

Topics in Regularity and Qualitative Properties of Solutions of Nonlinear Elliptic Equations

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Introduction

In these notes we describe the Alexandroff-Bakelman-Pucci estimate and the Krylov-Safonov Harnack inequality for solutions of $Lu = f(x)$, where L is a second order uniformly elliptic operator in nondivergence form

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u,$$

with bounded measurable coefficients in a domain of \mathbb{R}^n . These inequalities do not require any regularity of the coefficients of L , and this makes them powerful tools in the study of second order nonlinear elliptic equations. It is the purpose of these notes to present several of their applications in this field.

The first topic is the study of the maximum principle for the operator L and its applications to symmetry properties of positive solutions of semilinear problems

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Using the moving planes method, we prove the symmetry result of Gidas, Ni and Nirenberg [16], in the improved version of Berestycki and Nirenberg [7] which uses the maximum principle in domains of small measure. In [16, 7] the same method is used to prove symmetry results for some fully nonlinear elliptic equations

$$F(x, u, Du, D^2u) = 0.$$

Next, we present a short proof of several estimates and maximum principles (taken from [8] and [9]) for solutions in “narrow” domains. We discuss also recent work of Berestycki, Caffarelli and Nirenberg [5] on qualitative properties of positive solutions in some unbounded domains of cylindrical type.

The second topic that we treat is the regularity theory for solutions of fully nonlinear elliptic equations. Our presentation is only a first and short introduction to this topic; see [12, 17] for more detailed expositions. We start giving important examples of fully nonlinear elliptic equations: Bellman equations in stochastic control theory, Isaacs equations in differential games, the Monge-Ampère equation, and the equation of prescribed Gauss curvature. We prove a $C^{1,\alpha}$ estimate for classical solutions of fully nonlinear equations of the form

$$F(D^2u) = 0.$$

The main tool employed here is the C^α regularity for solutions of linear equations $Lu = 0$ with bounded measurable coefficients, which is a consequence of the Krylov-Safonov Harnack inequality.

Next, we introduce the notion of viscosity solution of a fully nonlinear elliptic equation and we give the basic properties of this class of solutions. Finally, we present Jensen's approximate solutions [19]. They constitute a key tool when proving uniqueness and regularity for viscosity solutions — a topic that we omit here. We also omit the important $C^{2,\alpha}$ regularity theory of Evans and Krylov for convex fully nonlinear equations (see [12, 17]).

The results presented in these notes are a sample from the vast literature on the maximum principle, symmetry properties and regularity theory for fully nonlinear equations. Some of them are fundamental results in these theories. Others have been selected to illustrate the main techniques used in these fields of research.

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1 The Alexandroff-Bakelman-Pucci estimate

Throughout these notes, L will denote an elliptic operator in a domain $\Omega \subset \mathbb{R}^n$, of the form

$$Lu = a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u + c(x)u$$

(where summation over repeated indices is understood). We assume that L is uniformly elliptic and that it has bounded measurable coefficients. That is, we suppose that there exist constants $0 < c_0 \leq C_0$, $b \geq 0$ and $\tilde{b} \geq 0$ such that

$$\begin{aligned} c_0|\xi|^2 &\leq a_{ij}(x)\xi_i\xi_j \leq C_0|\xi|^2 \\ (\sum b_i(x)^2)^{1/2} &\leq b \\ |c(x)| &\leq \tilde{b}. \end{aligned}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Hence, the matrix $A(x) = [a_{ij}(x)]$ (which is assumed to be symmetric) has all its eigenvalues in the interval $[c_0, C_0]$.

For a given function $f : \Omega \rightarrow \mathbb{R}$, we consider the linear equation $Lu = f(x)$. It is called a second order uniformly elliptic equation in nondivergence form with bounded measurable coefficients. Under no further assumptions on the coefficients of L , the following basic estimate (which we call APB estimate) was proven independently by Alexandroff, Bakelman and Pucci in the sixties [1, 2, 3, 25].

Theorem 1.1. (Alexandroff, Bakelman, Pucci) *We assume that Ω is a bounded domain of \mathbb{R}^n and that $c \leq 0$ in Ω . Let d be a constant such that $\text{diam}(\Omega) \leq d$. Let $u \in W_{\text{loc}}^{2,n}(\Omega)$ and $f \in L^n(\Omega)$ satisfy $Lu \geq f$ in Ω and $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$. Then*

$$\sup_{\Omega} u \leq C \text{diam}(\Omega) \|f\|_{L^n(\Omega)},$$

where $C = C(n, c_0, bd)$ is a constant depending only on n , c_0 and bd .

Here $W_{\text{loc}}^{2,n}(\Omega)$ denotes the Sobolev space of functions that, together with their second derivatives, belong to $L_{\text{loc}}^n(\Omega)$. Recall that n is the dimension of the space and that $W_{\text{loc}}^{2,n}(\Omega) \subset C(\Omega)$ — the space of continuous functions in Ω . If $u \in C(\bar{\Omega})$ then the condition $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$ means simply that $u \leq 0$ on $\partial\Omega$.

When $Lu \geq f$ we say that u is a *subsolution* of the equation $Lu = f$. If $Lu \geq f$ in Ω but the assumption $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$ is not satisfied, an estimate for $\sup_{\Omega} u$ may be obtained by applying Theorem 1.1 to $u - \limsup_{x \rightarrow \partial\Omega} u^+(x)$. We have

$$\sup_{\Omega} u \leq \limsup_{x \rightarrow \partial\Omega} u^+(x) + C \text{diam}(\Omega) \|f\|_{L^n(\Omega)},$$

where $u^+ = \max(u, 0)$ denotes the positive part of u . In what follows we will also denote $u^- = \max(-u, 0)$, so that $u = u^+ - u^-$.

Proof of Theorem 1.1.

Step 1. By a simple argument, we may further assume $c \equiv 0$. Indeed, we replace Ω by any connected component of $\tilde{\Omega} := \{x \in \Omega : u(x) > 0\} \subset \Omega$, and the operator L by $L_0 := a_{ij}(x)\partial_{ij} + b_i(x)\partial_i$ (which has no zero order terms). Then $L_0u = Lu - cu \geq Lu \geq f$ in $\tilde{\Omega}$, since $c \leq 0$. Note also that $\limsup_{x \rightarrow \partial\tilde{\Omega}} u(x) = 0$.

Next we make the assumption $b_i \equiv 0$. The proof in the general case is slightly more elaborate. For this, see Chapter 9 of [17] and our remark below, in Step 3. Finally, it is easy to reduce the proof to the case $u \in C^2(\Omega) \cap C(\bar{\Omega})$ by an approximation argument (see [17]).

Hence, from now on, we assume that $\sup_{\Omega} u > 0$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{cases} Lu = a_{ij}(x)\partial_{ij}u \geq f(x) & \text{in } \Omega \\ u \leq 0 & \text{on } \Omega. \end{cases}$$

Step 2. Let $x_0 \in \Omega$ be such that

$$M := \sup_{\Omega} u = u(x_0) > 0.$$

We define the upper contact set of u by

$$\Gamma_u := \{y \in \Omega : u(x) \leq u(y) + \nabla u(y) \cdot (x - y) \quad \forall x \in \Omega\}.$$

It is the set of points y in Ω such that the tangent hyperplane to the graph of u at y lies above u in all Ω . We claim that

$$B_{M/d}(0) \subset \nabla u(\Gamma_u) \quad (1)$$

(recall that d satisfies $\text{diam}(\Omega) \leq d$). To show (1), take any $p \in \mathbb{R}^n$ with $|p| < M/d$. Consider the family of parallel hyperplanes given by

$$l_p(x) = p \cdot x + a \quad \text{for } x \in \Omega,$$

where $a \in \mathbb{R}$ is any constant. If a is very big then $u < l_p$ in $\overline{\Omega}$. We let the constant a get smaller until the graph of l_p touches the graph of u for first time at some point (possibly one of many) $y \in \overline{\Omega}$. Let a_0 be such value of a , and l_p the hyperplane corresponding to $a = a_0$. This argument shows in a geometric way the following obvious fact. There exists a unique value a_0 of a , in fact given by the Legendre transform of u

$$a_0 = \sup_{x \in \overline{\Omega}} \{u(x) - p \cdot x\},$$

such that for $a = a_0$ we have

$$\begin{cases} u \leq l_p & \text{in } \Omega \\ u(y) = l_p(y) & \text{for some } y \in \overline{\Omega}. \end{cases} \quad (2)$$

Using $|p| < M/d$ we show that necessarily $y \in \Omega$. For this, the idea is that the hyperplanes l_p have constant ‘‘slope’’ smaller than $M/d = u(x_0)/d \leq u(x_0)/\text{diam}(\Omega)$ and hence they will touch (when we decrease the value of a) the graph of u at the point $(x_0, u(x_0))$ before touching it at a point $y \in \partial\Omega$. Formally, the argument is the following. Suppose that $y \in \partial\Omega$. Then $u(y) \leq 0$; using (2) we have

$$\begin{aligned} M = u(x_0) &\leq l_p(x_0) \\ &= l_p(y) + p \cdot (x_0 - y) \\ &= u(y) + p \cdot (x_0 - y) \\ &\leq p \cdot (x_0 - y) \leq |p| \text{diam}(\Omega) \\ &\leq |p|d < M, \end{aligned}$$

a contradiction.

Since $y \in \Omega$, (2) implies that

$$p = \nabla l_p(y) = \nabla u(y), \quad y \in \Gamma_u$$

and

$$D^2u(y) \leq 0$$

(i.e., $D^2u(y)$ is a nonpositive definite matrix).

In particular, $p \in \nabla u(\Gamma_u)$ and hence our claim (1) is proved. Considering the Lebesgue measure of the sets in (1), we deduce

$$\omega_n(M/d)^n \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp, \quad (3)$$

where $w_n = |B_1|$.

Step 3. To proceed, we compute the right hand side of (3) using the “change of variables”

$$p = \nabla u(x) \quad \text{for } x \in \Gamma_u.$$

We use the area formula (see Theorems 1 and 2 in Section 3.3 of [15]). It states that if $\phi : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz map, then

$$\int_{\mathbb{R}^n} \left\{ \sum_{x \in A, \phi(x)=p} g(x) \right\} dp = \int_A |\text{Jac } \phi(x)| g(x) dx$$

for any integrable function $g : A \rightarrow \mathbb{R}$; here $\text{Jac } \phi = \det D\phi$. We apply this formula with $\phi = \nabla u$, $A = \Gamma_u$ and $g = 1$. We obtain the inequality

$$\int_{\nabla u(\Gamma_u)} dp \leq \int_{\Gamma_u} \det(-D^2u(x)) dx, \quad (4)$$

where we have used that $|\text{Jac } \nabla u(x)| = |\det D^2u(x)| = \det(-D^2u(x))$ for $x \in \Gamma_u$.

At this point, we make two remarks. First, when the coefficients b_i are not identically zero, the proof proceeds by applying the area formula to $g(x) := \tilde{g}(\nabla u(x)) = \tilde{g}(p)$ for an appropriate \tilde{g} , instead of $g \equiv 1$ as in the case $b_i \equiv 0$ (see [17]).

Second, we have used the area formula applied to the map ∇u , which is not necessarily one-to-one on Γ_u . In an alternative way, (4) could have been obtained using the classical change of variables formula applied, for any $\epsilon > 0$, to the map $x \mapsto \nabla u(x) - \epsilon x$ (that can be shown to have nonzero Jacobian on Γ_u , and to be one-to-one on Γ_u) and then letting $\epsilon \rightarrow 0$. This alternative argument shows that, in fact, equality holds in (4).

Step 4. Combining (3) and (4), and using that all the eigenvalues of $A(x) = [a_{ij}(x)]$ are greater than or equal to c_0 , we deduce

$$\begin{aligned} w_n(M/d)^n &\leq \int_{\Gamma_u} \det(-D^2u(x)) dx \\ &\leq (1/c_0^n) \int_{\Gamma_u} \det[A(x)(-D^2u(x))] dx. \end{aligned}$$

We use now a simple fact from linear algebra. If A and B are symmetric matrices with $A \geq 0$ and $B \geq 0$ then

$$\det(AB) \leq \{\operatorname{tr}(AB)/n\}^n$$

— a generalization of the arithmetic and geometric means inequality. Here tr denotes the trace.

