AN EXPLICIT CONSTRUCTION FOR NEIGHBORLY CENTRALLY SYMMETRIC POLYTOPES

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Abstract. We give an explicit construction, based on Hadamard matrices, for an infinite series of $\lfloor \frac{1}{2}\sqrt{d} \rfloor$-neighborly centrally symmetric d-dimensional polytopes with $4d$ vertices. This appears to be the best explicit version yet of a recent probabilistic result due to Linial and Novik, who proved the existence of such polytopes with a neighborliness of $\frac{d}{400}$.

1. Introduction

A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if $P = -P$. A cs polytope P is k*-neighborly* if every set of k of its vertices, no two of which are antipodes, is the vertex set of a face of P.

In their recent paper [7], Linial and Novik give probabilistic constructions for highly neighborly cs polytopes. Namely, based on probabilistic techniques due to Garnaev and Gluskin [4], they construct k-neighborly d-dimensional cs polytopes with $2m = 2(n+d)$ vertices, such that $k = \Theta\big(\frac{d}{1+\log(m/d)}\big)$; moreover, they show that this value is asymptotically optimal. In the "diagonal" case $n = d$ they use a probabilistic result due to Kašin [5] to construct d-dimensional $\frac{d}{400}$-neighborly cs polytopes with $4d$ vertices, and ask if there exists an explicit construction of highly neighborly cs polytopes.

In this note, we provide such an explicit and non-probabilistic construction:

Theorem 1. *For each $d \geq 4$ such that there exists a Hadamard matrix of size d, there is an explicit construction for a $\lfloor \frac{1}{2}\sqrt{d} \rfloor$-neighborly cs d-polytope with $4d$ vertices.*

Hadamard matrices exist for every $d = 2^e$ with integer $e \geq 2$; see [8] for a survey.

Of course, Theorem 1 does not attain (by far) the bound given by Linial and Novik's probabilistic arguments, but to date no better explicit construction of highly neighborly cs polytopes seems to be known; see also our additional comments in Section 4. We refer to [7] for a (necessarily short) survey of the known results on neighborly cs polytopes.

To briefly outline the remaining contents of this note, we proceed to characterize k-neighborly cs d-polytopes with $2(n+d)$ vertices in terms of a certain linear projection (Proposition 3) and a certain matrix equation (Theorem 5). In Section 3, we then find very special solutions of this equation in the "diagonal" case $n = d$ and prove Theorem 1.

Date: June 29, 2006.

The author was supported by a *Juan de la Cierva* postdoctoral research grant from the Spanish Ministry of Education (MEC), and by projects MCYT BFM2003-00368 and MTM2005-08618-C02-01.

2. cs-transforms, polarity, and a matrix equation

The key to our construction is the following lemma due to Linial and Novik, which characterizes those point sets arising as McMullen and Shephard's *cs-transforms* [6] of cs polytopes:

Lemma 2 (Linial and Novik [7, Lemma 3.1])**.** *A cs set* $\overline{V} = \{\pm\overline{v}_1, \dots, \pm\overline{v}_m\} \subset \mathbb{R}^n$ *is a cs transform of the vertex set of a* k*-neighborly cs* d*-polytope with* $2m = 2(d+n)$ *vertices if and only if the set* $\overline{V}_+ = \{\overline{v}_1, \dots, \overline{v}_m\}$ *does not contain dominant subsets of size* k*.*

Here, Linial and Novik define a subset $\{\overline{v}_i : i \in I\}$ of $\overline{V}_+$ to be *dominant* if there exists $0 \neq u \in \mathbb{R}^n$ such that

$$\sum_{i\in I} \big|\langle \overline{v}_i, u\rangle\big| \ \geq\ \frac{1}{2}\sum_{j=1}^{m} \big|\langle \overline{v}_j, u\rangle\big| \,. \tag{1}$$

To interpret this characterization geometrically, let $\Diamond_m \subset \mathbb{R}^m$ be the standard cross-polytope, $C_{\pm a}(m)$ the m-dimensional cube $\{x \in \mathbb{R}^m : -a \leq x_i \leq a \text{ for } 1 \leq i \leq m\}$, and $E(m,k)$ the m-dimensional convex hull of all $0/\pm1$-vectors of length m with exactly k non-zero entries. Thus, $E(m,k) = C_{\pm1}(m) \cap k\Diamond_m$; equivalently, we obtain $E(m,k)$ by reflecting the standard m-dimensional hypersimplex $\Delta(m,k)$ in the coordinate hyperplanes of $\mathbb{R}^m$.

