

Labeling Generating Matrices

Figuerola-Centeno, R.M.
Mathematics Departament
University of Hawaii at Hilo
College Hall 4-A, 200 W. Kawili St.
Hilo, HI 96720-4091

R. Ichishima
College of Humanities and Sciences,
Nihon University, 3-25-40 Sakurajosui Setagaya-Ku
Tokyo 156-8550, Japan

F. A. Muntaner-Batle *
Facultat de Ciències Polítiques i Jurídiques
Universitat Internacional de Catalunya, c/ Immaculada 22
08017 Barcelona, Spain

M. Rius-Font †
Departament de Matemàtica Aplicada i Telemàtica
Universitat Politècnica de Catalunya, Jordi Girona Salgado 1
08034 Barcelona, Spain

Abstract

This paper is mainly devoted to generate (special)(super) edge-magic labelings of graphs using matrices. Matrices are used in order to find lower bounds for the number of non-isomorphic (special)(super) edge-magic labelings of certain types of graphs. Also new applications of graph labelings are discussed.

*wants to acknowledge Dr. Eduardo Canale since without the mathematical discussions and his very useful advice this work would have been impossible.

†Supported by the Spanish Research Council under project MTM2004-07891-C02-01 and by the Catalan Research Council under project 2001SGR00258.

1 Introduction

Graphs considered in this paper are not necessarily simple, that is to say, we may allow loops, however, in this paper we will not consider graphs with multiple edges. Also for most of the graph theory terminology and notation utilized here, the authors refer the reader to Chartrand and Lesniak [9]. However, in order to make the paper reasonable self contained, we mention that for a graph G we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively. If $|V(G)| = p$ and $|E(G)| = q$ we say that G is a (p, q) -graph. By the notation $G = (V, E)$ we mean a graph G with vertex set V and with edge set E . Also for a digraph D , we denote by $V(D)$ and $E(D)$ the sets of vertices and arcs of D respectively. By $D = (V, E)$ we mean a digraph D with vertex set V and arc set E .

The seminal paper in edge-magic labelings was published in 1970 by Kotzig and Rosa [19] who called these labelings magic valuations. These were later rediscovered by Ringel and Lladó [24] who coined one of the now popular terms: edge-magic (EM) labelings. More recently, they have been refereed to, as EM total labelings by Wallis [26]. For a (p, q) -graph $G = (V, E)$, a bijective function $f : V \cup E \rightarrow \{1, 2, \dots, p + q\}$ is an EM labeling of G if $f(u) + f(uv) + f(v)$ is a constant val_f (called the valence of f) for any edge $uv \in E$. A graph that admits such a labeling is an EM graph. Also, we take the opportunity at this point to introduce the concept of EM digraph, which will be of help in order to achieve the goals pertained in this paper. An EM digraph is a digraph for which its underlying graph is EM. In [10], Enomoto, Lladó and Ringel defined an EM labeling f of a graph G to be super edge-magic (SEM) if it has the extra property that $f(V(G)) = \{1, 2, \dots, p\}$. Thus, a SEM graph is a graph that admits a SEM labeling. Lately, SEM labelings and SEM graphs have been called by Wallis [26] strongly EM total labelings and strongly EM total graphs, respectively. In a similar way as we did in the case of EM labelings and EM graphs, we define in this paper the concept of SEM digraph to be a digraph for which its underlying graph is SEM. The next characterization found in [11] has proven to be very useful and therefore we state it as lemma 1.1.

Lemma 1.1 *A (p, q) -graph G is SEM if and only if there exists a bijective function $f : V(G) \rightarrow \{1, 2, \dots, p\}$ such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ consists of q consecutive integers. In such a case, f extends to a SEM labeling of G with valence $val_f = p + q + s$, where $s = \min(S)$ and*

$$S = \{val_f - (p + i)\}_{i=1}^q.$$

Therefore, it is clear that due to Lemma 1.1, it suffices to exhibit the vertex labeling in order to identify a SEM graph.

Next, let us define what in this paper we mean by the adjacency matrix of a digraph D , namely $A(D)$, with $V(D) = \{a_1 < a_2 < \dots < a_p\} \subset \mathbb{N}$. The rows and columns of

the matrix are labeled as shown in Figure 1.

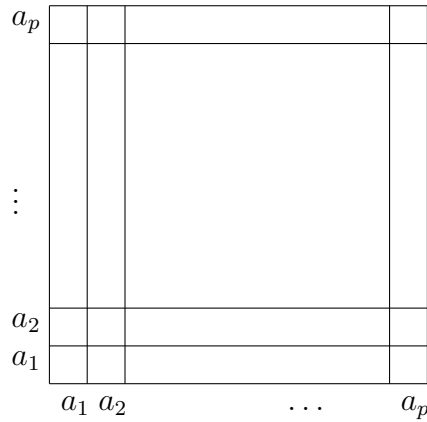


Figure 1: Adjacency matrix

By the position (i, j) we mean the position with row labeled with a_i and column labeled with a_j . The entry (i, j) in $A(D)$ is 1 if and only if the arc $(a_i, a_j) \in E(D)$, and 0 otherwise. The main diagonals in our matrix are the diagonals from top left to bottom right, as shown in Figure 2.

We note that if the longest no main diagonal in $A(D)$ contains no 1's, then the digraph D contains no loops.

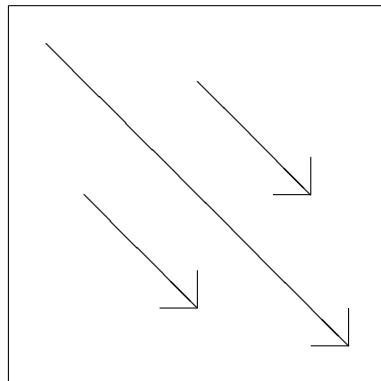


Figure 2: Main diagonals

The following lemma is an immediate consequence of lemma 1.1.

Lemma 1.2 *A digraph D is SEM if and only if the adjacency matrix (d_{ij}) of D obtained by relabeling the vertices of D after the corresponding labels of a SEM labeling, has the following two properties:*

1. $d_{ij} = d_{i'j'} = 1 \implies [(i, j) = (i', j') \text{ or } i + j \neq i' + j']$.
2. *The main diagonals $i + j = k$ of the adjacency matrix with some entry different from 0 are consecutive.*

The following definition will also prove to be useful for this paper.

Let G be a SEM (p, q) -graph, and let f be a SEM labeling of G . We define the complementary labeling of f to be the function $g(x) = p + 1 - f(x)$ if $x \in V(G)$. Then, g is also a SEM labeling.

Also in 2001, Muntaner [21] introduced a further restriction of SEM labelings, that he called special super edge-magic labelings (SSEM), which only makes sense for bipartite graphs. A SSEM labeling of a bipartite (p, q) -graph with bipartite sets V_1 and V_2 is a SEM labeling f with the extra property that $f(V_1) = \{1, 2, \dots, |V_1|\}$.

Notice that, as it happens in the case of SEM labelings, it is possible to redefine SSEM labelings of bipartite graphs in such a way that only the vertices of the graph are involved, and we do it next (See [21],[22] and [26]).

Lemma 1.3 *A bipartite (p, q) -graph $G = (V, E)$ with bipartite sets V_1 and V_2 is SSEM if and only if there exists a bijective function $f : V \longrightarrow \{1, 2, \dots, p\}$ such that the set $f(V_1) = \{1, 2, \dots, |V_1|\}$ and the set $\{f(u) + f(v) : uv \in E\}$ consists of q consecutive integers.*

SSEM labelings are of interest since there exists a close relationship among them and a particular type of graceful labelings called α -labelings. These two last concepts were first defined by Rosa in 1966 [24], in order to provide a different approach to the problem of decomposing complete graphs into isomorphic copies of a given tree. Since then, many papers dealing with such labelings have appeared in the literature. We mention that Rosa called graceful labelings β -valuations. The term "graceful" was first introduced by Golomb in [17] and broadly popularized by a paper of Martin Gardner in 1972 [16].