Note that

$$\begin{aligned} \operatorname{tr}\{A(x)(-D^2u(x))\} &= -a_{ij}(x)\partial_{ij}u \\ &= -Lu \leq -f(x) \leq |f(x)|. \end{aligned}$$

We conclude that

$$w_n \left(\frac{M}{d}\right)^n \leq \left(\frac{1}{nc_0}\right)^n \int_{\Gamma_u} |f|^n,$$

and hence

$$\begin{aligned} \sup_{\Omega} u = M &\leq \frac{1}{nc_0 w_n^{1/n}} d \|f\|_{L^n(\Omega)} \\ &= C(n, c_0) \operatorname{diam}(\Omega) \|f\|_{L^n(\Omega)}, \end{aligned}$$

which is the desired inequality. \square

We now introduce a standard terminology concerning the maximum principle.

Definition 1.2. We say that the maximum principle holds for the operator L in Ω if $u \in W_{\text{loc}}^{2,n}(\Omega)$, $\sup_{\Omega} u < \infty$,

$$Lu \geq 0 \text{ in } \Omega \quad \text{and} \quad \limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$$

imply $u \leq 0$ in Ω .

Note that, when Ω is bounded, the condition $\sup_{\Omega} u < \infty$ is automatically satisfied, since it is a consequence of the assumptions $u \in W_{\text{loc}}^{2,n}(\Omega)$ and $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$.

The following result is a well known sufficient condition for the maximum principle to hold. It is an immediate consequence of Theorem 1.1.

Corollary 1.3. *If Ω is bounded and $c \leq 0$ in Ω then the maximum principle holds for L in Ω .*

The condition $c \leq 0$ in Ω is, however, too restrictive for some applications, for instance when studying symmetry properties of positive solutions of nonlinear problems (see next section). Instead, the following maximum principle in domains of small measure does not make any assumption on the sign of $c(x)$, and it will be very useful in the study of symmetry properties.

Theorem 1.4. *Assume that Ω is bounded and $\text{diam}(\Omega) \leq d$ for a positive constant d . Then there exists a constant $\delta > 0$, depending only on n, c_0, b, \tilde{b} and d , such that the maximum principle holds for L in Ω if the measure of Ω , $|\Omega|$, satisfies*

$$|\Omega| \leq \delta.$$

In this maximum principle c may change sign, but the measure of $|\Omega|$ is required to be small depending on various quantities which include the upper bound \tilde{b} for $\|c\|_{L^\infty(\Omega)}$. In fact, the proof will show that the weaker assumption $c(x) \leq \tilde{b}$ in Ω suffices. Theorem 1.4 is a consequence of the ABP estimate that was first noted by Bakelman and later by Varadhan.

Proof of Theorem 1.4. Let u satisfy $Lu \geq 0$ in Ω and $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$. Let $c = c^+ - c^-$, and consider the operator $L_0 = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i$. Writing $Lu \geq 0$ in the form

$$(L_0 - c^-)u \geq -c^+u \geq -c^+u^+,$$

we may apply the ABP estimate to the operator $L_0 - c^-$ and obtain

$$\begin{aligned} \sup_{\Omega} u &\leq C(n, c_0, b, d) \|c^+u^+\|_{L^n(\Omega)} \\ &\leq C(n, c_0, b, \tilde{b}, d) |\Omega|^{1/n} \sup_{\Omega} u^+. \end{aligned}$$

If $C(n, c_0, b, \tilde{b}, d) |\Omega|^{1/n} \leq 1/2$, we conclude that $u \leq 0$ in Ω . □

In Section 4 we will prove other sufficient conditions for the maximum principle to hold. They will improve Theorem 1.4.

The ABP estimate can also be used to prove the following strong maximum principle for supersolutions in $W_{\text{loc}}^{2,n}(\Omega)$ (see Chapters 3 and 9 of [17]). Here, we make no assumption on the sign of c but we assume that $u \geq 0$ in Ω .

Proposition 1.5. *If $u \in W_{\text{loc}}^{2,n}(\Omega)$ satisfies $u \geq 0$ in Ω and $Lu \leq 0$ in Ω , then either $u \equiv 0$ or $u > 0$ in Ω .*

2 Symmetry properties of positive solutions in bounded domains

The goal of this section is to prove the following symmetry result for positive solutions of semilinear problems. It is taken from [7].

Theorem 2.1. (Berestycki-Nirenberg) *Let Ω be any bounded domain of \mathbb{R}^n (not necessarily smooth) which is convex in the x_1 direction and symmetric with respect to the hyperplane $\{x_1 = 0\}$. Let $u \in W_{\text{loc}}^{2,n}(\Omega) \cap C(\overline{\Omega})$ be a solution of the problem*

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that f is Lipschitz continuous. Then u is symmetric with respect to x_1 , i.e., $u(x_1, y) = u(-x_1, y)$ for any $(x_1, y) \in \Omega$. Moreover, the partial derivative of u with respect to x_1 satisfies

$$u_{x_1} < 0 \quad \text{for } x_1 > 0.$$

When Ω is a smooth domain, this symmetry result was already proven in the classical paper of Gidas, Ni and Nirenberg [16] in 1979. Their proof did not apply, however, to some nonsmooth domains such as cubes. Theorem 2.1 answers affirmatively the symmetry question in nonsmooth domains, including the case when Ω is a cube.

An immediate consequence of Theorem 2.1 is the radial symmetry of positive solutions when Ω is a ball. To prove it, one applies Theorem 2.1 to all hyperplanes passing through 0.

Corollary 2.2. (Gidas-Ni-Nirenberg) *Let $B_R = \{|x| < R\} \subset \mathbb{R}^n$ be a ball, and u be a positive solution in $C^2(\overline{B_R})$ of*

$$\Delta u + f(u) = 0 \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R.$$

If f is Lipschitz then u is radially symmetric (i.e., $u(x) = u(|x|)$) and $u_r < 0$ for $0 < r = |x| \leq R$.

The proof of these symmetry results uses the maximum principle and a method of Alexandroff called the moving planes method. The proof given in [16] used a version of the maximum principle — the Hopf boundary lemma — that did not allow some domains Ω with corners. We now present the improved method found in [7]. It replaces the use of the Hopf boundary lemma by the maximum principle in domains of small measure; in this way, the proof applies to nonsmooth domains.

Proof of Theorem 2.1. We denote points $x \in \mathbb{R}^n$ by $x = (x_1, y)$, $y \in \mathbb{R}^{n-1}$. It suffices to show

$$u(x_1, y) < u(x_1^*, y) \quad \text{if } -x_1 < x_1^* < x_1 \tag{5}$$

and

$$u_{x_1} < 0 \quad \text{if } x_1 > 0 \tag{6}$$

whenever $(x_1, y) \in \Omega$. Indeed, letting $x_1^* \rightarrow -x_1$ we get $u(x_1, y) \leq u(-x_1, y)$. The same result with the coordinate x_1 changed by $-x_1$ gives the symmetry: $u(x_1, y) = u(-x_1, y)$.

To show (5) and (6), we use the method of moving planes. Let $a = \sup_{\Omega} x_1$. For $0 < \lambda < a$, we consider the hyperplane T_λ and the set Σ_λ defined by

$$\begin{aligned} T_\lambda &= \{x_1 = \lambda\} \\ \Sigma_\lambda &= \{x \in \Omega : x_1 > \lambda\} \subset \Omega. \end{aligned}$$

For $x \in \mathbb{R}^n$ we denote by

$$x^\lambda = (2\lambda - x_1, y)$$

the reflection of x with respect to T_λ . We consider the reflection of Σ_λ ,

$$\Sigma'_\lambda = \{x^\lambda : x \in \Sigma_\lambda\} \subset \Omega,$$

which is contained in Ω by the assumptions of the theorem. Hence, the function

$$w_\lambda(x) := u(x) - u(x^\lambda) \quad \text{for } x \in \overline{\Sigma}_\lambda$$

is well defined.

Since the Laplacian is invariant under reflections, the function $x \mapsto u(x^\lambda)$ satisfies the same semilinear equation $\Delta v + f(v) = 0$. Thus, the difference w_λ satisfies the linear equation

$$\begin{aligned} 0 &= \Delta w_\lambda + f(u(x)) - f(u(x^\lambda)) \\ &= \Delta w_\lambda + c_\lambda(x)w_\lambda, \end{aligned}$$

where

$$c_\lambda(x) = \frac{f(u(x)) - f(u(x^\lambda))}{u(x) - u(x^\lambda)}.$$

Note that $\partial\Sigma_\lambda$ has two parts, one contained in T_λ and the other in $\partial\Omega$. Using that $u = 0$ on $\partial\Omega$ and $u > 0$ in Ω , we conclude

$$\begin{cases} \Delta w_\lambda + c_\lambda(x)w_\lambda = 0 & \text{in } \Sigma_\lambda \\ w_\lambda \leq 0 & \text{on } \partial\Sigma_\lambda, \quad w_\lambda \not\equiv 0. \end{cases} \quad (7)$$

Moreover, $|c_\lambda| \leq \tilde{b}$ for some constant \tilde{b} which we can take to be the Lipschitz constant of f on $[0, \sup_{\Omega} u]$.

To prove (5) and (6) it suffices to verify

$$w_\lambda < 0 \quad \text{for any } \lambda \in (0, a). \quad (8)$$

Indeed, it then follows from the Hopf lemma (see [17]) that on $T_\lambda \cap \Omega$, where $w_\lambda = 0$, we have $0 > (w_\lambda)_{x_1} = 2u_{x_1}$.

Now, if $a - \lambda$ is small then $\Sigma_\lambda \subset \Omega \cap \{\lambda < x_1 < a\}$, and hence Σ_λ has small measure. In particular, the maximum principle holds for the operator $\Delta + c_\lambda$ in Σ_λ if $a - \lambda$ is small (by Theorem 1.4). We deduce from (7) that $w_\lambda \leq 0$ in Σ_λ . Now, the strong maximum principle (Proposition 1.5) gives that $w_\lambda < 0$ in Σ_λ . We have proved (8) for $a - \lambda$ small.

Let (λ_0, a) be the largest open interval of parameters for which (8) holds. We want to show that $\lambda_0 = 0$. We suppose $\lambda_0 > 0$ and we show that it leads to contradiction. First, by continuity we have $w_{\lambda_0} \leq 0$ in Σ_{λ_0} and, by the strong maximum principle, $w_{\lambda_0} < 0$ in Σ_{λ_0} .

Next, let $\delta > 0$ be the constant of Theorem 1.4. Let $K \subset \Sigma_{\lambda_0}$ be a compact set such that $|\Sigma_{\lambda_0} \setminus K| \leq \delta/2$. We then have $w_{\lambda_0} \leq -\eta < 0$ in K for some constant η , since K is compact. Hence, $w_{\lambda_0-\epsilon} < 0$ in K and $|\Sigma_{\lambda_0-\epsilon} \setminus K| \leq \delta$ for $\epsilon > 0$ small enough.

