The Hierarchical Product of Graphs

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Abstract

A new operation on graphs is introduced and some of its properties are studied. We call it hierarchical product, because of the strong (connectedness) hierarchy of the vertices in the resulting graphs. In fact, the obtained graphs turn out to be subgraphs of the cartesian product of the corresponding factors. Some well-known properties of the cartesian product, such as a reduced mean distance and diameter, simple routing algorithms and some optimal communication protocols are inherited by the hierarchical product. We also address the study of some algebraic properties of the hierarchical product of two or more graphs. In particular, the spectrum of the binary hypertree T_m (which is the hierarchical product of several copies of the complete graph on two vertices) is fully characterized; turning out to be an interesting example of graph with all its eigenvalues distinct. Finally, some natural generalizations of the hierarchic product are proposed.

Key words: Graph operations, Hierarchical product, Diameter, Mean distance, Adjacency matrix, Eigenvalues, Characteristic polynomial.

1 Introduction

Complex systems, constituted by some basic elements which interact among them, appear in different fields such as technology, biology and sociology and recent research shows that many share similar organization principles. When considering graph theoretical models, the associated graphs have a low diameter (in most cases logarithmic with the order) and the degree distribution usually obeys either a power-law distribution (it is "scale-free") or an exponential distribution. Such properties are often associated to a modular or hierarchical structure of the system and the

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existence of nodes with a relatively high degree (hubs). Therefore, the characterization of graphs with these properties, and the study of related graph operations and constructions allowing their generation, has recently attracted much interest in the literature, see [8] and references therein.

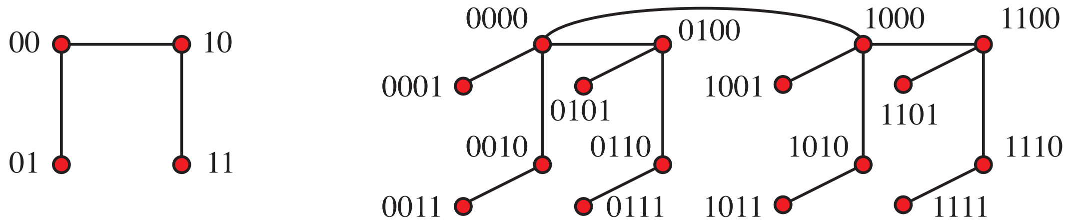
Classical graphs can also display a modular or hierarchical structure. One of the best known examples is the hypercube or n-cube, which can be seen as the cartesian (or direct) product of complete graphs on two vertices. The hypercube has been considered in parallel computers, (Ncube, iPSC/860, TMC CM-2, etc.) as it has a small diameter and degree and nice communication properties: it is a minimum broadcast graph which allows optimal broadcasting and gossiping under standard communication models. However it has a relatively large number of edges and many of them are not used in optimal communication schemes [4,12].

Here we introduce a new graph operation, called the hierarchical product, that turns out to be a generalization of the cartesian product and allows the construction of hierarchical graphs families. As a consequence, some of the well known properties of the direct product, such as a reduced diameter and simple routing algorithms, are also shared by our construction. The name has been inspired by the strong (connectedness) hierarchy of the vertices in the resulting graphs (see Fig. 1). An example of hierarchical product is the deterministic tree obtained by Jung, Kim and Kahng [6] which corresponds to the case when all the factors are star graphs; see Fig. 2. Also, the hierarchical graph constructions proposed in [9,10,11] can be related to hierarchical products of complete graphs.

In the following section we give the definition of the new hierarchical product and we present the main properties of this product. We study the vertex hierarchy and metric parameters like the radius, eccentricity, diameter and mean distance. We also show that communication schemes valid for direct products can also be used here. Section 3 is devoted to the study of algebraic properties of the hierarchical product, in particular the determination of the spectrum of the hierarchical power of a given graph. In particular, the spectrum of the binary hypertree T_m (which is the hierarchical product of several copies of the complete graph on two vertices) is fully characterized; turning out to be an interesting example of graph with all its eigenvalues distinct. Finally, in the last section a generalization of the product is given and other possible generalizations are suggested.

2 The hierarchical product

Let $G_i = (V_i, E_i)$ be N graphs with each vertex set V_i, $i = 1, 2, \ldots, N$, having a distinguished or *root* vertex, labeled 0. The *hierarchical product* $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is the graph with vertices the N-tuples $x_N \ldots x_3x_2x_1$, $x_i \in V_i$, and edges defined by



(a) (b)

Figure 1. The hierarchical products K_2^2 and K_2^4.

the adjacencies:

$$x_N \dots x_3 x_2 x_1 \sim \begin{cases} x_N \dots x_3 x_2 y_1 & \text{if} \quad y_1 \sim x_1 \text{ in } G_1, \\ x_N \dots x_3 y_2 x_1 & \text{if} \quad y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 = 0, \\ x_N \dots y_3 x_2 x_1 & \text{if} \quad y_3 \sim x_3 \text{ in } G_3 \text{ and } x_1 = x_2 = 0, \\ \quad \vdots & \quad \vdots \\ y_N \dots x_3 x_2 x_1 & \text{if} \quad y_N \sim x_N \text{ in } G_N \text{ and } x_1 = \cdots = x_{N-1} = 0. \end{cases} \tag{1}$$

Notice that the structure of the obtained product graph H heavily depends on the root vertices of the factors G_i. As an example, Fig. 1 shows the hierarchical products of two and four copies of the complete graph K_2. The chosen order of the factors, with their subindexes in decreasing order, is because when $V_i = \{0, 1, \dots, b-1\}$, the vertices $x_N \dots x_3 x_2 x_1$ of H represent the first b^N numbers in base b. Notice that, with $n_i = |V_i|$ and $m_i = |E_i|$, the number of vertices of H is $n_N \cdots n_3 n_2 n_1$. Moreover, the number of edges of $G_2 \sqcap G_1$ is $m_2 + n_2 m_1$; the number of edges of $G_3 \sqcap G_2 \sqcap G_1$ is $m_3 + n_3(m_2 + n_2 m_1) = m_3 + n_3 m_2 + n_3 n_2 m_1$; and so on.

Note also that the hierarchical product $G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is simply a subgraph of the classical cartesian product $G_N \,\square\, \cdots \,\square\, G_2 \,\square\, G_1$. (This fact suggested us to derive the notation "$\sqcap$" from "$\square$".) Although the cartesian product is both commutative and associative, the hierarchical product has only the second property, provided that the root vertices are conveniently chosen (in the natural way). Moreover, such a product is also distributive on the right with respect to the union of graphs.

Lemma 2.1 *The hierarchical product of graphs satisfies the following simple properties:*

(a) (Associativity) If the root vertices of $G_2 \sqcap G_1$ and $G_3 \sqcap G_2$ are chosen to be 00*,*

$$G_3 \sqcap G_2 \sqcap G_1 = G_3 \sqcap (G_2 \sqcap G_1) = (G_3 \sqcap G_2) \sqcap G_1; \tag{2}$$

(b) (Right-distributivity)

$$(G_3 \cup G_2) \sqcap G_1 = (G_3 \sqcap G_1) \cup (G_2 \sqcap G_1); \tag{3}$$

(*c*) (*Left-semi-distributivity*) *If the root vertex of* $G_2 \cup G_1$ *is chosen in* G_2,

$$G_3 \sqcap (G_2 \cup G_1) = (G_3 \sqcap G_2) \cup n_3 G_1, \tag{4}$$

where $n_3 G_1 = \overline{K}_{n_3} \sqcap G_1$ *is* n_3 *copies of* G_1.

