The Local Spectra of Regular Line Graphs ⋆

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Abstract

The local spectrum of a graph $G = (V, E)$, constituted by the standard eigenvalues of G and their local multiplicities, plays a similar role as the global spectrum when the graph is "seen" from a given vertex. Thus, for each vertex $i \in V$, the i-local multiplicities of all the eigenvalues add up to 1; whereas the multiplicity of each eigenvalue λ of G is the sum, extended to all vertices, of its local multiplicities.

In this work, using the interpretation of an eigenvector as a charge distribution on the vertices, we compute the local spectrum of the line graph LG in terms of the local spectrum of the regular graph G it derives from. Furthermore, some applications of this result are derived as, for instance, some results about the number of circuits of LG.

Key words: Graph spectrum; Eigenvalue; Local multiplicity; Line graph.

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1 Basic results

Throughout the paper, $G = (V, E)$ denotes a simple connected *graph* with *order* $n = |V|$ and *size* $m = |E|$. We label the vertices with the integers $1, 2, \ldots, n$. If i is adjacent to j; that is, $(i, j) \in E$, we sometimes write $i \sim j$. The *distance* between two vertices is denoted by $\operatorname{dist}(i, j)$. The set of vertices which are *ℓ-appart* from vertex i is $\Gamma_\ell(i) = \{j : \operatorname{dist}(i, j) = \ell\}$. Thus, the *degree* of vertex i is just $\delta_i := |\Gamma_1(i)| \equiv |\Gamma(i)|$. The *eccentricity* of a vertex is $\operatorname{ecc}(i) := \max_{1\le j\le n} \operatorname{dist}(i, j)$ and the *diameter* of the graph is $D = D(G) := \max_{1\le i\le n} \operatorname{ecc}(i)$. Whenever $\operatorname{ecc}(i) = D$, we say that i is a *diametral* vertex, and also that a pair of vertices i, j such that $\operatorname{dist}(i, j) = D$ is a *diametral pair*. Moreover, any shortest path between i and j is a *diametral path* of the graph. The graph is called *diametral* when all its vertices are diametral.

1.1 *Some algebraic-graph concepts*

Let us now recall some algebraic graph concepts and results. The *adjacency* matrix of a graph G, denoted by $\boldsymbol{A} = (a_{ij}) = \boldsymbol{A}(G)$, has entries $a_{ij} = 1$ if $i \sim j$ and $a_{ij} = 0$ otherwise. Then, the *characteristic* polynomial of G is just the characteristic polynomial of $\boldsymbol{A}$:

$$\phi_G(x) := \det(x\boldsymbol{I} - \boldsymbol{A}) = \prod_{l=0}^{d} (x - \lambda_l)^{m_l}.$$

Its roots, or eigenvalues of $\boldsymbol{A}$, constitute the *spectrum* of G, denoted by

$$\operatorname{sp} G := \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$$

where the superindices denote multiplicities. The different eigenvalues of G are represented by

$$\operatorname{ev} G := \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}.$$

It is well known that the diameter of G is lesser than the number of different eigenvalues; that is, $D(G) \le d$ (see, for instance, Biggs [1]). When $D(G) = d$ we say that G is an *extremal graph*.

1.2 *The spectral decomposition*

For each eigenvalue λ_l, $0 \le l \le d$, let $\boldsymbol{U}_l$ be the matrix whose columns form an orthonormal basis for the λ_l-eigenspace $\mathcal{E}_l := \operatorname{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$. The *(principal) idempotents* of $\boldsymbol{A}$ are the matrices $\boldsymbol{E}_l := \boldsymbol{U}_l \boldsymbol{U}_l^\top$ representing the orthogonal projections onto $\mathcal{E}_l$. Thus, in particular, $\boldsymbol{E}_0 = \frac{1}{\|\boldsymbol{v}\|^2} \boldsymbol{v}\boldsymbol{v}^\top$, where

$\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^\top$ denotes the normalized positive eigenvector. From their structure, it is readily checked that such matrices satisfy the following properties (see, for instance, Godsil [11]):

$(a.1)$ $\boldsymbol{E}_l \boldsymbol{E}_h = \begin{cases} \boldsymbol{E}_l & \text{if } l = h, \\ \boldsymbol{0} & \text{otherwise;} \end{cases}$
$(a.2)$ $\boldsymbol{A}\boldsymbol{E}_l = \lambda_l \boldsymbol{E}_l$;
$(a.3)$ $p(\boldsymbol{A}) = \sum_{l=0}^{d} p(\lambda_l)\boldsymbol{E}_l$, for any polynomial $p \in \mathbb{R}[x]$.