Next we provide the necessary definitions and we establish the relationship existing among α -labelings and SSEM labelings.

A function f is a graceful labeling of a graph $G = (V, E)$ if f is an injection from V to $\{0, 1, \dots, |E|\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$ then

the resulting edge labels are distinct. If f has the extra property that there exists an integer k such that for each edge uv either $f(u) \leq k < f(v)$ or $f(v) \leq k < f(u)$, then f is called either an α -valuation or an α -labelings of G .

The way of converting SSEM labelings of trees into α -labelings and viceversa is not hard and it is described either in [21] or [22]. However in order to make the paper easy to read we will describe the procedure in the proof of Lemma 1.4 .

Lemma 1.4 *A tree T is SSEM if and only if T admits an α -labeling.*

Proof.

Let V_1 and V_2 be the bipartite sets of T with $|V_1| = p_1$ and $|V_2| = p_2$. Assume that the function f is an α -valuation of T such that $f(u) < f(v) \forall u \in V_1, v \in V_2$. Then the new function g defined by the rule

$$g(u) = \begin{cases} p_1 - f(u) & \text{if } u \in V_1 \\ f(u) + 1 & \text{if } u \in V_2 \end{cases}$$

can be extended to a SSEM labeling of T .

Now, assume that g is a SSEM labeling of T . Then the new function f defined by the rule

$$f(u) = \begin{cases} p_1 - g(u) & \text{if } u \in V_1 \\ g(u) - 1 & \text{if } u \in V_2 \end{cases}$$

is an α -labeling of T . □

It is important to notice that two different SSEM labelings are transformed into two different α -valuation and viceversa, when using the transformations described previously.

Another labeling that will be considered in this paper, and that is similar in nature to SEM labelings, was introduced by Acharya and Hegde in [2] under the name of (k, d) -arithmetic labelings. A graph $G = (V, E)$ is (k, d) -arithmetic if there is a bijective function $f : V \rightarrow D$ where $D \subset \mathbb{N}$, such that the set $S = \{f(u) + f(v) : uv \in E(G)\}$ forms an arithmetic progression of $|E|$ terms with first term k and difference d . Then, f is called a (k, d) -arithmetic labeling. In this paper we define a (k, d) -arithmetic digraph to be a digraph for which its underlying graph is (k, d) -arithmetic.

From Lemma 1.1 it is trivial to observe that every SEM graph is also $(s, 1)$ -arithmetic, where s is defined as in Lemma 1.1.

Operations among graphs and digraphs will also be considered in this paper. We begin by defining the corona product, that was first introduced by Harary and Frucht [14] in 1970.

Assume that G and H are two graphs. Then the corona product of G and H , denoted by $G \odot H$, is the graph with

$$\left. \begin{aligned} V(G \odot H) &= V(G) \cup V(G \times H) \\ E(G \odot H) &= E(G) \cup \{(i, u)(i, v) : i \in V(G), uv \in E(H)\} \cup \{i(i, u) : i \in V(G), \\ &u \in V(H)\} \end{aligned} \right\}$$

In other words, the corona product of two graphs, G and H , where G has order p , is obtained by taking one copy of G , p copies of H and joining each vertex of a copy of H with a vertex of G . This is done for all the vertices of G .

Next, we define what we mean by the Kronecker product of matrices. Assume that B is any matrix and $A = (a_{ij})$ is an $m \times n$ matrix. Then the Kronecker product of A and B , denoted by $A \otimes B$, is the new matrix defined as follow:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \dots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix}$$

Also, in [4], Barrientos defined the concept of path-like tree as follows: we embed the path P_n as a subgraph of the 2-dimensional grid. Given such an embedding, we consider the ordered set of sub-paths L_1, \dots, L_k which are maximal straight segments in the embedding, and such that the end of L_i is the beginning of L_{i+1} . Suppose that $L_i \cong P_2$ for some i and that some vertex u of L_{i-1} is at distance 1 in the grid to a vertex v of L_{i+1} . An elementary transformation of the path consists in replacing the edge L_i by a new edge uv . We say that a tree T of order n is a path-like tree, when it can be obtained from some embedding of P_n in the grid by a sequence of elementary transformations.

It has been shown that path-like trees admit several types of labelings. For instance α -labelings [4], and edge-antimagic total labelings [3]. In this paper, we will also study the SEM properties of trees which are obtained from path like-trees, using the corona product.

To conclude this introduction, we will state the following result that can be found in [11]

Lemma 1.5 *Let f be a SEM labeling of a (p, q) -graph $G = (V, E)$. The valence, val_f , of f is given by the formula:*

$$val_f = \frac{\sum_{x \in V} (f(x)deg(x)) + \sum_{y \in E} f(y)}{q}.$$

In particular if $p = q$, that is to say if G is a 2-regular graph, by Lemma 1.5 we obtain:

Corollary 1.1 *Let $G = (V, E)$ be a 2-regular graph of order p , then:*

1. *If G is SEM then p is odd.*
2. *If f is a SEM labeling of G then:*
 - $val_f = \frac{5p+3}{2}$
 - $\min\{f(u) + f(v) : uv \in E\} = \frac{p+3}{2}$
 - $\max\{f(u) + f(v) : uv \in E\} = \frac{3p+1}{2}$

The reader interested in an account on graph labelings results, is addressed to [15] for a very detailed exposition of such results.

2 A new relation among labelings

SEM labelings are interesting, not only because of the beauty of the labelings themselves, but also because of the large number of relations that have been found among SEM labelings and other types of labelings. Some of such labelings have been deeply studied. This is the case, for instance, of graceful and harmonious labelings. In fact, SEM labelings are one of the most powerful links about labelings known by the authors. For a detailed account on these links the interested reader can consult [11]. In this section we establish a new relation among SEM labelings and (k, d) -arithmetic labelings, that we state next.

Proposition 2.1 *Let G be a (p, q) -SEM graph. Then $\forall d \in \mathbb{N}, \exists k \in \mathbb{N} : G$ is (k, d) -arithmetic.*

Proof.

Assume that the vertices of G are named after the labels of some SEM labelings of G . For a fix d we consider the labeling g of G with $g(i) = 1 + (i - 1)d, i \in \{1, \dots, p\}$. As $\{i + j : ij \in E(G)\}$ are consecutive numbers, the numbers $\{g(i) + g(j) : ij \in E(G)\} = \{2 + (i + j - 2)d : ij \in E(G)\}$ form an arithmetic progression with difference d . □

Figure 3 shows a SEM labeling of C_5 , as well as $(6, 2)$, $(8, 3)$ and $(10, 4)$ -arithmetic labelings of C_5 .

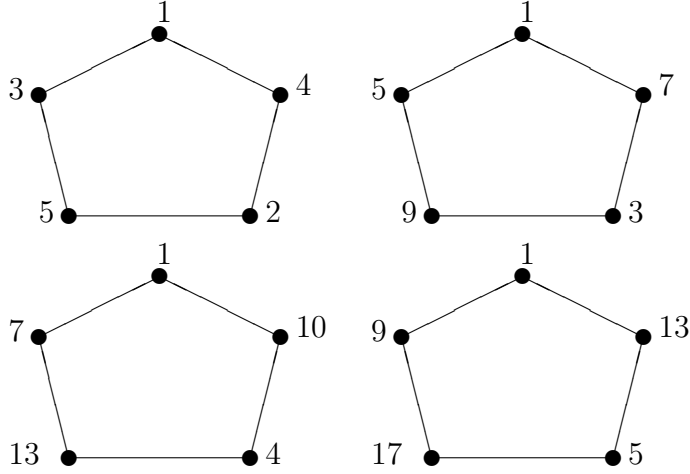


Figure 3: (k,d)-arithmetic graphs

3 A new digraph operation

In this section we define a new operation of digraphs that is in fact, in some sense, a generalization of the well studied Kronecker product of matrices. Also we will establish the relation of this product with (S)(S)EM graphs. We will use such relation in order to find lower bounds for the number of non-isomorphic labelings of certain families of graphs.