Proof. To prove the first equality in (2) (the other being similar), we only need to show that vertex $x_3(x_2x_1)$—with selfexplanatory notation—has the same adjacent vertices in $G_3 \sqcap (G_2 \sqcap G_1)$ as vertex $x_3x_2x_1$ has in $G_3 \sqcap G_2 \sqcap G_1$. Indeed,

$$x_3(x_2x_1) \sim \begin{cases} x_3(y_2y_1) & \text{if} \quad (y_2y_1) \sim (x_2x_1) \text{ in } G_2 \sqcap G_1\text{; that is,} \\ & \text{if} \begin{cases} y_1 \sim x_1 \text{ in } G_1 \text{ and } y_2 = x_2, \\ y_2 \sim x_2 \text{ in } G_2 \text{ and } y_1 = x_1 = 0, \end{cases} \\ y_3(x_2x_1) & \text{if} \quad y_3 \sim x_3 \text{ in } G_3 \text{ and } (x_2x_1) = 0\text{; that is, } x_2 = x_1 = 0. \end{cases}$$

Thus, the required isomorphism is simply $x_3(x_2x_1) \mapsto x_3x_2x_1$. □

As a consequence, since clearly $K_1 \sqcap G = G \sqcap K_1 = G$, the set of graphs with the binary operation $\sqcap$ is a *semigroup* with identity element K_1 (that is, a *monoid*). A simple consequence of the adjacency conditions (1) and the role of K_1 is the following lemma, whose (trivial) proof is omitted.

Lemma 2.2 *Let* $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$. *For a fixed string* $\boldsymbol{z}$ *of appropriate length* (*for instance* $\boldsymbol{z} = \mathbf{0} = 0 \ldots 0$), *let* $H\langle \boldsymbol{z} x_k \ldots x_1 \rangle$ *denote the subgraph of* H *induced by the vertex set* $\{\boldsymbol{z} x_k \ldots x_1 | x_i \in V_i, 1 \leq i \leq k\}$. *Let* $H\langle x_N \ldots x_k \boldsymbol{z} \rangle$ *be defined analogously. Then,*

(*a*) $H\langle \boldsymbol{z} x_k \ldots x_1 \rangle = G_k \sqcap \cdots \sqcap G_1$ *for any fixed* $\boldsymbol{z}$;

(*b*) $H\langle x_N \ldots x_k \mathbf{0} \rangle = G_N \sqcap \cdots \sqcap G_k$;

(*c*) $H\langle x_N \ldots x_k \boldsymbol{z} \rangle = (n_N \cdots n_k) K_1$ *for* $\boldsymbol{z} \neq \mathbf{0}$. □

Basic properties of the factor graphs that are inherited by their hierarchical product H are, among others, tree structure, bipartiteness and planarity. (We again skip the proofs, as they are straightforward consequences of our product definition.)

2.1 The vertex hierarchy

The reader will have already noticed from the definition, that there is a strong connectedness hierarchy of the vertices in the graphs obtained with this new graph product. More precisely, the more consecutive zeroes they have on the right side of its string labels, the more neighbors (adjacent vertices) they have. In the language of network theory such vertices with high degree act as hubs. A more detailed study follows.

Let G_i, $i = 1, 2, \ldots, N$, be N graphs whose root vertices have degrees $\delta_i = \delta_{G_i}(0)$. Then, the degree of a generic vertex of its hierarchical product $H = G_N \sqcap \cdots \sqcap G_1$, say $\boldsymbol{x} = x_N x_{N-1} \ldots x_k 00 \ldots 0$, $x_k \neq 0$, is

$$\delta_H(\boldsymbol{x}) = \sum_{i=1}^{k} \delta_i, \tag{5}$$

and there are $(n_k - 1) \prod_{i=k+1}^{N} n_i$ vertices of this type. Moreover, the degree of the root vertex $\mathbf{0} = 00 \ldots 0$ in H is

$$\delta_G(\mathbf{0}) = \sum_{i=1}^{N} \delta_i. \tag{6}$$

In particular, if $G_i = G$ and $\delta_i = \delta$ for every $i = 1, 2, \ldots, N$, then H is the hierarchical power $G^N = G \sqcap G \sqcap \cdots \sqcap G$, a graph on n^N vertices whose degrees follow a exponential-law distribution (see Fig. 3 for an example with $G = K_3$). That is, the probability of a randomly chosen vertex to have degree k is $\mathsf{P}(k) = \gamma^{-k}$ for some constant γ. Indeed, note that, for $k = 1, \ldots, N-1$, the power graph G^N contains $(n-1)n^{N-k}$ vertices with degree $k\delta$ and n vertices with degree $N\delta$.

For instance, if $G = K_2$, the hierarchical product $T_m = K_2^m$ (which hereafter will be called the *binary hypertree* or simply *m-tree*) has 2^{m-k} vertices of degree $k = 1, \ldots, m-1$, and two vertices of degree m. See again Fig. 1(b) for the case $N = 4$.

Now we show how the suppression of the root vertices results in a disconnected graph whose number of components increases as the number of such zeroes does.

Lemma 2.3 *Given a hierarchical product of graphs, $H = G_N \sqcap \cdots \sqcap G_1$, $N \geq 2$, let $H^* = H - \mathbf{0}$ denote the graph H after deleting its root vertex $\mathbf{0} = 00 \ldots 0$. Then*

$$(G_N \sqcap \cdots \sqcap G_2 \sqcap G_1)^* = \bigcup_{k=1}^{N} (G_k^* \sqcap G_{k-1} \sqcap \cdots \sqcap G_1).$$

In particular, if $G_k = K_2$ for all $1 \leq k \leq N$, we have

$$(K_2^N)^* = \bigcup_{k=0}^{N-1} K_2^k$$

where, by convention, $K_2^0 = K_1$.

Proof. To prove the first equality, simply apply recursively the formula $(H_2 \sqcap H_1)^* = (H_2^* \sqcap H_1) \cup H_1^*$ using the associativity of the product in Lemma 2.2(a). Then, the formula concerning the power graph K_2^N follows from the identity role of $K_2^* = K_1$. $\square$

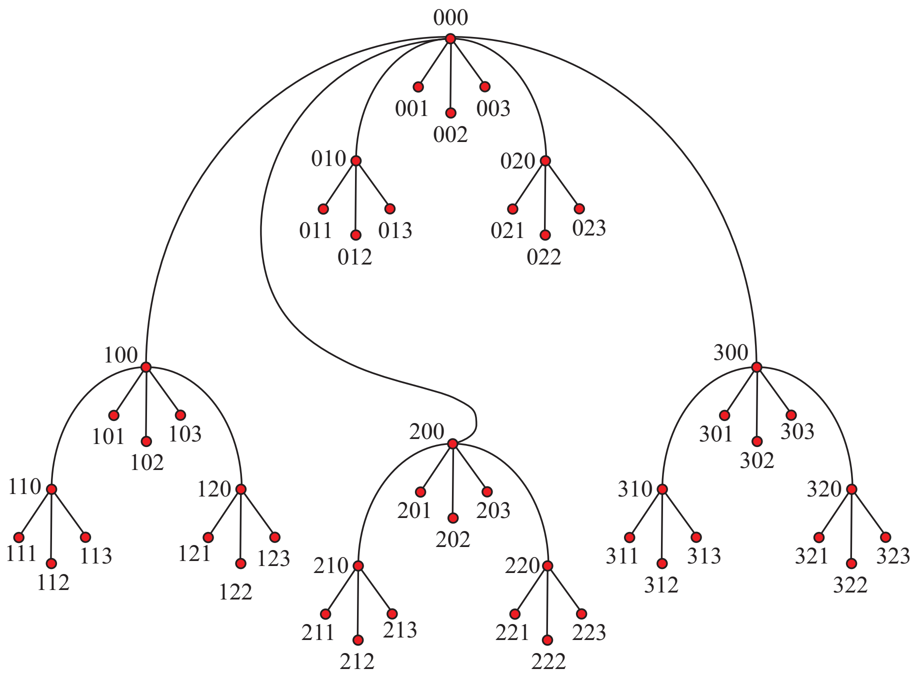


Figure 2. The hierarchical product $S_3 \sqcap S_2 \sqcap S_3$.