In particular, notice that if, in $(a.3)$, we take $p = 1$ and $p = x$ we obtain, respectively, $\sum_{l=0}^{d} \boldsymbol{E}_l = \boldsymbol{I}$ (as expected, since the sum of all orthogonal projections gives the original vector), and the so-called "Spectral Decomposition Theorem" $\sum_{l=0}^{d} \lambda_l \boldsymbol{E}_l = \boldsymbol{A}$. The following spectral decomposition of the canonical vectors is used below: $\boldsymbol{e}_i = \boldsymbol{z}_{i0} + \boldsymbol{z}_{i1} + \cdots + \boldsymbol{z}_{id}$ where $\boldsymbol{z}_{il} := \boldsymbol{E}_l \boldsymbol{e}_i \in \mathcal{E}_l$, $1 \le i \le n$, $0 \le l \le d$. Moreover,

$$\boldsymbol{z}_{i0} = \frac{\langle \boldsymbol{e}_i, \boldsymbol{v} \rangle}{\|\boldsymbol{v}\|^2} \boldsymbol{v} = \frac{v_i}{\|\boldsymbol{v}\|^2} \boldsymbol{v}. \tag{1}$$

In particular for regular graphs $\boldsymbol{z}_{i0} = (1/n)\boldsymbol{j}$ since, in this case $\boldsymbol{v} = (1/\sqrt{n})\,\boldsymbol{j}$, with $\boldsymbol{j}$ being the all-1 vector.

1.3 The local multiplicity

Given two vertices i, j and an eigenvalue λ_l, Garriga, Yebra and the first author, introduced in [6] the concept of *crossed* $(ij\text{-})$*local multiplicity* of λ_l as $m_{ij}(\lambda_l) := \langle \boldsymbol{z}_{il}, \boldsymbol{z}_{jl} \rangle$. Note that this corresponds to the ij-entry of the idempotent $\boldsymbol{E}_l$ since, using the symmetric character of $\boldsymbol{E}_l$ and property $(a.1)$,

$$\langle \boldsymbol{z}_{il}, \boldsymbol{z}_{jl} \rangle = \langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{E}_l \boldsymbol{e}_j \rangle = \langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{e}_j \rangle = (\boldsymbol{E}_l)_{ij}.$$

From the above properties of the idempotents we have that the crossed local multiplicities satisfy the following:

$(b.1)$ $\sum_{l=0}^{d} m_{ij}(\lambda_l) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$
$(b.2)$ $\sum_{j \sim i} m_{ij}(\lambda_l) = 2\sum_{(i,j)\in E} m_{ij}(\lambda_l) = \lambda_l m_{ii}(\lambda_l)$;
$(b.3)$ $a_{ij}^{\ell} = \sum_{l=0}^{d} m_{ij}(\lambda_l)\lambda_l^{\ell}$,

where $a_{ij}^{\ell} := (\boldsymbol{A}^{\ell})_{ij}$ is the number of ℓ-walks between vertices i and j (see Godsil [9,10]) including closed walks (when $i = j$). Under some assumptions, the local crossed multiplicities admit closed expressions. For instance, when

$\lambda = \lambda_0$, we have

$$m_{ij}(\lambda_0) = \langle \frac{v_i}{\|\boldsymbol{v}\|^2}\boldsymbol{v}, \frac{v_j}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\rangle = \frac{v_i v_j}{\|\boldsymbol{v}\|^2}. \tag{2}$$

Another example is given by the following result, see [6].

