathcal{G}(J;M)_q \to \mathcal{G}(I,J;M)_{p,q}$ is an isomorphism for all $p \geq 1$ and $\mathcal{G}(I,J;M)_{p,q} = \mathcal{G}(I; J^qM/J^{q+1}M)_p$, then $\overline{\pi}_{*,q} : \mathcal{G}(I) \otimes J^qM/J^{q+1}M \to \mathcal{G}(I; J^qM/J^{q+1}M)$ is an isomorphism for all $q \geq 1$. By Proposition 2.4, we have $\mathcal{R}(I) \otimes J^qM/J^{q+1}M \to \mathcal{R}(I; J^qM/J^{q+1}M)$ is an isomorphism for all $q \geq 1$. Since $I^p \otimes J^qM/J^{q+1}M \to \mathcal{R}(I; J^qM/J^{q+1}M)_p$ is an isomorphism for all $p, q \geq 1$ and $\mathcal{R}(I; J^qM/J^{q+1}M)_p = \mathcal{G}(J; I^pM)_q$, then $I^p \otimes \mathcal{G}(J;M) \to \mathcal{G}(J; I^pM)$ is an isomorphism for all $p \geq 1$. By Proposition 2.4 again, $I^p \otimes \mathcal{R}(J;M) \to \mathcal{R}(J;I^pM)$ is an isomorphism for all $p \geq 1$. So $\pi : \mathcal{R}(I) \otimes \mathcal{R}(J;M) \to \mathcal{R}(I,J;M)$ is an isomorphism and $\mathrm{Tor}_1(A/I^p, \mathcal{R}(J,M)) = 0$ for all $p \geq 1$. Since $\mathrm{Tor}_1(A/I^p, \mathcal{R}(J;M)) = 0$ for all $p \geq 1$ and $I^pM \cap J^qM = I^pJ^qM$ for all $p, q \geq 1$, then $\mathrm{Tor}_1(A/I^p, \mathcal{G}(J;M)) = 0$ for all $p \geq 1$. ∎

3 Some examples

Example 3.1 Let A be a noetherian local ring, I, J two ideals of A and M a finitely generated A-module. If $I = (x)$ is principal and x A-regular, then $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I+J;M)$ is an isomorphism if and only if x is a nonzero divisor in $\mathcal{R}(J;M)$ and in $\mathcal{G}(J;M)$. Indeed, let $\mathcal{K}(y;N)$ denote the Koszul complex of a sequence of elements $y = y_1, \ldots, y_m$ of A with respect to an A-module N and let $H_i(y;N)$ denote its i-th Koszul homology group. Then $\mathrm{Tor}_1(A/I, N) = H_1(\mathcal{K}(x;A) \otimes N) = H_1(x;N) = 0$ if and only if x is a non-zerodivisor in N.

Example 3.2 Let A be a noetherian local ring and let $I = (x)$ and $J = (y)$ be two principal ideals of A. If $(0 : x) \subset (y)$ and $(0 : y) \subset (x)$, then $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J) \to \mathcal{G}(I+J)$ is an isomorphism if and only if x, y is an A-regular sequence.

Example 3.3 Let R be a noetherian local ring and let z, t be an R-regular sequence. Let $A = R/(zt)$, $x = z + (zt)$, $y = t + (zt)$, $I = (x)$ and $J = (y)$. Then $\overline{\sigma} : \mathcal{G}(I,J) \to \mathcal{G}(I+J)$ is an isomorphism, but $\overline{\pi} : \mathcal{G}(I) \otimes \mathcal{G}(J) \to \mathcal{G}(I,J)$ is not an isomorphism.

An example of a pair of ideals I, J with the property $\mathrm{Tor}_1(A/I^p, A/J^q) = 0$ for all integers $p, q \geq 1$ arises from a product of affine varietes (see [**10**], pages 130 to 136, and specially Proposition 5.5.7). The next result is well known (see, for instance, 21.9 and 21.11 in [**4**]). We give here a proof for the sake of completeness.