2.2 Some metric parameters

In this section we study some of the more relevant metric parameters of the product graphs. Namely, radius, diameter and mean distance. With this aim, let $G_i = (V_i, E_i)$ be N graphs with root vertices having eccentricities $\varepsilon_i = \mathrm{ecc}_{G_i}(0)$, $1 \le i \le N$. Then, the eccentricity of the root vertex $\mathbf{0} = 00\ldots0$ in H is

$$\mathrm{ecc}_H(\mathbf{0}) = \sum_{i=1}^{N} \varepsilon_i. \tag{7}$$

With respect to the diameter and radius of the hierarchical product H, we have the following result:

Proposition 2.4 *Given any shortest path routings $\boldsymbol{\rho}_i$ of G_i, $i = 1, \ldots, N$, there exists an induced shortest path routing $\boldsymbol{\rho}$ of $H = G_N \sqcap \cdots \sqcap G_1$. Moreover, if the graph factor G_N has radius r_N and diameter D_N, then the radius and diameter of H are, respectively,*

$$r_H = r_N + \sum_{i=1}^{N-1} \varepsilon_i, \tag{8}$$

$$D_H = D_N + 2\sum_{i=1}^{N-1} \varepsilon_i. \tag{9}$$

Proof. Assume that we are given a shortest path routing $\boldsymbol{\rho}_i$ of G_i, for every $i = 1, \ldots, N$. Then, for every fixed vertices $v_N, v_{N-1}, \ldots, v_{i+1}$, we can naturally extend

$\boldsymbol{\rho}_i$ to the subgraph of H induced by the vertex subset $\{v_N v_{N-1} \dots v_{i+1} x_i 0 \dots 0 \,|\, x_i \in V_i\} \subset V_H$. For the sake of simplicity, these extensions will also be denoted by $\boldsymbol{\rho}_i$.

Given two arbitrary vertices $\boldsymbol{x}, \boldsymbol{y}$ of H, let $\boldsymbol{z} = x_N \dots x_{k+1}$ their maximum common prefix. (If $k = N$ then $\boldsymbol{z}$ is the empty string.) Hence, we have $\boldsymbol{x} = \boldsymbol{z} x_k \dots x_1$ and $\boldsymbol{y} = \boldsymbol{z} y_k \dots y_1$ with $x_k \neq y_k$.

This allows us to define the following (shortest path) routing from $\boldsymbol{x}$ to $\boldsymbol{y}$, where the symbol $\circ$ denotes concatenation of paths:

$$\begin{aligned}\boldsymbol{\rho}(\boldsymbol{x}, \boldsymbol{y}) = {} & \boldsymbol{\rho}_1(\boldsymbol{z} x_k \dots x_2 x_1, \boldsymbol{z} x_k \dots x_2 0) \circ \boldsymbol{\rho}_2(\boldsymbol{z} x_k \dots x_2 0, \boldsymbol{z} x_k \dots 00) \circ \cdots \\ & \cdots \circ \boldsymbol{\rho}_k(\boldsymbol{z} x_k 0 \dots 0, \boldsymbol{z} y_k 0 \dots 0) \circ \boldsymbol{\rho}_{k-1}(\boldsymbol{z} y_k 0 \dots 0, \boldsymbol{z} y_k y_{k-1} \dots 0) \circ \cdots \\ & \dots \circ \boldsymbol{\rho}_1(\boldsymbol{z} y_k \dots y_2 0, \boldsymbol{z} y_k \dots y_2 y_1).\end{aligned}$$

(Notice that some of the above subpaths could be empty.) That is, in terms of distances,

$$\mathrm{dist}_H(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{dist}_{G_k}(x_k, y_k) + \sum_{i=1}^{k-1} (\mathrm{dist}_{G_i}(x_i, 0) + \mathrm{dist}_{G_i}(0, y_i)). \tag{10}$$

Consequently, this algorithm provides the shortest path routing $\boldsymbol{\rho}$ and, as a by-product, gives a constructive proof of the results about the radius and diameter. Indeed, for a fixed vertex $\boldsymbol{x}$, it is clear by (10) that there exists a vertex $\boldsymbol{y}$ with $y_N \neq x_N$ ($k = N$) such that

$$\mathrm{ecc}_H(\boldsymbol{x}) = \mathrm{dist}_H(\boldsymbol{x}, \boldsymbol{y}) = \mathrm{ecc}_{G_N}(x_N) + \sum_{i=1}^{N-1} (\mathrm{dist}_{G_i}(x_i, 0) + \varepsilon_i). \tag{11}$$

Then, the minimum eccentricity (that is the radius) is attained when $\boldsymbol{x} = x_N 00 \dots 0$ and $\mathrm{ecc}_{G_N}(x_N) = r_N$, (such vertices constitute the center of H), proving (8). Moreover, the maximum eccentricity (the diameter) is attained for any vertex $x_N x_{N-1} \dots x_1$ satisfying $\mathrm{ecc}_{G_N}(x_N) = D_N$ and $\mathrm{dist}_{G_i}(x_i, 0) = \varepsilon_i$, $1 \leq i \leq N-1$, which proves (9). $\square$

Concerning the mean distance of the hierarchical product, we next give a result for the case of two factors, for simplicity reasons. The case of more factors can be solved by recursively applying such a result because of the associativity property (2). First, we introduce the following notation. For a given graph $G = (V, E)$ with order n and a root vertex, say 0, let d_G^0 denote the average distance between 0 and all the other vertices in G (including 0 itself); that is, $d_G^0 = \frac{1}{n} \sum_{v \in V} \mathrm{dist}_G(0, v)$.

Proposition 2.5 *Let G_i be two graphs with orders $n_i = |V_i|$, root vertices 0, local mean distances (from 0) $d_i^0 = d_{G_i}^0$ and (global standard) mean distances d_i, $i = 1, 2$. Then their hierarchical product $H = G_2 \sqcap G_1$, with order $n = n_1 n_2$ and root vertex 00, has the following parameters:*

$$d_H^{00} = d_1^0 + d_2^0, \tag{12}$$

$$d_H = \frac{1}{n-1}\left[(n_1-1)d_1 + n_1(n_2-1)(d_2+2d_1^0)\right]. \tag{13}$$

