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# Fast and Flexible Determination of Force-Closure Independent Regions to Grasp Polygonal Objects\*

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## Abstract

*Force-closure independent regions are parts of the object edges such that a grasp with a finger in each region ensures a force-closure grasp. These regions are useful to provide some robustness to the grasp in the presence of uncertainty as well as in grasp planning. Most of the approaches to the computation of these regions for  $N$  fingers work on the contact space, implying a  $N$ -dimensional problem. This paper presents a new approach to determine independent regions on polygonal objects considering  $N$  friction or frictionless contacts. The approach works on the object space, implying that it is always a two-dimensional problem and, since it is not necessary to compute all the force-closure space, it becomes a very fast approach. Besides, the approach is also flexible since constraints on the fingers placement can be easily introduced. Some graphical examples are included in the paper showing the simplicity of the methodology.*

## 1 Introduction

The obtention of grasps capable of ensuring the immobility of the object despite external disturbances has been a topic of great interest in grasping and manipulation of objects. These grasps are characterized by the properties of form-closure (the position of the fingers ensures the object immobility) or force-closure (the forces applied by the fingers ensure the object immobility) [1]. In order to select a grasp among all the possible force-closure grasps (hereafter FC grasps), algorithms that optimize a quality criterion (for instance, [2], [3], [4]) or algorithms based on heuristics criterions (for instance, [5], [6]) were developed. These algorithms determine fingertips or “precision” grasps, i.e., grasps formed by a set of contacts points on the object where the fingertips will be placed, and they require a good precision in the fingertip placements (in [2] and [3] the robustness in front of the fingers positioning errors is partially treated). In a real execution, the final grasp and the theoretical grasp may differ due to fingers positioning errors. A metric for measuring the sensitivity of a grasp to positioning errors can be found in [7]. In order to provide robustness to the grasp in front of these errors, Nguyen [8] introduced the concept of independent regions, i.e, regions on the object boundary such that a finger in each region ensures

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a FC grasp independently of the exact contact point, and he developed a geometrical approach to determine the maximum independent regions on polygonal objects using four frictionless contacts and two friction contacts. The problem of determining independent regions using four frictionless contacts was also treated in [9]. Ponce and Faverjon [10] and Ponce et al. [11] extended Nguyen’s approach to three finger grasps on polygonal objects and to four finger grasps on polyhedral objects, respectively. These works are based on a sufficient condition implying that only a subset of FC grasps are evaluated and they are specific for a given number of fingers. Liu [12] and Li, Yu and Tsujio [13] proposed algorithms to determine all the  $N$ -finger FC grasps on polygonal objects. These algorithms have not been used to compute independent regions, although in [13] the most stable grasp considering fingers positioning errors is determined. Recently, Pollard [14] presented an approach to determine independent regions on 3D objects based on initial examples, but the selection of a good initial example for a given object is a critical step.

This paper deals with the problem of determining independent regions on polygonal objects considering  $N$  friction or frictionless contacts. Since the space defined by all the FC grasps may be concave (this result can be obtained from [12] and [13] and it will be also shown here) it is decomposed into a set of convex subspaces, establishing each one a necessary and sufficient condition for the existence of a FC grasp and, in order to obtain it, at least one of these conditions must be satisfied. The computational cost of the decomposing algorithm for  $N$  fingers is  $O(N^3)$ . A condition to determine independent regions on each subspace is also presented and, using its geometrical interpretation, the problem of determining independent regions is reduced to find two particular points on the object space. The approach is very fast since it is applied on the object space and it is not necessary to compute the  $N$ -dimensional space of all the FC grasps, whose computational cost is at least  $O(N^3 \log N)$ . Besides, other constraints on the fingers placement can be easily introduced, providing flexibility to the algorithm. The approach developed here follows a previous work of the authors [15] where linear programming were used to obtain maximum independent region on the contact space, while in this paper a faster approach is presented.

The main assumptions considered in this work are:

1. Grasped objects are planar and polygonal-shaped.
2. The object edges where the fingers will contact are given.
3. Forces applied by the fingers act only against the object boundary.
4. The fingertip is a point.

Note that in this approach there is no constraint regarding the number of fingers neither the number of fingers per edge.

## 2 Wrench Space

### 2.1 Representation of forces and torques

Let  $\mathbf{f}_i$  be the maximum force exerted by each finger on the object boundary at each contact point. In the absence of friction,  $\mathbf{f}_i$  is the applied force normal to the object boundary and it produces a torque  $\tau_i$  with respect to the object’s center of mass. Considering that  $\mathbf{f}_i$  is normalized to be  $\|\mathbf{f}_i\| = 1$ , the components of  $\mathbf{f}_i$  (respect to the reference frame of the object) and  $\tau_i$  form the wrench vector  $\boldsymbol{\omega}_i = [\cos \theta_i \ \sin \theta_i \ \tau_i]^T$ , where  $\theta_i$  indicates the direction normal to the contact edge. Since  $\|\mathbf{f}_i\| = 1$ ,  $\tau_i$  is equal to the distance  $d_i$  from the object’s center of mass (CM) to the line of action of  $\mathbf{f}_i$  (see Fig. 1.a).