Proposition 3.4 *Let A be a noetherian local ring, I and J two ideals of A and M a finitely generated A-module. Let $x = x_1, \ldots, x_r$ be a system of generators of I and $y = y_1, \ldots, y_r$, $y_i = \overline{x}_i = x_i + J$, a system of generators of the ideal $\overline{I} = I + J/J$ of the quotient ring $\overline{A} = A/J$. If $\mathcal{G}(J)$ and $\mathcal{G}(J;M)$ are free $\overline{A}$-modules and y is an $\overline{A}$-regular sequence in $\overline{I}$, then x is an A-regular sequence in I and $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(J;M) \to \mathcal{G}(I+J;M)$ is an isomorphism.*

Proof. Since $J^qM/J^{q+1}M$ is $\overline{A}$-free for all $q \geq 1$ and y is an $\overline{A}$-regular sequence, then y is a $J^qM/J^{q+1}M$-regular sequence in $\overline{I}$ for all $q \geq 1$. In particular, x is a $J^qM/J^{q+1}M$-regular sequence in I for all $q \geq 1$. So x is an M/J^qM-regular sequence in I for all $q \geq 1$ and x is an M-regular sequence in I. Analogously, but using the hypothesis $\mathcal{G}(J)$ is $\overline{A}$-free, we deduce x is an A-regular sequence in I. Therefore $\mathrm{Tor}_i(A/I, M) = 0$ and $\mathrm{Tor}_i(A/I, M/J^qM) = 0$ for all $i, q \geq 1$. Using the long exact sequences in homology associated to the short exact sequences

$$0 \to J^qM \to M \to M/J^qM \to 0 \text{ and } 0 \to J^qM/J^{q+1}M \to M/J^{q+1}M \to M/J^qM \to 0,$$

we deduce $\mathrm{Tor}_1(A/I, \mathcal{R}(J;M)) = 0$ and $\mathrm{Tor}_1(A/I, \mathcal{G}(J;M)) = 0$. Since I^p/I^{p+1} is A/I-free, then $\mathrm{Tor}_1(I^p/I^{p+1}, \mathcal{R}(J;M)) = \mathrm{Tor}_1(A/I, \mathcal{R}(J;M)) \otimes I^p/I^{p+1} = 0$ and $\mathrm{Tor}_1(I^p/I^{p+1}, \mathcal{G}(J;M)) = \mathrm{Tor}_1(A/I, \mathcal{G}(J;M)) \otimes I^p/I^{p+1} = 0$. Applying the long exact sequences in homology to the short exact sequences $0 \to I^p/I^{p+1} \to A/I^{p+1} \to A/I^p \to 0$, we deduce $\mathrm{Tor}_1(A/I^p, \mathcal{R}(J;M)) = 0$ and $\mathrm{Tor}_1(A/I^p, \mathcal{G}(J;M)) = 0$ for all $p \geq 1$. ∎

4 Relation type of tensor products

Lemma 4.1 *Let U be a standard A-algebra and F a standard U-module. If M is an A-module, then $F \otimes M$ is a standard U-module and $\mathrm{rt}(F \otimes M) \leq \mathrm{rt}(F)$. If $\lambda : M \to N$ is an epimorphism of A-modules, then $1 \otimes \lambda : F \otimes M \to F \otimes N$ is a surjective graded morphism of standard U-modules. Moreover, for each integer $n \geq 1$, $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$. In particular, for each $n \geq 2$, there exists an epimorphism of A-modules $E(F \otimes M)_n \to E(F \otimes N)_n$ and $\mathrm{rt}(F \otimes N) \leq \mathrm{rt}(F \otimes M)$.*