Proof. In computing all the distances in H, we distinguish two cases, depending on whether the two vertices are in the same or different copy of G_1; that is, $k=1,2$ respectively. The, using (10) we have:

$$\begin{aligned}
d_H &= \frac{1}{\binom{n}{2}}\left(\sum_{x_2}\sum_{x_1,y_1}\operatorname{dist}_H(x_2x_1,x_2y_1) + \sum_{x_2\neq y_2}\sum_{x_1,y_1}\operatorname{dist}_H(x_2x_1,y_2y_1)\right)\\
&= \frac{1}{\binom{n}{2}} n_2 \sum_{x_1\neq y_1}\operatorname{dist}_{G_1}(x_1,y_1)\\
&+ \frac{1}{\binom{n}{2}} \sum_{x_2\neq y_2}\sum_{x_1,y_1}[\operatorname{dist}_{G_2}(x_2,y_2)+\operatorname{dist}_{G_1}(x_1,0)+\operatorname{dist}_{G_1}(0,y_1)]\\
&= \frac{1}{\binom{n}{2}}\left(n_2\binom{n_1}{2}d_1 + \sum_{x_2\neq y_2} n_1^2\operatorname{dist}_{G_2}(x_2,y_2) + \sum_{x_1,y_1}[\operatorname{dist}_{G_1}(x_1,0)+\operatorname{dist}_{G_1}(0,y_1)]\right)\\
&= \frac{1}{\binom{n}{2}}\left(n_2\binom{n_1}{2}d_1 + n_1^2\binom{n_2}{2}d_2 + \binom{n_2}{2}\sum_{x_1}[n_1\operatorname{dist}_{G_1}(x_1,0) + \sum_{y_1}\operatorname{dist}_{G_1}(0,y_1)]\right)\\
&= \frac{1}{\binom{n}{2}}\left(n_2\binom{n_1}{2}d_1 + n_1^2\binom{n_2}{2}d_2 + \binom{n_2}{2}n_1^2d_1^0 + \binom{n_2}{2}\sum_{x_1}n_1\operatorname{dist}_{G_1}(0,y_1)\right)\\
&= \frac{1}{\binom{n}{2}}\left(n_2\binom{n_1}{2}d_1 + n_1^2\binom{n_2}{2}d_2 + 2\binom{n_2}{2}n_1^2d_1^0\right),
\end{aligned}$$

which gives (13). $\square$

As a corollary of the preceding results, we get the following result concerning the N-th hierarchical power $G^N = G \sqcap G \sqcap \overset{N}{\cdots} \sqcap G$.

Corollary 2.6 *Let G be a graph on n vertices, having a root vertex 0 with eccentricity ε and local mean distance d^0. Let G have radius r, diameter D, and mean distance d. Then the N-th hierarchical power G^N, $N \geq 2$, with root vertex $\mathbf{0}$, has the following parameters:*

(*a*) *Eccentricity and local mean distance:* $\operatorname{ecc}_N(\mathbf{0}) = N\varepsilon$; $\;d_N^{\mathbf{0}} = Nd^0$;

(*b*) *Radius and diameter:* $r_N = r + (N-1)\varepsilon$; $\;D_N = D + 2(N-1)\varepsilon$;

(*c*) *Mean distance:* $d_N = d + 2\left(\frac{(N-1)n^N+1}{n^N-1} - \frac{1}{n-1}\right)d^0$.

Proof. The first equality in (*a*) and the result in (*b*) are direct consequences of (7) and Proposition 2.4. The second equality in (*a*) is also easily obtained by applying (12) recursively and using the associativity law. Finally, the same technique can be used for proving (*c*) from (13). Indeed, since $G^N = G^{N-1} \sqcap G$, Eq. (13) yields the following recursive formula (where $d_1 = d$ and $d_1^0 = d^0$):

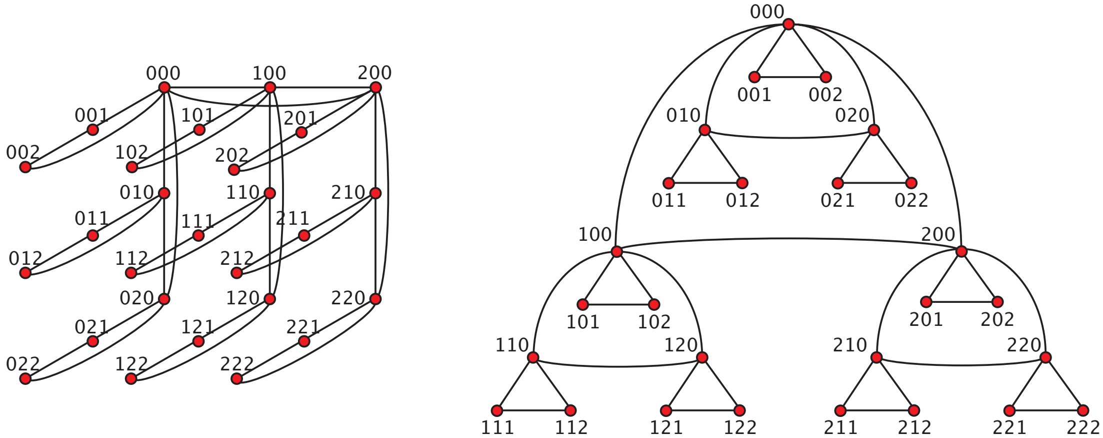


Figure 3. Two views of the hierarchical product K_3^3.

$$\begin{aligned}
d_N &= \frac{n(n^{N-1}-1)}{n^N-1}d_{N-1} + \frac{n-1}{n^N-1}d_1 + \frac{n(n^{N-1}-1)}{n^N-1}2d_1^0\\
&= \frac{n(n^{N-1}-1)}{n^N-1}\left(\frac{n(n^{N-2}-1)}{n^{N-1}-1}d_{N-2} + \frac{n-1}{n^{N-1}-1}d_1 + \frac{n(n^{N-2}-1)}{n^{N-1}-1}2d_1^0\right)\\
&+ \frac{n-1}{n^N-1}d_1 + \frac{n(n^{N-1}-1)}{n^N-1}2d_1^0 = \cdots\\
&= \frac{n^{N-1}(n-1)}{n^N-1}d_1 + \frac{n^{N-1}-1}{n^N-1}d_1\\
&+ \frac{1}{n^N-1}\left((N-1)n^N - n^{N-1} - n^{N-2} - \cdots - n\right)2d_1^0\\
&= d_1 + \left(\frac{(N-1)n^N+1}{n^N-1} - \frac{1}{n-1}\right)2d_1^0.
\end{aligned}$$

□

In particular, note that, when N increases the mean distance of such a N-th power graph is

$$d_N \sim d + 2d^0\left(N - \frac{n}{n-1}\right)$$

which, for large values of n, gives in turn a mean distance of the order of

$$d_N \sim d + 2Nd^0.$$

Example 2.7 *The complete graph K_2 has radius and diameter $r = D = 1$, local mean distance $d^0 = 1/2$ and mean distance $d = 1$. Then, for the hypertree $T_m = K_2^m$, with order 2^m and root vertex $\mathbf{0}$, the above parameters turn to be:*

(a) $\mathrm{ecc}_m(\mathbf{0}) = m$; $d_m^{\mathbf{0}} = m/2$;

(b) $r_m = m$; $D_m = 2m-1$;

(c) $d_m = \frac{m2^m}{2^m-1} - 1 \sim m-1$.

3 Algebraic properties

In this section we study some algebraic properties of the hierarchical product in terms of the corresponding properties of the factors. In particular, we derive results about the spectra of hierarchical products of two different graphs and the hierarchical power (repeated product) of a given graph.

Let us begin by recalling that the *Kronecker product* of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, usually denoted by $\boldsymbol{A} \otimes \boldsymbol{B}$, is the matrix obtained by replacing each entry a_{ij} of $\boldsymbol{A}$ by the matrix $a_{ij}\boldsymbol{B}$ for all i and j. Recall also that the Kronecker product is not commutative in general. However, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices, $\boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{B} \otimes \boldsymbol{A}$ are *permutation similar*, denoted by $\boldsymbol{A} \otimes \boldsymbol{B} \cong \boldsymbol{B} \otimes \boldsymbol{A}$; that is, there exist a permutation matrix $\boldsymbol{P}$ such that $\boldsymbol{A} \otimes \boldsymbol{B} = \boldsymbol{P}(\boldsymbol{B} \otimes \boldsymbol{A})\boldsymbol{P}^\top$. (In terms of graphs, assuming that both matrices are adjacency matrices, we would say that the corresponding graphs are isomorphic.) Here the Kronecker product allows us to give the adjacency matrix of the hierarchical product of two graphs:

Lemma 3.1 *Let G_i be two graphs on n_i vertices, $i = 1, 2$, and with adjacency matrices $\boldsymbol{A}_i$. Then, the adjacency matrix of its hierarchical product $H = G_2 \sqcap G_1$, under some appropriate labeling of its vertices, can be written as*

$$\begin{aligned} \boldsymbol{A}_H &= \boldsymbol{A}_2 \otimes \boldsymbol{D}_1 + \boldsymbol{I}_2 \otimes \boldsymbol{A}_1 && (14)\\ &\cong \boldsymbol{D}_1 \otimes \boldsymbol{A}_2 + \boldsymbol{A}_1 \otimes \boldsymbol{I}_2 && (15) \end{aligned}$$

where $\boldsymbol{D}_1 = \mathrm{diag}(1, 0, \ldots, 0)$ and $\boldsymbol{I}_2$ (the identity matrix) have size $n_1 \times n_1$ and $n_2 \times n_2$, respectively.

