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Synthesis of 4-Frictionless Optimal Grasps of Polygonal Objects *

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Abstract

The paper proposes a new approach to the problem of determining optimal form-closure grasps of polygonal objects using four frictionless contacts. A new set of grasp parameters is determined based only on the directions of the applied forces. These parameters are used to obtain a new formulation of the necessary and sufficient condition for the existence of four-finger frictionless form-closure grasps, as well as to determine the optimal grasp. Given a set of contact edges, using an analytical procedure a solution that is either the optimal one or is very close to it is obtained (only in this second case an iterative procedure is needed to find a root of a non-linear equation). This procedure is the used for an efficient determination of the optimal grasp on the whole object. The algorithms have been implemented and numerical examples are shown.

1 Introduction

Grasping and manipulation of objects using multi-finger mechanical hands has become a field of great interest in the two last decades. Good overviews of the state of the art in this field including the related problems were done by Bicchi (2000) and Shimoga (1996).

The obtention of grasps capable of ensuring the immobility of the object despite external disturbances has been a topic extensively studied in the literature. These grasps are characterized by one of the following properties: form-closure (the position of the fingers ensures the object immobility) or force-closure (the forces applied by the fingers ensure the object immobility) (Bicchi, 1995). Mishra et al. (1987) enunciated a necessary and sufficient condition that a form-closure grasp must satisfy. Nguyen (1988) determined a set of geometrical conditions that four frictionless contacts and two frictional contacts must satisfy to obtain force-closure grasps of 2D polygonal objects. Ponce and Faverjon (1995) and Ponce et al. (1997) extended Nguyen's approach to three finger grasps of 2D polygonal objects and to four finger grasps of 3D polyhedral objects, respectively, using a sufficient condition for force-closure. Li et al. (2003) enunciated necessary and sufficient conditions for three finger force-closure grasps of 2D and 3D objects. These works are specific for a given number of fingers. For any number of fingers, Chen and Burdick (1993) developed a qualitative test to determine if a set of contact points allows a force-closure grasp and Liu (1998) and Li et al. (2002) proposed algorithms to determine the set of all the force-closure grasps of 2D polygonal objects.

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Finding the optimal grasp among all the force/form-closure grasps is a common problem in grasp and fixture planning, and several criteria have been proposed for the grasp quality evaluation (Suárez et al., 2006). Some of these criteria consider only the geometrical aspect of the grasp (i.e., the immobility of the object is assured but without considering the magnitudes of the forces applied by the fingers). In this line, Ponce and Faverjon (1995) proposed a grasp quality criterion based on the minimization of the distance between the object's center of mass and the geometric center of the grasping points, criterion that was used for the grasp synthesis by Ding et al. (2001), among others. A more complete problem is the determination of the optimal force/form-closure grasp considering constraints on the fingers forces (a review of the most used constraints was done by Mishra (1995)). In this line, Ferrari and Canny (1992), and Kirkpatrick et al. (1992) proposed a criterion based on the maximum wrench that the grasp can safely resist in any direction considering that the forces applied by the fingers are limited, which is known as the criterion of the maximum ball. This criterion has been frequently used to evaluate force-closure grasps generated with different strategies (Pollard, 1996; Borst et al., 1999; Miller et al., 2003), but although these approaches provide good grasps they do not generate the optimum. The synthesis of optimal grasps considering this criterion and with a reasonable computational cost is a problem of great interest and it remains largely unsolved. The main drawback of the general approaches developed until now is the computational cost, implying that these approaches must be simplified when they are applied in systems with time constraints (Liu et al., 2004). Variations of this criterion were also used to obtain general procedures. Trinkle (1992) presented a variant that allows to obtain the final grasp with linear programming and Zhu et al. (2003) used a different norm to compute the module of the wrenches, nevertheless, the convergence to the optimal grasp is not guaranteed. In the field of fixture design it is common the use of heuristics and exhaustive search procedures to obtain the final fixture design (Kumar et al., 2000; Tan et al., 2004).

The quality according to the criterion of the maximum ball depends on the position of the origin of the reference system and requires the definition of a metric in the wrench space. A solution with physical sense is the selection of the center of mass as the origin of the reference system for torque measurement and the selection of the radius of gyration for the metric adjust between forces and torques. Other solutions were proposed by Mirtich and Canny (1994) decoupling forces and torques, and Teichmann (1996) defining an invariant metric. A comparison of the criteria proposed by Ferrari and Canny (1992), Mirtich and Canny (1994) and Ponce and Faverjon (1995) was done by Bone and Du (2001).

1.1 Contributions of this work

This paper presents a new procedure to determine the optimal form-closure grasp (hereafter FC grasp) of 2D polygonal objects using four frictionless contacts and the quality measure of the maximum ball. This implies that the final grasp is a basic solution with respect to the number of fingers (four is the minimum number of frictionless contacts that allow a FC grasp of 2D objects (Markenscoff et al., 1990)) and a conservative solution with respect to friction, whose existence increases the robustness of the solution (even when the exact friction coefficient is not known). The obtention of a procedure to solve this specific problem with a reasonable computational cost was presented as an open problem in the literature (Mishra, 1995) and, up to where we know, it has not been solved yet. The approach developed here follows a previous work (Cornellà and Suárez, 2003), where the particular case of determining the optimal position of a fourth finger given the positions of the other three was solved in a fully analytical way. In this paper, the determination of the optimal position of the four fingers is deeply analyzed and, as a result, a procedure to determine the optimal grasp without involving hard iterative search procedures is obtained. Specifically, the main contributions of this paper are:

Grasp analysis: Determination of a new set of intrinsic grasp parameters that depend only on the object shape. These parameters are used to obtain a new necessary and sufficient condition for the existence of a FC grasp and to identify different cases for the optimal grasp determination.

Grasp synthesis: Development of an efficient procedure to determine the optimal grasp in each case considering one of the most popular quality measures. This procedure obtains analytically a solution that is either the optimal solution or is very close to it. In this second case, an iterative procedure is needed to find a root of a non-linear equation.

The authors are not aware of any previous work that analytically determines the optimal grasp of 2D objects using the quality measure of the maximum ball. The approach presented by Jia (1995) identified equivalent cases for the optimal grasp although not all of them were solved. Here, a faster identification of each case is presented as well as the methodology to solve all of them. The proposed approach to determine FC grasps with four frictionless contacts is also of practical interest in the design of fixtures for 2D polygonal objects and some particular cases of 3D polyhedral objects (Brost and Goldberg, 1996; Wallack and Canny, 1996; Stappen et al., 2000).

The main general assumptions considered in this work are that the contact between the object and the fingertip is punctual, and that the forces applied by the fingers act only against the object boundary (positivity constraint).The vertices of the object are not considered as possible contact points even when concave vertices may be actually considered for grasping purpose.

There is no constraint regarding the number of fingers per edge. Thus, in this approach, it is possible to consider two fingers on the same edge (for polygonal objects a minimum of three edges must be contacted to allow a FC grasp).

1.2 Paper layout

The rest of the paper is organized as follows. Section 2 describes the constraint on the finger forces and the grasp quality measure used in this work. Section 3 presents the main algorithm to obtain the optimal grasp on the whole object and Section 4 presents an efficient procedure to obtain the optimal grasp on a given set of edges, which is the main contribution of the paper. Different examples of the proposed methodology are included in Section 5. Some concluding remarks and possible future research lines to extend this work are pointed out in Section 6. The paper also includes three Appendices: in the first one, the quality measure is further studied and some of its properties are detailed; in the second one, some geometrical reasonings that reduce the computational cost of the proposed algorithms are presented; and, in the third one, the proofs of all the propositions stated in the paper are included.

2 Grasp quality measure

The following subsections formally present the constraint on the finger forces and the quality measure used in this work.

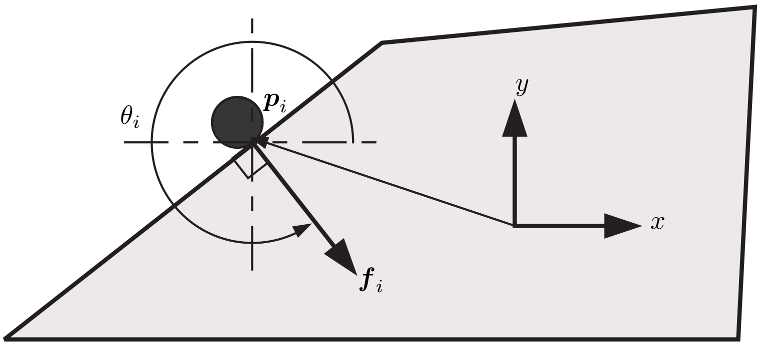


Figure 1: Force $\boldsymbol{f}_i$ applied by finger i at the contact point $\boldsymbol{p}_i$.

2.1 Constraint on the finger forces

The forces applied by the fingers can be subject to different constraints, depending on the characteristics of the grasp (an exhaustive analysis of the most used constraints with their physical and geometrical meanings was done by Mishra (1995)). The constraint used in this work is that the total force exerted by the fingers is limited, for instance, due to a maximum available power for all the finger actuators.

Let $\boldsymbol{p}_i$ be a contact point on the object boundary described with respect to the object center of mass, and let $\boldsymbol{f}_i = \alpha_i \widehat{\boldsymbol{f}}_i$, with $\alpha_i \geq 0$ and $\|\widehat{\boldsymbol{f}}_i\| = 1$, be the force exerted by the finger i at $\boldsymbol{p}_i$. In the absence of friction, $\widehat{\boldsymbol{f}}_i$ is normal to the object boundary, i.e. $\widehat{\boldsymbol{f}}_i = (\cos\theta_i \ \sin\theta_i)^T$, where θ_i indicates the inward direction normal to the contact edge (Fig. 1). The force exerted by each finger produces a torque with respect to the object center of mass $\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i$, and the components of $\boldsymbol{f}_i$ and τ_i form the wrench vector $\boldsymbol{\omega}_i = (\boldsymbol{f}_i^T \ \lambda\tau_i)^T$, where λ is a constant that defines the metric of the wrench space. In order for the metric to have a physical meaning in terms of energy, λ is considered to be the radius of gyration of the object. The proposed approach is valid independently of the value of λ; thus, for simplicity, from now on it is considered $\lambda = 1$ and therefore it is removed from the equations.