Proof. Clearly $F \otimes M$ is a standard U-module and $1 \otimes \lambda : F \otimes M \to F \otimes N$ is a surjective graded morphism of standard U-modules. Consider the symmetric presentation $\gamma : \mathbf{S}(U_1) \otimes F_0 \to F$ of F. If we tensor $0 \to \ker\gamma_i \to \mathbf{S}_i(U_1) \otimes F_0 \stackrel{\gamma_i}{\to} F_i \to 0$ by M we get exact sequences $\ker\gamma_i \otimes M \to \mathbf{S}_i(U_1) \otimes F_0 \otimes M \stackrel{\gamma_i \otimes 1}{\to} F_i \otimes M \to 0$. They induce natural epimorphisms $\ker\gamma_i \otimes M \to \ker(\gamma_i \otimes 1_M)$. On the other hand, if we tensor $U_1 \otimes \ker\gamma_{n-1} \to \ker\gamma_n \to E(\gamma)_n \to 0$ by M we get the exact sequence

$$U_1 \otimes \ker\gamma_{n-1} \otimes M \to \ker\gamma_n \otimes M \to E(\gamma)_n \otimes M \to 0\,.$$

We thus have the following commutative diagram of exact rows

$$\begin{array}{ccccccc}
U_1 \otimes \ker\gamma_{n-1} \otimes M & \longrightarrow & \ker\gamma_n \otimes M & \longrightarrow & E(\gamma)_n \otimes M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
U_1 \otimes \ker(\gamma_{n-1} \otimes 1_M) & \longrightarrow & \ker(\gamma_n \otimes 1_M) & \longrightarrow & E(\gamma \otimes 1_M)_n & \longrightarrow & 0\,,
\end{array}$$

from where we deduce epimorphisms $E(\gamma)_n \otimes M \to E(\gamma \otimes 1_M)_n$ for all $n \geq 1$. In particular, $\mathrm{rt}(F \otimes M) = s(E(\gamma \otimes 1_M)) \leq s(E(\gamma)) = \mathrm{rt}(F)$. Consider now, for each $n \geq 1$, the following commutative diagram of exact columns and rows:

$$\begin{array}{ccccccccc}
 & & & & (\ker\partial_n) \otimes M & \xrightarrow{1 \otimes \lambda} & (\ker\partial_n) \otimes N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & U_1 \otimes \ker(1_{F_{n-1}} \otimes \lambda) & \longrightarrow & U_1 \otimes F_{n-1} \otimes M & \xrightarrow{1 \otimes 1 \otimes \lambda} & U_1 \otimes F_{n-1} \otimes N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \partial_n \otimes 1_M & & \downarrow \partial_n \otimes 1_N & & \\
0 & \longrightarrow & \ker(1_{F_n} \otimes \lambda) & \longrightarrow & F_n \otimes M & \xrightarrow[1 \otimes \lambda]{} & F_n \otimes N & \longrightarrow & 0
\end{array}$$

Using a diagram chasing argument, one deduces $\ker(1_{F_n} \otimes \lambda) = U_1 \cdot \ker(1_{F_{n-1}} \otimes \lambda)$ for all $n \geq 1$. Consider the symmetric presentation $\gamma \otimes 1 : \mathbf{S}(U_1) \otimes F_0 \otimes M \to F \otimes M$ of $F \otimes M$. By Lemma 2.3 in [**8**], there exists an exact sequence of A-modules $E(\gamma \otimes 1)_n \to E((1 \otimes \lambda) \circ (\gamma \otimes 1))_n \to E(1 \otimes \lambda)_n \to 0$ for all $n \geq 1$. But $E(\gamma \otimes 1)_n = E(F \otimes M)_n$, $E((1 \otimes \lambda) \circ (\gamma \otimes 1))_n = E(F \otimes N)_n$ and $E(1 \otimes \lambda)_n = 0$ for all $n \geq 2$. Thus $E(F \otimes M)_n \to E(F \otimes N)_n$ is surjective for all $n \geq 2$ and $\mathrm{rt}(F \otimes N) \leq \mathrm{rt}(F \otimes M)$. ∎

Theorem 4.2 *Let A be a commutative ring, U and V two standard A-algebras and F a standard U-module and G a standard V-module. Then $U \otimes V$ is a standard A-algebra, $F \otimes G$ is a standard $U \otimes V$-module and $\mathrm{rt}(F \otimes G) \leq \max(\mathrm{rt}(F), \mathrm{rt}(G))$.*