For instance, let G be a graph of order N, and consider the product $H = G \sqcap K_n$. Then, using (15),

$$\boldsymbol{A}_H = \boldsymbol{D}_1 \otimes \boldsymbol{A}_G + \boldsymbol{A}_{K_n} \otimes \boldsymbol{I}_N = \begin{pmatrix} \boldsymbol{A}_G & \boldsymbol{I}_N & \cdots & \boldsymbol{I}_N \\ \boldsymbol{I}_N & \boldsymbol{0} & \cdots & \boldsymbol{I}_N \\ \vdots & \vdots & & \vdots \\ \boldsymbol{I}_N & \boldsymbol{I}_N & \cdots & \boldsymbol{0} \end{pmatrix}, \qquad (16)$$

which is an $n \times n$ matrix of $N \times N$ blocks.

The following result of Silvester is also used in our study:

Theorem 3.2 [14] *Let R be a commutative subring of $F^{n\times n}$, the set of all $n \times n$ matrices over a field F (or a commutative ring), and let $\boldsymbol{M} \in R^{m\times m}$. Then,*

$$\det_F \boldsymbol{M} = \det_F(\det_R \boldsymbol{M}),$$

(*the subindex indicates where the determinant is computed*).

This theorem gives a method to compute the determinant of a certain 2×2 block matrix:

Corollary 3.3 *Let* $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$ *be a block matrix where* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ *are* $n \times n$ *matrices over a field which all commute with each other. Then,*

$$\det \boldsymbol{M} = \det(\boldsymbol{AD} - \boldsymbol{BC}).$$

3.1 Spectral properties of $G \sqcap K_2^m$

In this section we study the characteristic polynomial and the spectrum of $G \sqcap K_2^m$.

Let us concentrate first on $H = G \sqcap K_2$. Then, if G has order n and adjacency matrix $\boldsymbol{A}$, the adjacency matrix of H is

$$\boldsymbol{A}_H = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{I}_n \\ \boldsymbol{I}_n & \boldsymbol{0} \end{pmatrix},$$

with characteristic polynomial

$$\phi_H(x) = \det(x\boldsymbol{I}_{2n} - \boldsymbol{A}_H) = \det \begin{pmatrix} x\boldsymbol{I}_n - \boldsymbol{A} & -\boldsymbol{I}_n \\ -\boldsymbol{I}_n & x\boldsymbol{I}_n \end{pmatrix}.$$

Then, using Corollary 3.3, we get

$$\begin{aligned} \phi_H(x) &= \det((x^2-1)\boldsymbol{I}_n - x\boldsymbol{A}) \\ &= \det(x[(x - \tfrac{1}{x})\boldsymbol{I}_n - \boldsymbol{A}]) \\ &= x^n \phi_G(x - \tfrac{1}{x}), \end{aligned} \tag{17}$$

where ϕ_G stands for the characteristic polynomial of G. In fact this result corresponds to a particular case ($r = 2$) of a theorem in [2] that gives the characteristic polynomial of the graph corresponding to our hierarchical product $G \sqcap S_r$, with $S_r = K_{1,r-1}$ (the *star graph* on r vertices).

Going back to our case, note that, since ϕ_G is monic polynomial of degree n, the coeficient of $1/x^n$ in $\phi_G(x - 1/x)$ is 1, and hence so it is the constant term of $\phi_H(x)$. Consequentely, 0 is never an eigenvalue of H. More precisely, we obtain the following result:

Proposition 3.4 *Let G be a graph on n vertices, with spectrum*

$$\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$$

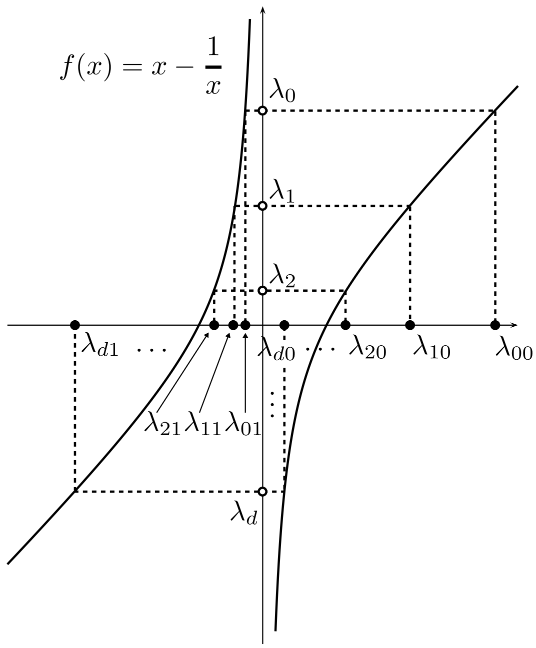


Figure 4. From $\circ \in \operatorname{sp} G$ to $\bullet \in \operatorname{sp}(G \sqcap K_2)$

where the supraindexes denote multiplicities ($m_0 = 1$ *if* G *is connected*) *and* $\lambda_0 < \lambda_1 < \cdots < \lambda_d$. *Then the spectrum of the hierarchical product* $H = G \sqcap K_2$ *is*

$$\operatorname{sp} H = \{\lambda_{00}^{m_0}, \lambda_{01}^{m_1}, \ldots, \lambda_{0d}^{m_d}, \lambda_{10}^{m_0}, \lambda_{11}^{m_1}, \ldots, \lambda_{1d}^{m_d}\},$$

where

$$\lambda_{0i} = f_0(\lambda_i) = \tfrac{\lambda_i - \sqrt{\lambda_i^2+4}}{2}, \qquad \lambda_{1i} = f_1(\lambda_i) = \tfrac{\lambda_i + \sqrt{\lambda_i^2+4}}{2} \qquad (0 \le i \le d), \tag{18}$$

and

$$\lambda_{00} < \lambda_{01} < \cdots < \lambda_{0d} < 0 < \lambda_{10} < \lambda_{11} < \cdots < \lambda_{1d}. \tag{19}$$

Proof. From (17) and the subsequent comments we get the following implications:

$$\lambda \in \operatorname{sp} H \Leftrightarrow \phi_H(\lambda) = \lambda^n \phi_G(\lambda - \tfrac{1}{\lambda}) = 0 \Leftrightarrow \lambda - \tfrac{1}{\lambda} \in \operatorname{sp} G.$$

Thus, for every $\lambda_i \in \operatorname{sp} G$ we can obtain two eigenvalues of H, which are the solutions $\lambda_{0i} = f_0(\lambda_i)$, $\lambda_{1i} = f_1(\lambda_i)$ of the second degree equation $\lambda^2 - \lambda_i\lambda - 1 = 0$ (coming from $\lambda_i = \lambda - 1/\lambda$), satisfying

$$\lambda_{0i} = \tfrac{\lambda_i - \sqrt{\lambda_i^2+4}}{2} < 0 < \tfrac{\lambda_i + \sqrt{\lambda_i^2+4}}{2} = \lambda_{1i}. \tag{20}$$