Now let $T : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map given by the real $(m \times n)$-matrix whose rows are $\overline{v}_1, \dots, \overline{v}_m$, so that $Tu = (\langle \overline{v}_1, u\rangle, \dots, \langle \overline{v}_m, u\rangle)^T$ for any $u \in \mathbb{R}^n$ (here and throughout, the superscript T denotes transpose; we trust that this will not cause confusion). We assume that T has full rank, and denote the image of T by L, a linear n-space in $\mathbb{R}^m$.

To express Linial & Novik's lemma in this language, write $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$ and note that

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{m} \big|\langle \overline{v}_j, u\rangle\big| \ &=\ \frac{1}{2}\max\left\{\sum_{j=1}^{m} \varepsilon_j \langle \overline{v}_j, u\rangle \ :\ \boldsymbol{\varepsilon} \in \operatorname{vert} C_{\pm1}(m)\right\} \\
&=\ \frac{1}{2}\max\Big\{\boldsymbol{\varepsilon}\big(\langle \overline{v}_1, u\rangle, \dots, \langle \overline{v}_m, u\rangle\big)^T : \boldsymbol{\varepsilon} \in \operatorname{vert} C_{\pm1}(m)\Big\} \\
&=\ \max\Big\{\langle z, Tu\rangle : z \in \operatorname{vert} C_{\pm\frac{1}{2}}(m)\Big\},
\end{aligned}$$

and analogously, for any subset $I \subset [m]$ of cardinality k,

$$\begin{aligned}
\sum_{i\in I} \big|\langle \overline{v}_i, u\rangle\big| \ &=\ \max\left\{\sum_{i\in I} \delta_i \langle \overline{v}_i, u\rangle \ :\ \delta_i = \pm1 \text{ for all } i \in I\right\} \\
&=\ \max\big\{\langle w, Tu\rangle \ :\ w \in \operatorname{vert} E(m,k)\big\}.
\end{aligned}$$

Thus, Lemma 2, condition (1) and the fact that the maximum of any linear function on a polytope is attained at one of the vertices together say that $\{\pm\overline{v}_1, \dots, \pm\overline{v}_m\}$ is a cs-transform of a k-neighborly cs d-polytope with $2m$ vertices if and only if

$$\max_{w\in E(m,k)} \langle w, v\rangle \ <\ \max_{z\in C_{\pm1/2}(m)} \langle z, v\rangle \qquad \text{for all } 0 \neq v \in L\,.$$

By dualizing — i.e., considering $\langle v, w\rangle$ instead of $\langle w, v\rangle$ — we can also read this condition as saying that for *any* non-zero vector $v \in L$, an affine hyperplane perpendicular to v that sweeps outward from the origin along v should have left behind all vertices of $E(m,k)$ before encountering the last vertex of $C_{\pm 1/2}(m)$. We have reached the following conclusion:

Proposition 3. *The set $\{\pm\overline{v}_1, \dots, \pm\overline{v}_m\} \subset \mathbb{R}^n$ is a cs-transform of a k-neighborly cs d-polytope with $2m$ vertices if and only if*

$$\operatorname{proj}_L E(m,k) \;\subset\; \operatorname{proj}_L C_{\pm 1/2}(m), \tag{2}$$

where proj_L denotes orthogonal projection to $L \subset \mathbb{R}^m$, the linear n-space that is the image of the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix has rows $\overline{v}_1, \dots, \overline{v}_m$.