Proof. Clearly $U \otimes V$ is a standard A-algebra and $F \otimes G$ is a standard $U \otimes V$-module. Take $\varphi : X \to F$ and $\psi : Y \to G$ two symmetric presentations of F and G, respectively. Then $\varphi \otimes \psi : X \otimes Y \to F \otimes G$

is a symmetric presentation of $F \otimes G$. Since $\varphi \otimes \psi = (\varphi \otimes 1_G) \circ (1_X \otimes \psi)$, then, for each integer $n \geq 2$, there exists an exact sequence of A-modules

$$E(1_X \otimes \psi)_n \to E(\varphi \otimes \psi)_n \to E(\varphi \otimes 1_G)_n \to 0 .$$

Since $\psi : Y \to G$ is a symmetric presentation of G, then $1_{X_0} \otimes \psi : X_0 \otimes Y \to X_0 \otimes G$ is a symmetric presentation of $X_0 \otimes G$ and $E(X_0 \otimes G)_n = E(1_{X_0} \otimes \psi)_n$ for all $n \geq 2$. Using Lemma 4.1, $\ker(1_{X_i} \otimes \psi_{n-i}) = U_1 \cdot \ker(1_{X_{i-1}} \otimes \psi_{n-i})$ for all $i \geq 1$. Then

$$E(1_X \otimes \psi)_n = \frac{\ker(1_X \otimes \psi)_n}{(U \otimes V)_1 \cdot \ker(1_X \otimes \psi)_{n-1}} =$$
$$\frac{\oplus_{i=0}^{n} \ker(1_{X_i} \otimes \psi_{n-i})}{(\oplus_{i=0}^{n-1} U_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i})) + (\oplus_{i=0}^{n-1} V_1 \cdot \ker(1_{X_i} \otimes \psi_{n-i}))} =$$
$$\frac{\ker(1_{X_0} \otimes \psi_n)}{V_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1})} \oplus \frac{\ker(1_{X_1} \otimes \psi_{n-1})}{U_1 \cdot \ker(1_{X_0} \otimes \psi_{n-1}) + V_1 \cdot \ker(1_{X_1} \otimes \psi_{n-2})} \oplus \ldots \oplus$$
$$\frac{\ker(1_{X_{n-1}} \otimes \psi_1)}{U_1 \cdot \ker(1_{X_{n-2}} \otimes \psi_1) + V_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} \oplus \frac{\ker(1_{X_n} \otimes \psi_0)}{U_1 \cdot \ker(1_{X_{n-1}} \otimes \psi_0)} = E(1_{X_0} \otimes \psi)_n .$$

Therefore $E(1_X \otimes \psi)_n = E(1_{X_0} \otimes \psi)_n = E(X_0 \otimes G)_n$ for all $n \geq 2$. Analogously, $E(\varphi \otimes 1_G)_n = E(\varphi \otimes 1_{G_0})_n = E(F \otimes G_0)_n$ for all $n \geq 2$. Hence there exists an exact sequence of A-modules

$$E(X_0 \otimes G)_n \to E(F \otimes G)_n \to E(F \otimes G_0)_n \to 0$$

for all $n \geq 2$ and, by Lemma 4.1, $\mathrm{rt}(F \otimes G) \leq \max(\mathrm{rt}(F \otimes G_0), \mathrm{rt}(X_0 \otimes G)) \leq \max(\mathrm{rt}(F), \mathrm{rt}(G))$. ∎

Remark 4.3 Let A be a commutative ring and let U and V be two standard A-algebras. If $\mathrm{Tor}_1^A(U, V) = 0$, then $E(U \otimes V) = E(U) \oplus E(V)$. This follows from the characterization $E(U) = H_1(A, U, A)$ (see Remark 2.3 in [**7**]) and Proposition 19.3 in [**1**].