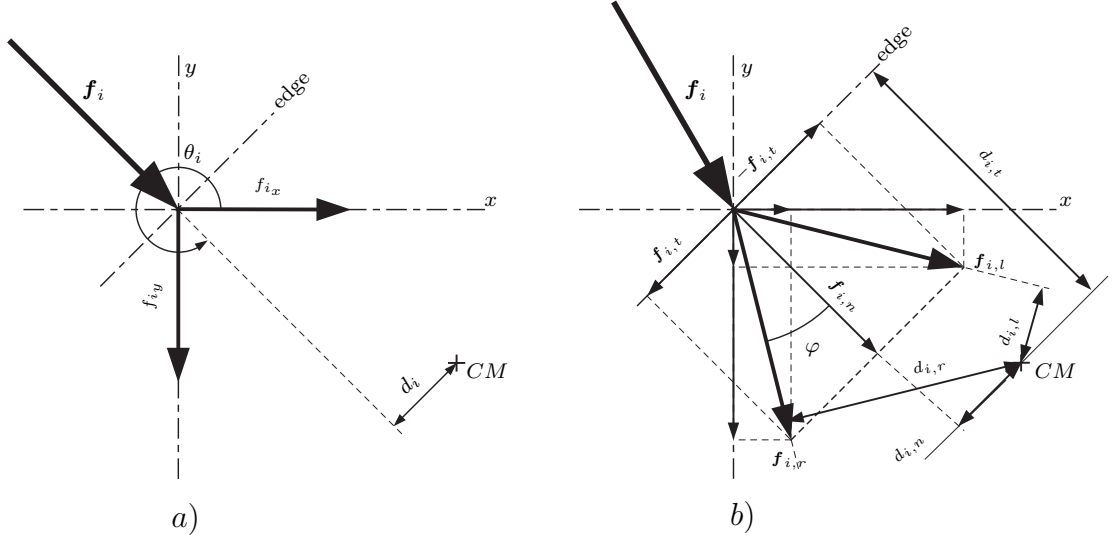


Figure 1: a) *Frictionless contact*; b) *Friction contact*, where  $\mathbf{f}_{i,l}$  and  $\mathbf{f}_{i,r}$  are the primitive forces,  $\mathbf{f}_{i,n}$  is the normal force and  $\mathbf{f}_{i,t}$  is the tangential force.

When friction is taken into account,  $\mathbf{f}_i$  can be decomposed in two components  $\mathbf{f}_{i,n}$  and  $\mathbf{f}_{i,t}$  which are respectively normal and tangent to the contact edge (see Fig. 1.b). In order to avoid that the finger slips on the edge, the Coulomb's law must be accomplished:  $|\mathbf{f}_{i,n}| \geq \mu |\mathbf{f}_{i,t}|$ , where  $\mu$  is the friction coefficient. This implies that  $\mathbf{f}_i$  can be applied in a range of directions around the normal of the contact edge, determining the friction cone. Then,  $\mathbf{f}_i$  can be expressed as a positive linear combination of two forces:

$$\mathbf{f}_i = \alpha_{i,l} \mathbf{f}_{i,l} + \alpha_{i,r} \mathbf{f}_{i,r} \quad (1)$$

where  $\mathbf{f}_{i,l}$  and  $\mathbf{f}_{i,r}$  are the forces along the boundaries of the friction cone, usually called primitive forces.

Then, the wrench produced in a contact point can be expressed as a linear combination of two primitive wrenches:

$$\boldsymbol{\omega}_i = \alpha_{i,l} \boldsymbol{\omega}_{i,l} + \alpha_{i,r} \boldsymbol{\omega}_{i,r} \quad (2)$$

with

$$\boldsymbol{\omega}_{i,r} = [\cos(\theta_i - \varphi) \quad \sin(\theta_i - \varphi) \quad \tau_{i,r}]^T \quad (3)$$

$$\boldsymbol{\omega}_{i,l} = [\cos(\theta_i + \varphi) \quad \sin(\theta_i + \varphi) \quad \tau_{i,l}]^T \quad (4)$$

where  $\theta_i$  indicates the direction normal to the contact edge,  $\varphi = \arctan \mu$  and  $\tau_{i,r}$  and  $\tau_{i,l}$  are the torques produced by  $\mathbf{f}_{i,r}$  and  $\mathbf{f}_{i,l}$ , respectively.

Considering that the torque component of each primitive wrench is produced by the corresponding force components tangential and normal to the edge (see Fig. 1.b), we obtain:

$$\tau_{i,r} = \tau_{i,n} + \tau_{i,t} \quad (5)$$

$$\tau_{i,l} = \tau_{i,n} - \tau_{i,t} \quad (6)$$

where  $\tau_{i,n}$  and  $\tau_{i,t}$  are the torques produced by  $\mathbf{f}_{i,n}$  and  $\mathbf{f}_{i,t}$ , respectively. Therefore, the relation between the two primitive torques is

$$\tau_{i,r} = \tau_{i,l} + 2\tau_{i,t} \quad (7)$$

## 2.2 Constraint on the finger forces

The forces applied by the fingers can be subject to different constraints [16]. The constraint considered in this work is that the total force exerted by all the fingers is limited, for instance, due to a maximum available power for all the finger actuators. Then, the applied forces can generate a resultant wrench

$$\boldsymbol{\omega} = \sum_{i=1}^N \alpha_i \boldsymbol{\omega}_i \quad \text{with} \quad \sum_{i=1}^N \alpha_i \leq 1 \quad (8)$$

where  $N$  is the number of contacts. If friction contacts are considered, then  $\alpha_i = \alpha_{i,l} + \alpha_{i,r}$  is a linear approximation.

Geometrically, the resultant wrench produced by a grasp can be any one inside the polyhedron  $\mathcal{P}_1$  defined in the wrench space as:

$$\mathcal{P}_1 = \text{ConvexHull}\left(\bigcup_{i=1}^N \{\boldsymbol{\omega}_i\}\right) \quad (9)$$

and, for a FC grasp,  $\mathcal{P}_1$  must contain the origin [17].

## 3 Force-closure space

**Definition 1** The *contact space* is the space defined by  $N$  parameters that represent the contact points on a given edges of an object.  $\diamond$

Since the direction of the normal force and the primitive forces are known when the contact edge is given, there is a univocal relation between the torque produced by these forces and the exact contact point. Thus, the parameters used in this paper to define the contact space will be the torques produced by the normal forces when frictionless contacts are considered and the torques produced by the primitive forces when friction contacts are considered.

**Definition 2** The *force-closure space*, *FC-space*, is the subset of the contact space where FC grasps are produced.  $\diamond$

A methodology to obtain the FC-space as the union of a set of convex subspaces is presented in this section. The obtained result is similar to the result in [13], although the initial considerations are different (in [13] there is not any constraint on the finger forces). Besides, the approach developed here determine additional information on the finger forces that is quite useful in the determination of the independent regions.

### 3.1 Convex FC-subspace using frictionless contacts

Consider four frictionless contacts applying forces  $\mathbf{f}_i$ ,  $i=1, \dots, 4$ . Let  $\mathcal{P}_f$  be the polygon defined by these forces (it coincides with the projection of  $\mathcal{P}_1$  on the force space). In order to obtain a FC grasp,  $\mathbf{0} \in \mathcal{P}_f$  must be satisfied.

The following terms will be used in the determination of the FC-subspaces.