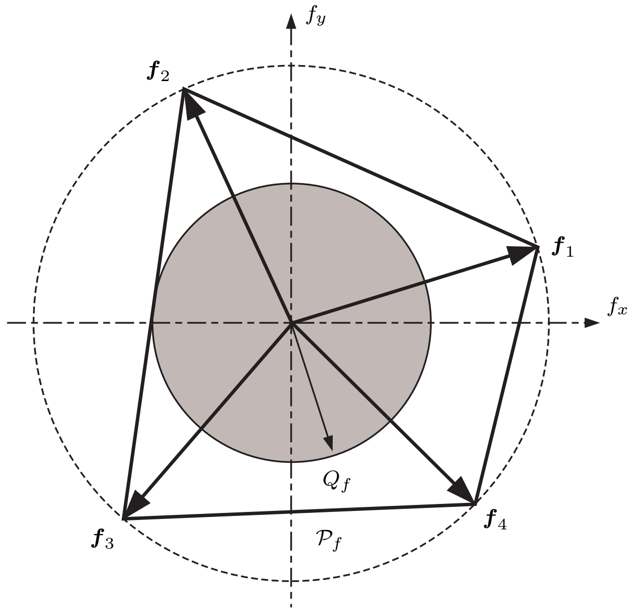
Considering that the total force exerted by the four fingers is limited by α_{max}, the resultant force $\boldsymbol{f}$ on the object is given by

$$\boldsymbol{f} = \sum_{i=1}^{4} \alpha_i \widehat{\boldsymbol{f}}_i = \alpha \widehat{\boldsymbol{f}} \qquad \text{with} \quad \sum_{i=1}^{4} \alpha_i \leq \alpha_{max} \tag{1}$$

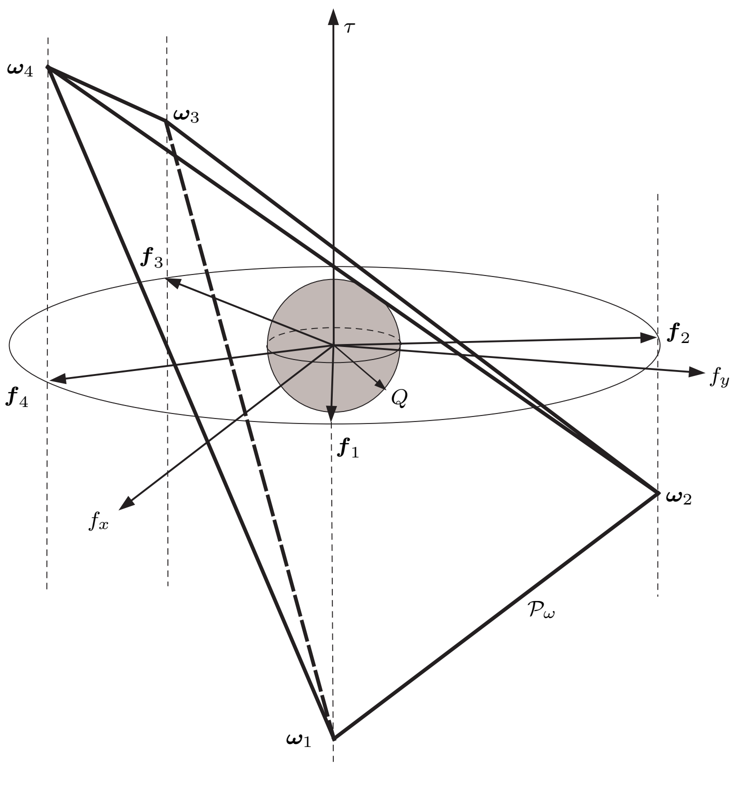
Geometrically, this constraint implies that the fingers can apply forces on the object that produce a resultant inside the polygon $\mathcal{P}_f$ defined in the force space as (Fig 2a),

$$\mathcal{P}_f = ConvexHull(\bigcup_{i=1}^{4}\{\boldsymbol{f}_i\}) \quad \text{with } \boldsymbol{f}_i = \alpha_{max}\widehat{\boldsymbol{f}}_i \tag{2}$$

Analogously, the resultant wrench applied on the object lies inside the polyhedron $\mathcal{P}_\omega$ defined



(a)



(b)

Figure 2: Constraint and quality measure on: (a) the force space (polygon $\mathcal{P}_f$ and circumference of radius Q_f); (b) the wrench space (polyhedron $\mathcal{P}_\omega$ and sphere of radius Q).

in the wrench space as (Fig 2b),

$$\mathcal{P}_\omega = \mathit{ConvexHull}(\bigcup_{i=1}^{4}\{\boldsymbol{\omega}_i\}) \quad \text{for } \boldsymbol{f}_i = \alpha_{max}\widehat{\boldsymbol{f}}_i \tag{3}$$

In a FC grasp, $\mathcal{P}_f$ and $\mathcal{P}_\omega$ must contain the origin of the force and wrench space respectively (Mishra et al., 1987).

In the rest of the paper, for simplicity and without loss of generality, we consider $\alpha_{max} = 1$ and therefore $\boldsymbol{f}_i$ will refer always to the maximum (unitary) possible applied force. Regardless of the application point on the object boundary, $\boldsymbol{f}_i = (\cos\theta_i \ \sin\theta_i)^T$ always represents a vector in the force space.

2.2 Quality measure definition

The quality Q of a FC grasp is given by the maximum wrench that the finger forces can generate in any direction of the wrench space (Ferrari and Canny, 1992), i.e.

$$Q = \min_{\boldsymbol{\omega}\in\partial\mathcal{P}_\omega} \|\boldsymbol{\omega}\| \tag{4}$$

where $\partial\mathcal{P}_\omega$ is the boundary of $\mathcal{P}_\omega$.

Geometrically, the quality measure Q is the radius of the maximum ball centered at the origin of the wrench space and fully contained inside $\mathcal{P}_\omega$, which is determined by the shortest distance from the origin to the faces of $\mathcal{P}_\omega$. Let D_{ijk} be the distance from the origin of the wrench space to the plane defined by $\boldsymbol{\omega}_i$, $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ (the wrenches produced by fingers i, j and k). Then, the quality measure can also be expressed as,

$$Q = \min_{i,j,k\in\{1,\ldots,4\}, i\neq j\neq k} \{D_{ijk}\} \tag{5}$$

The same concept can be applied to define a quality measure considering only the force space (Mirtich and Canny, 1994) as,

$$Q_f = \min_{\boldsymbol{f}\in\partial\mathcal{P}_f} \|\boldsymbol{f}\| \tag{6}$$

where $\partial\mathcal{P}_f$ is the boundary of $\mathcal{P}_f$.

Fig. 2 shows the geometrical interpretation of the constraints $\mathcal{P}_f$ and $\mathcal{P}_\omega$ and the quality measures Q_f and Q, respectively. Note that $\mathcal{P}_f$ is the projection of $\mathcal{P}_\omega$ on the force space and it is not possible to obtain a sphere fully contained in $\mathcal{P}_\omega$ with radius larger than Q_f. Therefore, Q_f is an *upper bound* for Q.

Some interesting properties of the quality measure that are useful in the followings sections are detailed in Appendix A.

3 Main algorithm

This section presents the procedure to obtain the optimal grasp over the whole object. The following terms will be used.

Definition 1. The *edge-optimal grasp*, G_e, is the set of four contact points that generates the optimal grasp on a given set of three or four contact edges. ◇

Definition 2. The *object-optimal grasp*, G_o, is the set of four contact points that generates the optimal grasp over the whole object (i.e. G_o is the best G_e). $\diamond$

Given a combination of three or four edges where the fingers will contact, the direction θ_i, $i = 1, ..., 4$, of the force applied by each finger is known, and from them $\boldsymbol{f}_i = (\cos\theta_i \ \sin\theta_i)^T$ and then $\mathcal{P}_f$ are directly obtained. Therefore, Q_f can be easily computed from equation (6) once the contact edge of each finger is given.

The object-optimal grasp G_o over the whole object is obtained with the following algorithm, which uses Q_f of a set of contact edges as an upper bound for the quality Q of any grasp produced on those edges.

Algorithm 1. (Computation of G_o). Let C be the set of possible different combinations of three and four edges:

1. Initialize $Q = 0$
2. Determine the subset C' of C with the combinations of edges that satisfy $\mathbf{0} \in \mathcal{P}_f$.
3. Compute Q_f for each combination of edges in C'.
4. Order C' from better to worse Q_f.
5. For each combination of edges in C' and following the order established in step 3, do:
   - 5.1) Determine G_e and its quality Q'.
   - 5.2) If $Q < Q'$ then $G_o = G_e$ and $Q = Q'$.
   - 5.3) If Q is greater that the value of Q_f of the next combination of edges then exit the loop.
6. Return G_o and its quality Q.

$\diamond$

The determination of G_e in step 5.1 is the critical operation in terms of computational cost. The rest of the paper deals with an efficient procedure to solve this problem, which is the key contribution of this work.

4 Optimal grasp for a set of contact edges

Considering a given combination of contact edges (either three or four), the goal now is the determination of G_e on these edges.

Geometrically, the determination of G_e is equivalent to determine the polyhedron $\mathcal{P}_\omega$ that contains the largest sphere centered at the origin (refer again to Fig. 2), which is completely defined by its four vertices $\boldsymbol{\omega}_i = (\boldsymbol{f}_i^T \ \tau_i)^T$, with $i = 1, ..., 4$. Given a set of contact edges, the directions θ_i and, therefore, the components $\boldsymbol{f}_i$ of the four wrenches are known, the problem being the determination of the values of τ_i that maximize Q (i.e. the radius of the sphere). Since $\boldsymbol{f}_i$ is known, the values of τ_i determine the positions of the contact points $\boldsymbol{p}_i$ on the corresponding edges. For this reason, from now on we will frequently refer to the problem of finding the optimal contact points $\boldsymbol{p}_i$ as the problem of finding the optimal values of τ_i.

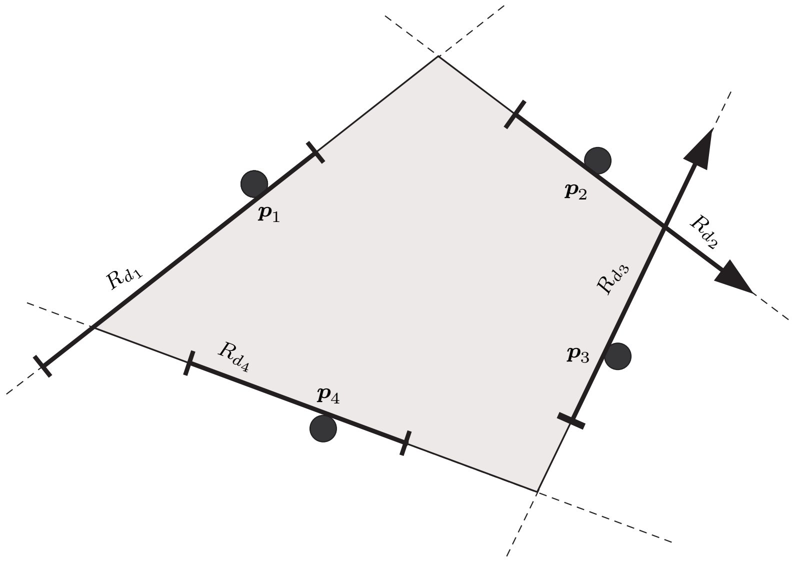


Figure 3: Example of a FC grasp and of the directional ranges.

4.1 Determination of Form-Closure Grasps

Based on the univocal relation between the contact point $\boldsymbol{p}_i$ and the torque component τ_i, the following concepts are defined.

Definition 3. The *real range* of τ_i, R_i, is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ applied at any point $\boldsymbol{p}_i$ on the contact edge E_i, i.e.

$$R_i = \{\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i \; / \; \boldsymbol{p}_i \in E_i\} \tag{7}$$

◇

Definition 4. The *directional range* of τ_i, R_{d_i}, is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ at any point $\boldsymbol{p}_i$ on the supporting line e_i of the contact edge E_i, that allows a FC grasp given any other three wrenches $\boldsymbol{\omega}_h$, $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ applied on the object, i.e.

$$R_{d_i} = \{\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i \; / \; \boldsymbol{p}_i \in e_i \text{ and } \boldsymbol{0} \in \mathcal{P}_\omega\} \tag{8}$$

with $\mathcal{P}_\omega$ described by equation (3). ◇

Note that R_{d_i} may include values of τ_i that are not physically possible due to the real edge length (R_{d_i} is obtained considering that the contact edge has infinite length).

From these two definitions, four contact points $\boldsymbol{p}_i$, with $i = 1, ..., 4$ allow a FC grasp if $\tau_i \in R_i \cap R_{d_i}$. Fig. 3 shows an example of a FC grasp and the directional ranges associated to each contact point.