5 Uniform bounds

Lemma 5.1 *Let $(A, \mathfrak{m})$ be a noetherian local ring and M be a finitely generated A-module. Let $\mathfrak{p}$ a prime ideal of A such that $A/\mathfrak{p}$ is regular local and $\mathcal{G}(\mathfrak{p})$ and $\mathcal{G}(\mathfrak{p}; M)$ are free $A/\mathfrak{p}$-modules. Then $\mathrm{rt}(\mathfrak{m}; M) \leq \mathrm{rt}(\mathfrak{p}; M)$.*

Proof. Since $A/\mathfrak{p}$ is regular local, there exists a sequence of elements $x = x_1, \ldots, x_r$ in A such that $y = y_1, \ldots, y_r$, defined by $y_i = x_i + \mathfrak{p}$, is a system of generators of $\mathfrak{m}/\mathfrak{p}$ and an $\overline{A}$-regular sequence. Let I be the ideal of A generated by x. In particular, $I + \mathfrak{p}/\mathfrak{p} = \mathfrak{m}/\mathfrak{p}$ and $I + \mathfrak{p} = \mathfrak{m}$. By Proposition 3.4, x is an A-regular sequence and $\mathrm{Tor}_1(A/I^p, \mathcal{R}(\mathfrak{p}; M)) = 0$ and $\mathrm{Tor}_1(A/I^p, \mathcal{G}(\mathfrak{p}; M)) = 0$ for all $p \geq 1$. By Theorem 2.5, $\overline{\varphi} : \mathcal{G}(I) \otimes \mathcal{G}(\mathfrak{p}; M) \to \mathcal{G}(\mathfrak{m}; M)$ is an isomorphism. By Theorem 4.2, $\mathrm{rt}(\mathcal{G}(\mathfrak{m}; M)) \leq \max(\mathrm{rt}(\mathcal{G}(I)), \mathrm{rt}(\mathcal{G}(\mathfrak{p}; M)))$. By Remark 2.7 in [**8**], $\mathrm{rt}(\mathcal{G}(J; M)) = \mathrm{rt}(J; M)$ for any ideal J of A. Since I is generated by a regular sequence, then $\mathrm{rt}(I) = 1$ (see, for instance, [**10**] page 30). Thus $\mathrm{rt}(\mathfrak{m}; M) \leq \mathrm{rt}(\mathfrak{p}; M)$. ∎

Next result is a slight generalization of a well known Theorem of Duncan and O'Carroll [**2**]. In fact the proof of our theorem is directly inspired by their's. We sketch it here for the sake of completeness.

Theorem 5.2 *Let A be an excellent (or $J-2$) ring and let M be a finitely generated A-module. Then there exists an integer $s \geq 1$ such that, for all maximal ideals $\mathfrak{m}$ of A, the relation type of $\mathfrak{m}$ with respect to M satisfies $\mathrm{rt}(\mathfrak{m}; M) \leq s$.*