**Definition 3** The *Real Range* of  $\tau_i$ ,  $R_i$ , is the set of values of  $\tau_i$  produced by the contact force  $\mathbf{f}_i$  that are physically possible due to the length of the contact edge.  $\diamond$

**Definition 4** The *Directional Range* of  $\tau_i$ ,  $R_{f_{c_i}}$ , is the set of values of  $\tau_i$  produced by the contact force  $\mathbf{f}_i$  that allow a FC grasp considering that the contact edge has infinite length (i.e. only the “direction” of the edge is considered).  $\diamond$

From the two above definitions, the existence of a FC grasp implies that  $R_i \cap R_{f_{c_i}} \neq \emptyset$ . Since  $R_i$  is known, the set of valid torques that produces a FC grasp can be determined by finding  $R_{f_{c_i}}$ .

Four frictionless contacts generate the minimum number of wrenches necessary to obtain a FC grasp [18], generating a convex hull  $\mathcal{P}_1$  with minimum number of faces. Since a FC grasp must satisfy  $\mathbf{0} \in \mathcal{P}_1$  [17] and  $\mathcal{P}_1$  is convex, the Directional Range  $R_{f_{c_i}}$ ,  $i = 1, \dots, 4$ , is a continuous set that has one or two finite extremes. Then, the type of Directional Range can be:

*Infinite:*  $R_{f_{c_i}}$  has only one finite extreme  $\tau_{i_m}$ . Then,  $R_{f_{c_i}} = [\tau_{i_m}, \infty)$  or  $R_{f_{c_i}} = (-\infty, \tau_{i_m}]$ .  
*Limited:*  $R_{f_{c_i}}$  have two finite extremes  $\tau_{i_m}$  and  $\tau_{i_{m'}}$ . Then,  $R_{f_{c_i}} = [\tau_{i_m}, \tau_{i_{m'}}]$  or  $R_{f_{c_i}} = [\tau_{i_{m'}}, \tau_{i_m}]$ .

**Proposition 1** Consider four applied contact forces  $\mathbf{f}_i$ ,  $i = 1, \dots, 4$ . The number of extremes and, therefore, the type of the Directional Range  $R_{f_{c_i}}$  of each torque  $\tau_i$ , can be determined knowing how many pairs of the following coefficients are non-positive:

$$\beta_{i,jk} = \frac{\sin(\theta_i - \theta_k)}{\sin(\theta_j - \theta_k)} \quad (10)$$

$$\beta_{i,kj} = \frac{\sin(\theta_j - \theta_i)}{\sin(\theta_j - \theta_k)} \quad (11)$$

where  $\theta_i$ ,  $\theta_j$  and  $\theta_k$  are the directions of the respective forces.  $\diamond$

**Proof:** Let  $\tau_{i_m}$  be an extreme of  $R_{f_{c_i}}$ . Since  $\mathcal{P}_1$  is convex, if  $\tau_{i_m}$  defines a vertex of  $\mathcal{P}_1$ , then  $\mathbf{0} \in \partial\mathcal{P}_1$ ,  $\partial\mathcal{P}_1$  being the boundary of  $\mathcal{P}_1$ , and  $\mathbf{0}$  can be expressed as a positive linear combination of the three vertices that define the face of  $\partial\mathcal{P}_1$  containing  $\mathbf{0}$ , i.e.  $\mathbf{0} = \alpha_i \boldsymbol{\omega}_{i_m} + \alpha_j \boldsymbol{\omega}_j + \alpha_k \boldsymbol{\omega}_k$  with  $\alpha_i, \alpha_j, \alpha_k \geq 0$  and  $\alpha_i + \alpha_j + \alpha_k = 1$ . Solving this expression for  $\boldsymbol{\omega}_{i_m}$  results

$$\cos \theta_i = \beta_{i,jk} \cos \theta_j + \beta_{i,kj} \cos \theta_k \quad (12)$$

$$\sin \theta_i = \beta_{i,jk} \sin \theta_j + \beta_{i,kj} \sin \theta_k \quad (13)$$

$$\tau_{i_m} = \beta_{i,jk} \tau_j + \beta_{i,kj} \tau_k \quad (14)$$

with  $\beta_{i,jk} \leq 0$  and  $\beta_{i,kj} \leq 0$  (since  $\boldsymbol{\omega}_{i_m} \neq \mathbf{0}$ ,  $\beta_{i,jk}$  and  $\beta_{i,kj}$  can not be simultaneously null). Then, there are three equalities with only two unknowns,  $\beta_{i,jk}$  and  $\beta_{i,kj}$ .

If the forces  $\mathbf{f}_j$  and  $\mathbf{f}_k$  are not parallel, equalities (12) and (13) are independent, and the two unknowns  $\beta_{i,jk}$  and  $\beta_{i,kj}$  can be obtained from them as equations (10) and (11). If  $\mathbf{f}_j$  and  $\mathbf{f}_k$  are parallel (i.e.  $\theta_j = \theta_k + t\pi$ ,  $t \in \{0, 1\}$ ), meaning that there are two fingers on the same edge or on parallel edges) equalities (12) and (13) have no solution for  $\beta_{i,jk}$  and  $\beta_{i,kj}$ , and therefore, from equation (14),  $\tau_{i_m}$  does not exist. Since no more than two forces can have the same direction in a four frictionless FC grasp, a minimum of one extreme always exist.

As a result, the number of extremes of  $R_{f_{c_i}}$  are determined using information related only with the applied forces without taking into account the values of the torques.  $\diamond$