We now apply the maximum principle in $\Sigma_{\lambda_0-\epsilon} \setminus K$. We have

$$\begin{cases} \Delta w_{\lambda_0-\epsilon} + c_{\lambda_0-\epsilon}(x)w_{\lambda_0-\epsilon} = 0 & \text{in } \Sigma_{\lambda_0-\epsilon} \setminus K \\ w_{\lambda_0-\epsilon} \leq 0 & \text{on } \partial(\Sigma_{\lambda_0-\epsilon} \setminus K); \end{cases}$$

note that $\partial(\Sigma_{\lambda_0-\epsilon} \setminus K)$ has one part contained in K , and we have used that $w_{\lambda_0-\epsilon} < 0$ in K . Since $|\Sigma_{\lambda_0-\epsilon} \setminus K| \leq \delta$, Theorem 1.4 and Proposition 1.5 give $w_{\lambda_0-\epsilon} < 0$ in $\Sigma_{\lambda_0-\epsilon} \setminus K$. Therefore, $w_{\lambda_0-\epsilon} < 0$ in $\Sigma_{\lambda_0-\epsilon}$, which contradicts the maximality of the interval (λ_0, a) . \square

We point out that the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_R \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial B_R \end{cases} \quad (9)$$

may admit solutions that change sign and are not radially symmetric. As a simple example, there exist eigenfunctions of the Laplacian in a ball:

$$\Delta u + \lambda u = 0 \text{ in } B_R, \quad u = 0 \text{ on } \partial B_R$$

which are not radially symmetric. Hence, the condition $u > 0$ in Ω in the previous theorems is, in general, necessary to conclude symmetry.

Obviously, if one knows that problem (9) has a unique solution u_0 then u_0 is necessarily radial. Indeed, the composition $u_0 \circ R$ of u_0 with any rotation is also a solution of (9) and, if there is uniqueness, it must coincide with u_0 . Hence u_0 is radially symmetric.

The following is a more interesting remark. For some nonlinearities f , the symmetry result of Gidas-Ni-Nirenberg (Corollary 2.2) may be used to prove that (9) has a unique

positive solution. The idea is that, by Corollary 2.2, one knows that any positive solution of (9) is radial. As a consequence, it suffices to show uniqueness among positive radial solutions — an easier task. As an example, this can be carried out to prove that

$$\begin{cases} \Delta u + u^p = 0 & \text{in } B_R, \quad p > 1, \\ u = 0 & \text{on } \partial B_R \end{cases}$$

has a unique positive solution (see Section 2.8 of [16]).

Remark 2.3. One can prove radial symmetry in a very simple way for *stable solutions* (not necessarily positive) of (9). We say that a solution u of (9) is stable if the first eigenvalue in B_R of the linearized operator $-\Delta - f'(u)$ of (9) at u , defined by

$$\lambda_1(\Delta + f'(u); \Omega = B_R) := \inf_{0 \neq v \in H_0^1(\Omega)} \frac{\int_{\Omega} \{|\nabla v|^2 - f'(u)v^2\} dx}{\int_{\Omega} v^2 dx}, \quad (10)$$

is positive.

We claim that any stable solution u of (9) is radially symmetric. Indeed, for any given couple of indices $i \neq j$, consider the vector field $\partial_t = x_i \partial_{x_j} - x_j \partial_{x_i}$, which is everywhere normal to the radial direction ∂_r . Defining $v := \partial_t u$, we see that v is a solution of the linearized equation of (9):

$$\begin{aligned} \Delta v &= \Delta(x_i u_{x_j} - x_j u_{x_i}) \\ &= x_i \Delta u_{x_j} + 2 \nabla x_i \cdot \nabla u_{x_j} - x_j \Delta u_{x_i} - 2 \nabla x_j \cdot \nabla u_{x_i} \\ &= x_i (\Delta u)_{x_j} - x_j (\Delta u)_{x_i} \\ &= -f'(u) \{x_i u_{x_j} - x_j u_{x_i}\} = -f'(u)v. \end{aligned}$$

Moreover, since $u = 0$ on ∂B_R and ∂_t is a tangential derivative on ∂B_R , we have that $v = 0$ on ∂B_R . Hence $v \in H_0^1(B_R)$; multiplying $-\Delta v - f'(u)v = 0$ by v and integrating by parts, we obtain $\int_{B_R} \{|\nabla v|^2 - f'(u)v^2\} dx = 0$. Since $\lambda_1(\Delta + f'(u); B_R) > 0$ by assumption, we deduce $v \equiv 0$. From this (and since the indices $i \neq j$ are arbitrary), we conclude that u is radial.

We refer to [16, 6, 7, 4] and references therein for symmetry results concerning more general equations, such as fully nonlinear elliptic equations

$$F(x, u, Du, D^2u) = 0,$$

and more general domains (for instance, some unbounded domains).

In [6] and [7] a new method was introduced — the sliding method — for equations in infinite and finite cylinders. In Section 5 we will discuss a more recent result from [5] that uses the moving planes method in infinite cylinders.

3 C^α estimate: the Krylov-Safonov Harnack inequality

Let $L = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i + c(x)$ be a uniformly elliptic operator in nondivergence form and with bounded measurable coefficients as described in the beginning of Section 1.

In 1979, Krylov and Safonov [23, 24] proved the following deep result — a Harnack inequality for the operator L under no regularity assumptions on its coefficients. We will use it extensively throughout these notes. We point out that here $c(x)$ may change sign.

Theorem 3.1. (Krylov-Safonov) *Let B_R be a ball of radius R in \mathbb{R}^n , and denote by B_{2R} the concentric ball of radius $2R$. Let $u \in W^{2,n}(B_{2R})$ and $f \in L^n(B_{2R})$ satisfy $u \geq 0$ in B_{2R} and $Lu = f$ in B_{2R} . Then*

$$\sup_{B_R} u \leq C \left\{ \inf_{B_R} u + R \|f\|_{L^n(B_{2R})} \right\},$$

where C is a constant depending only on n, c_0, C_0, bR and $\tilde{b}R^2$.

Roughly speaking, the inequality states that, for any nonnegative solution u , the value of u at one point controls the values of u in any given interior compact set. The proof of Theorem 3.1, that we omit, uses two ingredients: the ABP estimate (Theorem 1.1) and the Calderón-Zygmund cube decomposition; see [12] and [17] for the proof of Theorem 3.1.

An important consequence of the Krylov-Safonov Harnack inequality is the Hölder continuity of solutions of $Lu = f$.

Corollary 3.2. *Let $u \in W^{2,n}(B_1)$ and $f \in L^n(B_1)$ satisfy $Lu = f$ in B_1 .*

(i) *Suppose $c \equiv 0$. Then there exists a constant $0 < \mu < 1$, depending only on n, c_0, C_0 and b , such that*

$$\text{osc}_{B_{1/2}} u \leq \mu \text{osc}_{B_1} u + \|f\|_{L^n(B_1)},$$

where $\text{osc}_{B_R} u = \sup_{B_R} u - \inf_{B_R} u$ denotes the oscillation of u .

(ii) *For any $c \in L^\infty(B_1)$, we have that $u \in C^\alpha(\overline{B_{1/2}})$ and*

$$\|u\|_{C^\alpha(\overline{B_{1/2}})} \leq C \left\{ \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \right\},$$

where $0 < \alpha < 1$ and C depend only on n, c_0, C_0, b and \tilde{b} .

Proof. Let

$$\begin{aligned} M_1 &:= \sup_{B_1} u, & m_1 &:= \inf_{B_1} u, & o_1 &:= M_1 - m_1, \\ M_{1/2} &:= \sup_{B_{1/2}} u, & m_{1/2} &:= \inf_{B_{1/2}} u, & o_{1/2} &:= M_{1/2} - m_{1/2}. \end{aligned}$$

Theorem 3.1 applied to $u - m_1 \geq 0$ in B_1 and to $M_1 - u \geq 0$ in B_1 (here we assume $c \equiv 0$) gives

$$M_{1/2} - m_1 \leq C \left\{ m_{1/2} - m_1 + \frac{1}{2} \|f\|_{L^n(B_1)} \right\}$$

and

$$M_1 - m_{1/2} \leq C \left\{ M_1 - M_{1/2} + \frac{1}{2} \|f\|_{L^n(B_1)} \right\}.$$

Adding these two inequalities, we obtain

$$o_1 + o_{1/2} \leq C \left\{ o_1 - o_{1/2} + \|f\|_{L^n(B_1)} \right\}$$

and hence

$$o_{1/2} \leq \frac{C-1}{C+1} o_1 + \frac{C}{C+1} \|f\|_{L^n(B_1)},$$

which proves (i).

Part (ii) (in the general case $c \neq 0$) follows easily from (i), with the aid of a simple lemma of real analysis (see Lemma 8.23 and Corollary 9.24 of [17]). \square

While the Harnack inequality applies only to nonnegative solutions of $Lu = f$, there are related inequalities that apply to subsolutions and to nonnegative supersolutions. In fact, the proof of the Harnack inequality may be divided into two parts; the first applies to subsolutions (see Theorem 9.20 of [17] and Theorem 4.8(2) of [12]). The second part is more delicate to prove; it applies to nonnegative supersolutions and it is called the weak Harnack inequality. Its statement is the following.

Theorem 3.3. *Let $u \in W^{2,n}(B_{2R})$ and $f \in L^n(B_{2R})$ satisfy $u \geq 0$ in B_{2R} and $Lu \leq f$ in B_{2R} . Then*

$$\left(\frac{1}{|B_R|} \int_{B_R} u^\epsilon \right)^{1/\epsilon} \leq C \left\{ \inf_{B_R} u + R \|f\|_{L^n(B_{2R})} \right\},$$

where $\epsilon > 0$ and C are constants depending only on n , c_0 , C_0 , bR and $\tilde{b}R^2$.

This result is Theorem 9.22 of [17] and Theorem 4.8(1) of [12]. We will use a boundary version of Theorem 3.3 in the next section.

4 Maximum principle in “narrow” domains

In this section we present a maximum principle (Theorem 2.5 of [8]) that improves the maximum principle for domains of small measure (Theorem 1.4 of these notes). We

present a short proof of this new maximum principle following an idea from [9]. We also give an improved version of the ABP estimate (Theorem 1.4 of [9]) that applies in some unbounded domains.

We start recalling some useful facts about the maximum principle.

- (a) Suppose that Ω is bounded and that there exists a function $\phi \in W_{\text{loc}}^{2,n}(\Omega) \cap C(\overline{\Omega})$ such that $\phi > 0$ in $\overline{\Omega}$ and $L\phi \leq 0$ in Ω . Then the maximum principle holds for L in Ω .

This is a well-known sufficient condition for the maximum principle to hold; see, for instance, [8]. In fact, within the proof of Theorem 5.5 below, we will encounter the argument that shows (a).

- (b) Assume that $a_{ij} \in C(\Omega)$ and that Ω is a smooth bounded domain. Berestycki, Nirenberg and Varadhan [8] introduce the quantity λ_1 , also denoted by $\lambda_1(L, \Omega)$, defined as follows:

$$\lambda_1 = \sup\{\lambda : \exists \phi > 0 \text{ in } \Omega \text{ and } (L + \lambda)\phi \leq 0 \text{ in } \Omega\}.$$

λ_1 is called the principal eigenvalue of L in Ω . They show that the maximum principle holds for L in Ω if and only if $\lambda_1(L; \Omega) > 0$. In particular, it follows from Corollary 1.3 that $\lambda_1(L_0; \Omega) > 0$, where $L_0 = L - c(x) = a_{ij}(x)\partial_{ij} + b_i(x)\partial_i$.

They also prove that there always exists a positive eigenfunction associated to λ_1 . That is, there exists $\phi_1 \in W_{\text{loc}}^{2,n}(\Omega) \cap C(\overline{\Omega})$, $\phi_1 > 0$ in Ω satisfying

$$\begin{cases} (L + \lambda_1)\phi_1 = 0 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, ϕ_1 is unique up to a multiplicative constant (i.e., λ_1 is simple).

For all these results, and many others on the maximum principle for operators in nondivergence form (also in nonsmooth domains), see [8].

We point out that, when the operator L can also be written in divergence form, the principal eigenvalue λ_1 coincides with the first eigenvalue of L defined by the usual variational formulation. For instance, when $L = \Delta + c(x)$ then λ_1 coincides with the variational expression (10).

Next, we define a geometric quantity of the domain Ω that will play a key role in the rest of this section.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be a domain, not necessarily bounded. Given a constant $0 < \sigma < 1$, we define $R(\Omega)$ to be the smallest positive constant R such that

$$|B_R(x) \setminus \Omega| \geq \sigma |B_R(x)| \quad \forall x \in \Omega. \quad (11)$$

We define $R(\Omega)$ to be $+\infty$ if no such radius R exists.