Besides, notice that $\lambda_i < \lambda_{i+1}$ implies $\lambda_{0i} < \lambda_{0,i+1}$ and $\lambda_{1i} < \lambda_{1,i+1}$. (See Fig. 4 for a "proof without words" [7].) $\square$

In particular, if G is bipartite its spectrum is symmetric with respect to 0: $\lambda_i = -\lambda_{d-i}$, $i = 0, 1, \ldots, \lfloor d/2 \rfloor$, (see, for instance, [1]); and so it is $H = G \sqcap K_2$ with eigenvalues

$$\lambda_{0i} = \tfrac{\lambda_i - \sqrt{\lambda_i^2+4}}{2} = \tfrac{-\lambda_{d-i} - \sqrt{\lambda_{d-i}^2+4}}{2} = -\lambda_{1,d-i} \qquad (0 \le i \le \lfloor \tfrac{d}{2} \rfloor). \tag{21}$$

Let us now consider the case of the multiple product $H_m = G \sqcap K_2^m$ with $m \ge 0$ where, by convention, $H_0 = G$. Since, by Lemma 2, $H_m = H_{m-1} \sqcap K_2$, we can deal with this case by recursively applying m times (17) or Proposition 3.4. However, an alternative recurrence procedure to get the characteristic polynomial ϕ_m of H_m can be obtained from another simple consequence of Theorem 3.2. Namely,

Lemma 3.5 *If p and q are arbitrary polynomials, then*

$$\det \begin{pmatrix} p\boldsymbol{I}_n - q\boldsymbol{A} & -q\boldsymbol{I}_n \\ -q\boldsymbol{I}_n & p\boldsymbol{I}_n \end{pmatrix} = det((p^2 - q^2)\boldsymbol{I}_n - pq\boldsymbol{A}).$$

This leads to a double recurrence relation for $\phi_m(x)$, derived by recursively applying Lemma 3.5. With this aim, notice that the adjacency matrix of H_m is

$$\boldsymbol{A}_m = \begin{pmatrix} \boldsymbol{A}_{m-1} & \boldsymbol{I}_{m-1} \\ \boldsymbol{I}_{m-1} & \boldsymbol{0} \end{pmatrix} \tag{22}$$

for $m \ge 1$, with $\boldsymbol{A}_0 = \boldsymbol{A}$ being the adjacency matrix of $H_0 = G$ and $\boldsymbol{I}_m$ denoting the identity matrix of size $n2^m$ (the same as $\boldsymbol{A}_m$).

Proposition 3.6 *Let $\{p_i, q_i\}_{i\ge 0}$ be the family of polynomials satisfying the recurrence equations*

$$p_i = p_{i-1}^2 - q_{i-1}^2\,, \tag{23}$$
$$q_i = p_{i-1}q_{i-1}\,, \tag{24}$$

with initial conditions $p_0 = x$ and $q_0 = 1$. Then, for every $m \ge 0$, the characteristic polynomial of $H_m = G \sqcap K_2^m$ is

$$\phi_m(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right),$$

where ϕ_0 is the characteristic polynomial of G.

Proof. Since $q_0(x) = 1$ and $p_0(x) = x$, the result trivially holds for $m = 0$. Let $m \ge 1$. First we prove that, for every i, $0 \le i \le m$,

$$\phi_m = \det(p_i\boldsymbol{I}_{m-i} - q_i\boldsymbol{A}_{m-i}). \tag{25}$$

The proof is by induction on i. By definition of the characteristic polynomial,

$$\phi_m = \det(x\boldsymbol{I}_m - \boldsymbol{A}_m) = \det(p_0\boldsymbol{I}_{m-1} - q_0\boldsymbol{A}_{m-1}),$$

we see that (25) holds for $i = 0$. Assuming that it holds for $i-1$, and considering the structure of $\boldsymbol{A}_m$ in (22), we get

$$\begin{aligned}
\phi_m &= \det(p_{i-1}\boldsymbol{I}_{m-i+1} - q_{i-1}\boldsymbol{A}_{m-i+1}) \\
&= \det\left(p_{i-1}\boldsymbol{I}_{m-i+1} - q_{i-1}\begin{pmatrix} \boldsymbol{A}_{m-i} & \boldsymbol{I}_{m-i} \\ \boldsymbol{I}_{m-i} & \boldsymbol{0} \end{pmatrix}\right) \\
&= \det\begin{pmatrix} p_{i-1}\boldsymbol{I}_{m-i} - q_{i-1}\boldsymbol{A}_{m-i} & -q_{i-1}\boldsymbol{I}_{m-i} \\ -q_{i-1}\boldsymbol{I}_{m-i} & p_{i-1}\boldsymbol{I}_{m-i} \end{pmatrix} \\
&= \det((p_{i-1}^2 - q_{i-1}^2)\boldsymbol{I}_{m-i} - p_{i-1}q_{i-1}\boldsymbol{A}_{m-i}) \\
&= \det(p_i\boldsymbol{I}_{m-i} - q_i\boldsymbol{A}_{m-i}),
\end{aligned}$$

where we have used Lemma 3.5 and the recurrence relations for p_i and q_i.

In particular, the case $i = m$ gives

$$\begin{aligned}
\phi_m(x) &= \det(p_m(x)\boldsymbol{I}_0 - q_m(x)\boldsymbol{A}_0) = \det\left(q_m(x)\left(\frac{p_m(x)}{q_m(x)}\boldsymbol{I}_0 - \boldsymbol{A}_0\right)\right) \\
&= q_m(x)^n\phi_0\left(\frac{p_m(x)}{q_m(x)}\right).
\end{aligned}$$

This completes the proof. $\square$

3.2 The spectra of the binary hypertree

Now we study the case of the m-tree $T_m = K_2^m$, which is obtained when we consider $G = K_1$ in the above family of graphs, with $n = 1$ and $A_0 = (0)$ (the "adjacency matrix" of the singleton graph). As a consequence of Proposition 3.6 we have the following corollary.

Corollary 3.7 *Let $\{p_i, q_i\}_{i\geq 0}$ the family of polynomials obtained from the recurrence relations of Proposition 3.6. Then, the characteristic polynomials of the m-tree T_m, $m \geq 0$ (with $T_0 = K_1$), and the graph $T_m^* = T_m - \boldsymbol{0}$, (with $m \geq 1$ and $\boldsymbol{0} = 00\ldots0$) are, respectively,*

$$\phi_{T_m}(x) = p_m(x), \tag{26}$$
$$\phi_{T_m^*}(x) = q_m(x). \tag{27}$$

Proof. To prove (26), we simply apply Proposition 3.6 with $G = K_1$, which implies $n = 1$, $\phi_0 = x$, and

$$\phi_{T_m}(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right) = p_m(x).$$

Then, the equality in (27) follows from the above and the fact that, by applying recursively (24), we have

$$q_m(x) = \prod_{i=0}^{m-1} p_i(x).$$

(Recall that, from Lemma 2.3, $T_m^* = \bigcup_{i=0}^{m-1} T_i$.) $\square$

The following result is then a consequence of Proposition 3.4 when taking $G = K_1$.

Proposition 3.8 *The m-tree T_m, $m \geq 1$, has eigenvalues $\lambda_0^m \leq \lambda_1^m \leq \cdots \leq \lambda_{n-1}^m$, with $n = 2^m$, satisfying the following recurrence relation:*

$$\lambda_k^m = \frac{\lambda_k^{m-1} + \sqrt{(\lambda_k^{m-1})^2 + 4}}{2}, \tag{28}$$

$$\lambda_{n-k-1}^m = -\lambda_k^m, \tag{29}$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n-1$. Moreover, all the eigenvalues are distinct.