Let i, j be a pair of diametral vertices of an extremal graph G with normalized positive eigenvector $\boldsymbol{v}$. Then, the number of diametral paths between them and the crossed ij-local multiplicities are respectively given by

$$a_{ij}^d = \pi_0 \frac{v_i v_j}{\|\boldsymbol{v}\|^2}, \qquad m_{ij}(\lambda_l) = (-1)^l \frac{\pi_0}{\pi_l}\frac{v_i v_j}{\|\boldsymbol{v}\|^2} \quad (1 \le l \le d),$$

where $\pi_l := \prod_{h=0, h\neq l}^{d} |\lambda_l - \lambda_h| \quad (0 \le l \le d)$.

In particular under the above assumptions, when G is a regular graph and considering that $a_{ij}^\ell = 0$ for any $\ell \le d-1$, property $(b.3)$ yields:

$$\sum_{l=0}^{d} \frac{(-1)^l}{\pi_l}\lambda_l^\ell = 0 \quad (0 \le \ell \le d-1); \qquad \sum_{l=0}^{d} \frac{(-1)^l}{\pi_d}\lambda_l^d = 1. \tag{3}$$

Moreover, for distance-regular graphs a closed expression for local crossed multiplicities can be obtained. In a distance-regular graph G, the crossed ij-local multiplicities only depend on the distance $k = \mathrm{dist}(i,j)$ and we can write $m_{ij}(\lambda_l) = m_{kl}$, see Godsil [10]. As noted in [4], crossed ij-local multiplicities can be given in terms of the k-distance polynomial and the (global) multiplicity,

$$m_{kl} = \frac{m(\lambda_l)\, p_k(\lambda_l)}{n\, p_k(\lambda_0)} \quad (0 \le k, l \le d). \tag{4}$$

where $p_k(x)$, $0 \le k \le d$, is the k-distance polynomial of G.

1.4 The local spectrum

The crossed ij-local multiplicities seem to have a special relevance when $i = j$. In this case $m_{ii}(\lambda_l) = \|\boldsymbol{z}_{il}\|^2 \ge 0$, denoted also by $m_i(\lambda_l)$, is referred to as the *i-local multiplicity of* λ_l. (In particular, (2) yields $m_i(\lambda_0) = v_i^2/\|\boldsymbol{v}\|^2$.) In [5] it was noted that when the graph is "seen" from vertex i, the i-local multiplicities play a similar role as the standard multiplicities, so justifying the name. Indeed, by property $(b.1)$ note that, for each vertex i, the i-local multiplicities of all the eigenvalues add up to 1: $\sum_{l=0}^{d} m_i(\lambda_l) = 1$ whereas the multiplicity of each eigenvalue λ_l is the sum, extended to all vertices, of its local multiplicities since

$$m(\lambda_l) = \operatorname{tr} \boldsymbol{E}_l = \sum_{i=1}^{n} m_i(\lambda_l). \tag{5}$$

Moreover, property $(b.3)$ tells us that the number of closed walks of length ℓ going through vertex i, a_{ij}^{ℓ}, can be computed in a similar way as is computed the whole number of such walks in G by using the "global" multiplicities. Some closely related parameters are the Cvetković's *"angles"* of G, which are defined as the cosines $\cos \beta_{il}$, $1 \le i \le n$, $0 \le l \le d$, with β_{il} being the angle between $\boldsymbol{e}_i$ and the eigenspace $\operatorname{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$ (notice that $m_i(\lambda_l) = \cos^2 \beta_{il}$.) For a number of applications of these parameters, see for instance Cvetković, Rowlinson, and Simić [3].

By considering only the eigenvalues, say $\mu_0 (= \lambda_0) > \mu_1 > \cdots > \mu_{d_i}$, with non-null local multiplicities, we can now define the (i-)*local spectrum* as

$$\operatorname{sp}_i G := \{\lambda^{m_i(\lambda_0)}, \mu_1^{m_i(\mu_1)}, \ldots, \mu_{d_i}^{m_i(\mu_{d_i})}\}. \tag{6}$$

with (i-)*local mesh*, or set of distinct eigenvalues, $\mathcal{M}_i := \{\lambda_0 > \mu_1 > \cdots > \mu_{d_i}\}$. Then it can be proved that the eccentricity of i satisfies a similar upper bound as that satisfied by the diameter of G in terms of its distinct eigenvalues. More precisely, $\operatorname{ecc}(i) \le d_i = |\mathcal{M}_i| - 1$ (see [6].)