From now on, we will denote the underlying graph of a digraph D by the notation $und(D)$. Let D be a digraph and let $\Gamma = \{F_1, F_2, \dots, F_n\}$ be a family of digraphs such that $V(F_i) = V$ for every $i \in \{1, \dots, n\}$. Consider a function $h : E(D) \rightarrow \Gamma$, then the product $D \otimes_h \Gamma$ is a digraph with vertex set $V(D) \times V$ and $((a, b), (c, d)) \in E(D \otimes_h \Gamma) \iff (a, c) \in E(D) \wedge (b, d) \in E(h(a, c))$. The adjacency matrix of $D \otimes_h \Gamma$, $A(D \otimes_h \Gamma)$ is obtained by multiplying every 0 entry of $A(D)$ by the $|V| \times |V|$ nul square matrix and every 1 entry of $A(D)$ by $A(h(a, c))$. Notice that when h is constant, the matrix product that we have just defined coincides with the classical Kronecker product of matrices.

From now on, let S_p denote the set of all SEM 1-regular labeled digraphs of odd order p where each vertex takes the name of the label that has assigned.

Theorem 3.1 *Let D be an EM digraph with valence val_f and let $h : E(D) \rightarrow S_p$, then the graph $und(D \otimes_h S_p)$ is EM.*

Proof.

We rename the vertices of D and each element of S_p after labels of their corresponding

EM and SEM labelings respectively. We define the following labeling for the digraph $D \otimes_h S_p$:

1. If $(i, j) \in V(D \otimes_h S_p)$ we label the vertex with label $\hat{f}(i, j) = p(i - 1) + j$.
2. If $(i, j), (i', j') \in E_{D \otimes_h S_p}$ we label the arc with the label $val_{\hat{f}} - [p(i + i' - 2) + j + j']$ where

$$val_{\hat{f}} = p[val_f - 3] + \frac{3p + 1}{2} + 1.$$

We want to show that the following items with respect to labels of $D \otimes_h S_p$.

- A** All vertex label are distinct.
- B** All arc labels are distinct.
- C** No arc label and vertex label coincide.
- D** The maximum label used is $p(|V(D)| + |E(D)|)$.

Next we prove:

Proof A If $p(i - 1) + j = p(i' - 1) + j'$ then $0 \leq p|(i - i')| = |j' - j| < p$. Therefore, $j = j'$ and $i = i'$.

Proof B If an arc $((i, j), (i', j'))$ is labeled with the same labeled as the arc $((k, l), (k', l'))$ then $0 \leq p|(i + i' - (k + k'))| = |(l + l') - (j + j')| < \frac{3p+1}{2} - \frac{p+3}{2} = p - 1$ (by corollary 1.1). Therefore we obtain that $(j, l) = (j', l')$ and $(i, k) = (i', k')$.

Proof C If the label of the arc $((i, j), (i', j'))$ coincides with the label of the vertex (k, l) , then $p(k - 1) + l = Val_{\hat{f}} - [p(i + i' - 2) + j + j']$ and therefore we have that $p(k + i + i' - 3) + l + j + j' = Val_{\hat{f}} = p[val_f - 3] + \frac{3p+1}{2} + 1$. That is to say, $0 \leq p|(val_f - k - i - i')| = |l + j + j' - \frac{3p+1}{2} - 1| < p$ and $val_f = k + i + i'$, contradiction since D is EM and $(i, i') \in E(D)$.

Proof D Assume that M is the maximum label on the vertices of D . We consider two cases:

- If $M < |E(D)| + |V(D)|$, then the maximum label of D appears on an arc (i, i') for which $i + i'$ is the maximum sum taken over all arcs of D . Thus $val_f - (i + i') = |E(D)| + |V(D)|$. Therefore the maximum label of $D \otimes_h S_p$ is $val_{\hat{f}} - [p(i + i' - 2) + \frac{p+3}{2}] = p[|V(D)| - (i + i')] = p[|E(D)| + |V(D)|]$.

- Assume that $M = |E(D)| + |V(D)|$. Then the maximum label on an arc of $D \otimes_h S_p$ is of the form $p[V(D) - (i + i')]$. That is to say, the product of p a label or an arc of F , and therefore less than $p[|E(D)| + |V(D)|]$. \square

Similar results can be obtained for SEM and SSEM graphs as corollaries of the previous result.

Corollary 3.1 *Let D be a SEM digraph and let $h : E(D) \longrightarrow S_p$. Then $D \otimes_h S_p$ is SEM.*

Proof.

From the proof of Theorem 3.1, it is clear that if the graph D is SEM then the smallest labels will appear on the vertices of $D \otimes_h S_p$, under the labeling \hat{f} . \square

Corollary 3.2 *Let D be a SSEM digraph and let $h : E(D) \longrightarrow S_p$. Then $D \otimes_h S_p$ is SSEM.*

Proof.

It is clear that the product $D \otimes_h S_p$ preserves bipartiteness, provided that D is itself bipartite. Also the labeling \hat{f} , as defined in the proof of Theorem 3.1, preserves the special property if D admits a SSEM labeling. \square

Hence we have introduced a method that allows us to construct EM, SEM and SSEM graphs. The coming sections are mainly, although not exclusively, devoted to apply this method to different families of graph. By doing this, we will find lower bounds for the number of non-isomorphic labelings of certain families of graphs.

4 The union of bipartite graphs

In [12] Figueroa et al. proved that if a (S)EM graph is either bipartite or tripartite, then any odd disjoint union of copies of the graph is also a (S)EM graph. In fact, the proof provided in [12] also applies for SSEM graphs. However, in the paper it is not studied how many (S)(S)EM labelings of such unions exist. In this section we will provide lower bounds for the number of non-isomorphic (S)(S)EM labelings of such unions. We begin by studying the case of trees.

Let \sum_n be the set of all 1-regular digraphs of order n . Consider a digraph $D \in \sum_n$ and let $x \in V(D)$, we denote by $x +_{E(D)} 1$ the only vertex with $(x, y) \in E(D)$.

Lemma 4.1 *Let T be a rooted tree with root $a \in V(T)$. For each function $h : E(T) \longrightarrow \sum_n$ there exist n labelings of T , namely $lab(h)_i, i \in \{1, \dots, n\}$ such that:*

1. $lab(h)_i(x) \neq lab(h)_j(x)$ if $i \neq j$.
2. If $xy \in E(T)$ and $d(a, x) < d(a, y)$ then $(lab(h)_i(x), lab(h)_i(y)) \in h(xy) \forall i$.
3. For all $x \in V(T)$, $\cup_i lab(h)_i(x) = \{1, \dots, n\}$.
4. If $h \neq \hat{h}$ then there exists i with $lab(h)_i \neq lab(\hat{h})_j \forall j$.

Proof.

Fix a function $h : E(T) \longrightarrow \sum_n$. For each $i \in \{1, \dots, n\}$ the labeling $lab(h)_i$ of T defined recursively as follow:

1. $lab(h)_i(a) = i$.
2. Assume that we have labeled the vertices of T up to level $k - 1$, and that z belongs to level k with amb $yz \in E(T)$, then $lab(h)_i(z) = lab(h)_i(y) +_{E_h(yz)} 1$.

At this point we observe that each labeling $lab(h)_i$ is characterized by the label of vertex a . By construction, these labelings meet the two first properties requested in the conclusions of the lemma.

Observe that we can take the vertex x as a root (this proves (3)). If $h \neq \hat{h}$ then let $(r, s) \in E(h(x, y))$ and $(r, s) \notin E(\hat{h}(x, y))$. Let i with $lab(h)_i(x) = r$. Then $s = lab(h)_i(y) \neq lab(\hat{h})_j(y)$. If $i \neq j$ then $i = lab(h)_i(a) \neq lab(\hat{h})_j(a) = j$. \square

Lemma 4.2 *Lemma 4.1 is also true for acyclic graphs.*

Proof.