Once the constant $0 < \sigma < 1$ is fixed, the quantity $R(\Omega)$ depends only on the domain Ω . We claim that

$$R(\Omega) \leq C(n, \sigma)|\Omega|^{1/n},$$

for a constant $C(n, \sigma)$ depending only on n and σ . Indeed, defining R by the relation $(1 - \sigma)|B_R| = |\Omega|$ (in case $|\Omega| < \infty$) we have that $R = C(n, \sigma)|\Omega|^{1/n}$ and (11) is satisfied, since $|B_R(x) \setminus \Omega| \geq |B_R(x)| - |\Omega| = \sigma|B_R(x)|$. This proves the claim.

Obviously, $|\Omega|^{1/n} \leq C(n) \text{diam}(\Omega)$. The quantity $R(\Omega)$ is therefore a more precise geometric constant of Ω than the measure or the diameter of Ω . There exist domains with infinite measure for which the quantity $R(\Omega)$ is finite (or even small). This is the case, for example, when Ω is contained between two parallel hyperplanes, or when Ω is contained in, say,

$$\mathbb{R}^n \setminus \bigcup_{p \in \mathbb{Z}^n} B_{1/10}(p)$$

where \mathbb{Z} denotes the integer numbers. See [9] for some other examples, and [8, 9] for a more refined version of the quantity $R(\Omega)$.

The following is a maximum principle in domains (not necessarily bounded) for which $R(\Omega)$ is sufficiently small. Here, no assumption on the sign of $c(x)$ is made. It is essentially Theorem 2.5 of [8].

Theorem 4.2. (Berestycki-Nirenberg-Varadhan) *Let $0 < \sigma < 1$ be a constant. Then:*

- (i) *There exists a constant R^* , depending only on $n, c_0, C_0, b, \tilde{b}$ and σ , such that the maximum principle holds for L in Ω if $R(\Omega) \leq R^*$.*
- (ii) *Assume that Ω is a smooth bounded domain and that $a_{ij} \in C(\Omega)$. Consider the operator $L_0 = L - c(x)$. Then*

$$\lambda_1(L_0; \Omega) \geq \frac{\tau}{R(\Omega)^2}$$

(and in particular $\lambda_1(L_0; \Omega) \geq \tilde{\tau}|\Omega|^{-2/n}$), where τ is a positive constant depending only on $n, c_0, C_0, bR(\Omega)$ and σ .

This result is proved in [8] using a variant of the Krylov-Safonov Harnack inequality. Below we present a short proof of Theorem 4.2 following an idea from [9] that uses a boundary version of the Krylov-Safonov weak Harnack inequality (Theorem 3.3).

We will easily deduce Theorem 4.2 from the following improved ABP estimate. It applies in any domain (not necessarily bounded) satisfying $R(\Omega) < \infty$.

Theorem 4.3. ([9]) *Let $0 < \sigma < 1$ be a constant and let Ω be a domain such that $R(\Omega) < \infty$. We assume that $c \leq 0$ in Ω . Let $u \in W_{\text{loc}}^{2,n}(\Omega)$ and $f \in L^n(\Omega)$ satisfy $\sup_{\Omega} u < \infty$, $Lu \geq f$ in Ω and $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$. Then*

$$\sup_{\Omega} u \leq C R(\Omega) \|f\|_{L^n(\Omega)}$$

and

$$\sup_{\Omega} u \leq C R(\Omega)^2 \|f\|_{L^\infty(\Omega)}, \quad (12)$$

where C is a constant depending only on n , c_0 , C_0 , $bR(\Omega)$ and σ .

Proof. Considering any connected component Ω_0 of the set $\{x \in \Omega : u(x) > 0\}$ and the operator $L_0 = L - c(x)$ (as in Step 1 of the proof of the ABP estimate, Theorem 1.1), it is easy to reduce the problem to the case $c \equiv 0$, $u > 0$ in Ω and

$$\begin{cases} Lu \geq f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here it is important to note that $R(\Omega_0) \leq R(\Omega)$.

Suppose first that Ω is bounded. Then the supremum of u is achieved, so that

$$M := \sup_{\Omega} u = u(x_0) > 0$$

for some $x_0 \in \Omega$. To simplify notation, we write $R := R(\Omega)$ and $B_R := B_R(x_0)$. We know (see Definition 4.1) that

$$\frac{|B_R \setminus \Omega|}{|B_R|} \geq \sigma. \quad (13)$$

We consider the function

$$v = M - u,$$

which satisfies $0 \leq v \leq M$ in Ω , $v(x_0) = 0$, $v = M$ on $\partial\Omega$ and $Lv = -Lu \leq -f$ in Ω . We extend the function v to be identically M in $\mathbb{R}^n \setminus \Omega$, obtaining in this way a continuous function, still denoted by v , in all \mathbb{R}^n . We also extend f by zero outside Ω .

Note that the graph of the extended function v may have ‘‘corners’’ on $\partial\Omega$, and hence v may not belong to $W^{2,n}(B_{2R})$, since $B_{2R} \setminus \Omega \neq \emptyset$ by (13). However, since $0 \leq v \leq M$ in Ω and $v = M$ on $\partial\Omega$, the extended function v is still a ‘‘generalized’’ nonnegative supersolution of $Lv \leq -f$ in \mathbb{R}^n , in the sense that it satisfies the weak Harnack inequality (Theorem 3.3). See Theorem 9.27 of [17] for this boundary version of the weak Harnack inequality. Alternatively, the extended function v satisfies $Lv \leq -f$ in \mathbb{R}^n in the viscosity sense (see Section 8, Proposition 8.4).

Now, we conclude easily. Using (13), $v(x_0) = 0$ and Theorem 3.3 applied to v in B_{2R} , we have

$$\begin{aligned}
\sigma^{1/\epsilon} M &\leq \left(\frac{|B_R \setminus \Omega|}{|B_R|} \right)^{1/\epsilon} M \leq \left(\frac{1}{|B_R|} \int_{B_R \setminus \Omega} v^\epsilon \right)^{1/\epsilon} \\
&\leq \left(\frac{1}{|B_R|} \int_{B_R} v^\epsilon \right)^{1/\epsilon} \\
&\leq C \left\{ \inf_{B_R} v + R \|f\|_{L^n(B_{2R})} \right\} \\
&= CR \|f\|_{(B_{2R} \cap \Omega)},
\end{aligned}$$

where $\epsilon > 0$ and C depend only on n, c_0, C_0 and bR . This proves the desired inequalities.

In case that Ω is unbounded, the proof is the same with minor changes. We define $M := \sup_\Omega u$ (recall that $M < \infty$ by assumption) and we take, for any $\eta > 0$, a point $x_0 \in \Omega$ such that $M - \eta \leq u(x_0)$. We now have that $v(x_0) \leq \eta$. We proceed as before and we get the desired estimate by letting $\eta \rightarrow 0$ at the end of the proof. \square

Finally, we easily deduce the maximum principle of Theorem 4.2 from estimate (12).

Proof of Theorem 4.2. To show (i), we use the same idea as in the proof of the maximum principle for domains of small measure — Theorem 1.4. If $Lu \geq 0$ in Ω , $\limsup_{x \rightarrow \partial\Omega} u(x) \leq 0$ and $\sup_\Omega u < \infty$, we have

$$(L_0 - c^-)u \geq -c^+ u \geq -c^+ u^+.$$

By estimate (12) applied to the operator $L_0 - c^-$, we have

$$\begin{aligned}
\sup_\Omega u &\leq C R(\Omega)^2 \|c^+ u^+\|_{L^\infty(\Omega)} \\
&\leq C \tilde{b} R(\Omega)^2 \sup_\Omega u^+
\end{aligned}$$

where $C = C(n, c_0, C_0, bR(\Omega), \sigma)$. If $C \tilde{b} R(\Omega)^2 \leq C \tilde{b} (R^*)^2 \leq 1/2$ we conclude that $u \leq 0$ in Ω . Here the dependence of C on $bR(\Omega)$ may be replaced by dependence only on b , since $R(\Omega) \leq R^*$ and we can take $R^* \leq 1$.

To prove (ii), we know (see the beginning of this section) that $\lambda_1 = \lambda_1(L_0; \Omega) > 0$ and that there exists $\phi_1 > 0$ in Ω such that

$$\begin{cases} L_0 \phi_1 = -\lambda_1 \phi_1 & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Applying estimate (12) to this problem, we obtain

$$\sup_\Omega \phi_1 \leq C R(\Omega)^2 \lambda_1 \sup_\Omega \phi_1$$

with $C = C(n, c_0, C_0, bR(\Omega), \sigma)$. We conclude $\lambda_1 \geq \tau R(\Omega)^{-2}$. \square

5 Positive solutions in some unbounded domains

In this section we discuss some questions concerning a symmetry result in unbounded domains of cylindrical type, recently proved in [5]. We consider domains of the form

$$\Omega = \mathbb{R}^{n-j} \times \omega,$$

where $\omega \subset \mathbb{R}^j$ is a smooth bounded domain. We denote the points in Ω by $(x, y) = (x_1, \dots, x_{n-j}, y_1, \dots, y_j) \in \Omega$. We consider the semilinear problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega, \end{cases} \quad (14)$$

and we assume that $u \in C_{\text{loc}}^{2,\mu}(\Omega)$ for some $0 < \mu < 1$, and that f is globally Lipschitz. No assumption is made on the behavior of the solution u near infinity.

Note that, when $j = n - 1$, Ω is a cylinder whose $(n - 1)$ -dimensional cross section is bounded. If $j = 1$ then Ω is the domain contained between two parallel hyperplanes.

The symmetry result in [5] is the following.

Theorem 5.1. (Berestycki-Caffarelli-Nirenberg) *Assume that ω is convex in the y_1 direction and that it is symmetric with respect to the hyperplane $\{y_1 = 0\}$. Suppose that $j \geq 2$, or that $j = 1$ and $f(0) \geq 0$. Then, any solution u of (14) is symmetric in y_1 , and $u_{y_1} < 0$ for $y_1 > 0$.*

Therefore, the solution u satisfies $u(x, y_1, y_2, \dots, y_j) = u(x, -y_1, y_2, \dots, y_j)$. As in Section 2, this yields the radial symmetry of u when ω is a ball. That is, we have:

Corollary 5.2. *Suppose that $\omega = \{|y| < R\} \subset \mathbb{R}^j$ is a ball. Assume also that $j \geq 2$, or that $j = 1$ and $f(0) \geq 0$. Then u is radially symmetric in y (i.e., $u(x, y) = u(x, |y|)$), and $u_\rho < 0$ for $0 < \rho = |y| < R$.*

Theorem 5.1 is proved using the moving planes method (see Section 2). We do not present its entire proof but, following [5], we show in detail the preliminary results on the behavior of u near infinity (and on the maximum principle) needed to start the moving planes method. By “starting the moving planes method” we mean (using the notation of the proof of Theorem 2.1 with x_1 replaced by y_1) to verify (8) for $a - \lambda$ small enough.

The first result concerns the growth of u at infinity. Note that (14) may have solutions that grow exponentially at infinity. For example, the function $u(x_1, y_1) = e^{x_1} \cos y_1$ is positive and harmonic in $\Omega = \mathbb{R} \times (-\pi/2, \pi/2)$ and vanishes on $\partial\Omega$. The first result of [5] states that, in fact, any solution of (14) grows at most at an exponential

rate. Here, the condition $u > 0$ in Ω is important (see [5] for a changing sign solution that grows faster than any exponential).

Proposition 5.3. *If u is a solution of (14) then there exist positive constants α and C such that*

$$u(x, y) \leq Ce^{\alpha|x|} \text{ in } \Omega.$$

The main tool in the proof of this result is a new and useful boundary Harnack inequality found in [5]. It is an extension up to the boundary of the Krylov-Safonov Harnack inequality. Recall that another boundary version of the Harnack inequality was also very useful in Section 4. Also, we point out that very similar (in their statement but not in their proof!) interior and boundary Harnack inequalities hold for operators in divergence form, $\tilde{M}u = \partial_{x_i}(a_{ij}(x)\partial_{x_j}u)$, with bounded measurable coefficients; this is the DeGiorgi-Nash-Moser theory (see Chapter 8 of [17]).