Stappen et al. (2000) defined as critical grasps those grasps that separate the FC grasps from the non FC grasps and, therefore, that have $Q = 0$. Taking into account that R_{d_i} is defined for three other fixed wrenches, it is a continuous set whose extremes τ_{i_m} are the values of τ_i that produce a critical grasp, i.e., $\tau_i = \tau_{i_m}$ implies $Q = 0$ and either $\tau_i = \tau_{i_m} + \delta$ or $\tau_i = \tau_{i_m} - \delta$ (but not both at the same time) produce a FC grasp, with δ arbitrarily small.

Geometrically, τ_{i_m} implies that the origin of the wrench space belongs to a face of $\mathcal{P}_\omega$ whose vertices are $\boldsymbol{\omega}_i$ and two other wrenches $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$, i.e.

$$\mathbf{0} = \alpha_i \boldsymbol{\omega}_i + \alpha_j \boldsymbol{\omega}_j + \alpha_k \boldsymbol{\omega}_k \tag{9}$$

with $\alpha_i > 0$, $\alpha_j, \alpha_k \geq 0$ and $\alpha_i + \alpha_j + \alpha_k = 1$. Solving equation (9) for $\boldsymbol{\omega}_i$ and expanding its components results:

$$\cos\theta_i = \beta_{i,jk}\cos\theta_j + \beta_{i,kj}\cos\theta_k \tag{10}$$

$$\sin\theta_i = \beta_{i,jk}\sin\theta_j + \beta_{i,kj}\sin\theta_k \tag{11}$$

$$\tau_{i_m} = \beta_{i,jk}\tau_j + \beta_{i,kj}\tau_k \tag{12}$$

where $\beta_{i,jk} = -\frac{\alpha_j}{\alpha_i} \leq 0$ and $\beta_{i,kj} = -\frac{\alpha_k}{\alpha_i} \leq 0$ (note that $\beta_{i,jk}$ and $\beta_{i,kj}$ can not be simultaneously null because $\cos(\theta_i)$ and $\sin(\theta_i)$ can not be simultaneously null).

Given two known wrenches $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$, the corresponding extreme of R_{d_i} can be determined as follows:

1. Solving $\beta_{i,jk}$ and $\beta_{i,kj}$ from equations (10) and (11) as:

$$\beta_{i,jk} = \frac{\sin(\theta_i - \theta_k)}{\sin(\theta_j - \theta_k)} \tag{13}$$

$$\beta_{i,kj} = \frac{\sin(\theta_j - \theta_i)}{\sin(\theta_j - \theta_k)} \tag{14}$$

2. If $\beta_{i,jk} \leq 0$ and $\beta_{i,kj} \leq 0$ then the τ_{i_m} resulting from equation (12) is an extreme of R_{d_i} that produces $Q = 0$ (if either $\beta_{i,jk} > 0$ or $\beta_{i,kj} > 0$, the resulting τ_{i_m} from equation (12) makes that the plane defined by $\boldsymbol{\omega}_i$, $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ contains the origin, but with the origin outside the face of $\mathcal{P}_\omega$).

The exact determination of R_{d_i} is only possible when the other three applied wrenches are known (i.e. the positions of the other three contact points are given). Nevertheless, the number of finite extremes can be determined knowing how many pairs $\beta_{i,jk}$ and $\beta_{i,kj}$ from equations (13) and (14), respectively, have non-positive values for $i, j, k \in \{1, ..., 4\}$. Thus, the number of extremes of each directional range depends only on the directions of the applied forces. Taking into account the number of finite extremes, the directional range is classified in one of the following two types:

Limited: $R_{d_i} = [\tau_{i_1}, \tau_{i_2}]$, τ_{i_1} and τ_{i_2} being two finite extremes where $Q = 0$ (e.g., R_{d_1} and R_{d_4} in Fig. 3).

Infinite: $R_{d_i} = (-\infty, \tau_{i_1}]$ or $R_{d_i} = [\tau_{i_1}, \infty)$, τ_{i_1} being the unique finite extreme where $Q = 0$ while the quality for $\tau_i \to \pm\infty$ is a finite value L (L is given by equation (40) in the properties of the quality function in Appendix A) (e.g., R_{d_2} and R_{d_3} in Fig. 3).

Proposition 1. Given the three or four edges where the four fingers will contact and, therefore, the directions θ_i of the applied forces, the number of infinite directional ranges is:

General case: If all the angles between the applied forces are different from π, there are two infinite directional ranges that correspond to the torques generated by the two forces that lie between the negated of the other two (Fig. 4a and Fig. 4b).

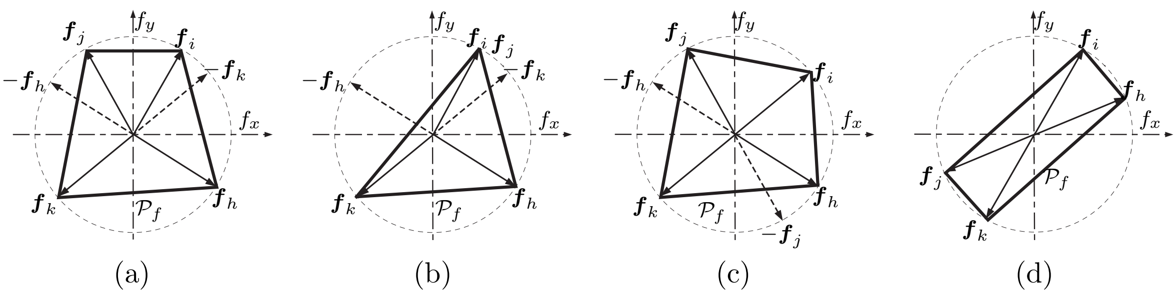


Figure 4: Examples of the determination of the types of directional ranges from the applied forces: a) General case: R_{d_i} and R_{d_j} are infinite and R_{d_h} and R_{d_k} are limited; b) General case (with two fingers on the same edge): R_{d_i} and R_{d_j} are infinite and R_{d_h} and R_{d_k} are limited; c) Particular case (with two opposite forces): R_{d_k} is limited and R_{d_h}, R_{d_i} and R_{d_j} are infinite; d) Particular case (with two pairs of opposite forces): all the directional ranges are infinite.

Particular cases: If the angle between two forces is π, there are three infinite directional ranges corresponding to the torques generated by the other two forces and the force that lies between them (Fig. 4c), and if the angles between two pairs of forces are π, the four directional ranges are infinite (Fig. 4d). $\diamond$

From Proposition 1, it always exists two wrenches whose force components define two consecutive vertices of $\mathcal{P}_f$ and whose torque components have infinite directional ranges. These wrenches take an special relevance in order to establish the following necessary and sufficient condition that a FC grasp must satisfy.

Proposition 2. (**Necessary and sufficient condition**) Four frictionless contacts allow a FC grasp iff for $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ defining two consecutive vertices of $\mathcal{P}_f$ and τ_i and τ_j having infinite directional ranges, the following condition is satisfied

$$sign(\Gamma_i) \neq sign(\Gamma_j) \tag{15}$$

where

$$\Gamma_\rho = \beta_{\rho,hk}\tau_h + \beta_{\rho,kh} - \tau_\rho \tag{16}$$

with $\rho \in \{i, j\}$ and $\beta_{\rho,hk}$ and $\beta_{\rho,kh}$ being determined from equations (13) and (14). $\diamond$

This necessary and sufficient condition can be interpreted as follows. When $\Gamma_\rho = 0$ equation (16) is equivalent to equation (12), implying that $\tau_\rho = \tau_{\rho 1}$ is the unique finite extreme of R_{d_ρ}. Then, equation (15) establishes that τ_i has to be greater than τ_{i_1} while τ_j has to be smaller than τ_{j_1}, or viceversa, in order to obtain a FC grasp (how greater or smaller τ_i and τ_j are with respect to τ_{i_1} and τ_{j_1} is irrelevant for the necessary and sufficient condition).

4.2 Optimal Grasp Cases

The four contact points define the polyhedron $\mathcal{P}_\omega$ in the wrench space, which has four faces and the grasp quality Q is the distance from the origin to one or more of these faces. Jia (1995) established a relation between the number of faces that are at a distance Q of the origin and the number of contact points that lie on an extreme of an edge, classifying the optimal grasp into one of the four possible cases:

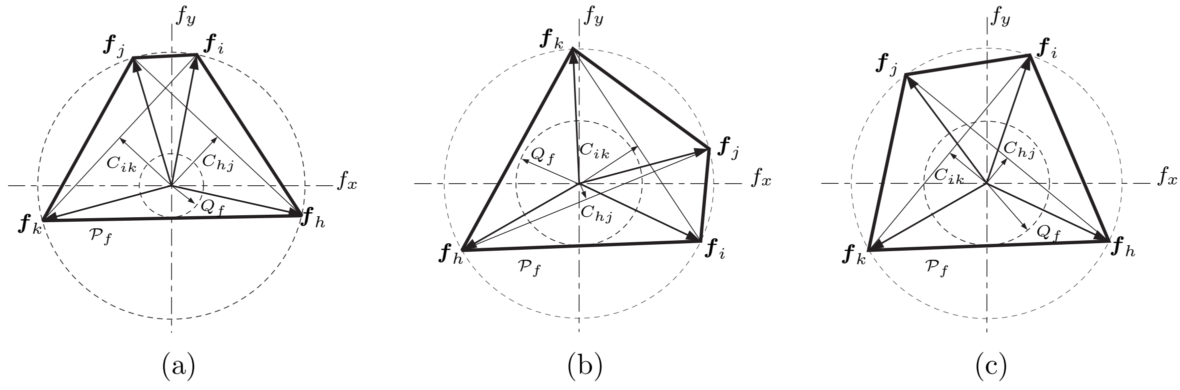


Figure 5: Examples of the internal bounds and of the three cases in the determination of the directional-optimal grasp of Proposition 4 (a) $C_{hj} \geq Q_f$ and $C_{ik} \geq Q_f$; (b) $C_{hj} < Q_f$ and $C_{ik} \geq Q_f$; (c) $C_{hj} < Q_f$ and $C_{ik} < Q_f$.

Case 1: If Q is the distance to one face of $\mathcal{P}_\omega$, then the four contact points lie on the extremes of the edges.

Case 2: If Q is the distance to two faces of $\mathcal{P}_\omega$, then at least two contact points lie on the extremes of the edges.

Case 3: If Q is the distance to three faces of $\mathcal{P}_\omega$, then at least one contact point lie on an extreme of the edge.

Case 4: If Q is the distance to the four faces of $\mathcal{P}_\omega$, then there may be no contact point lying on an extreme of the edge.

The approach presented by Jia (1995) determines the number of contact points that lie on an extreme of the edge, which simplify the problem of computing the optimal grasp. Nevertheless, given a set of contact edges, this approach does not determine which cases can really exist and which contact points lie on extremes of the edges, implying that all the possible combinations have to be checked. Moreover, the approach only solves the first and second cases.

In order to reduce computations, some useful parameters are introduced here; they are based on the directions of the given contact edges and allow to identify which of the four cases are possible and which contact points lie on extremes of the edges.

Definition 5. The *internal bounds*, C_{hj} and C_{ik}, of a FC grasp are the distances from the origin of the force space to each one of the segments determined by two non-consecutive vertices of $\mathcal{P}_f$ (e.g. $\overline{\boldsymbol{f}_h\boldsymbol{f}_j}$ and $\overline{\boldsymbol{f}_i\boldsymbol{f}_k}$ in Fig. 5). ◇

Definition 6. The *directional-optimal grasp*, G_d, is the set of four points on the supporting lines of some given grasping edges that generate the optimal grasp (i.e. the length of the edges is not considered and only the direction of the edges is relevant). ◇

Note that the points that determine G_d may not lie on the actual object boundary and, therefore, G_d can actually be unreachable.