From the i-local spectrum (6), it is natural to consider the function that is the analogue of the characteristic polynomial, which we call the *i-local characteristic function*, defined by:

$$\phi_i(x) := \prod_{l=0}^{d_i} (x - \mu_l)^{m_i(\mu_l)}. \tag{7}$$

As expected, such a function can be computed from the knowledge of the characteristics polynomials of G and $G \setminus i$.

Proposition 1 *Given a vertex i of a graph G, its i-local characteristic function is*

$$\phi_i(x) = e^{\int \phi_{G \setminus i}(x)/\phi_G(x)\, dx}. \tag{8}$$

Proof. First note that the characteristic polynomial $\phi_{G \setminus i}(x)$ is just, see [2], the ii-entry of the adjoint matrix of $x\boldsymbol{I} - \boldsymbol{A}$ which, in turn, can be written as

$$\begin{aligned} \det(x\boldsymbol{I} - \boldsymbol{A})(x\boldsymbol{I} - \boldsymbol{A})^{-1} &= \phi_G(x)(x\boldsymbol{I} - \boldsymbol{A})^{-1} \\ &= \phi_G(x) \sum_{l=0}^{d} \frac{1}{x - \lambda_l} \boldsymbol{E}_l, \end{aligned}$$

where we have used property $(a.3)$ extended to the continuity points of any rational function (in our case, $x \neq \lambda_l$). Hence, $\phi_{G\setminus i}(x) = \phi_G(x)\sum_{l=0}^{d}\frac{m_i(\lambda_l)}{x-\lambda_l}$. and, thus,

$$\frac{\phi_{G\setminus i}(x)}{\phi_G(x)} = \sum_{l=0}^{d}\frac{m_i(\lambda_l)}{x-\lambda_l} = \sum_{l=0}^{d_i}\frac{m_i(\mu_l)}{x-\mu_l} = \frac{\phi_i'(x)}{\phi_i(x)}. \tag{9}$$

Then, we obtain the claimed result integrating both sides with respect to x and isolating $\phi_i(x)$. □

As a by-product, note also that, from (9) and adding over all the vertices, we get the known result:

$$\sum_{i=1}^{n}\phi_{G\setminus i}(x) = \phi_G(x)\sum_{l=0}^{d}\sum_{i=1}^{n}\frac{m_i(\lambda_l)}{x-\lambda_l} = \phi_G(x)\sum_{l=0}^{d}\frac{m_l}{x-\lambda_l} = \phi_G'(x).$$

See, for instance, [11].

1.5 Eigenvectors in a graph

A very simple, yet surprisingly useful, idea is the interpretation of the eigenvectors and eigenvalues of a graph as a dynamic process of "charge displacement" (see, for instance, Godsil [11]). To this end, suppose that $\boldsymbol{A}$ is the adjacency matrix of a graph $G = (V, E)$ and $\boldsymbol{v}$ a right eigenvector of $\boldsymbol{A}$ with eigenvalue λ. If we think $\boldsymbol{v}$ as a function from V to the real numbers, we associate v_i to the "initial charge" (or weight) of vertex i. Since $\boldsymbol{A}$ is a 0-1 matrix, the equation $\boldsymbol{Av} = \lambda\boldsymbol{v}$ is equivalent to

$$(\boldsymbol{Av})_i = \sum_{j=1}^{n} a_{ij}v_j = \sum_{j\sim i} v_j = \lambda v_i \qquad \text{for all } i \in V. \tag{10}$$

Thus, the sum of the charges of the neighbors of i is λ times the charge of vertex i. In [11] it is shown how this idea can be extended to "vector charges", so leading to the important area of research in graph theory known as representation theory.

2 The spectra of line graphs

The *line graph* LG of a graph $G = (V, E)$ is defined as follows. Each vertex in LG represents an edge of G, $V_L = \{(i\cdot j) : (i, j) \in E\}$, and two vertices of LG are adjacent whenever the corresponding edges in G have one vertex in common.

Since the classical paper of Sachs [12], the spectra of line graphs have been studied extensively. In [7] the authors used the mentioned idea of interpreting the eigenvectors as a certain charge distributions to prove that, if a δ-regular graph G has the eigenvector $\boldsymbol{u}$ with eigenvalue $\lambda \neq -\delta$, then the vector $\boldsymbol{v}$ with entries $v_{(i\cdot j)} = u_i + u_j$, $(i\cdot j) \in V_L$, is a $(\lambda+\delta-2)$-eigenvector of LG. The same method can be used to derive the local spectrum of LG.