Here we consider an elliptic operator $Mu = a_{ij}(x)\partial_{ij}u$ with bounded coefficients, satisfying the uniform ellipticity condition of Section 1 with constants c_0 and C_0 . We assume that a_{ij} are continuous in $\bar{\Omega}$ (this will merely be a qualitative assumption since the estimates will not depend on the modulus of continuity of the a_{ij}).

Theorem 5.4. (Berestycki-Caffarelli-Nirenberg) *Let Ω be any domain of \mathbb{R}^n and let $\Sigma \subset \partial\Omega$ be a smooth open subset of $\partial\Omega$. Suppose that $u \in W_{loc}^{2,p}(\Omega \cup \Sigma)$, $p > n$, $u > 0$ in Ω , $u = 0$ on Σ , and*

$$|Mu| \leq A(|\nabla u| + u + \kappa) \text{ in } \Omega, \tag{15}$$

for some constants A and $\kappa \geq 0$. Let $K \subset \Omega$ be a compact subset of Ω , and let $G \subset \Omega \cup \Sigma$ be a compact subset of $\Omega \cup \Sigma$. Then

$$\sup_G u \leq C\{\inf_K u + \kappa\},$$

where C is a constant depending only on Ω , Σ , K , G , c_0 , C_0 and A .

For the proof of Theorem 5.4, see [5]. Using it, we easily deduce Proposition 5.3.

Proof of Proposition 5.3. We have that

$$\begin{aligned} |\Delta u| &= |f(u)| \leq |f(u) - f(0)| + |f(0)| \\ &\leq Au + |f(0)| = A(u + \kappa) \end{aligned}$$

for $\kappa := |f(0)|/A$, where A is the Lipschitz constant of f . Hence (15) is satisfied with $M = \Delta$. We fix a point $y_0 \in \omega$. Applying Theorem 5.4 with $\Sigma = \{|x| < 2\} \times \partial\omega$, $G = \{|x| \leq 1\} \times \bar{\omega}$ and $K = \{(0, y_0)\}$, we obtain

$$u(x, y) + \kappa \leq C\{u(0, y_0) + \kappa\} \text{ for } (x, y) \in \{|x| \leq 1\} \times \bar{\omega},$$

for some constant C . We fix any direction e_1 in \mathbb{R}^{n-j} , $|e_1| = 1$. Applying the previous inequality with u replaced by $u(x + e_1, y)$ (note that Ω is invariant by such a translation), we have

$$u(x, y) + \kappa \leq C\{u(e_1, y_0) + \kappa\} \quad \text{for } (x, y) \in \{|x - e_1| \leq 1\} \times \bar{\omega},$$

for the same constant C .

Putting both inequalities together, we obtain

$$u(x, y) + \kappa \leq C^2\{u(0, y_0) + \kappa\} \quad \text{for } (x, y) \in \{|x - e_1| \leq 1\} \times \bar{\omega}.$$

It is now easy to deduce, by induction, that

$$u(x, y) + \kappa \leq C^{m+1}\{u(0, y_0) + \kappa\} \quad \text{for } (x, y) \in \{|x - me_1| \leq 1\} \times \bar{\omega}.$$

This inequality yields at most exponential growth in the direction e_1 . Since e_1 is arbitrary, we obtain the conclusion. \square

To start the moving planes method for problem (14), we consider $a = \sup_{\omega} y_1$, $\omega_{\lambda} = \{y \in \omega : y_1 > \lambda\}$, $\Sigma_{\lambda} = \mathbb{R}^{n-j} \times \omega_{\lambda} \subset \Omega$, and $w_{\lambda}(x, y) = u(x, y) - u(x, y^{\lambda})$ where y^{λ} is the reflection of y in the plane $\{y_1 = \lambda\}$. The function w_{λ} satisfies $\Delta w_{\lambda} + c_{\lambda}(x, y)w_{\lambda} = 0$ in Σ_{λ} and $w_{\lambda} \leq 0$ on $\partial\Sigma_{\lambda}$. To start the method we need a maximum principle in cylinders Σ_{λ} with section ω_{λ} of small measure. Such a maximum principle has been proved in the previous section for bounded functions w_{λ} since, using the notation of that section, $R(\Sigma_{\lambda})$ is small if $|\omega_{\lambda}|$ is small. However, we cannot apply here such maximum principle since the function w_{λ} may be unbounded; in fact, we know that it may grow exponentially.

The starting point for the moving planes method is accomplished with the following maximum principle for subsolutions with at most exponential growth in ‘‘cylinders’’ with cross section of small measure; it is Theorem 1.6 of [5].

Theorem 5.5. *Let $\Omega = \mathbb{R}^{n-j} \times \omega$, where $\omega \subset \mathbb{R}^j$ is a smooth bounded domain. Let $w \in W_{\text{loc}}^{2,n}(\Omega) \cap C(\bar{\Omega})$ (here w is not necessarily bounded) satisfy*

$$\begin{cases} \Delta w + c(x, y)w \geq 0 & \text{in } \Omega \\ w \leq 0 & \text{on } \partial\Omega, \end{cases}$$

with $c \leq \tilde{b}$, and

$$w \leq Ce^{\alpha|x|} \quad \text{in } \Omega$$

for some positive constants \tilde{b} , α and C . Then there exists a constant $\delta > 0$, depending only on n , j , \tilde{b} and α , such that

$$|\omega| \leq \delta$$

implies $w \leq 0$ in Ω .

Proof. The function w satisfies

$$w \leq C e^{\mu'(|x_1| + \dots + |x_{n-j}|)}$$

for some $\mu' > 0$. Since ω is smooth, we can take a domain $\tilde{\omega}$, such that $\bar{\omega} \subset \tilde{\omega}$, with measure arbitrarily close to $|\omega|$. Since $|\tilde{\omega}| \approx |\omega| \leq \delta$ is small, we know that the principal eigenvalue of Δ_y in $\tilde{\omega}$ is large. Indeed, by Theorem 4.2 (ii), we have

$$\lambda_1 := \lambda_1(\Delta_y, \tilde{\omega}) \geq \frac{\tilde{\tau}(j)}{|\tilde{\omega}|^{2/j}} \geq \frac{\tilde{\tau}(j)}{2|\omega|^{2/j}} \geq \frac{\tilde{\tau}(j)}{2\delta^{2/j}}.$$

Hence, we can choose $\delta = \delta(n, j, \tilde{b}, \alpha) > 0$ sufficiently small such that $\lambda_1 > (\mu')^2(n - j) + \tilde{b}$.

Let β be a constant such that

$$\beta > \mu' \quad \text{and} \quad \beta^2(n - j) + \tilde{b} - \lambda_1 < 0.$$

Let ϕ_1 be the principal eigenfunction of Δ_y in $\tilde{\omega}$ (see Section 4):

$$\begin{cases} \Delta_y \phi_1(y) = -\lambda_1 \phi_1 & \text{in } \tilde{\omega} \\ \phi_1 = 0 & \text{on } \partial\tilde{\omega} \\ \phi_1 > 0 & \text{in } \tilde{\omega}. \end{cases}$$

We consider the function

$$g(x, y) = \phi_1(y) \cosh(\beta x_1) \dots \cosh(\beta x_{n-j}).$$

It satisfies $g > 0$ in $\bar{\Omega}$ (since $\bar{\omega} \subset \tilde{\omega}$) and

$$\begin{aligned} (\Delta + c(x, y))g &= \{\beta^2(n - j) + c - \lambda_1\}g \\ &\leq \{\beta^2(n - j) + \tilde{b} - \lambda_1\}g < 0 \text{ in } \Omega \end{aligned}$$

(compare this with condition (a) mentioned in the beginning of Section 4). To prove $w \leq 0$, we consider the function

$$z := \frac{w}{g}.$$

Using the classical maximum principle for z , we show that $z \leq 0$ and hence $w \leq 0$. Indeed, we have $g^2 \nabla z = g \nabla w - w \nabla g$, and hence $g^2 \Delta z + 2g \nabla g \cdot \nabla z = \operatorname{div}(g^2 \nabla z) = g \Delta w - w \Delta g \geq -c g w - w \Delta g = -g[(\Delta + c)g]z$. Therefore, z satisfies

$$\begin{cases} \Delta z + 2g^{-1} \nabla g \cdot \nabla z + g^{-1}[(\Delta + c)g]z \geq 0 & \text{in } \Omega \\ z \leq 0 & \text{on } \partial\Omega \\ \limsup_{|x| \rightarrow \infty} z(x, y) \leq 0, \end{cases}$$

where we have used that $\beta > \mu'$ to deduce the last inequality.

Note that the zero order coefficient, $g^{-1}(\Delta+c)g$, is negative in Ω . We then conclude, by the classical maximum principle, that $z \leq 0$ in Ω (and hence $w \leq 0$ in Ω). Indeed, if z was positive somewhere, it would achieve its supremum at an interior point — a contradiction with the elliptic inequality satisfied by z . \square

To start the moving planes method, we apply Theorem 5.5 to $\Sigma_\lambda = \mathbb{R}^{n-j} \times \omega_\lambda$ when $a - \lambda$ is small enough. Note that ω_λ has corners; however, the proof of Theorem 5.5 still applies to Σ_λ since ω_λ has an ϵ -neighborhood with small measure.

To continue moving the plane until λ reaches 0, a delicate analysis is needed since Σ_λ is not compact (see [5]). It is here where the condition $f(0) \geq 0$, if $j = 1$, enters.

6 Fully nonlinear equations: definitions and examples

We consider equations of the form

$$F(D^2u, x) = f(x),$$

where x belongs to a bounded domain Ω of \mathbb{R}^n , D^2u denotes the Hessian of the function $u : \Omega \rightarrow \mathbb{R}$, and $F(M, x)$ is a real valued function defined on $\mathcal{S}_n \times \Omega$. Here \mathcal{S}_n denotes the space of real $n \times n$ symmetric matrices.

We assume that F and f are continuous in x , and that F is a uniformly elliptic operator; that is:

Definition 6.1. We say that F is uniformly elliptic if there exist two constants (called the ellipticity constants) $0 < c_0 \leq C_0$ such that

$$c_0 \|N\| \leq F(M + N, x) - F(M, x) \leq C_0 \|N\|$$

for any $x \in \Omega$ and any pair M, N of symmetric matrices with $N \geq 0$. Here, $N \geq 0$ means that N is nonnegative definite, and $\|N\|$ denotes the largest eigenvalue of N (i.e., the spectral radius or (L^2, L^2) -norm of N).

We recall that any $M \in \mathcal{S}_n$ can be uniquely decomposed as $M = M^+ - M^-$, where $M^+ \geq 0$, $M^- \geq 0$ and $M^+M^- = 0$. Using this, it is easy to check that F is uniformly elliptic if and only if

$$F(M_0 + M, x) \leq F(M_0, x) + C_0 \|M^+\| - c_0 \|M^-\|$$

for any $x \in \Omega$, $M_0 \in \mathcal{S}_n$ and $M \in \mathcal{S}_n$.

In particular, any uniformly elliptic operator F is a monotone increasing and Lipschitz continuous function of $M \in \mathcal{S}_n$. Here we consider the usual order in \mathcal{S}_n , i.e., $M_1 \leq M_2$ if $M_2 - M_1 \geq 0$.

Suppose now that F is of class C^1 . We extend F to the space of all real $n \times n$ matrices, for instance by $F(A, x) = F((A + A^t)/2, x)$. Then F is a function of $n \times n$ real variables a_{ij} , and of $x \in \Omega$. We consider the first derivative of F with respect to a_{ij} and we denote it by F_{ij} , i.e.,

$$F_{ij}(A, x) = \frac{\partial F}{\partial a_{ij}}(A, x).$$

It is clear that if M and N are symmetric then $DF(M, x) \cdot N = F_{ij}(M, x)N_{ij}$ does not depend on the previous extension of F (since this is a directional derivative of F in a direction given by a symmetric matrix).