Proposition 3. Let $\boldsymbol{\omega}_h$, $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$ be three known wrenches (i.e. three wrenches produced by three contact points already placed on some given edges of the object) and let $\boldsymbol{\omega}_k$ be a wrench whose torque component τ_k is unknown. The optimal value τ_{d_k} (without considering the real range R_k) that produces the optimal grasp, can be analytically determined knowing the upper bound Q_f, the internal bounds C_{hj} and C_{ik}, and the type of the directional range R_{d_k}, according the following cases:

1. If R_{d_k} is infinite and $C_{hj} \geq Q_f$, then
   $\tau_{d_k} \rightarrow \pm\infty$ according to R_{d_k}
2. Else (i.e. R_{d_k} is limited or $C_{hj} < Q_f$)
   (a) If $C_{ik} \geq Q_f$, then τ_{d_k} is the solution of:
   $$D_{hik} = D_{hjk} \tag{17}$$
   where D_{hik} and D_{hjk} are the distances from the origin of the wrench space to the faces of $\mathcal{P}_\omega$ defined by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}$ and $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$, such that the triangles defined by $\{\boldsymbol{f}_h, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ and $\{\boldsymbol{f}_h, \boldsymbol{f}_j, \boldsymbol{f}_k\}$ intersect with the circumference of radius Q_f in the force space.
   (b) Else (i.e. $C_{hj} < Q_f$) τ_{d_k} is the solution of:
   $$D_{hik} = D_{hjk} \tag{18}$$
   $$D_{hik} = D_{ijk} \tag{19}$$
   $$D_{hjk} = D_{ijk} \tag{20}$$
   where D_{hik}, D_{hjk} and D_{ijk} are the distances from the origin of the wrench space to the faces of $\mathcal{P}_\omega$ defined by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}$, $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$ and $\{\boldsymbol{\omega}_i, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$ (i.e. the three faces of $\mathcal{P}_\omega$ that contain $\boldsymbol{\omega}_k$).

$\diamond$

This Proposition refines the results presented by Cornellà and Suárez (2003).

In order to obtain G_d, Proposition 3 is applied considering all the possible relations between the upper bound, the internal bounds and the types of directional ranges for the four contact points, obtaining the following result.

Proposition 4. Let $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_i$ be the wrenches whose force components determine the upper bound Q_f, and let $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ be the other two wrenches. The directional-optimal grasp, G_d, can be determined according to the values of Q_f and the internal bounds C_{hj} and C_{ik} as follows:

- If $C_{hj} \geq Q_f$ and $C_{ik} \geq Q_f$, then
  $\tau_{d_j} \rightarrow \pm\infty$ and $\tau_{d_k} \rightarrow \mp\infty$
  τ_{d_h} and τ_{d_i} are determined from:
  $$Max \quad D_{hij} \tag{21}$$
  $$\text{subject to} \quad D_{hij} = D_{hik} \tag{22}$$
- If $C_{hj} < Q_f$ and $C_{ik} \geq Q_f$, then
  $\tau_{d_j} \rightarrow \pm\infty$
  τ_{d_h}, τ_{d_i} and τ_{d_k} are determined from:
  $$Max \quad D_{hij} \tag{23}$$
  $$\text{subject to} \quad D_{hij} = D_{hik} = D_{hjk} \tag{24}$$

- If $C_{hj} < Q_f$ and $C_{ik} < Q_f$, then
  τ_{d_h}, τ_{d_i}, τ_{d_j} and τ_{d_k} are determined from:

$$Max \quad D_{hij} \tag{25}$$

$$\text{subject to} \quad D_{hij} = D_{hik} = D_{hij} = D_{ijk} \tag{26}$$

◇

Fig. 5 shows examples of the force directions that produce each case in Proposition 4. Note that although the type of directional range is useful in Proposition 3, it is not really necessary for the computation of G_d when the four contact points are unknown.

The optimization problems presented in Proposition 4 are unbounded, since the optimal grasp is obtained when the torques tends to infinite (satisfying in each case the corresponding constraints). In order to obtain the reachable optimal grasp, the positions of some optimal contact points lie on extremes of the edges, as it is stated in the following proposition.

Proposition 5. If $\tau_{d_i} \rightarrow \pm\infty$, then the optimal reachable torque is the extreme of R_i closest to τ_{d_i}. ◇

Note that when $C_{hj} < Q_f$ and $C_{ik} < Q_f$ there is also at least one point on an extreme (otherwise the optimization problem is unbounded), but it is not possible to determine which one.

From Propositions 4 and 5, the use of the upper bound and the internal bounds (parameters that depend only on the directions of the applied forces) allows an easy identification of which optimal contact points for sure lie on the extremes of the edges, and which of the distances from the origin to the faces of $\mathcal{P}_\omega$ are equal to Q in the optimal case. Then, one of the cases from those presented by Jia (1995) is also identified.

Since the real range of all the contact points have not been considered yet, this optimal case may not be actually reachable, this implies that the reachable optimal solution will have other contact points lying on an extreme of the edges. Then, in this situation, all the cases with more contact points than those initially identify are also possible and must be considered in the search of G_e.

4.3 Computation of G_e

The edge-optimal grasp, G_e, was introduced in Definition 1 (Section 3) as the set of four contact points that generates the optimal grasp on some given contact edges. On the wrench space, the determination of G_e is equivalent to the determination of four wrenches $\boldsymbol{\omega}_{e_i} = (\boldsymbol{f}_i^T \ \tau_{e_i})^T$, $i = 1, ..., 4$, that fix the vertices of the polyhedron $\mathcal{P}_\omega$ to contain the largest possible sphere with $\tau_{e_i} \in R_i$ (i.e. τ_{e_i} being actually reachable).

From Propositions 4 and 5, the optimal positions of some points lie on extremes of the edges while the optimal positions of the others are the solution of one of the described optimization problems. These optimization problems can be expressed in a generic form as follows:

$$Max \quad D_{hij} \tag{27}$$

$$\text{subject to} \quad \boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{28}$$

where $\boldsymbol{\mathcal{C}}_s$ is a constraint vector that includes $s = 1, ..., 3$ constraints depending on the optimization problem that is considered. Note that the number of unknown torques is always $s + 1$.

Since the constraints of the optimization problem defined by equations (27) and (28) are equalities, this problem can be translated into a system of equations using the Lagrange Theorem (Luenberger, 1973). Let $\boldsymbol{\mathcal{L}} = [\mathcal{L}_1 ... \mathcal{L}_s]^T$ be the Lagrange multipliers vector, the solution of the optimization problem can be determined by solving the following system of equations:

$$\nabla D_{hij} + \boldsymbol{\mathcal{L}}^T \nabla \boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{29}$$

$$\boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{30}$$

where ∇ is the gradient operator. Since there are s constraints and $s+1$ unknown torques, equations (29) and (30) represent a system of $2s+1$ equations with $2s+1$ unknowns (including the torques and the Lagrange multipliers).

Equation (29) represents $s+1$ linear equations with respect to the Lagrange multipliers. Since the determination of the Lagrange multipliers is not necessary, an evaluation function $\mathcal{F}$ can be obtained from equation (29) by eliminating the Lagrange multipliers. For instance, considering the optimization problem described by equations (21) and (22) with two unknown torques, $\mathcal{F}$ is:

$$\mathcal{F} = \frac{\frac{\partial D_{hij}}{\partial \tau_h}}{\frac{\partial (D_{hij} - D_{hik})}{\partial \tau_h}} - \frac{\frac{\partial D_{hij}}{\partial \tau_i}}{\frac{\partial (D_{hij} - D_{hik})}{\partial \tau_i}} \tag{31}$$

Equation (30) represents s non-linear equations with respect to $s+1$ unknown torques. Analytically, it is possible to solve a maximum of two constraints with two unknowns (see Appendix B for details). Taking into account this mathematical characteristic and the evaluation function $\mathcal{F}$, the following algorithm allows to efficiently determine the edge-optimal grasp G_e on a given set of edges.

Algorithm 2. (Computation of G_e). Given a set of contact edges, G_e can be determined with the following steps:

1. Determine Q_f, C_{hj} and C_{ik}.
2. Obtain the s constraints that form $\boldsymbol{\mathcal{C}}_s$ and the contact points whose optimal positions lie on extremes of the edges (Propositions 4 and 5).
3. Depending on s, do:
   (a) If $s = 1$ or $s = 2$, solve $\boldsymbol{\mathcal{C}}_s = \mathbf{0}$ from eq. (30) for, respectively, each of the four and eight systems resulting from fixing the position of each unknown torque on an extreme of the corresponding edge.
   (b) If $s = 3$, solve the four resulting subsystems of two constraints of $\boldsymbol{\mathcal{C}}_s$ resulting from fixing the positions of each pair of unknown torques on two extremes of the corresponding edges.
4. As a result of step 3:
   (a) If at least one of the computed sets of torques is reachable, take as initial reachable solution that with largest Q.
   (b) If none of the sets of torques is reachable:
      i. If there is only one unknown torque, its optimal value is on the edge extreme closest to the value computed in step 3. Then, G_e has all the contact points lying on extremes of the edges and the algorithm ends. Return G_e.

ii. Else fix the position of each unknown torque on an extreme of an edge and obtain new constraints $\boldsymbol{\mathcal{C}}_s$ applying Proposition 3 for the remaining unknown torques (note that $\boldsymbol{\mathcal{C}}_s$ is independent of the selected extremes). Return to step 3.

5. Obtain the evaluation function $\mathcal{F}$ and evaluate the initial reachable solution.

6. If $\mathcal{F} \neq 0$, determine in which direction the contact points fixed on extremes of the edges in step 3 have to be moved in order to make $\mathcal{F} \rightarrow 0$:

   (a) If the points have to be moved inside the edge, an iterative procedure is applied in order to obtain the solution that satisfies $\boldsymbol{\mathcal{C}}_s = \mathbf{0}$ and $\mathcal{F} = 0$.

   (b) Else the initial reachable solution can not be improved.

7. If $s = 3$, determine in which direction the contact points fixed on extremes of the edges in step 3 have to be moved in order to make the distances from the origin to the four faces of $\mathcal{P}_\omega$ be the same.

   (a) If the points have to be moved inside the edges, an iterative procedure is applied in order to obtain this solution.

   (b) Else the initial reachable solution can not be improved

8. Return as G_e the best of the results computed in steps 4a, 6a or 7a.

◇

As a difference from the approach proposed by Jia (1995), where the cases 3 and 4 were not solved, Algorithm 2 is complete, since it always finds the optimal grasp taking into account all the possible cases. Moreover, this algorithm is also really efficient, since in many cases the initial reachable solution obtained in step 4 either is G_e or is very close to it, completely avoiding or at least decreasing the number of iterations in steps 6 and 7. Even in this case, these are not hard iterative procedures since they are function of only one torque and they can be easily solved using the Bolzano theorem.

5 Examples

Numerical examples of the proposed methodology are presented in this section using the object shown in Fig. 6. The initial data of the object are the directions normal to the edges and the real ranges R_i of possible actual torques (Table 1). Since the optimal grasp has always at least one contact point on an extreme of the real range, in order to avoid placing a contact point on a vertex of the object, the real ranges were slightly reduced considering a security distance from the vertices of the object.

The Algorithm 1 described in Section 3 is applied to obtain the object-optimal grasp G_o. Since the object has eight edges, the total number of possible sets of three and four edges is 238. There are 95 sets whose $\mathcal{P}_f$ contains the origin. Considering the upper bounds Q_f of these sets, only 26 of them have been evaluated by Algorithm 1 to obtain G_o.