2.1 The local spectrum of a regular line graph

The following result tells us how to compute the local spectrum of a line graph from the local spectrum of the (regular) graph it derives from.

Theorem 2 *Let G be a δ-regular graph, with eigenvalue λ, multiplicity $m(\lambda)$, and (crossed) local multiplicities $m_{ij}(\lambda)$, $i,j \in V$. Then, the crossed local multiplicities of the eigenvalues $\lambda' = \lambda+\delta-2$, $\lambda \neq -\delta$, and $\lambda' = -2$ in the line graph LG, are given by the expressions:*

$$m_{(i\cdot j)(k\cdot h)}(\lambda') = \frac{m_{ik}(\lambda)+m_{ih}(\lambda)+m_{jk}(\lambda)+m_{jh}(\lambda)}{\delta+\lambda} \quad (\lambda \neq -\delta), \qquad (11)$$

$$m_{(i\cdot j)(k\cdot h)}(-2) = \alpha - \sum_{\lambda\neq-\delta} m_{(i\cdot j)(k\cdot h)}(\lambda), \qquad (12)$$

where $\alpha = 0$ if $(i\cdot j) \neq (k\cdot h)$ and $\alpha = 1$ otherwise.

Proof. Assume first that $\lambda \neq -\delta$, and let U be the set of $m(\lambda)$ column vectors of the matrix $\boldsymbol{U}$ (recall that these vectors constitute an orthonormal basis of the corresponding eigenspace $\mathcal{E} = \mathrm{Ker}(\boldsymbol{A}-\lambda\boldsymbol{I})$). Then, given $\boldsymbol{u} \in U$, vector $\boldsymbol{v}$ with components $v_{(i\cdot j)} = u_i + u_j$ is a $\lambda'(= \lambda+\delta-2)$-eigenvector of LG, see [7]. Notice that, since

$$\sum_{(i,j)\in E} (u_i+u_j)^2 = \sum_{i\in V} \delta u_i^2 + \sum_{(i,j)\in E} 2u_iu_j = \delta + \langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}\rangle = \delta+\lambda,$$

the corresponding normalized vector has components $\frac{u_i+u_j}{\sqrt{\delta+\lambda}}$. Then, the crossed $(i\cdot j)(k\cdot h)$-local multiplicity of λ' is

$$\begin{aligned} m_{(i\cdot j)(k\cdot h)}(\lambda') &= \sum_{\boldsymbol{u}\in U} \frac{(u_i+u_j)(u_k+u_h)}{\delta+\lambda} \\ &= \frac{1}{\delta+\lambda} \sum_{\boldsymbol{u}\in U} (u_iu_h + u_iu_k + u_ju_k + u_ju_h) \\ &= \frac{m_{ik}(\lambda)+m_{ih}(\lambda)+m_{jk}(\lambda)+m_{jh}(\lambda)}{\delta+\lambda}. \end{aligned}$$

Finally, the crossed local multiplicity of the eigenvalue $\lambda' = -2$ is obtained by using property $(b.1)$. $\square$

Notice that, in particular, the local multiplicities of λ' are $m_{(i\cdot j)}(\lambda') = m_{(i\cdot j)(i\cdot j)}(\lambda')$, which gives:

$$m_{(i\cdot j)}(\lambda') = \frac{m_i(\lambda) + 2m_{ij}(\lambda) + m_j(\lambda)}{\delta + \lambda} \quad (\lambda \neq -\delta); \tag{13}$$

$$m_{(i\cdot j)}(-2) = 1 - \sum_{\lambda \neq -\delta} \frac{m_i(\lambda) + 2m_{ij}(\lambda) + m_j(\lambda)}{\delta + \lambda}. \tag{14}$$

Then, as expected,

$$\begin{aligned}
\sum_{(i\cdot j)\in V_L} m_{(i\cdot j)}(\lambda') &= \frac{1}{\delta+\lambda} \sum_{(i,j)\in E} (m_i(\lambda) + 2m_{ij}(\lambda) + m_j(\lambda)) = \\
&= \frac{1}{\delta+\lambda} \left(\sum_{i\in V} \delta m_i(\lambda) + \sum_{i\sim j} m_{ij}(\lambda) \right) = \\
&= \frac{1}{\delta+\lambda} \sum_{i\in V} (\delta+\lambda) m_i(\lambda) = m(\lambda).
\end{aligned}$$

where the last two equalities come from property $(b.2)$ and equality (5), respectively.