It is easy to verify that if F is uniformly elliptic, with ellipticity constants c_0 and C_0 , then

$$c_0|\xi|^2 \leq F_{ij}(M, x)\xi_i\xi_j \leq C_0|\xi|^2 \quad \forall (M, x, \xi) \in \mathcal{S}_n \times \Omega \times \mathbb{R}^n. \quad (16)$$

On the other hand, if (16) is satisfied then F is uniformly elliptic (as in Definition 6.1) with ellipticity constants c_0 and nC_0 .

For a uniformly elliptic functional F (not necessarily of class C^1), we say that F is concave (respectively, convex) if F is a concave (resp., convex) function of $M \in \mathcal{S}_n$, i.e., $F((M_1 + M_2)/2, x) \geq \{F(M_1, x) + F(M_2, x)\}/2$ (resp., \leq) for any $M_1, M_2 \in \mathcal{S}_n$ and any $x \in \Omega$.

The following are important examples of fully nonlinear elliptic equations.

1. *Pucci's equations.* For any fixed constants $0 < c_0 \leq C_0$ and for $M \in \mathcal{S}_n$, we define

$$\mathcal{M}^-(M) = \mathcal{M}^-(M; c_0, C_0) = c_0 \sum_{e_i > 0} e_i + C_0 \sum_{e_i < 0} e_i$$

and

$$\mathcal{M}^+(M) = \mathcal{M}^+(M; c_0, C_0) = C_0 \sum_{e_i > 0} e_i + c_0 \sum_{e_i < 0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M . Now let $A = (a_{ij})$ be a symmetric matrix with all its eigenvalues in $[c_0, C_0]$, i.e., such that $c_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq C_0|\xi|^2$ for any $\xi \in \mathbb{R}^n$. We say in this case that $A \in \mathcal{A}_{c_0, C_0}$. Consider the linear functional L_A on \mathcal{S}_n defined by

$$L_A M = a_{ij}M_{ij} = \text{tr}(AM) \quad \text{for } M \in \mathcal{S}_n,$$

where tr denotes the trace. Alternatively, we may consider L_A acting on functions:

$$L_A u = L_A(D^2 u) = a_{ij}\partial_{ij}u.$$

Using that $M = ODO^t$, where O is an orthogonal matrix and D is diagonal with diagonal elements equal to the eigenvalues e_i of M , it is easy to verify that

$$\mathcal{M}^-(M) = \inf_{A \in \mathcal{A}_{c_0, C_0}} L_A M$$

and

$$\mathcal{M}^+(M) = \sup_{A \in \mathcal{A}_{c_0, C_0}} L_A M.$$

Using these expressions, we easily deduce that \mathcal{M}^- and \mathcal{M}^+ are uniformly elliptic operators with ellipticity constants c_0 and nC_0 . Moreover, \mathcal{M}^- is concave (since it is the infimum of linear functionals) and \mathcal{M}^+ is convex; see Lemma 2.10 of [12]. These expressions also show that \mathcal{M}^- and \mathcal{M}^+ are extremal with respect to all linear operators with fixed ellipticity constants c_0 and C_0 . They are called Pucci's extremal operators.

The corresponding fully nonlinear equations are

$$\mathcal{M}^-(D^2u) = f(x) \quad \text{and} \quad \mathcal{M}^+(D^2u) = f(x).$$

2. *Bellman equations.* These are the equations for the optimal cost in a stochastic control problem. They are of the form

$$F(D^2u, x) := \inf_{\alpha \in \mathcal{A}} \{L_\alpha u(x) - f_\alpha(x)\} = 0,$$

where \mathcal{A} is any set, f_α is a real function in Ω , and $L_\alpha u = a_{ij}^\alpha(x) \partial_{ij} u$ is, for each $\alpha \in \mathcal{A}$, a uniformly elliptic operator with bounded measurable coefficients and with given ellipticity constants c_0 and C_0 . It is easy to check that the Bellman operator is uniformly elliptic and concave.

Note that if all a_{ij}^α and f_α are constant functions then the corresponding Bellman equation is of the form $F(D^2u) = 0$.

3. *Isaacs equations.* These equations arise in the theory of differential games. They are of the form

$$F(D^2u, x) := \sup_{\beta \in \mathcal{B}} \inf_{\alpha \in \mathcal{A}} \{L_{\alpha\beta} u(x) - f_{\alpha\beta}(x)\} = 0,$$

where $L_{\alpha\beta}$ is an arbitrary family of elliptic operators (with fixed ellipticity constants) as in the previous example. Isaacs equations are still uniformly elliptic, but no longer concave nor convex.

4. *The Monge-Ampère equation.* This equation is

$$\det D^2u = f(x). \tag{17}$$

The functional in consideration is $F(M) = \det M$. Hence $F_{ij}(M)$ is the cofactor of the i, j -entry M_{ij} of M . Thus $F_{ij} = (\det M)M^{ij}$ (by Cramer's rule) where M^{ij} are the entries of the inverse of M (in case it exists). It follows that (17) is elliptic only for positive definite matrices M ; equivalently, (17) is elliptic on the set of strictly convex functions u . Note that, for a strictly convex solution u of (17) to exist, we must have f positive.

In this case, we write (17) in the form

$$G(D^2u) := \log \det D^2u = \log f(x).$$

We have that $G_{ij}(M) = M^{ij}$. Hence

$$\sum_r G_{ir} M_{rs} = \delta_{is}$$

and

$$\sum_r G_{ir,kl} M_{rs} + G_{ik} \delta_{sl} = 0,$$

where $G_{ir,kl}$ denote the second partial derivatives of G . We deduce that

$$G_{ij,kl} + M^{ik} M^{jl} = \sum_{r,s} G_{ir,kl} M_{rs} M^{sj} + \sum_s G_{ik} \delta_{sl} M^{sj} = 0,$$

and thus $G_{ij,kl} = -M^{ik} M^{jl}$. We obtain

$$G_{ij,kl}(M)N_{ij}N_{kl} \leq 0 \quad \forall M, N \in \mathcal{S}_n,$$

and hence that $G(M) = \log \det M$ is a concave operator in the cone of positive definite matrices.

Even that the Monge-Ampère equation is not uniformly elliptic in all \mathcal{S}_n , many of the methods for concave uniformly elliptic operators may be adapted to equation (17) when $f > 0$.

5. *The equation of prescribed Gauss curvature.* Given a function $K(x) > 0$ in Ω , we look for a function $u \in C^2(\Omega)$ such that $K(x)$ is the Gauss curvature of the graph of u at the point $(x, u(x))$. We recall that the Gauss curvature is the product of all the principal curvatures. It follows that u satisfies

$$F(D^2u, Du, x) := \det D^2u - K(x)(1 + |Du|^2)^{(n+2)/2} = 0.$$

This is an elliptic operator on the set of strictly convex functions u . Here, F depends also on Du .

To simplify our exposition we limit ourselves to the case $F = F(D^2u, x)$, but the results presented below can be easily generalized to the case $F(D^2u, Du, x)$ (see [17]). For more details and references on these equations, see [17, 12].

7 $C^{1,\alpha}$ estimate for classical solutions of $F(D^2u) = 0$

For a solution of a second order elliptic equation one expects, in general, to control the second derivatives of the solution by the oscillation of the solution itself. More precisely, the following $C^{2,\alpha}$ and $W^{2,p}$ interior “a priori” estimates hold. Let u be a solution of a linear uniformly elliptic equation of the form

$$a_{ij}(x)\partial_{ij}u = f(x) \quad \text{in } B_1 \subset \mathbb{R}^n.$$

Then we have:

- (a) *Schauder’s estimates*: if a_{ij} and f belong to $C^\alpha(\overline{B}_1)$, for some $0 < \alpha < 1$, then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and $\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{C^\alpha(\overline{B}_1)})$, where C depends on the ellipticity constants and the $C^\alpha(\overline{B}_1)$ -norm of a_{ij} ; see Chapter 6 of [17].
- (b) *Calderón-Zygmund estimates*: if $a_{ij} \in C(\overline{B}_1)$ and $f \in L^p(B_1)$, for some $1 < p < \infty$, then $u \in W^{2,p}(B_{1/2})$ and $\|u\|_{W^{2,p}(B_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)})$, where C depends on the ellipticity constants and the modulus of continuity of the coefficients a_{ij} ; see Chapter 9 of [17].

These statements should be understood as regularity results for appropriate linear small perturbations of the Laplacian. Indeed, these estimates are proven by regarding the equation $a_{ij}(x)\partial_{ij}u = f(x)$ as

$$a_{ij}(x_0)\partial_{ij}u = [a_{ij}(x_0) - a_{ij}(x)]\partial_{ij}u + f(x).$$

One then applies to this equation the corresponding estimates for the constant coefficients operator $a_{ij}(x_0)\partial_{ij}$ (that one can think of as the Laplacian), observing that the factor $a_{ij}(x_0) - a_{ij}(x)$ is small (locally around x_0) in some appropriate norm, due to the regularity assumptions made on a_{ij} . Thus, the key point is to prove $C^{2,\alpha}$ and $W^{2,p}$ estimates for Poisson’s equation $\Delta u = f(x)$.

The goal is to extend these regularity theories to fully nonlinear elliptic equations $F(D^2u, x) = f(x)$. As we will explain in more detail below, this can be accomplished for any uniformly elliptic operator $F(M, x)$ which is concave (or convex) in M .

The previous discussion shows that one should start considering the case of equations with constant “coefficients” $F(D^2u) = f(x)$ (here, we think of $F(D^2u)$ as being equal to $F(D^2u(x), x_0)$ for a fixed x_0). In fact, the key ideas already appear by considering the simpler equation

$$F(D^2u) = 0.$$

In this section we prove $C^{1,\alpha}$ estimates (for some $0 < \alpha < 1$) for any uniformly elliptic equation of the form $F(D^2u) = 0$; here no concavity or convexity assumption

on F is needed. The tool that we use is the Krylov-Safonov Harnack inequality and its corollary on Hölder continuity of solutions of elliptic equations in nondivergence form with measurable coefficients (see Section 3).

Indeed, suppose that $u \in C^3(\overline{B}_1)$ satisfies $F(D^2u) = 0$, with $F \in C^1$. Differentiate this equation with respect to a direction x_k . Writing $u_k = \partial_k u$, we have

$$F_{ij}(D^2u(x)) \partial_{ij} u_k = 0 \quad \text{in } B_1.$$

This can be regarded as a linear equation $Lu_k = 0$ for the function u_k , where $L = a_{ij}(x)\partial_{ij}$ and $a_{ij}(x) = F_{ij}(D^2u(x))$. By (16), we know that L is uniformly elliptic. Note that a regularity hypothesis on the coefficients $a_{ij}(x)$ would mean to make a regularity assumption on the second derivatives of u — that we need to avoid. The key point is that the Krylov-Safonov theory makes no assumption on the regularity of a_{ij} . Hence, from Corollary 3.2 (ii) applied to the equation $Lu_k = 0$, we obtain $\|u_k\|_{C^\alpha(\overline{B}_{1/2})} \leq C\|u_k\|_{L^\infty(B_1)}$, where $0 < \alpha < 1$ and C depend only on n , c_0 and C_0 . Thus, we have the $C^{1,\alpha}$ estimate for u

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C\|u\|_{C^1(\overline{B}_1)}. \quad (18)$$

This “a priori” estimate may be improved in the following way.

Theorem 7.1. *Let F be uniformly elliptic (see Definition 6.1) and assume that $F \in C^1$. Let $u \in C^2(\overline{B}_1)$ be a solution of $F(D^2u) = 0$ in B_1 . Then there exist constants $0 < \alpha < 1$ and C , depending only on n , c_0 and C_0 , such that*

$$\|u\|_{C^{1,\alpha}(\overline{B}_{1/2})} \leq C\{\|u\|_{L^\infty(B_1)} + |F(0)|\}.$$

This result may be obtained from a version of (18) involving more refined (in fact, weighted) Hölder norms and from an interpolation inequality (see [17]).