Let $\{a_1, \dots, a_l\}$ be the roots of l rooted trees. We consider the labelings $lab(h)_i$, that have the following properties: $lab(h)_i(a_j) = i \forall j \in \{1, \dots, l\}$ and the second item of the previous proof. \square

Let F be any acyclic graph. We denote by \vec{F} any digraph such that $und(\vec{F}) = F$.

Theorem 4.1 *Let T be a tree. Consider any function $h : E(\vec{T}) \longrightarrow \sum_n$. Then, $und(\vec{T} \otimes_h \sum_n) = nT$*

Proof.

Fix a function $h : E(\vec{T}) \rightarrow \sum_n$. Also, for the sake of brevity, we will use the notation \vec{T}_h in order to denote the digraph $\vec{T} \otimes_h \sum_n$. Now let us assume that $(x, j)_{n\vec{T}}$ is a vertex of $n\vec{T}$ if and only if $x \in V(\vec{T})$ and $j \in \{1, \dots, n\}$. Also the edges of $n\vec{T}$ are of the form $((x, j)_{m\vec{T}}, (y, j)_{m\vec{T}})$ where $(x, y) \in E(\vec{T})$. Next denote by $(x, j)_{\vec{T}_h}$, where $x \in E(T)$ and $j \in \{1, \dots, n\}$, the vertices of the digraph \vec{T}_h . The arcs of \vec{T}_h are of the form $((x, \text{lab}(h)_j(x))_{\vec{T}_h}, (y, \text{lab}(h)_j(y))_{\vec{T}_h})$ with $(x, y) \in E(\vec{T})$. Notice that the correspondence $(x, \text{lab}(h)_j(x))_{\vec{T}_h} \longleftrightarrow (x, j)_{n\vec{T}}$ is in fact an isomorphism among the digraphs $n\vec{T}$ and \vec{T}_h . The correspondence is a bijection by previous lemma and because of the fact that $((x, \text{lab}(h)_i(x))_{\vec{T}_h}, (y, \text{lab}(h)_j(y))_{\vec{T}_h}) \in E(\vec{T}_h) \Leftrightarrow (x, y) \in E(\vec{T})$ and $i = j \iff ((x, i), (y, i)) \in E(n\vec{T})$. \square

We say that the set $\cup_{x \in V(T)} (x, \text{lab}(h)_i(x))$ is the i^{th} component of $\vec{T} \otimes_h \sum_n$.

In fact Theorem 4.1 can be generalized in the following way:

Theorem 4.2 *Let F be an acyclic graph. Consider any function $h : E(\vec{F}) \rightarrow \sum_n$. Then $\text{und}(\vec{F} \otimes_h \sum_n) = nF$.*

Theorems 3.1 and 4.1 allow us to generate (S)(S)EM labelings of an odd number of copies of a given (S)(S)EM tree T . Given $h : E(\vec{T}) \rightarrow \sum_n$ with $h(E(\vec{T})) \subseteq S_n$, the label of the vertex $(x, \text{lab}(h)_j(x))$ is $\text{lab}(h)(x, \text{lab}(h)_j(x)) = n(x - 1) + \text{lab}(h)_j(x)$ (by Theorem 3.1). Assume that the root of the tree that we take in Lemma 4.1 is labeled by 1. Then the labelings $\text{lab}(h)_j$ satisfy $\text{lab}(h)_j(1) = j \forall j$ and the label $\text{lab}(h)$ satisfies $\forall h \text{lab}(h)(1, \text{lab}(h)_j(1)) = j$.

Let G be a (S)(S)EM graph. We say that two labelings f_1 and f_2 are isomorphic if there exists a G -automorphism, ϕ , such that $\forall x, y \in V(G) \phi(x) = y$ if and only if $f_1(x) = f_2(y)$.

Lemma 4.3 *Let T be any labeled tree, and let $h, \hat{h} : E(\vec{T}) \rightarrow S_n$. Using the previous notation, $\text{lab}(h)$ and $\text{lab}(\hat{h})$ are isomorphic labelings of nT if and only if $\text{lab}(h) = \text{lab}(\hat{h})$.*

Proof.

We see our tree as a rooted tree, and we let the root to be the vertex labeled by 1. We want to show that the automorphism g defined by the rule $g((x, \text{lab}(h)_j(x))) = (y, \text{lab}(h)_l(y))$ if and only if $\text{lab}(h)((x, \text{lab}(h)_j(x))) = \text{lab}(\hat{h})((y, \text{lab}(h)_l(y)))$ is the identity function. The two labelings coincide on the roots of the trees. We also assume that the labels coincide on the vertices which are at distance at most $k - 1$ from the root of its component of the forest. Let $(x, \text{lab}(h)_j(x))$ be a vertex at distance k from the root of its component, and let $(y, \text{lab}(h)_l(y))$ be a vertex with

$lab(h)(x, lab(h)_j(x)) = lab(\hat{h})(y, lab(\hat{h})_l(y))$. Since g is an automorphism, it follows that $(x, lab(h)_j(x)), (y, lab(h)_l(y))$ belong to the same component ($j = l$) and they are at the same distance from the root. On the other hand since $lab(h)(x, lab(h)_j(x)) \in (n(x-1), nx]$ and $lab(\hat{h})(y, lab(\hat{h})_j(y)) \in (n(y-1), ny]$ we have that $x = y$. \square

Corollary 4.1 *Lemma 4.3 is true if we replace the tree T by a forest with components T_1, \dots, T_k , roots a_1, \dots, a_k and we consider the labelings of Lemma 4.2.*

At this point we let $\sharp(n) = |S_n|$. For example, for $n = 7$ we have that $\sharp(7) = 28$. Table 1 shows all possible SEM labeled 2-regular digraphs of order 7, where each component has been oriented cyclically. There are 28 such digraphs;

$C_5 \cup C_1 \cup C_1$	$1 - 4 - 7 - 2 - 6 \cup 3 \cup 5$	two possible orientations
$C_6 \cup C_1$	$1 - 6 - 3 - 2 - 4 - 7 \cup 5$	two possible orientations
$C_6 \cup C_1$	$1 - 4 - 6 - 5 - 2 - 7 \cup 3$	two possible orientations
$C_3 \cup C_3 \cup C_1$	$1 - 5 - 6 \cup 2 - 3 - 7 \cup 4$	four possible orientations
C_7	$1 - 5 - 2 - 6 - 3 - 7 - 4$	two possible orientations
C_7	$1 - 6 - 5 - 3 - 7 - 2 - 4$	two possible orientations
C_7	$1 - 7 - 3 - 6 - 5 - 2 - 4$	two possible orientations
C_7	$1 - 4 - 3 - 7 - 2 - 6 - 5$	two possible orientations
C_7	$1 - 7 - 2 - 3 - 4 - 6 - 5$	two possible orientations
C_7	$1 - 6 - 4 - 7 - 2 - 3 - 5$	two possible orientations
C_7	$1 - 6 - 2 - 3 - 7 - 4 - 5$	two possible orientations
C_7	$1 - 5 - 2 - 3 - 6 - 3 - 7$	two possible orientations
C_7	$1 - 6 - 5 - 4 - 2 - 3 - 7$	two possible orientations

Table 1: Orientation for 1-regular digraphs

Then we obtain the following corollary:

Corollary 4.2 *Let F be a (S)(S)EM acyclic graph of order m with p components, and let n be an odd positive integer. Then the graph nF admits at least $[\sharp(n)]^{(m-p)}$ non-isomorphic (S)(S)EM labelings.*

Proof.

By Corollary 4.1 there exist at least as many (S)(S)EM labelings as functions $h : E(\vec{F}) \rightarrow S_n$. That is to say $[\sharp(n)]^{(m-p)}$ functions. \square

At this point we consider the case of an odd disjoint union of the complete graph on two vertices.