**Proposition 2** Given four applied forces, there are the following number of each type of Directional Ranges:

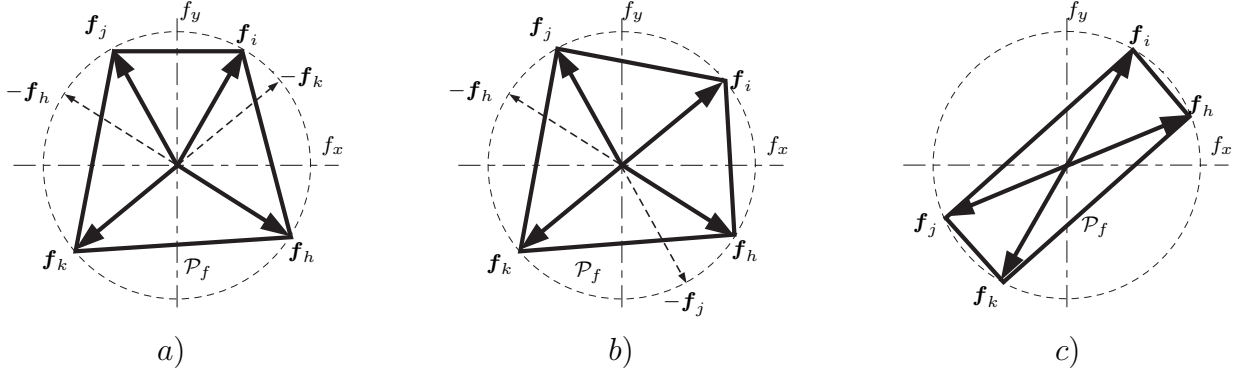


Figure 2: *Examples of the determination of the types of Directional Ranges from the applied forces: a) General case:  $R_{fc_i}$  and  $R_{fc_j}$  are Infinite and  $R_{fc_h}$  and  $R_{fc_k}$  are Limited; b) Particular case:  $R_{fc_k}$  is Limited and  $R_{fc_h}$ ,  $R_{fc_i}$  and  $R_{fc_j}$  are Infinite; c) Particular case: the four Directional Ranges are Infinite.*

General case: If all the angles between the applied forces are different from  $\pi$ , there are two Infinite and two Limited Directional Ranges (Fig. 2a).

Particular cases: If the angle between two forces is  $\pi$ , there are three Infinite and one Limited Directional Ranges (Fig. 2b), and if the angles between two pairs of forces are  $\pi$ , the four Directional Ranges are Infinite (Fig. 2c).  $\diamond$

**Proof:** From Proposition 1 the type of Directional Range of  $\tau_i$  is determined by the number of pairs of coefficients  $\beta_{i,jk}$  and  $\beta_{i,kj}$  that are non-positive, for  $\{i, j, k\} \in \{1, 2, 3, 4\}$  and  $i \neq j \neq k$ . From equations (10) and (11), these coefficients depend on the directions of three applied forces, which also define the coefficients  $\beta_{j,ik}$  and  $\beta_{j,ki}$ , and  $\beta_{k,ji}$  and  $\beta_{k,ij}$ , with the following relations between them

$$\beta_{i,jk} = \frac{1}{\beta_{j,ik}} = -\frac{\beta_{k,ji}}{\beta_{k,ij}} \quad (15)$$

$$\beta_{i,kj} = -\frac{\beta_{j,ki}}{\beta_{j,ik}} = \frac{1}{\beta_{k,ij}} \quad (16)$$

These relations imply that if one pair of coefficients is non-positive so are the other two pairs, determining one extreme for  $R_{fc_i}$ ,  $R_{fc_j}$  and  $R_{fc_k}$ . Then, a valid subset of three forces generates three extremes. Since each force appears in three from the four possible subsets of forces, two subsets of forces determine the extremes, otherwise there would be a Directional Range with more than two extremes or without any extreme, which is not possible.

In the general case, the two subsets of forces generate six different finite extremes for the four Directional Ranges. Therefore, there are two Directional Ranges with two extremes (so they are Limited), and two Directional Ranges with one extreme (so they are Infinite). In the particular case that the angle between two forces is  $\pi$ , the two subsets of forces generate five different finite extremes and one infinite extreme. Therefore, there are one Directional Range with two finite extremes (so it is Limited), and three Directional Ranges with one extreme (so they are Infinite). In the particular case that the angle between two pairs of forces is  $\pi$ , the two subsets of forces

generate four different finite extremes and two infinite extremes. Therefore, the four Directional Ranges have one finite extreme (so they are Infinite).  $\diamond$

Knowing the directions of four applied forces, it can be easily identified which torques have Limited and which ones have Infinite Directional Ranges (it can be checked from equations (10) and (11)): in the general case, the two Infinite Directional Ranges correspond to the torques generated by the two forces that lie between the negated of the other two (as in Fig 2a), and in the particular case that the angle between to forces is  $\pi$ , the three Infinite Directional Ranges correspond two the torques generated by the other two forces and the force that lies between them (as in Fig 2b).

**Lemma 1** Let  $R_{fc_i}$  and  $R_{fc_j}$  be two Infinite Directional Ranges with  $\mathbf{f}_i$  and  $\mathbf{f}_j$  defining two consecutive vertices of  $\mathcal{P}_f$ . If  $R_{fc_i}$  tends to  $\pm\infty$  then  $R_{fc_j}$  tends to  $\mp\infty$ .  $\diamond$

**Proof:** Consider first the general case with two Infinite and two Limited Directional Ranges. Let  $R_{fc_k}$  be one of the two Limited Directional Ranges. It is not known a priori if  $R_{fc_k} = [\tau_{k_1}, \tau_{k_2}]$  or  $R_{fc_k} = [\tau_{k_2}, \tau_{k_1}]$ , then the two cases must be considered. If  $R_{fc_k} = [\tau_{k_1}, \tau_{k_2}]$  then  $\tau_{k_1} \leq \tau_k \leq \tau_{k_2}$ . Substituting  $\tau_{k_1}$  and  $\tau_{k_2}$  by their expressions derived from equation (14), we obtain

$$\beta_{k,hj}\tau_h + \beta_{k,jh}\tau_j \leq \tau_k \leq \beta_{k,ih}\tau_i + \beta_{k,hi}\tau_h \quad (17)$$

If  $\tau_i$  and  $\tau_j$  are solved from equation (17), then

$$\tau_i \leq \frac{1}{\beta_{k,ih}}(\tau_k - \beta_{k,hi}\tau_h) \quad (18)$$

$$\tau_j \geq \frac{1}{\beta_{k,jh}}(\tau_k - \beta_{k,hj}\tau_h) \quad (19)$$

Therefore,  $\tau_i$  has an upper bound while  $\tau_j$  has a bottom bound implying that  $R_{fc_i}$  tends to  $-\infty$  and  $R_{fc_j}$  tends to  $+\infty$ . If  $R_{fc_k} = [\tau_{k_2}, \tau_{k_1}]$  then, with the same reasoning, equivalent equations to (18) and (19) are obtained with swapped inequalities. Then,  $\tau_i$  has a bottom bound while  $\tau_j$  has an upper bound implying that  $R_{fc_i}$  tends to  $+\infty$  and  $R_{fc_j}$  tends to  $-\infty$ .