The following examples shows the determination of G_o as well as the determination of the edge-optimal grasp G_e for other combinations of edges that were also computed when Algorithm 1 was applied. The optimal grasp was also computed using the brute force method taking 50 sample points per edge and evaluating all the possible contact combinations. The results were always coincident (up to the sample resolution).

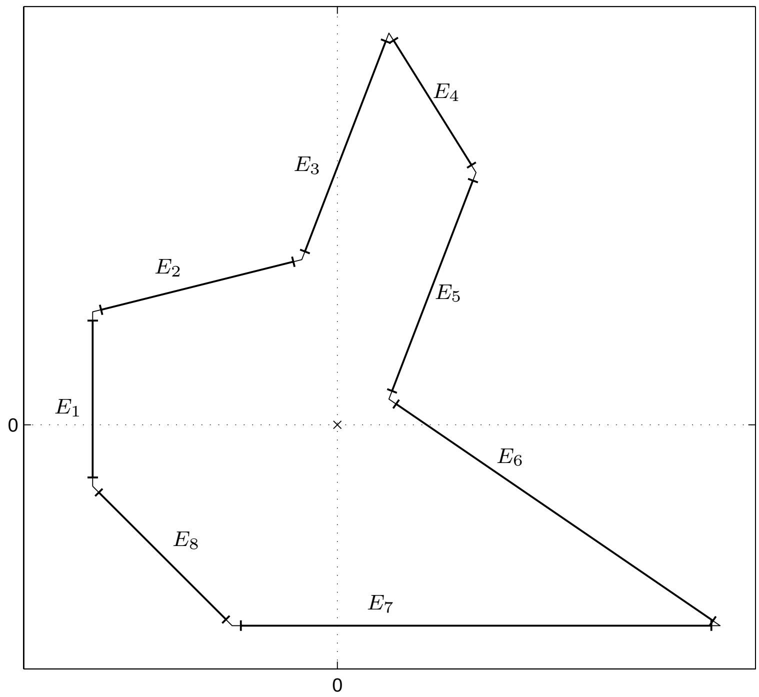


Figure 6: Polygonal object used in the examples.

Table 1: Initial object data

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
Normal direction (θ_i)	0	4.9574	5.9160	3.7002	2.7744	4.1123	1.5708	0.7854
Minimum torque (τ_{min_i})	-0.5979	0.0179	-2.1544	0.8559	0.2943	-2.4130	-0.5539	-0.6937
Maximum torque (τ_{max_i})	0.3021	1.1548	-0.8615	1.6993	1.5872	-0.2108	2.1461	0.3376

In all the examples the contact points are numbered such that the normal forces define consecutive vertices of $\mathcal{P}_f$ and the upper bound is determined by $\overline{\boldsymbol{f}_1\boldsymbol{f}_2}$.

Example 1 (edges E_1, E_4, E_6 and E_7). The edge-optimal grasp, G_e, on this set of edges is the object-optimal grasp, G_o. This is the eleventh evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_6$, $\boldsymbol{p}_2 \in E_1$, $\boldsymbol{p}_3 \in E_7$ and $\boldsymbol{p}_4 \in E_4$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4665$, $C_{13} = 0.2956$, $C_{24} = 0.2756$.

2. $C_{13} < Q_f$ and $C_{24} < Q_f$. Then, $s = 3$,
   $\boldsymbol{\mathcal{C}}_s = (D_{123} - D_{124} \;\; D_{123} - D_{134} \;\; D_{123} - D_{234})^T$ and it is not possible to determine which contact points lie on an extreme of the edge.

3. $s = 3$, then the four subsystems of two constraints of $\boldsymbol{\mathcal{C}}_s$ are solved fixing the positions of

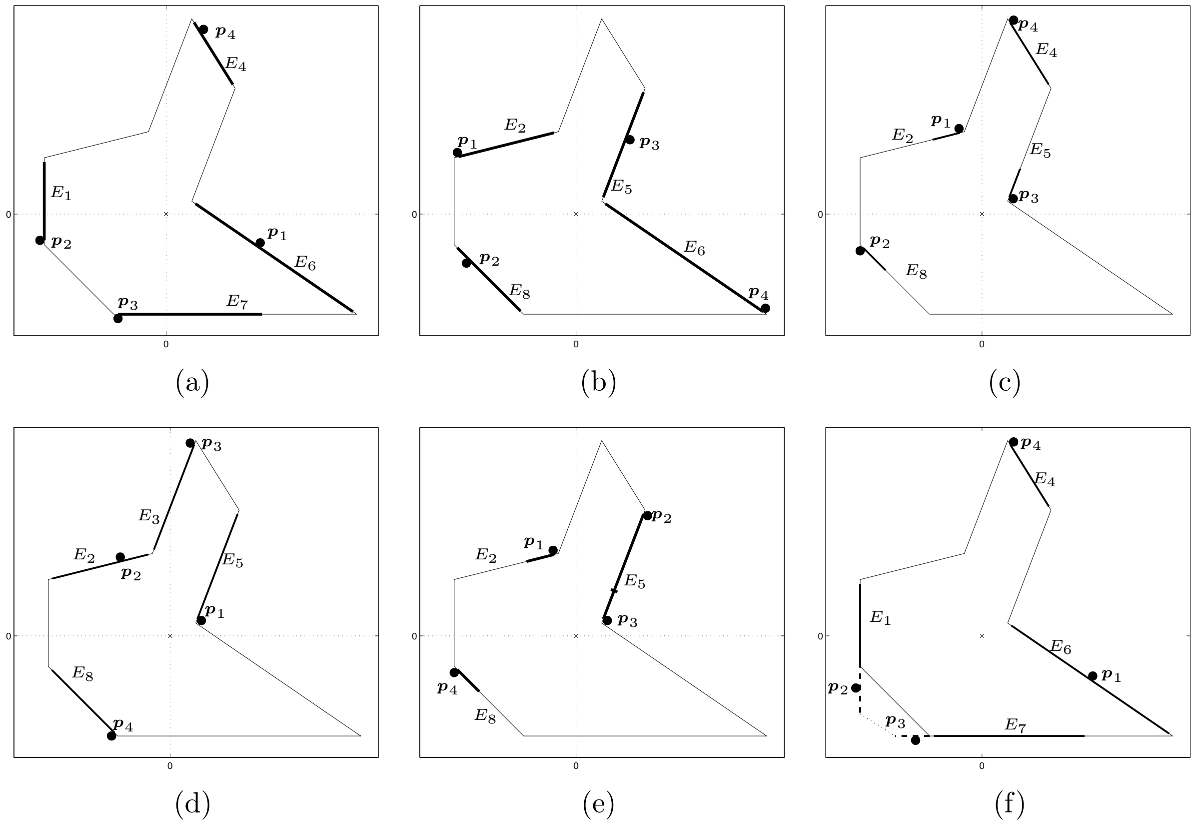


Figure 7: Edge-optimal grasps and the intersection of the directional range with the real range for each contact point (bold segments); Case (a) is the object-optimal grasp; Case (f) considers a slight different object.

two unknown contacts on extremes of the edges.

4. Step 3 produced reachable solutions, and the best one is produced when $\tau_2 = \tau_{max_2}$ and $\tau_3 = \tau_{min_3}$. In this case:

$$\tau_1 = -1.0819; \quad \tau_2 = 0.3021;$$
$$\tau_3 = -0.5539; \quad \tau_4 = 1.5742.$$

with $Q = D_{124} = D_{134} = D_{234} = 0.4040$ and $D_{123} = 0.4309$.

5. Evaluation of the initial reachable grasp in $\mathcal{F}$: If τ_2 is fixed, then $\mathcal{F} = 0.0062$. If τ_3 is fixed, then $\mathcal{F} = -0.0349$.

6. $\mathcal{F} \to 0$ for values greater than τ_{max_2} or smaller than τ_{min_3}. Then, the solution of step 4 can not be improved.

7. The distances to the four faces of $\mathcal{P}_\omega$ tend to be equal for values greater than τ_{max_2}. Then the solution can not be improved.

8. As a result, G_e is the grasp obtained in step 4.

Fig. 7a shows the resulting contact points.

Example 2 (edges E_2, E_5, E_6 and E_8). The quality of G_e on this set of edges is close to the quality of G_o. This is the fourth evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2$, $\boldsymbol{p}_2 \in E_8$, $\boldsymbol{p}_3 \in E_5$ and $\boldsymbol{p}_4 \in E_6$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4927$, $C_{13} = 0.4612$, $C_{24} = 0.0925$.

2. $C_{13} < Q_f$ and $C_{24} < Q_f$. Then, $s = 3$,
   $\boldsymbol{\mathcal{C}}_s = (D_{123} - D_{124} \;\; D_{123} - D_{134} \;\; D_{123} - D_{234})^T$ and it is not possible to determine which contact points lie on an extreme of the edge.

3. $s = 3$, then the four subsystems of two constraints of $\boldsymbol{\mathcal{C}}_s$ are solved fixing the positions of two unknown contacts on extremes of the edges.

4. Step 3 produced reachable solutions, and the best one is produced when $\tau_1 = \tau_{max_1}$ and $\tau_4 = \tau_{min_4}$. In this case,

   $$\tau_1 = 1.1548; \quad \tau_2 = -0.4940;$$
   $$\tau_3 = 1.0218; \quad \tau_4 = -2.4130.$$

   with $Q = D_{123} = D_{124} = D_{234} = 0.3999$ and $D_{134} = 0.6120$.

5. Evaluation of the initial reachable grasp in $\mathcal{F}$: If τ_1 is fixed, then $\mathcal{F} = -0.0222$.
   If τ_4 is fixed, then $\mathcal{F} = -0.0001$.

6. $\mathcal{F} \to 0$ for values greater than τ_{max_1} or smaller than τ_{min_4}. Then, the solution of step 4 can not be improved.

7. The distances to the four faces of $\mathcal{P}_\omega$ tend to be equal for values smaller than τ_{min_4}. Then the solution can not be improved.

8. As a result, G_e is the grasp obtained in step 4.

Fig. 7b shows the resulting contact points.

Example 3 (edges E_2, E_4, E_5 and E_8). This set of edges produces one of the worst cases in the determination of the optimal grasp because none of the constraints of the optimization algorithms can be satisfied. This is the third evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2$, $\boldsymbol{p}_2 \in E_8$, $\boldsymbol{p}_3 \in E_5$ and $\boldsymbol{p}_4 \in E_4$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4927$, $C_{13} = 0.4612$, $C_{24} = 0.1132$.

2. $C_{13} < Q_f$ and $C_{24} < Q_f$. Then, $s = 3$,
   $\boldsymbol{\mathcal{C}}_s = (D_{123} - D_{124} \;\; D_{123} - D_{134} \;\; D_{123} - D_{234})^T$ and it is not possible to determine which contact points lie on an extreme of the edge.

3. $s = 3$, then the four subsystems of two constraints of $\boldsymbol{\mathcal{C}}_s$ are solved fixing the positions of two unknown contacts on extremes of the edges.

4. None of the results computed in step 3 is reachable. Then, the number of constraints included in $\boldsymbol{\mathcal{C}}_s$ is reduced, obtaining the best solution when all the contact points lie on extremes of the edges. In this case,

$$\tau_1 = 0.0179; \quad \tau_2 = -0.6937;$$
$$\tau_3 = 0.2943; \quad \tau_4 = \ \ 1.6993.$$

with $Q = D_{123} = 0.0986$, $D_{124} = 0.2145$, $D_{134} = 0.3842$ and $D_{234} = 0.3281$.

As a result, this grasp is G_e and the algorithm ends.

Fig. 7c shows the resulting contact points.