Lemma 4.4 *Two different (S)(S)EM labelings of mK_2 , result into two non-isomorphic (S)(S)EM labelings of $mK_2 \otimes_h S_n$ for any h .*

Proof.

Let $\{e, E\}$ and $\{e, E'\}$ ($1 \leq e, E, E' \leq 2n$) two different labelings of the one component K_2 of mK_2 . When applying the product $mK_2 \otimes_h S_n$ the first labeling generates n different copies of $\{e, E\}$ with labels in the sets $\{n(e-1) + j : j \in \{1, \dots, n\}\}$ and $\{n(E-1) + j : j \in \{1, \dots, n\}\}$ respectively. Also, the second labeling generates n different copies of $\{e, E'\}$ with labels in the sets $\{n(e-1) + j : j \in \{1, \dots, n\}\}$ and $\{n(E'-1) + j : j \in \{1, \dots, n\}\}$. Therefore the labelings are non-isomorphic. \square

Let n be an odd positive integer and denote by $N(n)$ the number of non-isomorphic (S)(S)EM labelings of the graph nK_2 . We observe that by Corollary 4.2 $N(n) \geq [\sharp(n)]$. Then we obtain the following result.

Theorem 4.3 *Let m, n be two odd positive integers. Then the graph $(nm)K_2$ admits at least $\max\{N(n)[\sharp(m)]^n, N(m)[\sharp(n)]^m\}$ non-isomorphic (S)(S)EM labelings.*

Proof.

For every (S)(S)EM labeling of mK_2 we know, by Corollary 4.2, that the graph $n(mK_2)$ admits at least $[\sharp(n)]^m$ non-isomorphic labelings. By Lemma 4.4 we also know that there are at least $N(m)[\sharp(n)]^m$ non-isomorphic labelings of $n(mK_2)$. Finally, we notice that if we interchange the role of n and m we obtain the desired result. \square

Corollary 4.3 *Let l be an odd positive integer, and let the set $B(l) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : mn = l\}$. Then the graph lK_2 admits at least $\max_{(m,n) \in B(l)} \{N(n)[\sharp(m)]^n\}$ non-isomorphic (S)(S)EM labelings*

We enumerate in Table 2 all 16 SSEM labelings of $9K_2$, obtained by combining all SSEM labelings of $3K_2$ with all elements of S_3 .

Next, we turn our attention to the union of bipartite graphs in general.

Theorem 4.4 *Let $G = (V = V_1 \cup V_2, E)$ be a bipartite graph with bipartite sets V_1 and V_2 and let \vec{G} be a digraph such that:*

1. $Und(\vec{G}) = G$.
2. If $(u, v) \in E(\vec{G})$ then $u \in V_1, v \in V_2$.

Also, for every $n \in \mathbb{N}$, let $h : E(\vec{G}) \rightarrow \sum_n$ be a function that assigns the same element of \sum_n to each element of $E(\vec{G})$. Then $\text{und}(\vec{G} \otimes_h \sum_n) = nG$.

Proof.

Denote by (x, j) with $x \in V(G)$ and $j \in \{1, \dots, n\}$ the vertices of nG and $(x, j)(y, j) \in E(nG)$ if and only if $xy \in E(G)$ and consider the graph $\text{und}(\vec{G} \otimes_h \sum_n)$. The isomorphism $i : V(nG) \rightarrow V(G \otimes_h \sum_n)$ is defined by the rule:

$$i(x, j) = \begin{cases} (x, j), & \text{if } x \in V_1 \\ (x, j +_{E(h(x,y))} 1), & \text{if } x \in V_2. \end{cases}$$

□

1	14	15	14	14	14	15	15	15	17	17	17	18	17	18	18	18
2	15	13	15	15	15	13	13	13	18	18	18	16	18	16	16	16
3	13	14	13	13	13	14	14	14	16	16	16	17	16	17	17	17
4	17	17	18	17	18	17	18	18	11	12	11	11	12	12	11	12
5	18	18	16	18	16	18	16	16	12	10	12	12	10	10	12	10
6	16	16	17	16	17	16	17	17	10	11	10	10	11	11	10	11
7	11	11	11	12	12	12	11	12	14	14	15	14	15	14	15	15
8	12	12	12	10	10	10	12	10	15	15	13	15	13	15	13	13
9	10	10	10	11	11	11	10	11	13	13	14	13	14	13	14	14

Table 2: SSEM labelings of $3K_2$

Corollary 4.4 *Let G be a bipartite $(S)(S)EM$ graph and let n be an odd positive integer. Then every labeling of G generates at least $\sharp(n)$ non-isomorphic $(S)(S)EM$ labelings of nG .*

Proof.

The result is an immediate consequence of Theorem 4.4, Theorem 3.1, and Corollaries 3.1 and 3.2. We observe that changing $h : E(G) \rightarrow S_n$ all the labelings that we obtain are non-isomorphic since all the copies of V_1 have the same labels for every h . □

Notice that in all the proofs considered in this section all the components of the 2-regular graphs have been oriented in a cyclic way. Also we always take the same orientation of all the bipartite graphs under consideration. Changing these orientations substantially changes the resulting graph, in general. However we feel that this study should be considered in a future work.

5 Generating labelings for graphs of the form $G \odot K_n$

The goals in this section are to find a lower bound for the number of non-isomorphic SEM labelings of graphs obtained using the corona product.

Lemma 5.1 *Let m be an odd positive integer. Then the graph $mK_{1,n}$ admits at least $[\sharp(m)]^n$ non isomorphic SSEM labelings. Furthermore, if f is one of these labelings, then f assigns an integer in the set $\{1, 2, \dots, m\}$ to the central vertex in each component, and $\min\{f(u) + f(v) : uv \in E(mK_{1,n})\} = \frac{3m+3}{2}$.*

Proof.

The Lemma follows as a particular case of Corollary 4.2. The labeling of Theorem 3.1 assigns $\{1, 2, \dots, m\}$ to the central vertex and the valence given in Lemma 1.5 allows us to calculate the minimum edge induced sum. \square

Let G be a graph of odd order m , and let $\natural(G)$ denote the number of non-isomorphic SEM labelings of G with the property that

$$\max\{f(u) + f(v) : uv \in E(G)\} = \frac{3m + 1}{2}. \quad (1)$$

If G is a graph of even order m , then we denote by $\natural(G)$ the number of non-isomorphic SEM labelings of G with the property that

$$\max\{f(u) + f(v) : uv \in E(G)\} = \frac{3m}{2}. \quad (2)$$

Next we will state and prove a main result of this section.

Theorem 5.1 *Let G be a graph that satisfies equation (1). The graph $H \cong G \odot \bar{K}_n$ admits at least $2 \cdot \natural(G) \cdot [\sharp(m)]^n$ non-isomorphic SEM labelings.*

Proof.

Let f be a SEM labeling of G such that $\max\{f(u) + f(v) : uv \in E(G)\} = \frac{3m+1}{2}$. Next, consider the graph $mK_{1,n}$ with an SSEM labeling that to each central vertex of each component assigns a number in the set $\{1, 2, \dots, m\}$. By Lemma 5.1 we know that there are at least $[\sharp(m)]^n$ non-isomorphic such labelings. At this point, relabel the vertices of G and of $mK_{1,n}$ in such a way that each vertex is named after the label of a SEM labeling and a SSEM labeling of G and $mK_{1,n}$ respectively. Next, construct the graph $H \cong G \odot \bar{K}_n$ by identifying vertex i of $V(G)$ with the vertex i of $V(mK_{1,n})$. Let g be the resulting labeling. Since $\{g(u) + g(v) : uv \in E(G \odot \bar{K}_n)\}$ is a set of consecutive numbers, it follows that g is a SEM labeling. Also, since $mK_{1,n}$ admits at least $[\sharp(m)]^n$ non-isomorphic such SSEM labelings and there are $\natural(G)$ SEM labelings

of G that we can use, it follows that there are at least $\mathfrak{h}(G) \cdot [\mathfrak{h}(m)]^n$ SEM labelings of $H \cong G \odot \bar{K}_n$. Finally, notice that these labelings assign to the vertices with degree strictly greater than 1, the numbers from 1 up to m . Therefore, the complementary labelings are all different, to these labelings just constructed, since they assign the largest labels to the vertices with degree strictly greater than 1. Also all complementary labelings are mutually different. Therefore, we conclude that $H \cong G \odot \bar{K}_n$ admits at least $2 \cdot \mathfrak{h}(G) \cdot [\mathfrak{h}(m)]^n$ non-isomorphic SEM labelings. \square

Next, we will show how the results developed in this section can be applied to different families of graphs. First of all, we consider the set $T \odot \bar{K}_n$, where T is any path-like tree of odd order. In [4], Barrientos proved that path-like trees admit α -labelings. In fact, the labelings exhibit by Barrientos for path-like trees of odd order m , can be transformed into a SEM labelings f satisfying equation (1). Hence, using the previous theorem, we obtain the following result.