Here we present a simple proof of Theorem 7.1 found in [10]. It uses the technique of increments and, of course, the Krylov-Safonov theory. It may be adapted to viscosity solutions and also to the case when F is not C^1 (see [10, 12]). Note that it is interesting to cover nondifferentiable functionals F , in order to include Pucci’s, Bellman’s and Isaacs’ equations (note that these operators, presented in the previous section, are not differentiable in general).

Proof of Theorem 7.1. Clearly we have

$$\begin{aligned} -F(0) &= F(D^2u(x)) - F(0) \\ &= [F(tD^2u(x))]_{t=0}^1 = \left[\int_0^1 F_{ij}(tD^2u(x)) dt \right] \partial_{ij} u(x) \\ &=: a_{ij}(x) \partial_{ij} u(x). \end{aligned}$$

Note that a_{ij} are uniformly elliptic. The Krylov-Safonov theory, Corollary 3.2 (ii), yields

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C\{\|u\|_{L^\infty(B_1)} + |F(0)|\} =: CK, \quad (19)$$

where $0 < \alpha < 1$ and C (as well as all other constants C in the proof) depend only on n, c_0 and C_0 . The constant α will be the same throughout all the proof. To simplify notation, we have denoted $\|u\|_{L^\infty(B_1)} + |F(0)|$ by K .

We fix a direction $e \in \mathbb{R}^n$, $|e| = 1$, and consider the function $u_h(x) = u(x + he)$ for $h > 0$ small enough. We have that both u and u_h satisfy the same nonlinear equation $F(D^2u) = 0 = F(D^2u_h)$, and hence the difference $u_h - u$ satisfies a linear equation. Indeed,

$$\begin{aligned} 0 &= [F((1-t)D^2u(x) + tD^2u_h(x))]_{t=0}^1 \\ &= \left[\int_0^1 F_{ij}((1-t)D^2u(x) + tD^2u_h(x)) dt \right] \partial_{ij}(u_h - u)(x) \\ &=: \tilde{a}_{ij}(x) \partial_{ij}(u_h - u)(x). \end{aligned}$$

Hence, $\tilde{a}_{ij}(x) \partial_{ij}[(u_h - u)/h^\alpha] = 0$. Using Corollary 3.2 (ii) again (now rescaled, and after a covering argument to apply it in $B_{1/4} \subset B_{1/3}$) we deduce

$$\begin{aligned} \|h^{-\alpha}(u_h - u)\|_{C^\alpha(\overline{B}_{1/4})} &\leq C \|h^{-\alpha}(u_h - u)\|_{L^\infty(B_{1/3})} \\ &\leq C \|u\|_{C^\alpha(\overline{B}_{1/2})} \\ &\leq CK \end{aligned}$$

for h small enough, by (19). This $C^\alpha(\overline{B}_{1/4})$ estimate for $h^{-\alpha}(u_h - u)$ (uniform in h) implies

$$\|u\|_{C^{2\alpha}(\overline{B}_{1/5})} \leq CK$$

with the aid of an easy lemma of real analysis — Lemma 3.1 of [10] or Lemma 5.6 of [12].

The same argument applied now to $h^{-2\alpha}(u_h - u)$ (which also solves $\tilde{a}_{ij}(x) \partial_{ij}[(u_h - u)/h^{2\alpha}] = 0$) gives

$$\begin{aligned} \|h^{-2\alpha}(u_h - u)\|_{C^\alpha(\overline{B}_{1/7})} &\leq C \|h^{-2\alpha}(u_h - u)\|_{L^\infty(B_{1/6})} \\ &\leq C \|u\|_{C^{2\alpha}(\overline{B}_{1/5})} \leq CK \end{aligned}$$

uniformly in h . We deduce a $C^{3\alpha}$ estimate for u .

Iterating this procedure a finite number of times (which depends only on n, c_0 and C_0), we arrive at $\|u\|_{C^1(\overline{B}_\delta)} \leq CK$ for some $\delta = \delta(n, c_0, C_0) > 0$. Applying the same

argument to $h^{-1}(u_h - u)$, we obtain a $C^{1,\alpha}(\overline{B}_{\delta/2})$ estimate for u in terms of K . This inequality, applied in a family of balls (of sufficiently small radius) that cover $B_{1/2}$, leads to the desired $C^{1,\alpha}(\overline{B}_{1/2})$ estimate. \square

Remark 7.2. A perturbation method due to Caffarelli [11] (see also Chapter 8 of [12]) extends the $C^{1,\alpha}$ estimate of Theorem 7.1 to equations $F(D^2u, x) = f(x)$ under appropriate dependence of F and f in x (and without any concavity hypothesis on $F(\cdot, x)$).

Remark 7.3. When the operator F is concave or convex, Evans [14] and Krylov [20, 21, 22] have shown in 1982 that classical solutions of $F(D^2u) = 0$ satisfy the $C^{2,\alpha}$ estimate

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C \{ \|u\|_{L^\infty(B_1)} + |F(0)| \},$$

where $0 < \alpha < 1$ and C depend only on n , c_0 and C_0 (see [12]). Recall that Pucci's equations are concave (or convex), Bellman's equations are concave, and the Monge-Ampère operator is log-concave.

The proof of this $C^{2,\alpha}$ estimate is based on a delicate application of the Krylov-Safonov weak Harnack inequality to $C - u_{kk}$, where u_{kk} denote the pure second derivatives of u . Note that, differentiating $F(D^2u) = 0$ twice with respect to x_k , we have

$$\begin{aligned} 0 &= F_{ij}(D^2u(x))\partial_{ij}u_{kk} + F_{ij,rs}(D^2u(x))(\partial_{ij}u_k)(\partial_{rs}u_k) \\ &\leq F_{ij}(D^2u(x))\partial_{ij}u_{kk} \end{aligned}$$

(by the concavity of F) and hence u_{kk} are subsolutions of a linear equation.

In 1989, Caffarelli [11] generalized the Calderón-Zygmund and Schauder theories to the context of fully nonlinear equations $F(D^2u, x) = f(x)$. Under the assumption of concavity of F and appropriate hypothesis on the dependence of F in x (see Chapters 7 and 8 of [12]), he proved the following. If $f \in L^p$ with $n < p < \infty$, then $u \in W^{2,p}$ in the interior and there is a $W^{2,p}$ estimate. If $f \in C^\alpha$, with $0 < \alpha < 1$ small enough depending on the ellipticity constants, then $u \in C^{2,\alpha}$ in the interior. The proofs also apply to viscosity solutions and to nondifferentiable functionals.

8 Viscosity solutions and Jensen's approximate solutions

In 1983 Crandall and Lions [13] developed a theory of weak solutions (so called viscosity solutions) for nonlinear partial differential equations. They are very useful when proving existence of solutions. For fully nonlinear equations, these weak solutions take the place that energy (or H^1) solutions have in the divergence form theory.

Definition 8.1. Let u be a continuous function in Ω . We say that u is a *viscosity subsolution* of $F(D^2u, x) = f(x)$ in Ω (or that u satisfies $F(D^2u, x) \geq f(x)$ in the viscosity sense in Ω) if the following condition holds: whenever $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ and $u - \phi$ has a local maximum at x_0 , then

$$F(D^2\phi(x_0), x_0) \geq f(x_0).$$

The definition of viscosity supersolution is analogous, replacing “local maximum” by “local minimum” and the inequality \geq by \leq . We say that u is a *viscosity solution* when it is both a viscosity subsolution and supersolution.

It is useful to think of this definition in the following way.

Proposition 8.2. *The following are equivalent:*

- (a) u is a viscosity subsolution of $F(D^2u, x) = f(x)$ in Ω .
- (b) Whenever $x_0 \in \Omega$, A is an open neighborhood of x_0 , $\phi \in C^2(A)$ and

$$\begin{cases} u \leq \phi & \text{in } A \\ u(x_0) = \phi(x_0), \end{cases} \quad (20)$$

then $F(D^2\phi(x_0), x_0) \geq f(x_0)$.

- (c) Same property as (b), with “ $\phi \in C^2(A)$ ” replaced by “ ϕ is a paraboloid” (i.e., a polynomial of at most degree 2).

Proof. The proofs (a) \implies (b) and (b) \implies (c) are trivial. To prove (c) \implies (a), let $\phi \in C^2(\Omega)$ be such that $u - \phi$ has a local maximum at $x_0 \in \Omega$. For any $\epsilon > 0$, we consider the paraboloid

$$P_\epsilon(x) = u(x_0) + D\phi(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^t D^2\phi(x_0)(x - x_0) + \frac{\epsilon}{2}|x - x_0|^2.$$

Then $u(x_0) = P_\epsilon(x_0)$ and $u \leq P_\epsilon$ in an open neighborhood of x_0 . Since we assume (c), we have $F(D^2\phi(x_0) + \epsilon I, x_0) \geq f(x_0)$, where I denotes the identity matrix. Letting $\epsilon \rightarrow 0$, we conclude $F(D^2\phi(x_0), x_0) \geq f(x_0)$ since $F(M, x)$ is a Lipschitz function of M (see the remarks after Definition 6.1). \square

We say that ϕ touches u from above at x_0 whenever there exists an open neighborhood A of x_0 such that (20) holds.

The idea in the definition of viscosity solution is to take the maximum principle itself as definition of solution. That is, the definition of viscosity solution requires the maximum principle to hold whenever $u \in C(\Omega)$ is “tested” against $C^2(\Omega)$ subsolutions and supersolutions.

Recall that a classical solution u of $F(D^2u, x) = f(x)$ in Ω is a $C^2(\Omega)$ function u that satisfies this equation pointwise. The following result states that for $C^2(\Omega)$ functions the classical and the viscosity notions of solution coincide. Its simple proof is similar to the previous one.

Proposition 8.3. *Assume that $u \in C^2(\Omega)$. Then u is a viscosity subsolution (resp., solution) of $F(D^2u, x) = f(x)$ in Ω if and only if $F(D^2u(x), x) \geq f(x)$ (resp., $F(D^2u(x), x) = f(x)$) for any $x \in \Omega$.*

The next two results are very useful when trying to prove existence or estimates for viscosity solutions. They show the “flexibility” of this weak notion of solution. Their proofs are simple (see [12]).

Proposition 8.4. *If u and v are viscosity subsolutions of $F(D^2u, x) = f(x)$ in Ω then $\sup(u, v)$ is also a viscosity subsolution of this equation in Ω . The same statement holds for supersolutions, now with respect to $\inf(u, v)$.*

Proposition 8.5. *Let $\{F_k\}_{k \geq 1}$ be a sequence of uniformly elliptic operators with ellipticity constants c_0 and C_0 . Let $\{u_k\}_{k \geq 1} \subset C(\Omega)$ satisfy $F_k(D^2u_k, x) \geq f_k(x)$ in the viscosity sense in Ω . Assume that F_k converges uniformly in compact sets of $\mathcal{S}_n \times \Omega$ to F , and that u_k and f_k converge uniformly to u and f , respectively, in compact sets of Ω . Then $F(D^2u, x) \geq f(x)$ in the viscosity sense in Ω .*

Next, we present a very important tool in the theory of viscosity solutions: Jensen’s approximate solutions. They constitute an essential technique to prove existence and uniqueness results for the Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

and also to extend the estimates of the previous section to viscosity solutions.

Let u be a continuous function in Ω and let H be an open set such that $\overline{H} \subset \Omega$. For $\epsilon > 0$, we define the *upper ϵ -envelope* of u (with respect to H) by

$$u^\epsilon(x_0) = \sup_{x \in \overline{H}} \left\{ u(x) + \epsilon - \frac{1}{\epsilon} |x - x_0|^2 \right\} \quad \text{for } x_0 \in H.$$

Explained in a geometric way, the graph of u^ϵ is the envelope of the graphs of the family $\{P_x^\epsilon\}_{x \in \overline{H}}$ of concave paraboloids with vertex $(x, u(x) + \epsilon)$ and Hessian equal to $-(2/\epsilon)I$.