Example 4 (edges E_2, E_3, E_5 and E_8). This is the fourteenth evaluated set of edges considering the order based on on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_5$, $\boldsymbol{p}_2 \in E_2$, $\boldsymbol{p}_3 \in E_3$ and $\boldsymbol{p}_4 \in E_8$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4612$, $C_{13} = 0$, $C_{24} = 0.4927$.

2. $C_{13} < Q_f$ and $C_{24} > Q_f$. Then, $s = 2$,
   $\boldsymbol{\mathcal{C}}_s = (D_{123} - D_{124} \ \ D_{123} - D_{134})^T$ and the optimal position of $\boldsymbol{p}_3$ is on an extreme of the edge.

3. $s = 2$, then the constraints of $\boldsymbol{\mathcal{C}}_s$ are solved for the six combinations resulting from fixing the position of each unknown contact point on each extreme of an edge .

4. None of the results computed in step 3 is reachable. Then, the best solution is obtained when $\tau_1 = \tau_{min_1}$, $\tau_3 = \tau_{min_3}$ and $\tau_4 = \tau_{max_4}$. In this case,

$$\tau_1 = \ \ 0.2943; \quad \tau_2 = 0.3369;$$
$$\tau_3 = -2.1544; \quad \tau_4 = 0.3376.$$

   with $Q = D_{123} = D_{124} = 0.3228$, $D_{134} = 0.3726$ and $D_{234} = 0.5496$.

5. Evaluation of the initial reachable grasp in $\mathcal{F}$, obtaining $\mathcal{F} = 0.1343$.

6. $\mathcal{F} \to 0$ for values smaller than τ_{max_1}. Then, the solution of step 4 can not be improved.

7. Since $s = 2$ this step is not applicable.

8. As a result, G_e is the grasp obtained in step 4.

Fig. 7d shows the resulting contact points.

Example 5 (edges E_2, E_5 with two contact points and E_8). This is the seventeenth evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2$, $\boldsymbol{p}_2 \in E_5$, $\boldsymbol{p}_3 \in E_5$ and $\boldsymbol{p}_4 \in E_8$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4612$, $C_{13} = 0.4612$, $C_{24} = 0.5449$.

2. $C_{13} \geq Q_f$ and $C_{24} \geq Q_f$. Then, $s = 1$,
   $\boldsymbol{\mathcal{C}}_s = (D_{124} - D_{134})$ and the optimal positions of $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ are on an extreme of the edge.

3. $s = 1$, then the constraint of $\mathcal{C}_s$ is solved for the four combinations resulting from fixing the position of each unknown contact point on each extreme of an edge.

4. None of the results computed in step 3 are reachable. Then, the best solution is obtained when all the contact are on extremes. In this case,

$$\tau_1 = 0.0179; \quad \tau_2 = 0.2943;$$
$$\tau_3 = 1.5872; \quad \tau_4 = -0.6937.$$

with $Q = D_{124} = 0.0986$, $D_{123} = 0.46115$, $D_{134} = 0.1832$ and $D_{234} = 0.5449$.

As a result, G_e is this grasp and the algorithms ends.

Fig. 7d shows the resulting contact points (note that there is a mark on E_5 limiting the intersection of the directional range with the real range for each of the two contact points on this edge).

Example 6 (edges E_4, E_6 and modified edges E_1 and E_7). The obtention of G_e such that only one point is on an extreme is not very frequent (for instance, the considered object does not have any combination of edges that allows this case), but when it happens the quality of these grasps is very high. Then, the iterative procedures of steps 6 and 7 of the algorithm 2 are not frequently necessary. For illustrative purposes, the edges E_1 and E_7 are slightly enlarged in order to make this case possible and show all the steps of Algorithm 2.

The edges E_1 and E_7 have been enlarged such that $\tau_{max_1} = 0.9$ and $\tau_{min_7} = -1$. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_6$, $\boldsymbol{p}_2 \in E_1$, $\boldsymbol{p}_3 \in E_7$ and $\boldsymbol{p}_4 \in E_4$. Given the contact edges, Algorithm 2 is applied to obtain G_e:

1. Determination of Q_f, C_{13} and C_{24}:
   $Q_f = 0.4665$, $C_{13} = 0.2956$, $C_{24} = 0.2756$.

2. $C_{13} < Q_f$ and $C_{24} < Q_f$. Then, $s = 3$,
   $\mathcal{C}_s = (D_{123} - D_{124} \;\; D_{123} - D_{134} \;\; D_{123} - D_{234})^T$ and it is not possible to determine which contact points lie on an extreme of the edge.

3. $s = 3$, then the four subsystems of two constraints of $\mathcal{C}_s$ are solved fixing the positions of two unknown contacts on extremes of the edges.

4. Step 3 produced reachable solutions, and the best one is produced when $\tau_2 = \tau_{max_2}$ and $\tau_4 = \tau_{max_4}$. In this case:

$$\tau_1 = -1.7268; \quad \tau_2 = 0.9;$$
$$\tau_3 = -0.9132; \quad \tau_4 = 1.6993.$$

with $Q = D_{123} = D_{124} = D_{234} = 0.4134$ and $D_{134} = 0.4340$.

5. Evaluation of the initial reachable grasp in $\mathcal{F}$: If τ_2 is fixed, then $\mathcal{F} = 0.0190$.
   If τ_4 is fixed, then $\mathcal{F} = 0.0131$.

6. $\mathcal{F} \to 0$ for values smaller than τ_{max_2}. Then, the solution of step 4 can be iteratively improved obtaining that $\mathcal{F} \simeq 0$ when:

$$\tau_1 = -1.3147; \quad \tau_2 = 0.6002;$$
$$\tau_3 = -0.7644; \quad \tau_4 = 1.6993.$$

with $Q = D_{123} = D_{124} = D_{234} = 0.4147$ and $D_{134} = 0.4181$.

7. $s = 3$, and the distances to the four faces of $\mathcal{P}_\omega$ tend to be equal for values smaller than τ_{max_2}. Then, step 4 can also be improved iteratively making the four distances tend to be equal; the result is:

$$\tau_1 = -1.2437; \quad \tau_2 = 0.5440;$$
$$\tau_3 = -0.7341; \quad \tau_4 = 1.6993.$$

   with $Q = D_{123} = D_{124} = D_{134} = D_{234} = 0.4146$

8. As a result, the grasp computed in step 6 is the one with maximum quality ($Q = 0.4147$). Then, this grasp is G_e.

Fig. 7f shows the resulting contact points.

6 Conclusions and future works

This paper provides a new approach to determine the optimal form-closure grasp on polygonal objects using the quality measure of the maximum ball. As a result of the problem analysis some intrinsic grasp parameters have been determined: the upper bound, the internal bounds and the type of directional range. These parameters can be easily determined since they depend only on the directions of the applied forces. The type of directional range is used to obtain a new necessary and sufficient condition for the existence of form-closure grasps and the upper bound and the internal bounds are used to identify the case in the determination of the object-optimal grasp. The upper bound is also used as bound in the search of the object-optimal grasp. The main advantage of this proposed approach is that it is not necessary to obtain the edge-optimal grasp for all the sets of edges to find the object-optimal grasp, and that there are cases in the computation of the edge-optimal grasp where some contact points can be analytically determined.

The paper also introduces a new concept: the directional-optimal grasp, defined considering virtual edges with infinity lengths. Although the directional-optimal grasp is actually unreachable, it is useful to determine which cases of optimal grasp can take place and which optimal contact points lie on an extreme of the edges. With this information, an initial solution is obtained in a fully analytical way, that is either the edge-optimal grasp or very close to it. In this second case an iterative procedure, function of only one unknown, is used to obtain the edge-optimal grasp.

We would like to remark that the main concepts used in this approach depend only on the applied forces and can be easily determined. This is a very interesting characteristic that encourages to extend this work considering frictional contacts, non-polygonal objects and 3D objects in future works. In the ongoing work, the necessary and sufficient condition has been extended considering frictional contacts (Cornellà and Suárez, 2005b) and non-polygonal objects (Cornellà and Suárez, 2005a). Nevertheless, the determination of the optimal grasp using the methodology developed here is still a problem under development.

Appendices

A Quality measure properties

Let $Q(\tau_i)$ be the quality measure as a function of the torque τ_i of $\boldsymbol{\omega}_i$ given the other three wrenches, and let D_{ijk} be the distance from the origin of the wrench space to the plane defined by $\boldsymbol{\omega}_i$, $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$. From equation (5), $Q(\tau_i)$ is a function defined by pieces of $D_{ijk}(\tau_i)$, with $i, j, k \in \{1, ..., 4\}$ and $i \neq j \neq k$, being:

$$D_{ijk}(\tau_i) = \left| \frac{k_1 + k_2\tau_i}{\sqrt{(k_3+k_4\tau_i)^2+(k_5+k_6\tau_i)^2+k_7^2}} \right| \tag{32}$$

where:

$$\begin{aligned}
k_1 &= \sin(\theta_i - \theta_j)\tau_k + \sin(\theta_k - \theta_i)\tau_j && (33)\\
k_2 &= \sin(\theta_j - \theta_k) && (34)\\
k_3 &= (\sin\theta_j - \sin\theta_i)\tau_k + (\sin\theta_i - \sin\theta_k)\tau_j && (35)\\
k_4 &= \sin\theta_k - \sin\theta_j && (36)\\
k_5 &= (\cos\theta_i - \cos\theta_j)\tau_k + (\cos\theta_k - \cos\theta_i)\tau_j && (37)\\
k_6 &= (\cos\theta_j - \cos\theta_k) && (38)\\
k_7 &= \sin(\theta_j - \theta_i) + \sin(\theta_i - \theta_k) + \sin(\theta_k - \theta_j) && (39)
\end{aligned}$$

The function $D_{ijk}(\tau_i)$ has five relevant properties (Fig. 8):

1. It is a continuous function. The denominator can only be zero if all the forces are in the same direction, and then the FC grasp is not possible.
2. It has only one zero at $\tau_{i_0} = -k_1/k_2$.
3. It tends to a finite value L when $\tau_i \to \pm\infty$,

$$L = \lim_{\tau_i \to \pm\infty} D_{ijk}(\tau_i) = \frac{k_2}{\sqrt{(k_4^2 + k_6^2)}} \tag{40}$$

   Geometrically, L is the distance from the origin to the straight line defined by $\overline{\boldsymbol{f}_j \boldsymbol{f}_k}$.

4. It has only one maximum M at

$$\tau_{i_M} = \frac{(k_3^2 + k_7^2 + k_5^2)k_2 - (k_3k_4 + k_5k_6)k_1}{-k_2(k_3k_4 + k_5k_6) + k_1(k_4^2 + k_6^2)} \tag{41}$$

   Geometrically, M is the distance from the origin to the straight line defined by $\overline{\boldsymbol{\omega}_j \boldsymbol{\omega}_k}$.

5. It is constant when $\theta_j = \theta_k$ (if $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ have the same direction then $k_2 = k_4 = k_6 = 0$).

$Q(\tau_i)$, as a function defined by pieces of $D_{ijk}(\tau_i)$, has the following properties :

1. $Q(\tau_i)$ is defined by monotones pieces of $D_{ijk}(\tau_i)$. The maximum and the zero of $D_{ijk}(\tau_i)$ can not define $Q(\tau_i)$ since the first would be the radius of a sphere not contained in $\mathcal{P}_\omega$ (see property 4 of $D_{ijk}(\tau_i)$), and the second determines a critical grasp with $Q = 0$. Therefore, ruling out these two points any other continuous piece of $D_{ijk}(\tau_i)$ is monotone (see Fig. 9).