Proposition 5.1 *Let T be any path-like tree of odd order m . Then for every α -labeling, $T \odot \bar{K}_n$ admits at least $2 \cdot [\mathfrak{h}(m)]^n$ non-isomorphic SEM labelings.*

In [1], Abraham and Kotzig, found a method to construct α -labelings of paths. In fact, for paths of orders 5, 11, 13, 15 they found 1, 12, 35 and 84 α -labelings respectively. All these α -labelings can be transformed into SEM labelings of paths satisfying equation (1). Hence once again using Theorem 5.1 we obtain:

Proposition 5.2 *Let n be odd. There exist at least $2 \cdot \mathfrak{h}(P_n) \cdot [\mathfrak{h}(m)]^n$ non-isomorphic SEM labelings of $P_n \odot \bar{K}_m$. In particular, there exist at least*

1. $2 \cdot [\mathfrak{h}(5)]^m$ SEM labelings of $P_5 \odot \bar{K}_m$.
2. $24 \cdot [\mathfrak{h}(11)]^m$ SEM labelings of $P_{11} \odot \bar{K}_m$.
3. $70 \cdot [\mathfrak{h}(13)]^m$ SEM labelings of $P_{13} \odot \bar{K}_m$.
4. $168 \cdot [\mathfrak{h}(15)]^m$ SEM labelings of $P_{15} \odot \bar{K}_m$.

Lemma 5.2 ([13],[18]) *Let f be a SEM labeling of G with*

$$\begin{cases} M = \max\{f(u) + f(v) : uv \in E(G)\}. \\ m = \min\{f(u) + f(v) : uv \in E(G)\}. \end{cases}$$

Let $u, v \in V(G) : uv \notin E(G)$ and for which $f(u) + f(v) \in \{M + 1, m - 1\}$, then the new graph G^+ defined as follows

$$\begin{cases} V(G^+) = V(G) \\ E(G^+) = E(G) \cup \{uv\} \end{cases}$$

is a SEM graph.

Thanks to Lemma 5.2 we can convert all the SEM labelings of the paths of odd order, obtained by Abraham and Kotzig in [1] into SEM labelings of odd cycles. Once again, for each labeling f of C_m (m odd) we have that equation (1) is satisfied. Therefore, Proposition 5.2 is also true replacing the paths by cycles.

Theorem 5.2 *Let G be any graph that satisfies equation (2). Assume that H is the graph obtained from G by attaching the same number n of pendant edges to each vertex of G except for the vertex labeled m . Then H admits at least $2 \cdot \mathfrak{h}(G) \cdot [\mathfrak{h}(m)]^n$ non-isomorphic SEM labelings.*

Proof.

Let G be any graph that satisfies the conditions of the hypothesis of the theorem and let \vec{G} be any digraph such that $\text{und}(\vec{G}) = G$. Also let $S_{m-1} = \{D_1, D_2, \dots, D_s\}$. At this point, rename the vertices of \vec{G} and of D_1, D_2, \dots, D_s after the labels of their corresponding SEM labelings. Let $A(\vec{G}) = (g_{ij})$ and $A(D_k) = (d_{ij}^k)$; $k \in \{1, 2, \dots, s\}$ be the adjacency matrices of \vec{G} and D_k respectively. Next, for every $k \in \{1, 2, \dots, s\}$ define the new $(m-1) \times m$ -matrix $E_k = (e_{ij}^k)$ as follows:

$$e_{ij}^k = \begin{cases} d_{ij}^k & \text{if } j < m \\ 0 & \text{if } j = m \end{cases}$$

Let $\beta_{m-1} = \{E_1, E_2, \dots, E_s\}$, and let $\epsilon_1, \epsilon_2, \dots, \epsilon_s$ be n not necessarily distinct elements of β_{m-1} . At this point, we define the new matrix $M(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$ as the following Kronecker product:

$$M(\epsilon_1, \epsilon_2, \dots, \epsilon_s) = (1, \overbrace{0, \dots, 0}^n) \otimes \begin{pmatrix} \epsilon_s \\ \vdots \\ \epsilon_2 \\ \epsilon_1 \\ A(\vec{G}) \end{pmatrix}$$

From the matrix $M(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$ define the square matrix $M^*(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$ consisting of the first $mn + m - n$ rows and columns of $M(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$. Then it is clear that $M^*(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$ is the adjacency matrix of some digraph with underlying graph H . Thus we only need to show that H is a SEM graph. We will show that all such main diagonals with exactly one entry being 1 are consecutive. Let $M^*(\epsilon_1, \epsilon_2, \dots, \epsilon_s) = (m_{ij}^*)$.

We have that $\min\{i+j : m_{ij}^* = 1, m+1 \leq i \leq m+(m-1)\} = \frac{3m}{2} + 1$ and for every $k \in \{1, 2, \dots, n-1\}$, we have that $\min\{i+j : m_{ij}^* = 1, m+k(m-1)+1 \leq i \leq m+(k+1)(m-1)\} = \frac{3m}{2} + 1 + k(m-1)$. Now $\max\{i+j : m_{ij}^* = 1, i \leq m\} = \frac{3m}{2}$, and for every $k \in \{0, 1, 2, \dots, n-1\}$ we have that $\max\{i+j : m_{ij}^* = 1, m+k(m-1)+1 \leq i \leq m+(k+1)(m-1)\} = \frac{3m}{2} + (m-1)(k+1)$. All main diagonals of $M^*(\epsilon_1, \epsilon_2, \dots, \epsilon_s)$ either have all their entries 0, or all the entries are 0 except for exactly one entry which is 1. This proves that the graph H is a (S)EM graph. \square

Some families of graphs that satisfy the hypothesis of the Theorem 5.2 are, for instance, paths and paths like-trees, all of them of even order.

6 Results on 2-regular graphs

In this section we will find lower bounds for the number of non-isomorphic (S)EM labelings of certain types of two regular graphs. We start by introducing the following result:

Theorem 6.1 *Assume that \vec{C}_m is a strong orientation on the cycle C_m . Let h be a function that assigns to each of the arcs of the digraph \vec{C}_m the same strong orientation of the cycle C_n . Then $\text{und}(C_m \otimes_h S_n) = \text{GCD}(m, n)C_{\text{lcm}[m, n]}$*

Proof.