The two particular cases can be tackle as limits of the general case. Adding  $\delta\theta$  arbitrarily small to one of the aligned forces, the particular cases are transformed into the general case. Then, the above procedure can be applied obtaining the same results when  $\delta\theta \rightarrow 0$ .  $\diamond$

From Lemma 1, the following necessary and sufficient condition for the existence of a FC grasp can be enunciated.

**Necessary and sufficient condition:** Four frictionless contacts allow a FC grasp if and only if

$$\text{sign}(\sigma) \neq \text{sign}(\epsilon) \quad (20)$$

with

$$\sigma = \beta_{i,hk}\tau_h + \beta_{i,kh}\tau_k - \tau_i \quad (21)$$

$$\epsilon = \beta_{j,hk}\tau_h + \beta_{j,kh}\tau_k - \tau_j \quad (22)$$

where  $\tau_i$  and  $\tau_j$  have Infinite Directional Ranges and  $\mathbf{f}_i$  and  $\mathbf{f}_j$  define two consecutive vertices of  $\mathcal{P}_f$   $\diamond$



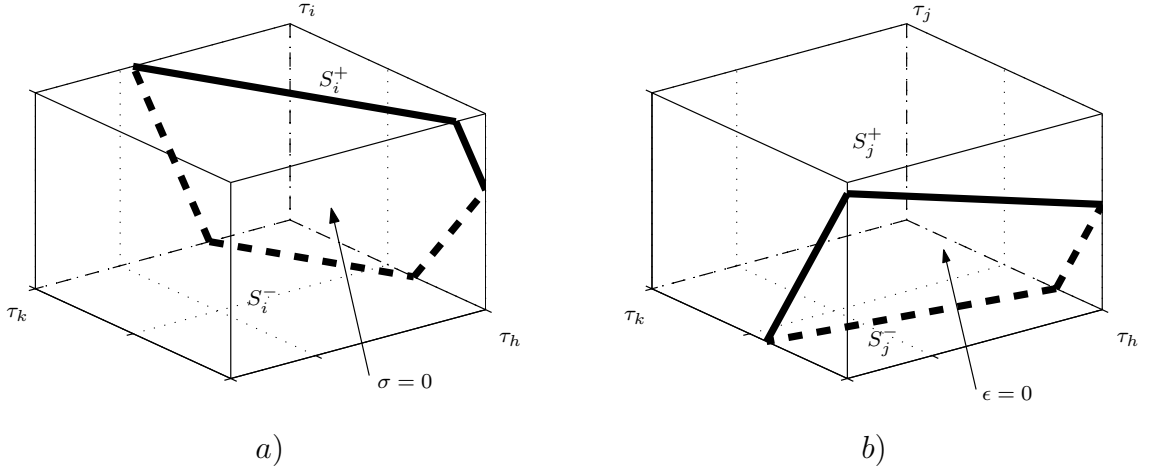


Figure 3: Polyhedrons resulting from the projection of: a)  $S_i^+$  and  $S_i^-$  on the subspace  $\{\tau_h, \tau_k, \tau_i\}$ ; b)  $S_j^+$  and  $S_j^-$  on the subspace  $\{\tau_h, \tau_k, \tau_j\}$ .

Considering the Real Range of each torque, the geometrical interpretation of this necessary and sufficient condition is:

$$(S_i^+ \cap S_j^-) \cup (S_i^- \cap S_j^+) \neq \emptyset \quad (23)$$

where  $S_i^+$ ,  $S_i^-$ ,  $S_j^+$  and  $S_j^-$  are the following polytopes:

$$\begin{aligned} S_i^+ &= \{ \{ \tau_h, \tau_i, \tau_j, \tau_k \} \mid \tau_h \in R_h, \tau_i \in R_i, \tau_k \in R_k, \sigma \geq 0 \} \\ S_i^- &= \{ \{ \tau_h, \tau_i, \tau_j, \tau_k \} \mid \tau_h \in R_h, \tau_i \in R_i, \tau_k \in R_k, \sigma \leq 0 \} \\ S_j^+ &= \{ \{ \tau_h, \tau_i, \tau_j, \tau_k \} \mid \tau_h \in R_h, \tau_j \in R_j, \tau_k \in R_k, \epsilon \geq 0 \} \\ S_j^- &= \{ \{ \tau_h, \tau_i, \tau_j, \tau_k \} \mid \tau_h \in R_h, \tau_j \in R_j, \tau_k \in R_k, \epsilon \leq 0 \} \end{aligned}$$

These polytopes can be represented as polyhedrons in two different 3-dimensional subspaces defined by  $\{\tau_h, \tau_i, \tau_k\}$  and  $\{\tau_h, \tau_j, \tau_k\}$ , as in Fig. 3. By construction  $S_i^+ \cap S_j^-$  and  $S_i^- \cap S_j^+$  are convex sets. Therefore, four applied forces determine two convex FC-subspaces.

Equations (21) and (22) have another useful geometrical property on the object space: the lines of action of  $\mathbf{f}_h$ ,  $\mathbf{f}_i$  and  $\mathbf{f}_k$  intersect at the same point when  $\sigma = 0$ , and the lines of action of  $\mathbf{f}_h$ ,  $\mathbf{f}_j$  and  $\mathbf{f}_k$  intersect at the same point when  $\epsilon = 0$ . Grasps with this property was called critical grasps in [9] since it separates the FC grasps from the non-FC grasps. In [9] it was considered that any intersection of three forces may determine a critical grasp. Equations (21) and (22) restrict this result determining only two intersections and the three forces that intersect in each one.

### 3.2 Convex FC-subspaces using friction contacts.

The procedure developed to obtain the necessary and sufficient condition for frictionless contacts is based on the knowledge of four applied forces. Since the primitive forces that define the friction cones are also known given the contact edges, the procedure developed in the previous subsection can also be applied with four primitive forces when friction is considered. Note that the primitive torques are not independent of each other and they must satisfy equation (7) to obtain a real FC grasp.