The previous result can be used to compute,in the line graph LG, the number of ℓ-circuits rooted at vertex $(i\cdot j)$.

Proposition 3 *Let G be a δ–regular graph, $\delta > 2$, with spectrum* $\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$ *and crossed (ij)-local multiplicities $m_{ij}(\lambda_l)$, $i, j \in V$ $\;0 \leq l \leq d$. Then, the number of circuits of length ℓ, $\ell > 1$, rooted at vertex $(i\cdot j)$ in the line graph LG, is given by*

$$(\boldsymbol{A}_{LG}^{\ell})_{(i\cdot j)(i\cdot j)} = (-2)^{\ell} + \sum_{\lambda_l \neq -\delta} \sum_{p=0}^{\ell-1} K_p \lambda_l^{\ell-p-1} (m_i(\lambda_l) + m_j(\lambda_l) + 2m_{ij}(\lambda_l)) \tag{15}$$

where, for each ℓ, the coefficient of $\lambda_l^{\ell-p-1}$ is

$$K_p := \sum_{r=0}^{p} \binom{\ell}{r} \binom{\ell-r-1}{p-r} \delta^{p-r} (-2)^r. \tag{16}$$

Proof. We assume $\delta > 2$ in order to have $\lambda' = -2$ as an eigenvalue of LG, although we only would need to exclude odd cycles. It follows from property $(b.3)$ and Eq. (11) that,

$$(\boldsymbol{A}^{\ell}_{LG})_{(i\cdot j)(i\cdot j)} = \sum_{\lambda'_l \in \mathrm{sp}\, LG} (\lambda'_l)^{\ell}\, m_{(i\cdot j)}(\lambda'_l) =$$

$$= \sum_{\lambda_l \neq -\delta} (\lambda_l + \delta - 2)^{\ell} \frac{m_i(\lambda_l) + 2m_{ij}(\lambda_l) + m_j(\lambda_l)}{\lambda_l + \delta} + (-2)^{\ell}\, m_{(i\cdot j)}(-2) =$$

$$= \sum_{\lambda_l \neq -\delta} (\lambda_l + \delta - 2)^{\ell} \frac{m_i(\lambda_l) + 2m_{ij}(\lambda_l) + m_j(\lambda_l)}{\lambda_l + \delta} +$$

$$+ (-2)^{\ell} \left[1 - \sum_{\lambda \neq -\delta} \frac{m_i(\lambda_l) + 2m_{ij}(\lambda_l) + m_j(\lambda_l)}{\lambda_l + \delta} \right] =$$

$$= (-2)^{\ell} + \sum_{\lambda_l \neq -\delta} \frac{(\lambda_l + \delta - 2)^{\ell} - (-2)^{\ell}}{\lambda_l + \delta} (m_i(\lambda_l) + 2m_{ij}(\lambda_l) + m_j(\lambda_l)) =$$

$$= (-2)^{\ell} +$$

$$+ \sum_{\lambda_l \neq -\delta} \left[\binom{\ell}{0} (\lambda_l + \delta)^{\ell-1} + \quad \cdots + \binom{\ell}{\ell-1} (-2)^{\ell-1} \right] (m_i(\lambda_l) + 2m_{ij}(\lambda_l) + m_j(\lambda_l)).$$

Collecting coefficients of $\lambda_l^{\ell-p-1}$, $0 \leq p \leq \ell - 1$, we obtain the result. □

We may rewrite the expression of the number of circuits rooted at a given vertex $(i \cdot j)$ in the line graph LG, as a function of the number of circuits rooted at vertex i, circuits rooted at vertex j and walks that contain edge (i, j) in the original graph G.