Through this proof the notation $+_i$ will be used in order to denote the sum taken modulo i . Assume that the vertices of \vec{C}_m have been labeled with the elements of the group \mathbb{Z}_m following an increasing orientation. Also assume that the cycle $h(E(\vec{C}_m))$ is the digraph \vec{C}_n labeled with the vertices of $\{1, \dots, n\}$ following an increasing order. Let h^- be the restriction of h to $E(\vec{P}_m) = E(\vec{C}_m) \setminus \{(m-1, 0)\}$, then the graph $\text{und}(\vec{P}_m \otimes_{h^-} S_n) \cong nP_m$ by Theorem 4.1. We label vertex j ($j \in \mathbb{Z}_m$) of the i^{th} copy ($i \in \mathbb{Z}_n$) of \vec{P}_m by $(j, i +_n j)$. We observe that $((x, y), (x', y')) \in E(\vec{C}_m \otimes_h S_n)$ if and only if

$$\begin{cases} x' = x +_m 1 \\ y' = y +_n 1 \end{cases}$$

Notice that in order to convert the digraph $\vec{P}_m \otimes_{h^-} S_n$ into the digraph $\vec{C}_m \otimes_h S_n$, we need to introduce arcs of the form $((m-1, i +_n m-1), (0, i +_n m))$. Fix a component i , then we introduce arcs going to the vertices of the set $\{(0, i +_n m), (0, i +_n 2m), \dots, (0, i +_n (k-1)m)\}$ with $i +_n km \equiv i \pmod{n}$, thus $k = \frac{\text{lcm}[m, n]}{m}$. Hence there are k copies of \vec{P}_n that form a cycle of order $km = \text{lcm}[m, n]$, which implies that the number of copies of the resulting digraph is $\frac{mn}{km} = \text{GCD}(m, n)$. Therefore $\text{und}(\vec{C}_m \otimes_h S_n) \cong \text{GCD}(m, n)C_{\text{lcm}[m, n]}$ \square

Corollary 6.1 *The graph $GCD(m, n)C_{lcm[m, n]}$ admits at least two times as many (S)EM labelings as the graph C_n .*

Proof.

The results is an immediate consequence of Theorem 6.1, Theorem 3.1 and Corollary 3.1. \square

The following number theoretical result provides a way of generating (S)EM labelings for the graph $GCD(m, n)C_{lcm[m, n]}$.

Lemma 6.1 *Let $\alpha = \gamma\beta$ be a natural number with $\beta = \prod_{j=1}^k \beta_j^{i_j}$ where $\beta_j^{i_j}$ are prime numbers. Then the number of different couples of natural numbers with greatest common divisor γ and least common multiple α is given by 2^{k-1} .*

Proof.

It is clear that α, γ is one of these couples. The other couples are obtained dividing α by $\beta_{j_1}^{i_{j_1}} \cdots \beta_{j_l}^{i_{j_l}}$ and multiplying γ by the same expression, where $\{j_1, \dots, j_l\} \subseteq \{1, \dots, k\}$. There are as many of these expressions as subsets of the set $\{1, \dots, k\}$. That is to say, there are 2^k such expressions. However by this procedure each couple appears twice, therefore we have 2^{k-1} such couples. \square

Lemma 6.1 allows us to generate (S)EM labelings in 2^{k-1} different ways. However, we still do not know in general how to prove that all such labelings constructed in this way are in fact non-isomorphic, although we strongly suspect that they are. In fact, the only result that we have in this direction is stated and proved next.

Proposition 6.1 *Assume that $G \in S_n$ and $H \in S_m$ where $GCD(m, n) = 1$ and $m, n > 1$. Also consider the functions $h : E(G) \rightarrow S_m$ and $\hat{h} : E(H) \rightarrow S_n$. Then $G \otimes_h S_m \neq H \otimes_{\hat{h}} S_n$.*

Proof.

We proceed by contradiction. Assume to the contrary that there exist functions $h : E(G) \rightarrow S_m$ and $\hat{h} : E(H) \rightarrow S_n$ and graphs $G \in S_n$, with adjacency matrix (g_{ij}) and $H \in S_m$ such that $G \otimes_h S_m = H \otimes_{\hat{h}} S_n$ and that $1 < n < m$. We reduce the adjacency matrix of $G \otimes_h S_m$ to a vector $B = (b_i)$ where $B(i) = b_i$ if and only if the position (i, b_i) in the adjacency matrix $G \otimes_h S_m$ is 1. We observe that $1 \leq b_i \leq mn$ and that if $i \neq j$ then $b_i \neq b_j$. Let $j = km + l$ ($0 \leq k \leq (n-1)$, $1 \leq l < m$) with $b_j = mn$. We have:

- *If $0 \leq k < (n-1)$ then we have that $|b_{(k+1)m+1} - b_{(k+1)m}| < n-1 < m-1$ and therefore $\{b_i : (k+1)m \leq i < (k+2)m\} = (m(n-2), m(n-1)]$ and $g_{k+2, n-1} = 1$. Since $g_{k+1, n} = 1$, it follows that this contradicts the fact that G is SEM.*

- If $k = n - 1$ then $g_n = 1$, and in G there is an arc with induced sum $2n > \frac{3n+1}{2}$ when $n > 1$, that contradicts Corollary 1.1.

□

Finally, in [21] Muntaner proved that every SEM labeling of a 2-regular graph can be transformed into a SSEM labeling of a 1-regular graph. Furthermore, two SEM labelings of two 2-regular graphs are transformed into two non-isomorphic SSEM labelings of 1-regular graphs. Therefore, the results of this section can also be used in order to construct SSEM labelings of 1-regular graphs.

7 Graphs with chords

Through this section, the symbol \mathbf{G} will be used to denote the set of all (p, q) -graphs which are either 1-regular, 2-regular or 3-regular. If H is a graph, we denote by $\sharp(H)$ the set of all non-isomorphic SEM labelings of H . Also let G be any (p, q) -graph with $q \neq \binom{p}{2}$. We denote by G^C the set $\{(p, q + 1)$ -graphs $H : G$ is a subgraph of $H\}$

Next we introduce the following new labeling, that will prove to be very useful to develop the results of this section.

Let G be any (p, q) -graph. Then, we say that a bijective function $f : V(G) \rightarrow \{1, 2, \dots, p\}$ is a jump of G , if the set $\{f(u) + f(v) : uv \in E(G)\}$ is of the form

$$\{\beta, \beta + 1, \dots, \beta + l, \beta + l + 2, \beta + l + 3, \dots, \beta + q\}$$

where β is a fixed element of \mathbf{N} and l is a fixed element of the set $\{0, \dots, q - 2\}$.

Lemma 7.1 *Let $G \in \mathbf{G}$. Then G cannot admit both, a SEM labeling and a jump.*

Proof.

Assume to the contrary, that G admits both, a SEM labeling M and a jump J . Consider the sets $S_M = \{M(u) + M(v) : uv \in E(G)\} = \{\alpha, \alpha + 1, \dots, \alpha + q - 1$ for some $\alpha \in \mathbf{N}\}$ and $S_J = \{J(u) + J(v) : uv \in E(G)\} = \{\beta, \beta + 1, \dots, \beta + l, \beta + l + 2, \beta + l + 3, \dots, \beta + q : \beta \in \mathbf{N}$ and $l \in \{1, \dots, q - 2\}\}$.

Since G is an r -regular graph, it follows that $\sum_{x \in S_M} x = \sum_{y \in S_J} y$. Now, $\sum_{x \in S_M} x = q\alpha + \sum_{i=1}^{q-1} i$, $\sum_{y \in S_J} y = q\beta + \sum_{i=1}^q i - (l + 1)$ and hence, $q(1 + \beta - \alpha) = l + 1$. Therefore $q|(l + 1)$ which is impossible since $(l + 1) \in \{1, \dots, q - 1\}$. □

Lemma 7.2 *Let a (p, q) -graph $G \in \mathbf{G}$ and $H \in G^C$ be graphs that admit SEM labelings. If f is a SEM labeling of H , then $f|_{V(G)}$ can be extended to an exactly one SEM labeling of G .*

Proof.