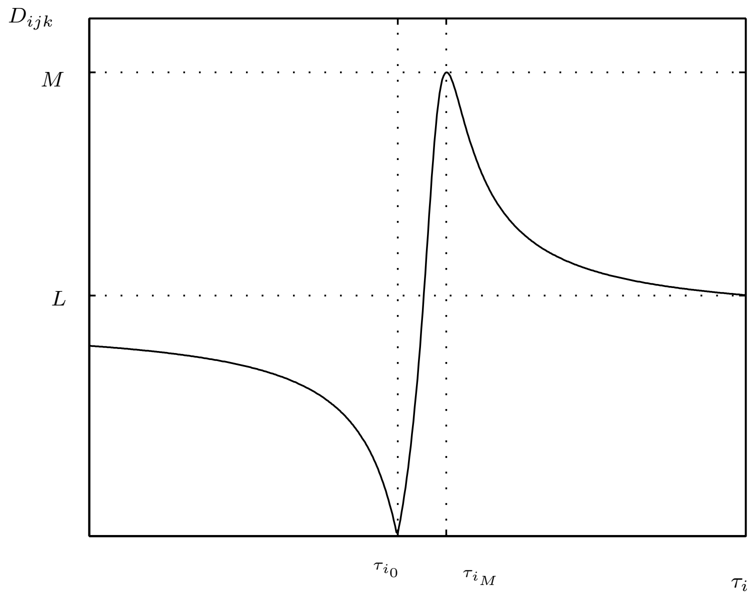


Figure 8: Qualitative shape of distance $D_{ijk}(\tau_i)$.

2. If the triangle defined by $\boldsymbol{f}_i$, $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$, (i.e. the projection on the force space of the face of $\mathcal{P}_\omega$ defined by $\boldsymbol{\omega}_i$, $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$) does not intersect with the circumference of radius Q_f (i.e. the upper bound of Q), then $D_{ijk} > Q_f$, implying that D_{ijk} can not determine Q (see examples in Fig. 5).

B Computational Aspects

Considering the distance from the origin to the faces of $\mathcal{P}_\omega$, the constraints included in equation (30) are four-order equations. Nevertheless, the order of these constraints can be reduced using the following geometrical property.

Consider the constraint $D_{hij} = D_{hik}$ (the same reasoning can be applied to the other constraints) and let Π_{hij}, Π_{hik} and Π_{hi0} be the planes defined in the wrench space by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_j\}$, $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}$ and $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{0}\}$ as:

$$\Pi_{hij} : \quad \boldsymbol{n}_{hij}\boldsymbol{\omega} + d_{hij} = 0 \tag{42}$$

$$\Pi_{hik} : \quad \boldsymbol{n}_{hik}\boldsymbol{\omega} + d_{hik} = 0 \tag{43}$$

$$\Pi_{hi0} : \quad \boldsymbol{n}_{hi0}\boldsymbol{\omega} = 0 \tag{44}$$

where $\boldsymbol{n}_{hij}$, $\boldsymbol{n}_{hik}$ and $\boldsymbol{n}_{hi0}$ are the vectors normal to the planes and d_{hij} and d_{hik} are the independent terms. These normal vectors and independent terms are linear functions of τ_h, τ_i, τ_j and τ_k.

The constraint $D_{hij} = D_{hik}$ implies that Π_{hi0} is a bisector plane of Π_{hij} and Π_{hik}, and any vector normal to Π_{hi0} intersects Π_{hij} and Π_{hik} at points located at the same distance from Π_{hi0}. Then, selecting the normal vector that passes through the origin, the distances D'_{hij} and D'_{hik}

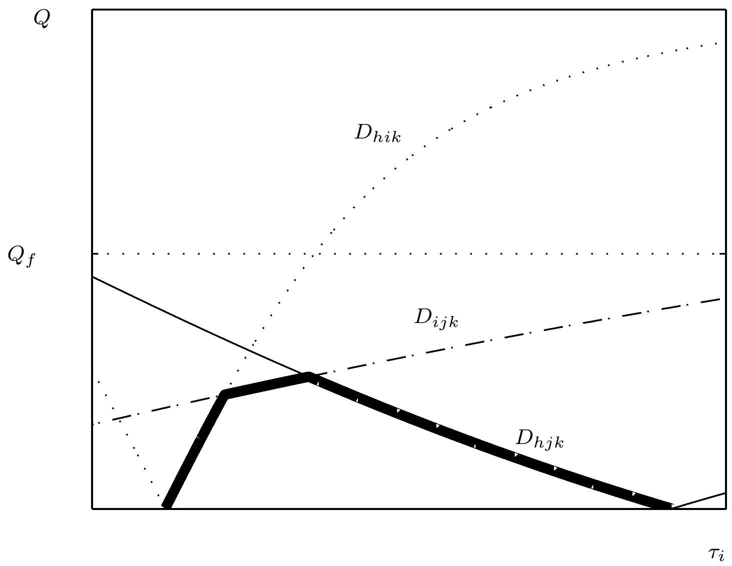


Figure 9: Quality measure (bold line) defined as pieces of distances $D_{ijk}(\tau_i)$ and upper bound Q_f.

from the origin to Π_{hij} and Π_{hik}, respectively, satisfy $D'_{hij} = -D'_{hik}$ and can be used instead $D_{hij} = D_{hik}$ (see Fig. 10). Using equations (42), (43) and (44) $D'_{hki} = D'_{hkj}$ can be expressed as

$$d_{hik}(\boldsymbol{n}_{hij} \cdot \boldsymbol{n}_{hi0}) = -d_{hij}(\boldsymbol{n}_{hik} \cdot \boldsymbol{n}_{hi0}) \tag{45}$$

Since $\boldsymbol{n}_{hij}$, d_{hij}, $\boldsymbol{n}_{hik}$, d_{hik} and $\boldsymbol{n}_{hi0}$ are linear functions of τ_h, τ_i, τ_j and τ_k, equation (45) expressed as a function of τ_h or τ_i is a three order equation, while the same equation expressed as a function of τ_j or τ_k is a linear equation. Using equation (45) to represent the constraints, a system of two constraints with two unknowns torques can be solved in a fully analytical way.

C Proofs of the Propositions

Proof of Proposition 1: The type of a directional range, R_{d_i}, is determined by the number of finite extremes that it has, which is equivalent to know how many pairs of coefficients $\beta_{i,jk}$ and $\beta_{i,kj}$, defined by equations (13) and (14), are non-positive, with $i, j, k \in \{1, 2, 3, 4\}$ and $i \neq j \neq k$.

The coefficients $\beta_{i,jk}$ and $\beta_{i,kj}$ are defined by the directions of three applied forces, $\boldsymbol{f}_i$, $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$, which also define the coefficients $\beta_{j,ik}$ and $\beta_{j,ki}$, and $\beta_{k,ji}$ and $\beta_{k,ij}$. Since there are four forces, there are four different subsets of three forces, with each force belonging to three of the four subsets.

When there is no opposite forces (general case), any three forces $\boldsymbol{f}_i$, $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ either span the force space (i.e. each force strictly lies between the negated of the other two) implying that the corresponding three pairs of coefficients $\{\beta_{i,jk}, \beta_{i,kj}\}$, $\{\beta_{j,ik}, \beta_{j,ki}\}$ and $\{\beta_{k,ji}, \beta_{k,ij}\}$ are negative and define a finite extreme for R_{d_i}, R_{d_j} and R_{d_k}, or do not span the force space, implying that some coefficients are positive and therefore do not define any extreme at all.

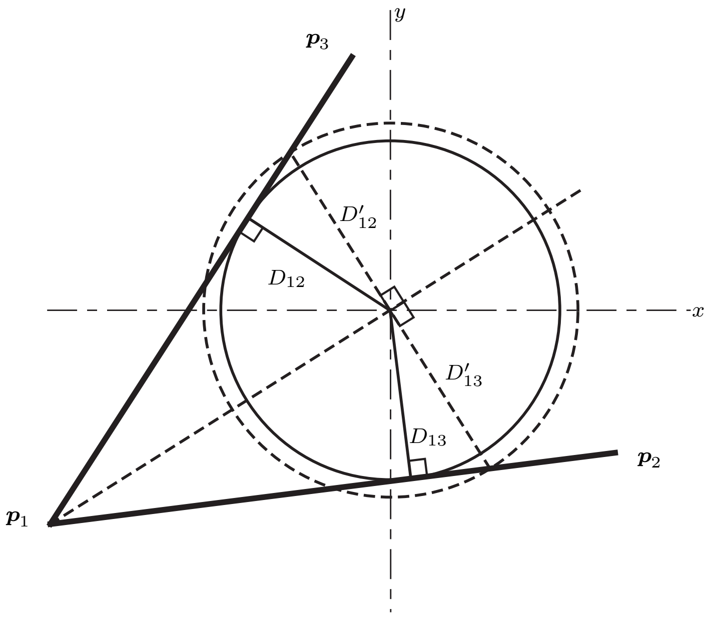


Figure 10: 2D qualitative example showing that $D'_{12} = D'_{13}$ when $D_{12} = D_{13}$.

Since a directional range can have only one or two finite extremes, then, only two of the four subsets of three forces can generate valid pairs of coefficients (note that if more than two subsets or only one subset were valid one of the four directional range would have more than two extremes or none at all, respectively, which is not possible). Then, there are six pairs of coefficients determining six finite extremes for the four directional ranges. This means that two directional ranges have two finite extremes and they are limited (those of the two torques generated by the forces that appear in the two valid subsets of forces), while the directional ranges of the other two torques have only one finite extreme and they are infinite (those generated by the forces that appear in only one of the two subsets of forces). Analyzing the signs produced by the relative values of θ_i, $i = 1, ..., 4$, in equations (13) and (14) the forces that appear in only one of the subsets of forces have to lie between the negated of the other two, therefore the two infinite directional ranges are those corresponding to the two torques produced by the two forces that lie between the negated of the other two (see Fig 4a and Fig 4b).

When there is one pair of opposite forces they must determine non-consecutive vertices of $\mathcal{P}_f$ in order to $\mathbf{0} \in \mathcal{P}_f$. Without loss of generality consider that the two opposite forces are $\boldsymbol{f}_i$ and $\boldsymbol{f}_k$, i.e. $\theta_k = \theta_i + \pi$. Using equations equations (13) and (14) with the proper subindexes for the possible combinations of forces, we obtain from the subset $\{\boldsymbol{f}_h, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ the coefficients $\beta_{i,hk} = 0$, $\beta_{i,kh} = -1$, $\beta_{k,hi} = 0$, $\beta_{k,ih} = -1$, while $\beta_{h,ik}$ and $\beta_{h,ki}$ are undefined (division by zero), obtaining therefore two finite extremes, one for R_{d_i} and R_{d_k}. In the same way, from the subset $\{\boldsymbol{f}_j, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ we obtain the coefficients $\beta_{i,jk} = 0$, $\beta_{i,kj} = -1$, $\beta_{k,ji} = 0$, $\beta_{k,ij} = -1$, while $\beta_{j,ik}$ and $\beta_{j,ki}$ are undefined (division by zero), obtaining therefore one finite extreme for R_{d_i} and another for R_{d_k}; but, since in this case $\beta_{i,hk} = \beta_{i,jk}$, $\beta_{i,kh} = \beta_{i,kj}$, $\beta_{k,hi} = \beta_{k,ji}$, and $\beta_{k,ih} = \beta_{k,ij}$ the two finite extremes of R_{d_i} and R_{d_k} are actually the same, and then the two sets of forces that includes the two opposite forces define only two different finite extremes, one for the directional range of each of the opposite forces. In the other two subsets of forces there are not opposite forces and therefore, as in the general case, they will produce either three valid or three invalid pairs of coefficients (and the same number of finite extremes). Since each of them contains one of the opposite forces, only one of these two subsets of forces has the opposite force between the negated of the other two, producing three new finite extremes. As a result, there will be five finite extremes, two for the directional range of the opposite force that lies between the negated of the non-opposite forces, so it is limited, and one extreme for each of the other directional ranges, so they are infinite (see Fig 4c).