Proof. The recurrence (28) and the symmetry property (29) follow from (18) and (21), respectively, since $G = K_1$ is trivially bipartite. Moreover, as K_1 has only the eigenvalue 0 with multiplicity $m_0 = 1$, the eigenvalue multiplicities of the subsequent iterations $T_1 = K_2^1 = K_1 \sqcap K_2$, $T_2 = K_2^2 = K_1 \sqcap K_2 \sqcap K_2$, etc. are also equal to 1 (Proposition 3.4), and so all eigenvalues are different. $\square$

Note that, with $\lambda_0^0 = 0$ (the unique eigenvalue of $K_2^0 = K_1$, the above result applies even for $m = 1$, giving the eigenvalues of K_2: $\lambda_0 = -1$ and $\lambda_1 = 1$. This suggests the following more appealing notation: Let denote the above eigenvalue λ_i^m of T_m just as $\lambda_{\boldsymbol{i}}$, where $\boldsymbol{i} = i_{m-1} \ldots i_1 i_0$ is the number i written in base two. Then, the higher the number i the greater the eigenvalue $\lambda_{\boldsymbol{i}}$, which is obtained from (18) by applying successively one of the functions f_0, f_1 to $\lambda_0^0 = 0$, as follows:

$$\lambda_{\boldsymbol{i}} = (f_{m-1} \circ \cdots \circ f_1 \circ f_0)(0). \tag{30}$$

To visualize the eigenvalue distribution obtained by the above procedure, Fig. 5 shows the spectra of T_m for $0 \leq m \leq 6$ and how "each parent gives birth to two off-spring". Note that the maximum eigenvalue (in absolute value) increases unboundedly with m, whereas the minimum eigenvalue tends to zero and the successive eigenvalues distribute along the real line filling it.

More precisely, we have the following result:

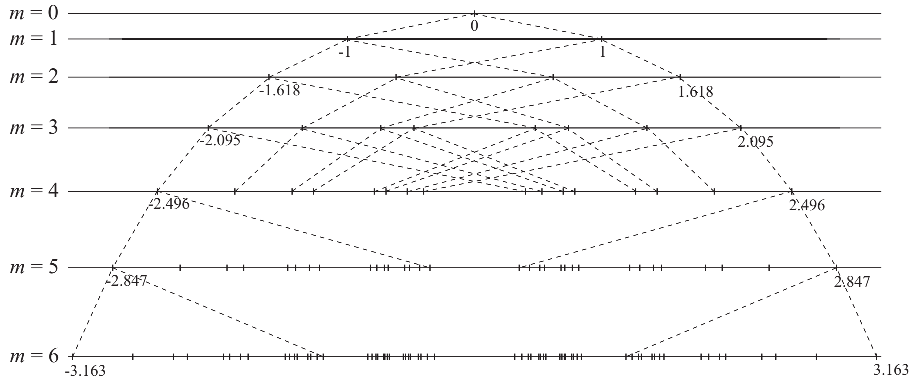


Figure 5. The distinct eigenvalues of the hypertree T_m for $0 \leq m \leq 6$.

Proposition 3.9 *The assimptotic behaviors, as $k \to \infty$, of the maximum eigenvalue λ_k (spectral radius), the second largest eigenvalue θ_k, and the minimum (in absolute value) eigenvalue τ_k of the hypertree T_k are:*

$$\lambda_k \sim \sqrt{2k}, \qquad \theta_k \sim \sqrt{2k}, \qquad \tau_k \sim 1/\sqrt{2k}.$$

Proof. According to (30), the above eigenvalues correspond to:

$$\lambda_k = \lambda_{111\ldots1}, \qquad \theta_k = \lambda_{11\ldots10}, \qquad \tau_k = \lambda_{10\ldots00}.$$

Thus, for $k > 1$ the two maximum eigenvalues λ_k and θ_k verify the recurrence $\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$. This function tends to a power law, $\lambda_k = \alpha k^\beta$ for $k \to \infty$, for some constants α and β. Indeed, if we put this expression of λ_k in the equation, we get:

$$\alpha(k+1)^\beta \sim \frac{\alpha k^\beta + \sqrt{\alpha^2 k^{2\beta} + 4}}{2} \qquad \Rightarrow \qquad \alpha^2(k+1)^\beta[(k+1)^\beta - k^\beta] \sim 1,$$

and it is easy to check that $\lambda_k = \sqrt{2k}$ is a solution when $k \to \infty$ since,

$$2(k+1)^{\frac{1}{2}}[(k+1)^{\frac{1}{2}} - k^{\frac{1}{2}}] = \frac{2(k+1)^{\frac{1}{2}}}{(k+1)^{\frac{1}{2}} + k^{\frac{1}{2}}} \to 1.$$

The behavior of τ_k is proved similarly or noting that $\lambda_k \tau_k = 1$. $\square$

In Fig. 6 we plot λ_k and θ_k in terms of k for all T_k such that $1 \leq k \leq 128$. Note that both scales are logarithmic, so that the straight line asymptote is, in both cases, $y = \frac{1}{2}\log 2 + \frac{1}{2}\log x$.

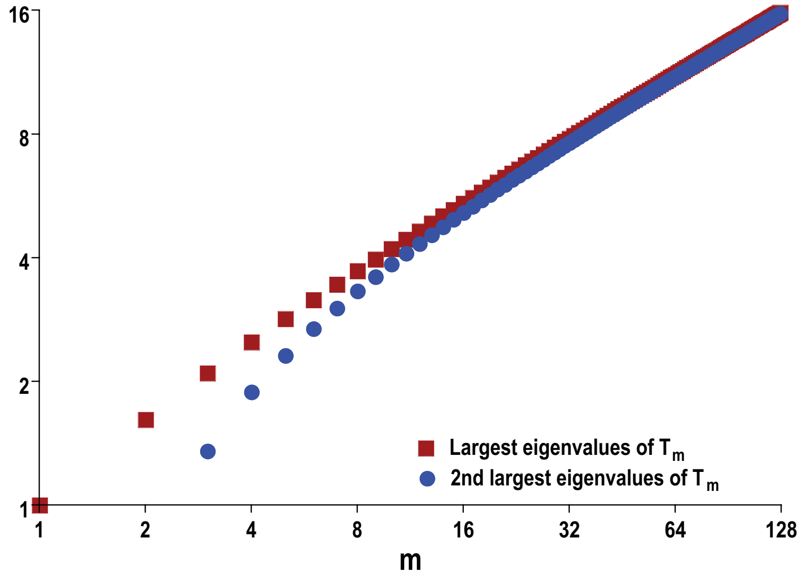


Figure 6. The two largest eigenvalues of T_m for $1 \leq m \leq 128$.