Let  $\mathbf{f}_{i,p} \in \{\mathbf{f}_{i,l}, \mathbf{f}_{i,r}\}$  and  $\tau_{i,p} \in \{\tau_{i,p}, \tau_{i,l}\}$  be a primitive force and a primitive torque, respectively. The following necessary and sufficient condition to obtain a FC grasp can be enunciated for friction contacts.

**Necessary and sufficient condition:** Considering friction contacts, a FC grasp exists if and only if four primitive torques satisfy equation (7) and

$$\text{sign}(\sigma) \neq \text{sign}(\epsilon) \quad (24)$$

with

$$\sigma = \beta_{i,hk}\tau_{h,p} + \beta_{i,kh}\tau_{k,p} - \tau_{i,p} \quad (25)$$

$$\epsilon = \beta_{j,hk}\tau_{h,p} + \beta_{j,kh}\tau_{k,p} - \tau_{j,p} \quad (26)$$

where  $\tau_{i,p}$  and  $\tau_{j,p}$  have Infinite Directional Ranges and  $\mathbf{f}_{i,p}$  and  $\mathbf{f}_{j,p}$  define two consecutive vertices of  $\mathcal{P}_f$   $\diamond$

The geometrical interpretation of this necessary and sufficient condition without considering equation (7) is identical to the geometrical interpretation of the necessary and sufficient condition for frictionless contacts but, in this case, considering the primitive forces. Then,  $(S_{i,p}^+ \cap S_{j,p}^-) \cup (S_{i,p}^- \cap S_{j,p}^+) \neq \emptyset$ . Equation (7) is a 2-dimensional subspace of the contact space, and its intersection with  $(S_{i,p}^+ \cap S_{j,p}^-)$  and  $(S_{i,p}^- \cap S_{j,p}^+)$  determines the convex FC-subspaces.

Considering friction contacts, a critical grasp is obtained when three primitive forces intersect at the same point [19] and, as well as in the case of frictionless contacts, it happens when  $\sigma = 0$  or  $\epsilon = 0$ .

### 3.3 Decomposition of the FC-space

The previous subsections determine two convex FC-subspaces limited by critical grasps considering four normal forces (frictionless contacts) or four primitive forces (friction contacts). When there are more than four normal or primitive forces, there may be several convex FC-subspaces limited by critical grasps. In order to obtain the combinations of four normal or primitive forces that determine convex FC-subspaces the following algorithm is used (the algorithm is described using the nomenclature of friction contacts but it can also be applied considering frictionless contacts changing the primitive forces by normal forces).

**Algorithm 1** Let  $n_f$  be the number of primitive forces and consider that  $\mathbf{f}_{h,p}$  and  $\mathbf{f}_{k,p}$  are two primitive forces that generate torques whose Directional Ranges are Limited.

The following steps are applied for each combination of two primitive forces:

1. Consider the two forces as  $\mathbf{f}_{h,p}$  and  $\mathbf{f}_{k,p}$ .
2. Obtain  $\mathbf{f}_{i,p} \in [-\mathbf{f}_{h,p}, -\mathbf{f}_{k,p}]$  for  $i = 1, \dots, n_f$  and  $i \neq h \neq k$ .
3. Let  $n_i$  be the number of  $\mathbf{f}_{i,p}$  that satisfy step 2:
  - 3.1. If  $n_i \leq 1$  Then discard this combination of forces.
  - 3.2. If  $n_i > 1$  Then two convex FC-subspaces are obtained with each combination of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and any two forces from the  $n_i$  obtained in step 2.  $\diamond$

As a result, all the convex FC-subspaces and the type of Directional Range of each torque are determined with a computational cost of  $O(N^3)$ . The FC-space is the union of these subspaces and it may be concave.

## 4 Independent regions

Since the independent regions are segments on the object boundary such that a finger in each segment ensures a FC grasp, they define a  $N$ -parallelepiped fully contained in the FC-space. Then, the problem of determine the independent regions is equivalent to the problem of finding a  $N$ -parallelepiped fully contained in the FC-space. Since the FC-space may be concave, it is not possible to assure that a  $N$ -parallelepiped is fully contained in it just by testing if its vertices belong to the FC-space, as it is done in [10] (remember that in [10] a sufficient condition is used, therefore the total FC-space is not considered). Using the convex FC-subspaces determined in the previous section and the following proposition (it is enunciated considering friction contacts) the complexity of the problem is reduced.

**Proposition 3** Consider a convex FC-subspace limited by the planes represented in equation (25) with  $\sigma=0$  and equation (26) with  $\epsilon=0$  (thus,  $\tau_{i,p}$  and  $\tau_{j,p}$  have Infinite Directional Ranges and  $\mathbf{f}_{i,p}$  and  $\mathbf{f}_{j,p}$  define two consecutive vertices of have  $\mathcal{P}_f$ ). The set  $(\tau_{\nu,p}^-, \tau_{\nu,p}^+)$ , with  $\nu \in \{h, i, j, k\}$ , is an independent region in the Directional Range  $R_{fc_\nu}$  if the primitive torques satisfy equation (7) and one of the two following conditions is satisfied:

- 1)  $\tau_{h,p}^+$ ,  $\tau_{k,p}^+$  and  $\tau_{i,p}^+$  in equation (25) make  $\sigma=0$  and  $\tau_{h,p}^-$ ,  $\tau_{k,p}^-$  and  $\tau_{j,p}^-$  in equation (26) make  $\epsilon=0$ .
- 2)  $\tau_{h,p}^-$ ,  $\tau_{k,p}^-$  and  $\tau_{i,p}^-$  in equation (25) make  $\sigma=0$  and  $\tau_{h,p}^+$ ,  $\tau_{k,p}^+$  and  $\tau_{j,p}^+$  in equation (26) make  $\epsilon=0$ .  $\diamond$

**Proof:** The meaning of equation (7) has been discussed in Section 2, therefore only conditions 1 and 2 need to be proved here. Proposition 1 determines that the coefficients of equations (25) and (26) are non-positive, implying that these two equations represent two planes with negative slope (see Fig. 4).