It turns out that u^ϵ is a good regularization of u when dealing with fully nonlinear equations. In fact, we have:

Theorem 8.6. (Jensen)

- (a) $u^\epsilon \in C(H)$ and $u^\epsilon \downarrow u$ uniformly in H as $\epsilon \rightarrow 0$.
- (b) Let $\epsilon > 0$ be fixed. Then, for almost every $x_0 \in H$ there exists a paraboloid P (i.e., a polynomial of at most degree 2) such that $u^\epsilon(x) = P(x) + o(|x - x_0|^2)$ as $x \rightarrow x_0$ (i.e., $|x - x_0|^{-2}|u^\epsilon(x) - P(x)| \rightarrow 0$ as $x \rightarrow x_0$). In this case, we define $D^2u^\epsilon(x_0) = D^2P$.
- (c) Suppose that u is a viscosity subsolution of $F(D^2u) = 0$ in Ω and that H_1 is an open set such that $\overline{H_1} \subset H$. Then, for ϵ sufficiently small, u^ϵ is also a viscosity subsolution of $F(D^2v) = 0$ in H_1 . In particular, $F(D^2u^\epsilon(x)) \geq 0$ for a.e. $x \in H_1$.

Using convex paraboloids, we can define in a similar way the *lower ϵ -envelope* u_ϵ of u . We have that $u_\epsilon \uparrow u$ uniformly in H , and that $F(D^2u_\epsilon(x)) \leq 0$ a.e. in H_1 if $F(D^2u) \leq 0$ in the viscosity sense in Ω .

To prove Theorem 8.6, we will use the following properties of u^ϵ .

Lemma 8.7. Let $x_0, x_1 \in H$. Then

- (i) $\exists x_0^* \in \overline{H}$ such that $u^\epsilon(x_0) = u(x_0^*) + \epsilon - |x_0^* - x_0|^2/\epsilon$.
- (ii) $u^\epsilon(x_0) \geq u(x_0) + \epsilon$.
- (iii) $|u^\epsilon(x_0) - u^\epsilon(x_1)| \leq (3/\epsilon) \text{diam}(H) |x_0 - x_1|$.
- (iv) $0 < \epsilon < \epsilon' \implies u^\epsilon(x_0) \leq u^{\epsilon'}(x_0)$.
- (v) $|x_0^* - x_0|^2 \leq \epsilon \text{osc}_H u$.
- (vi) $0 < u^\epsilon(x_0) - u(x_0) \leq u(x_0^*) - u(x_0) + \epsilon$.

Proof. (i), (ii), (iv) and (vi) are clear. To show (iii), let $x \in \overline{H}$ and note that

$$\begin{aligned} u^\epsilon(x_0) &\geq u(x) + \epsilon - \frac{1}{\epsilon}|x - x_0|^2 \\ &\geq u(x) + \epsilon - \frac{1}{\epsilon}|x - x_1|^2 - \frac{1}{\epsilon}|x_1 - x_0|^2 - \frac{2}{\epsilon}|x - x_1||x_1 - x_0| \\ &\geq u(x) + \epsilon - \frac{1}{\epsilon}|x - x_1|^2 - \frac{3}{\epsilon} \text{diam}(H) |x_1 - x_0|. \end{aligned}$$

Taking the supremum over $x \in \overline{H}$, we conclude (iii).

To prove (v), note that we have

$$\frac{1}{\epsilon}|x_0^* - x_0|^2 = u(x_0^*) + \epsilon - u^\epsilon(x_0) \leq u(x_0^*) - u(x_0),$$

by (i) and (ii). □

To prove Theorem 8.6 we will also need the following result about convex functions. Roughly speaking, it states that any convex function has second derivatives at almost every point. Recall that, by definition, a continuous function v is convex in a ball B whenever $v((x+y)/2) \leq (v(x) + v(y))/2$ for any $x, y \in B$.

Theorem 8.8. (Alexandroff, Buselman, Feller) *Let v be a convex function in a ball B . Then, for almost every $x_0 \in B$ there exists a paraboloid P such that*

$$v(x) = P(x) + o(|x - x_0|^2) \quad \text{as } x \rightarrow x_0, \quad (21)$$

i.e., $|x - x_0|^{-2}|v(x) - P(x)| \rightarrow 0$ as $x \rightarrow x_0$.

For the proof of this result, see Theorem 1 in Section 6.4 of [15]. Now we use it to give the:

Proof of Theorem 8.6. The assertions in (a) follow easily from Lemma 8.7. To show (b), for any fixed $x_0 \in H$ we have

$$P_0(x) := u(x_0^*) + \epsilon - \frac{1}{\epsilon}|x - x_0^*|^2 \leq u^\epsilon(x) \quad \forall x \in H$$

(by the definition of u^ϵ) and $P_0(x_0) = u^\epsilon(x_0)$ (by (i) in Lemma 8.7). That is, the paraboloid P_0 touches u^ϵ from below at x_0 in all H . In particular

$$\Delta_h^2 u^\epsilon(x_0) \geq \Delta_h^2 P_0(x_0) = -\frac{2}{\epsilon},$$

for any $x_0 \in H$, $h > 0$ and $e \in \mathbb{R}^n$ ($|e| = 1$) such that $x_0 + he \in H$ and $x_0 - he \in H$. Here $\Delta_h^2 u^\epsilon(x_0) = h^{-2}\{u^\epsilon(x_0 + he) + u^\epsilon(x_0 - he) - 2u^\epsilon(x_0)\}$ denotes a second incremental quotient of u^ϵ at x_0 .

Thus, the function $v^\epsilon(x) = u^\epsilon(x) + |x|^2/\epsilon$ satisfies $\Delta_h^2 v^\epsilon(x_0) \geq 0$ for any such x , h and e . This implies that v^ϵ is a convex function in any ball contained in H . By Theorem 8.8, we deduce that v^ϵ has second order derivatives a.e. in H (in the sense of Theorem 8.8). In particular, the same is true for u^ϵ , since $u^\epsilon(x) = v^\epsilon(x) - |x|^2/\epsilon$.

Finally, note that at a point x_0 where (21) holds, the paraboloid P is uniquely determined by (21). Hence $D^2 u^\epsilon(x_0) = D^2 P$ is a consistent definition.

To prove (c), let $x_0 \in H_1$ and let P be a paraboloid that touches u^ϵ from above at x_0 (we are using criterion (c) of Proposition 8.2 to check that $F(D^2 u^\epsilon) \geq 0$ in the viscosity sense). By property (v) of Lemma 8.7 and since $x_0 \in H_1$, we have that $x_0^* \in H$ for ϵ sufficiently small.

Take any $x \in H$ sufficiently close to x_0^* , so that $x + x_0 - x_0^* \in H$. Then, by definition of u^ϵ , we have

$$u(x) \leq u^\epsilon(x + x_0 - x_0^*) + \frac{1}{\epsilon}|x_0 - x_0^*|^2 - \epsilon.$$

Therefore, again for x close enough to x_0^* ,

$$u(x) \leq P(x + x_0 - x_0^*) + \frac{1}{\epsilon}|x_0 - x_0^*|^2 - \epsilon =: Q(x)$$

and $u(x_0^*) = Q(x_0^*)$ (since $P(x_0) = u^\epsilon(x_0)$). Hence, the paraboloid Q touches u from above at x_0^* . Since $F(D^2u) \geq 0$ in the viscosity sense in Ω , we get

$$0 \leq F(D^2Q) = F(D^2P).$$

We have proved that u^ϵ is a viscosity subsolution of $F(D^2v) = 0$ in H_1 . Finally, by (b) we know that for a.e. $x_0 \in H$ there exists a paraboloid P such that

$$u^\epsilon(x) = P(x) + o(|x - x_0|^2) \quad \text{as } x \rightarrow x_0.$$

Moreover, we have defined $D^2u^\epsilon(x_0) = D^2P$. Fix such a point x_0 and some $\delta > 0$. Then $P(x) + \delta|x - x_0|^2/2$ touches u^ϵ from above at x_0 . Hence $F(D^2P + \delta I) \geq 0$, since u^ϵ is a viscosity subsolution. Letting $\delta \rightarrow 0$, we conclude $F(D^2u^\epsilon(x_0)) = F(D^2P) \geq 0$. \square

The proof of the $C^{1,\alpha}$ estimate (Theorem 7.1) relied on the fact that both u and $u_h = u(\cdot + he)$ were classical solutions of $F(D^2w) = 0$ and, hence, the difference $u - u_h$ solved a linear uniformly elliptic equation.

To extend this estimate to viscosity solutions (and thus to prove the $C^{1,\alpha}$ regularity of viscosity solutions of $F(D^2u) = 0$), note that we have $F(D^2u_h) = 0$ in the viscosity sense. The key point is to prove that the difference $u - u_h$ solves (in a generalized or viscosity sense) a linear equation. More precisely, one can prove (see Theorem 5.3 of [12]) the following.

Theorem 8.9. *Let u be a viscosity subsolution of $F(D^2w) = 0$ in Ω and v be a viscosity supersolution of $F(D^2w) = 0$ in Ω . Then $u - v$ satisfies $\mathcal{M}^+(D^2(u - v); c_0/n, C_0) \geq 0$ in the viscosity sense in Ω , where \mathcal{M}^+ denotes the extremal Pucci operator (see Section 6).*

To prove this theorem, one uses the Alexandroff-Bakelman-Pucci method (see Theorem 1.1) applied to $u^\epsilon - v_\epsilon$, where u^ϵ and v_ϵ denote, respectively, the upper and lower ϵ -envelopes of u and v . By property (c) in Theorem 8.6, $F(D^2u^\epsilon) \geq 0$ and $F(D^2v_\epsilon) \leq 0$ pointwise almost everywhere. This is the key point to obtain that $\mathcal{M}^+(D^2(u^\epsilon - v_\epsilon)) \geq 0$ in the viscosity sense. Then, letting $\epsilon \rightarrow 0$ and using the stability of viscosity subsolutions (Proposition 8.5), we obtain $\mathcal{M}^+(D^2(u - v)) \geq 0$ in the viscosity sense (see the proof of Theorem 5.3 in [12]).

Note also that Theorem 8.9 is trivial when at least one of the functions u and v is C^2 ; in this case the theorem follows from the definition of viscosity solutions.

The conclusion of Theorem 8.9 states that $\mathcal{M}^+(D^2(u - v)) \geq 0$ in the viscosity sense. In this case we say that $u - v \in \underline{\mathcal{S}}$ and we call $\underline{\mathcal{S}}$ the class of viscosity subsolutions (see Chapter 2 of [12]). The idea is that we have replaced any particular linear equation with given ellipticity constants by certain extremal inequalities given by Pucci's operators corresponding to these ellipticity constants.

Our previous proof of Theorem 7.1, together with Theorem 8.9, shows the following. To conclude $C^{1,\alpha}$ estimates for viscosity solutions it remains to extend the Krylov-Safonov theory to functions in the class $\underline{\mathcal{S}}$. This is explained in Chapters 3 and 4 of [12]. In this way one finally proves that any viscosity solution of $F(D^2u) = 0$ is $C^{1,\alpha}$ in the interior, where $0 < \alpha < 1$ depends only on n , c_0 and C_0 (see Corollary 5.7 of [12]).

Moreover, if F is concave (or convex), then any viscosity solution u is $C^{2,\alpha}$ (see Theorem 6.6 of [12]).

Theorem 8.9 has another important consequence, first proven by Jensen [19]. It gives the uniqueness of viscosity solution in $C(\overline{\Omega})$ for the Dirichlet problem

$$\begin{cases} F(D^2u) = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (22)$$

for any uniformly elliptic operator F (not necessarily concave nor convex). Existence of a viscosity solution for (22) was proved by Ishii [18] using Perron's method and Jensen's uniqueness result. Therefore, using the notion of viscosity solution we have an existence and uniqueness theory for problem (22), even for nonconcave functionals F for which $C^{2,\alpha}$ estimates are not available.

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