Corollary 4 *Under the same hypothesis of Proposition 3,*

$$(\boldsymbol{A}^{\ell}_{LG})_{(i\cdot j)(i\cdot j)} = (-2)^{\ell} + \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{ii} + \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{jj} + 2 \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{ij}$$

Proof. Again we use property $(b.3)$ both in G and LG

$$(\boldsymbol{A}^{\ell}_{LG})_{(i\cdot j)(i\cdot j)} = (-2)^{\ell} + \sum_{\lambda_l \neq -\delta} \sum_{p=0}^{\ell-1} K_p \lambda_l^{\ell-p-1} (m_i(\lambda_l) + m_j(\lambda_l) + 2m_{ij}(\lambda_l)) =$$

$$= (-2)^{\ell} + \sum_{p=0}^{\ell-1} K_p \sum_{\lambda_l \neq -\delta} \lambda_l^{\ell-p-1} (m_i(\lambda_l) + m_j(\lambda_l) + 2m_{ij}(\lambda_l)) =$$

$$= (-2)^{\ell} + \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{ii} + \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{jj} + 2 \sum_{p=0}^{\ell-1} K_p (\boldsymbol{A}_G^{\ell-p-1})_{ij}.$$

□

In particular, since for $p = 0, 1, 2$ we have

$$K_0 = 1, \quad K_1 = (\ell - 1)\delta - 2\ell, \quad K_2 = \frac{(\ell-1)(\ell-2)}{2} \delta^2 - 2\delta\, \ell(\ell - 2) + 2\, \ell(\ell - 1).$$

the number of circuits rooted at vertex $(i{\cdot}j)$ in the line graph, is

$$\begin{aligned}(\boldsymbol{A}_{LG}^2)_{(i\cdot j)(i\cdot j)} &= 4\delta - 2\\ (\boldsymbol{A}_{LG}^3)_{(i\cdot j)(i\cdot j)} &= (\boldsymbol{A}_G^2)_{ii} + (\boldsymbol{A}_G^2)_{jj} + 2(\boldsymbol{A}_G^2)_{ij} + 2\delta^2 - 8\delta + 4\end{aligned}$$

respectively, for $\ell = 2$ and $\ell = 3$.

Furthermore, using a similar reasoning, a general expression for the number of ℓ-walks between any vertices $(i{\cdot}j)$ and $(k{\cdot}h)$ in LG holds,

$$(\boldsymbol{A}_{LG}^\ell)_{(i\cdot j)(k\cdot h)} = \alpha(-2)^\ell + \sum_{\lambda_l \neq -\delta} \sum_{p=0}^{\ell-1} K_p \lambda_l^{\ell-p-1}(m_{ik}(\lambda_l) + m_{ih}(\lambda_l) + m_{jk}(\lambda_l) + m_{jh}(\lambda_l)) \tag{17}$$

In terms of the number of walks between vertices in G

$$(\boldsymbol{A}_{LG}^\ell)_{(i\cdot j)(k\cdot h)} = \alpha(-2)^\ell + \sum_{p=l0}^{\ell} K_{p-1}\left((\boldsymbol{A}_G^{\ell-p})_{ik} + (\boldsymbol{A}_G^{\ell-p})_{ih} + (\boldsymbol{A}_G^{\ell-p})_{jk} + (\boldsymbol{A}_G^{\ell-p})_{jh}\right) \tag{18}$$

In both cases, $\alpha = 0$ if $(i{\cdot}j) \neq (k{\cdot}h)$ and $\alpha = 1$ otherwise and coefficients K_p are as in (16).

2.2 Local multiplicities in a cycle C_n

Let us consider C_n the cycle of order n. As the line graph of a cycle C_n is itself, $LC_n = C_n$, the number of ℓ–circuits rooted at vertex i in C_n, and at vertex $(i{\cdot}j)$ in LC_n are the same. This fact allow us to derive simple relations between the crossed and local multiplicities of each eigenvalue. Taking into account that, as $\delta = 2$, $\lambda' = \lambda + \delta - 2 = \lambda$ and using (11),

$$m_i(\lambda) = m_{(i\cdot j)(i\cdot j)}(\lambda') = \frac{m_i(\lambda) + 2m_{ij}(\lambda) + m_j(\lambda)}{\lambda + 2}$$