3.3 The spectrum of a generic two-term product

Theorem 3.10 *Let G_1 and G_2 be two graphs on n_i vertices, with adjacency matrix $\boldsymbol{A}_i$ and characteristic polynomial $\phi_i(x)$, $i = 1, 2$. Let G_1 have root vertex 0 and consider the graph $G_1^* = G_1 - 0$, with adjacency matrix $\boldsymbol{A}_1^*$ and characteristic polynomial ϕ_1^*. Then the characteristic polynomial $\phi_H(x)$ of the hierarchical product $H = G_2 \sqcap G_1$ is:*

$$\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left(\frac{\phi_1(x)}{\phi_1^*(x)} \right). \tag{31}$$

Proof. Without loss of generality, index the rows (and columns) of $\boldsymbol{A}_i$ as $0, 1, 2, \ldots$, and assume that the vertices adjacent to the root vertex 0 in G_1, with degree δ, say, are $1, 2, \ldots, \delta$. Then, using (15), the adjacency matrix of H can be written as an $n_1 \times n_1$ block matrix, where each block has size $n_2 \times n_2$, as follows:

$$\boldsymbol{A}_H = \boldsymbol{D}_1 \otimes \boldsymbol{A}_2 + \boldsymbol{A}_1 \otimes \boldsymbol{I}_2 = \begin{pmatrix} \boldsymbol{A}_2 & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{A}_1^* \otimes \boldsymbol{I}_2 \end{pmatrix},$$

(To be compared with (16)), where

$$\boldsymbol{B} = \left(\boldsymbol{I}_2 \,\middle|\, \overset{(\delta)}{\cdots\cdots} \,\middle|\, \boldsymbol{I}_2 \,\middle|\, \boldsymbol{0} \,\middle|\, \boldsymbol{0} \,\middle|\, \cdots\cdots \,\middle|\, \boldsymbol{0} \right).$$

Thus, the characteristic polynomial of H is

$$\phi_H(x) = \det(x\boldsymbol{I} - \boldsymbol{A}_H) = \det \begin{pmatrix} x\boldsymbol{I}_2 - \boldsymbol{A}_2 & -\boldsymbol{B} \\ -\boldsymbol{B}^\top & (x\boldsymbol{I}_1^* - \boldsymbol{A}_1^*) \otimes \boldsymbol{I}_2 \end{pmatrix}$$

where the n_1^2 blocks are of three types: One $x\boldsymbol{I}_2 - \boldsymbol{A}_2$, $n_1 - 1$ like $x\boldsymbol{I}_2$, and several (as many as edges of G_1) like $-\boldsymbol{I}_2$. Consequently, Theorem 3.2 applies (since every block commutates with each other) and we can obtain ϕ_H by computing the determinant in $\mathbb{R}^{n_2 \times n_2}$ expanding by the first row (the blocks of $\boldsymbol{B}$):

$$\begin{aligned} \phi_H(x) &= \det([x\boldsymbol{I}_2 - \boldsymbol{A}_2]\phi_1^*(x)\boldsymbol{I}_2 + \phi_1(x)\boldsymbol{I}_2 - x\boldsymbol{I}_2\phi_1^*(x)) \\ &= \det(\phi_1(x)\boldsymbol{I}_2 - \phi_1^*(x)\boldsymbol{A}_2) \\ &= \det\left(\phi_1^*(x)\left[\frac{\phi_1(x)}{\phi_1^*(x)}\boldsymbol{I}_2 - \boldsymbol{A}_2\right]\right) \\ &= \phi_1^*(x)^{n_2}\phi_2\left(\frac{\phi_1(x)}{\phi_1^*(x)}\right), \end{aligned}$$

as claimed. □

In fact, a much more lengthy proof of this result was first given by Schwenk [13] in another context, without mentioning the underlying graph operation.

An interesting particular case occurs when G_1 is a walk-regular graph [5] since then the so-called *local characteristic function* [3] is the same for every vertex and

$$\phi_1^*(x) = \frac{1}{n_1}\phi_1'(x),$$

where, as usual, the prime indicates derivative.

Corollary 3.11 *With the same notation as above, if G_1 is a walk-regular graph, we have*

$$\phi_H(x) = \left(\frac{\phi_1'(x)}{n_1}\right)^{n_2} \phi_2\left(\frac{n_1\phi_1(x)}{\phi_1'(x)}\right). \tag{32}$$

Some well-known instances of walk-regular graphs are the vertex-transitive graphs and the distance-regular graphs.

A particular case of the above is when G_2 is the (walk-regular) complete graph K_n.

Corollary 3.12 *Let G be a graph of order $n_2 = N$ and characteristic polynomial ϕ_G. Then the characteristic polynomial of $H = G \sqcap K_n$ is:*

$$\phi_H(x) = (x+1)^{N(n-2)}(x-n+2)^N\phi_G\left(\frac{(x+1)(x-n+1)}{(x-n+2)}\right).$$

Proof. Apply Eq. 32 with $n_1 = n$, $\phi_1 = (x-n+1)(x+1)^{n-1}$ (the characteristic polynomial of K_n) and $\phi_1' = (x+1)^{n-1} + (n-1)(x-n+1)(x+1)^{n-2}$. □

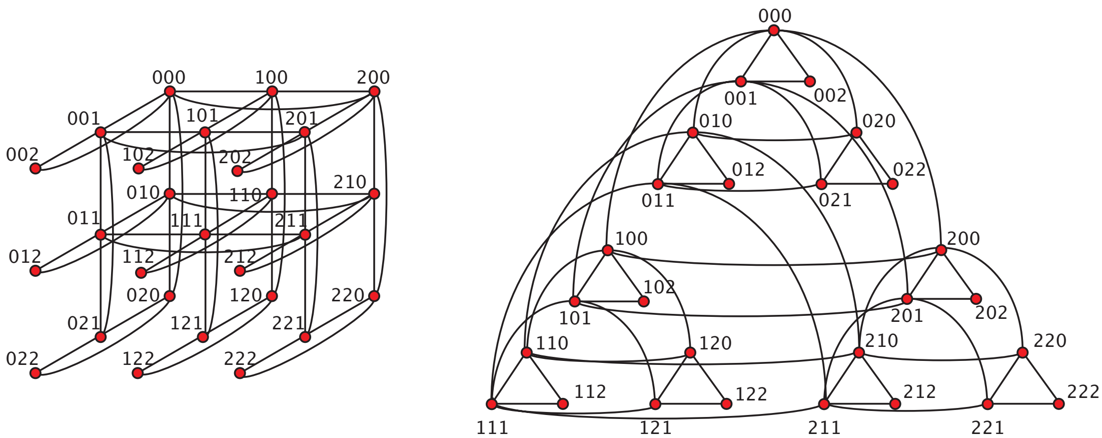


Figure 7. Two views of a generalized hierarchical product K_3^3 with $U_1 = U_2 = \{0, 1\}$.

4 Generalizations of the hierarchical product

A natural generalization of the hierarchical product is as follows. Given N graphs $G_i = (V_i, E_i)$ and (nonempty) vertex subsets $U_i \subseteq V_i$, $i = 1, 2, \ldots, N-1$, the *generalized hierarchical product* $H = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_1(U_1)$ is the graph with vertex set $V_N \times \cdots V_2 \times V_1$, as above, and adjacencies:

$$x_N \ldots x_3 x_2 x_1 \sim \begin{cases} x_N \ldots x_3 x_2 y_1 & \text{if} \quad y_1 \sim x_1 \text{ in } G_1, \\ x_N \ldots x_3 y_2 x_1 & \text{if} \quad y_2 \sim x_2 \text{ in } G_2 \text{ and } x_1 \in U_1 \\ x_N \ldots y_3 x_2 x_1 & \text{if} \quad y_3 \sim x_3 \text{ in } G_3 \text{ and } x_i \in U_i,\ i = 1, 2 \\ \quad\vdots & \quad\vdots \\ y_N \ldots x_3 x_2 x_1 & \text{if} \quad y_N \sim x_N \text{ in } G_N \text{ and } x_i \in U_i,\ i = 1, 2, \ldots, N-1. \end{cases}$$

In particular, notice that the two "extreme" cases are when all U_i are singletons, in which case we obtain the (standard) hierarchical product; and when $U_i = V_i$ for all $1 \leq i \leq N-1$, and the resulting graph is the cartesian product of the G_i's.

Another possible generalization of the hierarchical product is obtained by using an appropriate factorization of the graphs G_i. In this way, some properties such as a higher connectivity or a specific (scale-free) degree distribution, can be obtained in the hierarchical product.

Notice also that the hierarchical product can be defined for infinite graphs and digraphs.

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