Consider condition 1. If  $\tau_{h,p}^+$ ,  $\tau_{k,p}^+$  and  $\tau_{i,p}^+$  are the maximum torques generate on their respective independent regions and they make  $\sigma=0$  in equation (25), then it is not possible to obtain other values of  $\tau_{h,p}$ ,  $\tau_{k,p}$  and  $\tau_{i,p}$  that belong to the independent region (i.e., values smaller than  $\tau_{h,p}^+$ ,  $\tau_{k,p}^+$  and  $\tau_{i,p}^+$ ) that make  $\sigma=0$ , since the slope of the plane represented in equation (25) is negative. In the same way, if  $\tau_{h,p}^-$ ,  $\tau_{k,p}^-$  and  $\tau_{j,p}^-$  are the minimum torques generate on their respective independent region and they make  $\epsilon=0$  in equation (26), then it is not possible to obtain other values of  $\tau_{h,p}$ ,  $\tau_{k,p}$  and  $\tau_{j,p}$  that belong to the independent region (i.e., values bigger than  $\tau_{h,p}^-$ ,  $\tau_{k,p}^-$  and  $\tau_{j,p}^-$ ) that make  $\epsilon=0$ , since the slope of the plane represented in equation (26) is negative. The same reasoning is applied considering condition 2.

As a result, the two conditions of this Proposition assure that  $\tau_{\nu,p} \in (\tau_{\nu,p}^-, \tau_{\nu,p}^+)$ , with  $\nu \in \{h, i, j, k\}$ , cannot make  $\sigma=0$  neither  $\epsilon=0$ , implying that critical grasps can not belong to  $(\tau_{\nu,p}^-, \tau_{\nu,p}^+)$ . Therefore, since the subspace is convex, either condition 1 or condition 2 determine the extremes of the independent regions.  $\diamond$

On the object space,  $\sigma = 0$  (equation (25)) and  $\epsilon = 0$  (equation (26)) imply that the lines of action of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and  $\mathbf{f}_{i,p}$ , and the lines of action of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and  $\mathbf{f}_{j,p}$ , intersect, respectively, at the same point. As a consequence, the condition 1 of Proposition 3 is equivalent to say that the lines of action of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and  $\mathbf{f}_{i,p}$  intersect at the same point when they are applied on the extremes of their respective independent regions where the maximum torque is produced, and the lines of action of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and  $\mathbf{f}_{j,p}$  intersect at the same point when they are applied on the extremes of their respective independent regions where the minimum torque is produced. The same reasoning can be applied for condition 2. Note that the independent regions determined according

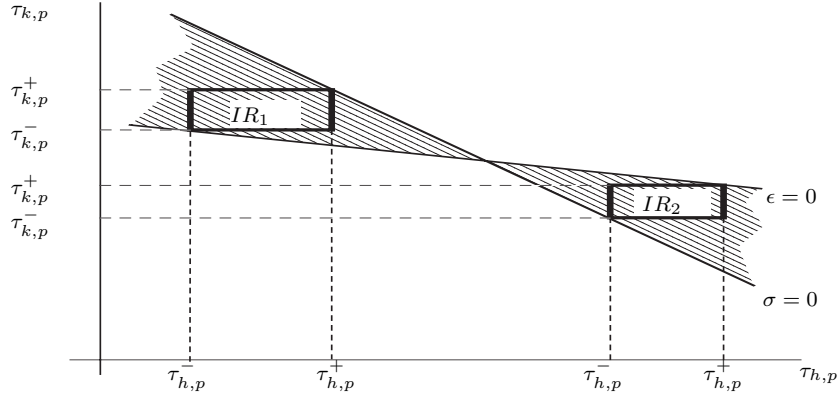


Figure 4: *Two dimensional slice of a contact convex FC-subspace where  $\tau_{h,p}$  and  $\tau_{k,p}$  have Limited Directional Range and the other torques are constants. The independent region  $IR_1$  satisfies condition 1 of Proposition 3, and the independent region  $IR_2$  satisfies condition 2 of Proposition 3.*

to Proposition 3 are defined on the Directional Ranges, so  $(\tau_{\nu,p}^-, \tau_{\nu,p}^+) \cap R_{\nu,p} \neq \emptyset$ ,  $\nu \in \{h, i, j, k\}$ , must be satisfied to obtain the independent region on the real edge. The following algorithm is used to obtain the independent regions on the object edges:

**Algorithm 2** Consider a convex FC-subspace limited by the planes represented in equation (25) with  $\sigma=0$  and equation (26) with  $\epsilon=0$ .

1. Select two arbitrary points on the corresponding object edges where  $\tau_{h,p}^+$  and  $\tau_{k,p}^+$  will be produced.
2. Determine the point where  $\tau_{i,p}^+$  is produced (the lines of action of the primitive forces applied on the two points selected in step 1 and on this point intersect at the same point).
3. Check if  $R_{f_{c_{i,p}}} \cap R_{i,p} \neq \emptyset$ . From the initial consideration  $R_{f_{c_{i,p}}}$  is Infinite and its finite extreme has been determined in step 2. If this condition is not satisfied a FC grasp is not possible and the position of the points must be adjusted in order to satisfy this condition.
4. Determine the region where the intersection of the lines of action of  $\mathbf{f}_{h,p}$ ,  $\mathbf{f}_{k,p}$  and  $\mathbf{f}_{j,p}$  must lie. This region must satisfy the following conditions:  $\tau_{h,p}^+ > \tau_{h,p}^-$ ,  $\tau_{k,p}^+ > \tau_{k,p}^-$ ,  $R_{f_{c_{j,p}}} \cap R_{j,p} \neq \emptyset$  and equation (7). If these conditions are incompatible, then the independent regions do not exist for the initial selected points.
5. Select an arbitrary point of this region and project it on the edges, obtaining the extremes where  $\tau_{h,p}^-$ ,  $\tau_{k,p}^-$  and  $\tau_{j,p}^-$  are produced.
6. Intersect the independent regions with the real edge to obtain the actual independent regions (by construction these intersections are always not null).  $\diamond$

In analogous way, the algorithm can be applied selecting in the first step  $\tau_{h,p}^-$  and  $\tau_{k,p}^-$ . Criteria to select the points in steps 1 and 5 are not discussed here, but it can be done considering other constraints on the finger placements (for instance, kinematics constraints or task requirements), providing flexibility to the algorithm.

Proposition 3 and Algorithm 2 are also applicable considering frictionless contacts, exchanging the primitive forces for the normal forces and without considering equation (7). Figure 5 shows examples considering frictionless contacts and friction contacts. It can be checked in these figures that it is not possible to intersect the lines of action of three forces when they are applied on an independent region, so critical grasps are not possible.