we get the crossed local multiplicity $m_{ij}(\lambda)$ corresponding to adjacent vertices i, j in C_n and $\lambda \neq -2$. Let us notice that, since cycles ara a particular case of walk-regular graphs, local multiplicities do not depend on the vertex they are referred to, see Godsil [11], thus property 5 and property (**??**) reads: $m_i(\lambda) = \frac{m(\lambda)}{n}$., thus

$$m_i(\lambda) = \frac{2m_i(\lambda) + 2m_{ij}(\lambda)}{\lambda + 2}, \quad m_{ij}(\lambda) = \frac{\lambda\, m_i(\lambda)}{2} = \frac{\lambda\, m(\lambda)}{2n}. \tag{19}$$

When $\lambda = -2$, i.e. for even cycles, the local multiplicity follows from (12),

$$m_i(-2) = 1 - 2\sum_{\lambda\neq -2} \frac{m_i(\lambda) + m_{ij}(\lambda)}{\lambda + 2}. \tag{20}$$

Crossed local multiplicities corresponding to non-adjacent vertices can also be computed. In order to simplify the notation, for a given eigenvalue λ of the cycle graph C_n and vertices i, j with $\mathrm{dist}(i, j) = \ell$, we will denote $m_\ell(\lambda) := m_{ij}(\lambda)$, or simply m_ℓ if there is no confusion about the eigenvalue we are referring to.

Let p, q, i, j, r, s vertices of the cycle graph C_n, such that $i \sim j$ and $r \sim s$. Let us suppose $\mathrm{dist}(p, q) = \ell = \mathrm{dist}((i \cdot j), (r \cdot s))$, $0 \leq \ell < \lfloor \frac{n}{2} \rfloor$. Then, as above, the equality $m_{pq}(\lambda) = m_{(i\cdot j)(r\cdot s)}(\lambda)$ and (11) leeds to

$$m_\ell = \frac{m_{\ell+1} + 2m_\ell + m_{\ell-1}}{\lambda + 2},$$

that gives

$$m_1 = \frac{\lambda}{2} m_0,$$

and

$$\lambda\, m_\ell = m_{\ell-1} + m_{\ell+1} \quad (0 < \ell < \lfloor \frac{n}{2} \rfloor). \tag{21}$$

Notice that the last equality is a Chebyshev's recurrence. Thus, the crossed local multiplicity corresponding to eigenvalue λ and vertices at distance ℓ, is related to the Chebyshev polynomial of the first kind of degree ℓ in $\frac{\lambda}{2}$, as follows

$$m_\ell(\lambda) = T_\ell\left(\frac{\lambda}{2}\right) m_0. \tag{22}$$

Using a different point of view, we may write equality (21) in matricial notation,

$$\begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} m_\ell \\ m_{\ell-1} \end{pmatrix} = \begin{pmatrix} m_{\ell+1} \\ m_\ell \end{pmatrix}$$

or equivaletnly

$$\begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}^\ell \begin{pmatrix} m_1 \\ m_0 \end{pmatrix} = \begin{pmatrix} m_{\ell+1} \\ m_\ell \end{pmatrix}$$

With some linear algebra, we can derive explicit expressions for the crossed multiplicities m_ℓ as function of the corresponding local multiplicity m_0. For each eigenvalue λ of G, define $\Phi_1 = \frac{\lambda+\sqrt{\lambda^2-4}}{2}$, $\Phi_2 = \frac{\lambda-\sqrt{\lambda^2-4}}{2}$, then

$$m_{\ell+1} = \frac{m_0}{\Phi_1 - \Phi_2}\left[\frac{\lambda}{2}\left(\Phi_1^{\ell+1} - \Phi_2^{\ell+1}\right) + \Phi_1^{\ell} - \Phi_2^{\ell}\right], \tag{23}$$

$0 < \ell < \lfloor \frac{n}{2} \rfloor$. As expected, we get the Chebyshev polynomials evaluated at $\lambda/2$:

$$m_1 = \frac{\lambda}{2}\, m_0, \quad m_2 = \frac{\lambda^2 - 2}{2}\, m_0, \quad m_3 = \frac{1}{2}\lambda(\lambda^2 - 3)\, m_0, \ldots$$

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