Let H be any SEM graph in the set G^C , and let f be a SEM labeling of H . Hence the set $S = \{f(x) + f(y) : xy \in E(H)\}$ is a set of $q + 1$ consecutive numbers. If $uv \in E(H) \setminus E(G)$ and $S \setminus \{f(u) + f(v)\}$ is not a set of q consecutive integers, then the function $f|_{V(G)}$ is a jump of G in contradiction with Lemma 7.1. \square

Theorem 7.1

$$\sum_{H \in C_{2k+1}^C} \#(H) = \begin{cases} k \cdot \#(C_{2k+1}) & \text{if } k \text{ is even} \\ (k-1) \cdot \#(C_{2k+1}) & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

We know by Lemma 7.2 that if f is a SEM labeling of $H \in C_{2k+1}^C$ then $f|_{V(C_{2k+1})}$ is also a SEM labeling of C_{2k+1} . Next, let M be a SEM labeling of C_{2k+1} and consider the set $S = \{M(u) + M(v) : uv \in E(C_{2k+1})\}$. Let $\delta = \min(S)$, $\gamma = \max(S)$. Also, by Lemma 1.5 we obtain that the valence of M , val_M , is $5k + 4$. Now, $val_M = \delta + (p + q)$ and $\gamma = \delta + (q - 1)$ thus, we have that $\delta = k + 2$ and $\gamma = 3k + 2$.

Next, notice that any two vertices u, v of $V(C_{2k+1})$ with $M(u) + M(v) \in \{k + 1, 3k + 3\}$ cannot be adjacent. Also if k is odd there are exactly $k - 1$ such pairs and if k is even there are exactly k such pairs.

Therefore, since any SEM labeling of any graph in C_{2k+1}^C is obtained from a SEM labeling of C_{2k+1} , we obtain the desired result. \square

Theorem 7.2

$$\#(P_4 \cup (2k - 1)K_2) = \begin{cases} (3k + 2) \cdot \#((2k + 1)K_2) & \text{if } k \text{ is even.} \\ (3k + 1) \cdot \#((2k + 1)K_2) & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

It is well known that the graph $(2k + 1)K_2$ is SEM. Furthermore, by Lemma 1.5 it is easy to obtain that the valence of any SEM labeling of $(2k + 1)K_2$ is $val = 9k + 6$.

Next, let M be any SEM labeling of $(2k + 1)K_2$, and let S be the set defined by $S = \{M(u) + M(v) : uv \in E((2k + 1)K_2)\}$. Let $\delta = \min S$ and $\gamma = \max S$, then $val = \delta + (p + q)$, $\delta = 3k + 3$ and $\gamma = \delta + (q - 1) = 5k + 3$. Hence all pairs of vertices

in $V((2k+1)K_2)$ with $M(u) + M(v) \in \{3k+2, 5k+4\}$ are not adjacent in $(2k+1)K_2$. If k is even, there are exactly $3k+2$ of such pairs, and if k is odd there are exactly $3k+1$ of such pairs. Next, notice that $((2k+1)K_2)^C = \{P_4 \cup (2k-1)K_2\}$ and that every SEM labeling of $(2k+1)K_2$ induces exactly one SEM labeling of $P_4 \cup (2k-1)K_2$, if we join exactly one pair of vertices u, v such that $M(u) + M(v) \in \{3k+2, 5k+4\}$. Now, if we put all the above together with Lemma 7.2 the Theorem follows. \square

8 A new related problem

Graph labelings appeared as an alternative way of attacking the well known Kotzig-Ringel conjecture [23] which states that the complete graph K_{2n+1} can be decomposed into $(2n+1)$ isomorphic trees of size n . Since then many applications of graph labelings have appeared. For instance we can find graph labelings showing up in radars, x-ray crystallography, coding theory, etc. For a detailed exposition of graph labeling applications, the interested reader can consult [5], [6], [7] and [25]. Also in algorithmics, we can find graph labelings showing up, since in [20] it has been proved that the problem of deciding whether or not a given graph admits a harmonious labeling is NP-complete. A similar result for equitable labelings has been established in [8].

In this section we introduce a new problem, that has been motivated by trying to find lower bounds for the number of non-isomorphic SEM labelings of the graph $(2k+1)K_2$. In fact, the problem, as stated in this section, is even more general, and the bounds found in this paper for the number of non-isomorphic SEM labelings of the graph $(2k+1)K_2$, solves just a very small portion of the problem.

Open problem: Assume that we have a set of n weights such that each weight is a natural number. Also there are k persons that are interested to transport the set of weights to another location so that each person takes k_i weights at a time and only one trip is made by each person. How many ways are there to distribute the weights among the persons so that each person carries the same total weight?

In this paper we have found lower bounds for the case in which $n = 3l$; l is an odd, non prime, natural number and there are l participants, such that each participant takes exactly three weights at a time.

References

- [1] J. Abraham and A. Kotzig, Exponential lower bounds for the number of graceful numbering of snakes. Congr. Numer. **72** (1990) 163-174.
- [2] B. D. Acharya and S. M. Hegde, Arithmetic graphs, J. Graph Theory, **14** (1990) 257-299.

- [3] M. Baca, Y. Lin, F. A. Muntaner-Batle, Super edge antimagic labelings of the path-like trees, *Utilitas Math.* To appear.
- [4] C. Barrientos, Difference vertex labelings, Ph.D. Thesis. Universitat Politècnica de Catalunya (2004).
- [5] G.S. Bloom and S.W. Golomb. Numbered complete graph, unusual rulers and assorted applications, in *Theory and Applications of Graphs, Lecture Notes in Math.* **642**, Springer.
- [6] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, *Proc. IEEE*, **65** (1977) 562-570.
- [7] G.S. Bloom and D.F. Hsu, On graceful digraphs and a problem in network addressing. *Congressus Numerantium*, **35** (1982) 91-103.
- [8] N. Cairnie and K. Edwards. The computational complexity of cordial and equitable labeling, *Discrete Math.*, **216** (2000) 29-34.
- [9] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, second edition. Wadsworth & Brooks/Cole Advanced Books and Software, Monterey (1986).
- [10] H. Enomoto, A. Lladó, T. Nakamigawa and G. Ringel, Super edge-magic graphs, *SUT J. Math.* **34** (1998), 105-109.
- [11] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, The place of super edge-magic labelings among other classes of labelings, *Discrete Math.* **231**(1-3) (2001), 153-168.
- [12] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On super edge-magic labelings of certain disjoint unions of graphs, *Australas. J. Combin.* **32** (2005), 225-242.
- [13] R.M. Figueroa-Centeno, R. Ichishima and F.A. Muntaner-Batle, On super edge-magic graphs, *Ars Combin.* **64** (2002), 81-95.
- [14] R. Frucht and F. Harary, In the corona of two graphs, *Aequationes Math.*, **4** (1970) 322-325.
- [15] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **5** (2004), #DS6.
- [16] M.Gardner, Mathematical games: The graceful graphs of Solomon Golomb, or how to number a graph parsimoniously, *Scientific American*, **226**, 3 (1972) 108-112; **226** 4 (1972) 104; **226** 6 (1972) 118.
- [17] S. W. Golomb, How to number a graph, in *graph theory and computing*, R.C. Read, ed., Academic Press, New York (1972) 23-37.

- [18] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, *SIAM J. Alg. Discrete Math.***1**(1980) 382-404.
- [19] A. Kotzig and A. Rosa, Magic valuations of finite graphs, *Canad. Math. Bull.* **13** (1970), 451-461.
- [20] A.K. Krishnaa, M.S. Dulawat and G.S. Rathore, Computational complexity in decision problems, presented in conf. of Raj. Parishad, Dec. 14-15, 2001, Udaipur, India.
- [21] F.A. Muntaner-Batle, Special super edge-magic labelings of bipartite graphs, *J. Combin. Math. Combin. Comput.* **39** (2001) 107-120.
- [22] F.A. Muntaner-Batle, Magic graphs, Ph.D. thesis. Universitat Politècnica de Catalunya (2001).
- [23] G. Ringel, Problem 25, in *Theory of Graphs and its Applications. Proc. Symposium Smolenice 1963 Praga* (1964) 162.
- [24] A. Rosa. On certain valuations of the vertices of a graph. *Theory of graphs* (Internet. Symposium, Rome, July 1966).
- [25] M. Sutton, Summable graphs labellings and their applications, Ph.D. thesis, Dep. Computer Science, The University of Newcastle, 2001.
- [26] W.D. Wallis, *Magic Graphs*, Birkhäuser, Boston (2001).