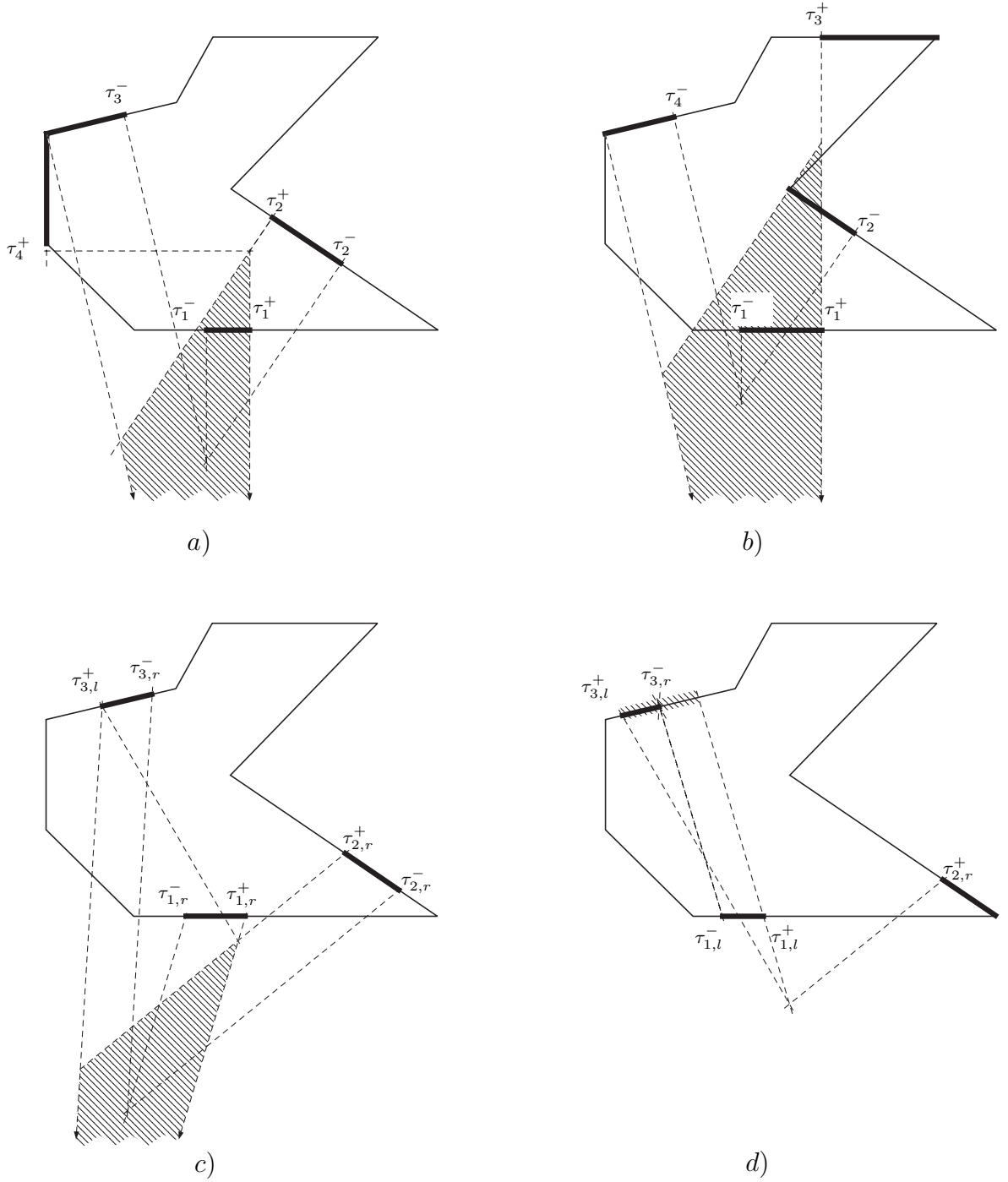


Figure 5: *Examples of the determination of independent regions (black segments) on the edges of an object and the region determined in step 4 of algorithm 2 (shaded region): a) Four frictionless contacts where  $R_{fc_1}$  and  $R_{fc_2}$  are Limited and  $R_{fc_3}$  and  $R_{fc_4}$  are Infinite; b) Four frictionless contacts where  $R_{fc_1}$  is Limited and  $R_{fc_2}$ ,  $R_{fc_3}$  and  $R_{fc_4}$  are Infinite; c) Three friction contacts where  $R_{fc_{1,r}}$  and  $R_{fc_{2,r}}$  are Limited and  $R_{fc_{3,r}}$  and  $R_{fc_{3,l}}$  are Infinite (the primitive forces considered in this case are  $\mathbf{f}_{1,r}$ ,  $\mathbf{f}_{2,r}$ ,  $\mathbf{f}_{3,r}$  and  $\mathbf{f}_{3,l}$ ); d) Three friction contacts where  $R_{fc_{1,l}}$  and  $R_{fc_{3,l}}$  are Limited and  $R_{fc_{2,r}}$  and  $R_{fc_{3,r}}$  are Infinite (the primitive forces considered in this case are  $\mathbf{f}_{1,l}$ ,  $\mathbf{f}_{3,l}$ ,  $\mathbf{f}_{2,r}$  and  $\mathbf{f}_{3,r}$  and the region is the shaded part of the contact edge).*

## 5 Conclusions and future works

In this paper a new approach to determine independent regions on 2D polygonal objects that allow a force-closure grasp considering any number of fingers has been presented. Since the FC-space may be concave, it is decomposed in a set of convex FC-subspaces establishing each one a necessary and sufficient condition for the existence of a FC grasp. A condition to obtain independent regions in each FC-subspace is also presented and using its geometrical interpretation the problem of determining independent region is reduced to find two particular points on the object space. The main advantages of the proposed algorithm are that it is applied on the object space, therefore the problem is always two-dimensional, and that it is not necessary to compute the  $N$ -dimensional FC-space. Besides, the algorithm is flexible and other constraints on the finger placements can be easily introduced.

The simplicity of the methodology and the fact that the algorithm is applied on the object space encourage to extend this work to non-polygonal objects and to 3D objects as future works. All the methodology presented here is based on the relative positions of the applied forces, and we consider that this concept can also be applied in the future work.

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