Proof. For every $\mathfrak{p} \in \operatorname{Spec}(A)$, let us construct a non-empty open subset $U(\mathfrak{p})$ of $V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(A) \mid \mathfrak{q} \supseteq \mathfrak{p}\} \simeq \operatorname{Spec}(A/\mathfrak{p})$. Remark that $A/\mathfrak{p}$ is a noetherian domain, $\mathcal{G}(\mathfrak{p})$ is a finitely generated $A/\mathfrak{p}$-algebra and $\mathcal{G}(\mathfrak{p}; M)$ is a finitely generated $\mathcal{G}(\mathfrak{p})$-module. By Generic Flatness (Theorem 22.A in [**6**]), there exist $f, g \in A - \mathfrak{p}$ such that $\mathcal{G}(\mathfrak{p})_f$ is an $(A/\mathfrak{p})_f$-free module and $\mathcal{G}(\mathfrak{p}; M)_g$ is an $(A/\mathfrak{p})_g$-free module. Since A is $J-2$, the set $\operatorname{Reg}(A/\mathfrak{p}) = \{\mathfrak{q} \in V(\mathfrak{p}) \mid (A/\mathfrak{p})_{\mathfrak{q}} \text{ is regular local}\}$ is a non-empty open subset of $V(\mathfrak{p})$. Define $U(\mathfrak{p})$ as the intersection $D(f) \cap D(g) \cap \operatorname{Reg}(A/\mathfrak{p}) = \{\mathfrak{q} \in V(\mathfrak{p}) \mid \mathfrak{q} \not\ni f, \mathfrak{q} \not\ni g, (A/\mathfrak{p})_{\mathfrak{q}} \text{ is regular local}\}$, which is a non-empty open subset of $V(\mathfrak{p})$. Remark that for all $\mathfrak{q} \in U(\mathfrak{p})$, $(A/\mathfrak{p})_{\mathfrak{q}}$ is regular local and $\mathcal{G}(\mathfrak{p})_{\mathfrak{q}}$ and $\mathcal{G}(\mathfrak{p}; M)_{\mathfrak{q}}$ are free $\mathcal{G}(\mathfrak{p})_{\mathfrak{q}}$-modules. By Lemma 5.1, $\operatorname{rt}(\mathfrak{q}A_{\mathfrak{q}}; M_{\mathfrak{q}}) \leq \operatorname{rt}(\mathfrak{p}A_{\mathfrak{q}}; M_{\mathfrak{q}}) \leq \operatorname{rt}(\mathfrak{p}; M)$ for all $\mathfrak{q} \in U(\mathfrak{p})$. In particular, $\operatorname{rt}(\mathfrak{m}; M) \leq \operatorname{rt}(\mathfrak{p}; M)$ for all maximal ideals $\mathfrak{m} \in U(\mathfrak{p})$. For each minimal prime $\mathfrak{p}_i$ of A, let $V(\mathfrak{p}_i) - U(\mathfrak{p}_i) = V(\mathfrak{p}_{i,1}) \cup \ldots \cup V(\mathfrak{p}_{i,r_i})$ be the decomposition into irreducible closed subsets of the proper closed subset $V(\mathfrak{p}_i) - U(\mathfrak{p}_i)$, $\mathfrak{p}_{i,j} \in \operatorname{Spec}(A)$, $\mathfrak{p}_{i,j} \supsetneq \mathfrak{p}_i$. Then $\operatorname{Spec}(A) = \cup_i V(\mathfrak{p}_i) = \cup_i U(\mathfrak{p}_i) \cup (V(\mathfrak{p}_i) - U(\mathfrak{p}_i)) = \cup_{i,j} U(\mathfrak{p}_i) \cup V(\mathfrak{p}_{i,j}) = \cup_{i,j} U(\mathfrak{p}_i) \cup U(\mathfrak{p}_{i,j}) \cup (V(\mathfrak{p}_{i,j}) - U(\mathfrak{p}_{i,j}))$. Since A is noetherian, one deduces that $\operatorname{Spec}(A)$ can be covered by finitely many locally closed sets of type $U(\mathfrak{p})$, i.e., there exists a finite number of prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$, such that $\operatorname{Spec}(A) = \cup_{i=1}^m U(\mathfrak{q}_i)$. Hence, $\operatorname{rt}(\mathfrak{m}; M) \leq \max\{\operatorname{rt}(\mathfrak{q}_i; M) \mid i = 1, \ldots, m\}$ for any maximal ideal $\mathfrak{m}$ of A. ∎

Using Theorem 2 in [**8**] we deduce the result of Duncan and O'Carroll in [**2**].

Corollary 5.3 [**2**] *Let A be an excellent (or $J-2$) ring and let $N \subseteq M$ be two finitely generated A-modules. Then there exists an integer $s \geq 1$ such that, for all integers $n \geq s$ and for all maximal ideals $\mathfrak{m}$ of A, $\mathfrak{m}^n M \cap N = \mathfrak{m}^{n-s}(\mathfrak{m}^s M \cap N)$.*

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