In the case of two pairs of opposite forces (e.g. $\theta_k = \theta_i + \pi$ and $\theta_h = \theta_j + \pi$, following the same reasonings above the subsets $\{\boldsymbol{f}_h, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ and $\{\boldsymbol{f}_j, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ produce one finite extreme for R_{d_i} and one for R_{d_k}, and $\{\boldsymbol{f}_i, \boldsymbol{f}_j, \boldsymbol{f}_h\}$ and $\{\boldsymbol{f}_k, \boldsymbol{f}_j, \boldsymbol{f}_h\}$ produce one finite extreme for R_{d_j} and one for R_{d_h}. As a result each of the four directional ranges has only one finite extreme and they all are infinite. ◇

Proof of Proposition 2 (necessary and sufficient condition): Consider first the general case with two infinite and two limited directional ranges. Let $R_{d_k} = [\tau_{k_1}, \tau_{k_2}]$ be one of the two limited directional ranges, i.e. $\tau_{k_1} \leq \tau_k \leq \tau_{k_2}$. Substituting τ_{k_1} and τ_{k_2} by the expressions derived from equation (12), we obtain (note that the subscripts i and j could be swapped):

$$\beta_{k,hj}\tau_h + \beta_{k,jh}\tau_j \leq \tau_k \leq \beta_{k,ih}\tau_i + \beta_{k,hi}\tau_h \tag{46}$$

If τ_i and τ_j are solved from equation (46), then

$$\tau_i \leq \frac{1}{\beta_{k,ih}}(\tau_k - \beta_{k,hi}\tau_h) \tag{47}$$

$$\tau_j \geq \frac{1}{\beta_{k,jh}}(\tau_k - \beta_{k,hj}\tau_h) \tag{48}$$

Therefore, τ_i has an upper bound while τ_j has a bottom bound implying that R_{d_i} tends to $-\infty$ and R_{d_j} tends to $+\infty$. Equations (47) and (48) can be converted to equalities subtracting Γ_i and adding Γ_j, respectively. As result, in a FC grasp the signs of Γ_i and Γ_j must be different (signs are swapped if the subscripts i and j are swapped).

The two particular cases can be tackle as limits of the general case. Adding $\delta\theta$ arbitrarily small to the direction of one of the aligned forces, the particular cases are transformed into the general case. Then, the above reasoning can be applied obtaining the same results when $\delta\theta \to 0$. ◇

Proof of Proposition 3: From property 1 of $Q(\tau_k)$ (Appendix A) the pieces of $D_{hik}(\tau_k)$, $D_{hjk}(\tau_k)$ and $D_{ijk}(\tau_k)$ that define $Q(\tau_k)$ are monotones.

When R_{d_k} is infinite and $C_{hj} \geq Q_f$ (case 1) there is only one extreme τ_{k_1} where $Q(\tau_{k_1}) = 0$ and the distances $D_{hik}(\tau_k)$, $D_{hjk}(\tau_k)$ and $D_{ijk}(\tau_k)$, tend to values greater or equal to Q_f when $\tau_k \to \pm\infty$ (property 3 of the distance). As a consequence, all the pieces of $D_{hik}(\tau_k)$, $D_{hjk}(\tau_k)$ and $D_{ijk}(\tau_k)$ that define $Q(\tau_k)$ increase monotonously when $R_{fc_k} = [\tau_{k_1}, \infty)$ or decrease monotonously when $R_{fc_4} = (-\infty, \tau_{k_1}]$, obtaining the maximum value of $Q(\tau_k)$ when $\tau_{d_k} \to \pm\infty$ according to the unbounded direction of R_{d_k} (see Fig 11).

When either R_{d_k} is limited or $C_{hj} < Q_f$ (case 2), either there are two points, τ_{k_1} and τ_{k_2}, where $Q(\tau_{k_1}) = 0$ and $Q(\tau_{k_2}) = 0$, or the distance D_{hjk} tends to a value smaller than Q. As a result in both situations, some of the pieces of $D_{hik}(\tau_k)$, $D_{hjk}(\tau_k)$ and $D_{ijk}(\tau_k)$ that define $Q(\tau_k)$ increase while other decrease, and the maximum $Q(\tau_k)$ is obtained at the intersection of

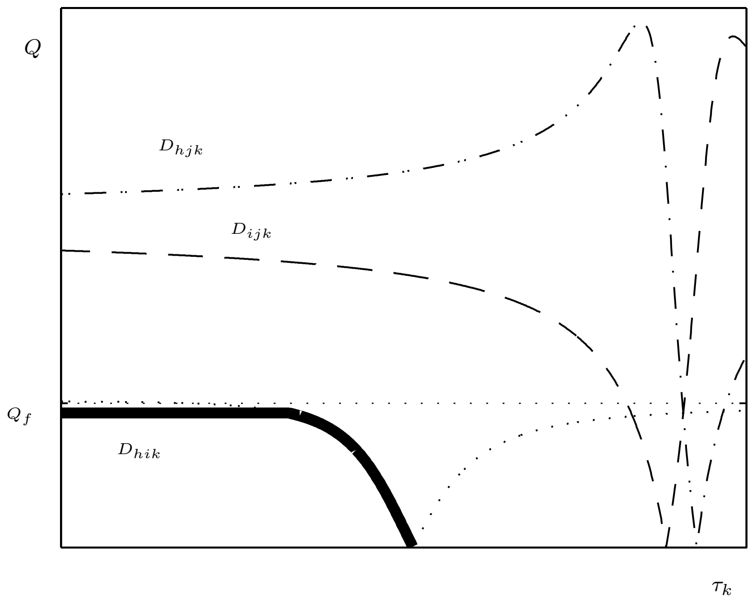


Figure 11: Quality measure Q as a function of τ_k (case where R_{d_k} is infinite and $C_{hj} \geq Q_f$).

two of them (see Fig 12). In order to identify the intersections that may determine the optimal value of $Q(\tau_k)$, property 2 of $Q(\tau_k)$ is used (see Appendix A): $D_{hik}(\tau_k)$, $D_{hjk}(\tau_k)$ and $D_{ijk}(\tau_k)$ may define $Q(\tau_k)$ only if the triangles formed on the force space by $\{\boldsymbol{f}_h, \boldsymbol{f}_i, \boldsymbol{f}_k\}$, $\{\boldsymbol{f}_h, \boldsymbol{f}_j, \boldsymbol{f}_k\}$ and $\{\boldsymbol{f}_i, \boldsymbol{f}_j, \boldsymbol{f}_k\}$, respectively, intersect with the circumference of radius Q_f. The number of triangles that intersect with this circumference is determined by C_{ik}, obtaining:

(a) If $C_{ik} \geq Q_f$, two of the three triangles intersect with the circumference of radius Q_f. Then, the optimal value is obtained at the intersection of the two corresponding associated distances.

(b) If $C_{ik} < Q_f$, the three triangles intersect with the circumference of radius Q_f. Then, any intersection between two of the corresponding distances may determine the optimal value. $\diamond$

Proof of Proposition 4: Given three wrenches, the conditions that the optimal value of the fourth torque must satisfy are determined by its directional range, the upper bound and the internal bounds (Proposition 3). Since these parameters do not depend on the values of the torques, the conditions neither depend on them when the four torques are variable. Then, the optimal value of each torque satisfies these conditions even when their exact values are unknown. Consider that $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_i$ are the wrenches whose force components determine the upper bound, and $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}_j$ are the other two wrenches. In order to obtain the conditions that G_d must satisfy when the four wrenches are unknown, all the possible combinations between directional ranges, internal bounds and upper bound considering the four torques are checked, obtaining that all the cases can be grouped in the three ones presented in the proposition. The demonstration of all the cases is large and tedious, but the procedure is similar; for this reason we only include here the proof of the first case.

When $C_{hj} \geq Q_f$ and $C_{ik} \geq Q_f$, the forces $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ are always between the negated of $\boldsymbol{f}_h$

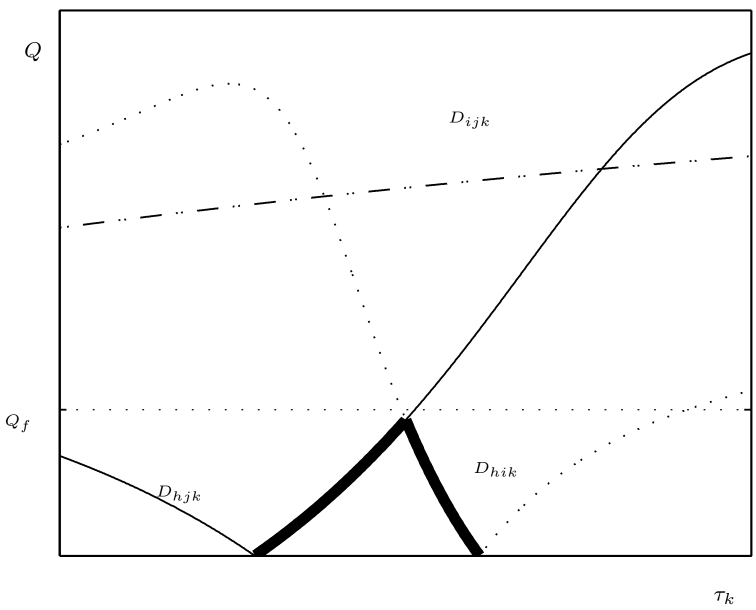


Figure 12: Quality measure Q as a function τ_k (case where R_{d_k} is limited).

and $\boldsymbol{f}_i$. Then, from Proposition 1, R_{d_h} and R_{d_i} are always limited, while R_{d_j} and R_{fc_k} are always infinite. Proposition 3 is applied for each torque, obtaining the following result:

τ_h: R_{d_h} is limited and $C_{ik} \geq Q_f$, then τ_{d_h} is the result of $D_{hij} = D_{hik}$.

τ_i: R_{d_i} is limited and $C_{hj} \geq Q_f$, then τ_{d_i} is the result of $D_{hij} = D_{hik}$.

τ_j: R_{d_j} is infinite and $C_{ik} \geq Q_f$, then the optimal $\tau_{d_j} \rightarrow \pm\infty$ according the unbound side of R_{d_j}.

τ_k: R_{d_k} is infinite and $C_{hj} \geq Q_f$, then $\tau_{d_k} \rightarrow \pm\infty$ according the unbound side of R_{d_k}.

As a result, $\tau_j \rightarrow \pm\infty$ and $\tau_k \rightarrow \mp\infty$ (from Proposition 2, the signs of these torques have to be different to ensure a FC grasp). The other two torques, τ_h and τ_i, have to be determined from the same equation ($D_{hij} = D_{hik}$), and therefore there are more unknowns that equations. Then, an optimization is introduced such that the two distances that determines the constraint are maximized. Similar reasoning can be applied in Cases 2 and 3.

Proof of Proposition 5: If $\tau_{d_i} \rightarrow \pm\infty$, then R_{d_i} is infinite and $C_{hj} \geq Q_f$ (see Proposition 3). In this case $Q(\tau_i)$ increase or decrease monotonously according R_{d_i}. Therefore, τ_{e_i} is the value $\tau_i \in R_i$ closest to τ_{d_i}, i.e. the closest extreme of R_i. $\diamond$

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