To proceed, we take advantage of the following duality (Lemma 4) that relates the section of a polytope $P \subset \mathbb{R}^m$ by a linear subspace L to the projection of P^Δ to L. Recall that the *polar set* of P is

$$P^\Delta \;=\; \big\{x \in \mathbb{R}^m : \langle x, y\rangle \le 1 \text{ for all } y \in P\big\}, \tag{3}$$

and that $(P^\Delta)^\Delta = P$ if $0 \in P$.

Lemma 4. *Let $P \subset \mathbb{R}^m$ be any polytope such that $0 \in P$, let $L \subset \mathbb{R}^m$ be any linear subspace, and denote the orthogonal projection of $\mathbb{R}^m$ to L by proj_L. Then*

$$\operatorname{proj}_L(P^\Delta) \;=\; (P \cap L)^\Delta \cap L. \tag{4}$$

We learned about this lemma from [10]; the proof is elementary and follows from the definition (3) of a polar set.

By substituting (4) with $P = E(m,k)^\Delta$, respectively $P = C_{\pm 1/2}(m)^\Delta$, into (2), we obtain

$$\big(E(m,k)^\Delta \cap L\big)^\Delta \cap L \;\subset\; \big(C_{\pm 1/2}(m)^\Delta \cap L\big)^\Delta \cap L.$$

We now restrict to the subspace L and polarize. Because both polytopes contain the origin and are full-dimensional in L (and therefore polarizing reverses inclusion, and $(P^\Delta)^\Delta = P$), we obtain the equivalent condition

$$C_{\pm 1/2}(m)^\Delta \cap L \;\subset\; E(m,k)^\Delta \cap L.$$

This in turn is satisfied if and only if the 2^m facet-defining inequalities of the polytope $C_{\pm 1/2}(m)^\Delta = 2\lozenge_m$, together with some fixed set of $d = m - n$ equations defining L, imply the $2^k\binom{m}{k}$ facet-defining inequalities of $E(m,k)^\Delta$.

To find a linear subspace L that achieves this, we represent L as the kernel of the matrix $(\mathrm{I}_d\,|A)$, where I_d is the $(d\times d)$ identity matrix and $A = (a_{ij})$ a real $(d\times n)$ matrix. Moreover, we pass to homogeneous coordinates, which means to express each point $x \in \mathbb{R}^m$ as $(1,x) \in \mathbb{R}\times\mathbb{R}^m$, and each inequality $ax \le a_0$, for $a \in (\mathbb{R}^m)^*$ and $a_0 \in \mathbb{R}$, as $(a_0, a) \in (\mathbb{R}^{m+1})^*$.

Phrased in this language, we must express each vertex $(1, e)$ of $\{1\} \times E(m, k) \in \mathbb{R} \times \mathbb{R}^m$ as a linear combination of the following form:

$$
\begin{array}{c|ccccccc}
\mu_1^e & 2 & -\delta_{1,1}^e & \cdots & -\delta_{1,d}^e & -\delta_{1,d+1}^e & \cdots & -\delta_{1,d+n}^e \\
\cdots & & & \cdots & & & \cdots & \\
\mu_{2^m}^e & 2 & -\delta_{2^m,1}^e & \cdots & -\delta_{2^m,d}^e & -\delta_{2^m,d+1}^e & \cdots & -\delta_{2^m,d+n}^e \\
\nu_1^e & 0 & 1 & \cdots & 0 & a_{1,1} & \cdots & a_{1,n} \\
\cdots & & & \cdots & & & \cdots & \\
\nu_d^e & 0 & 0 & \cdots & 1 & a_{d,1} & \cdots & a_{d,n} \\
\varepsilon & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline\hline
 & 1 & & & & e & &
\end{array}
\tag{5}
$$

In this table, the μ's, ν's and ε are understood to multiply the adjacent row vectors, and the result of this linear combination is the row vector $(1, e)$. Specifically, $\varepsilon \geq 0$, $\mu_i^e \geq 0$, and $\nu_i^e \in \mathbb{R}$ for all relevant indices, and not all of these coefficients are required to be non-zero; moreover, $\delta_{ij}^e = \pm 1$, and the reason for introducing the minus signs for the δ's will become clear in a moment. We will also use the notation $M_j^e = \sum_{i \in I_e} \mu_i^e \delta_{ij}^e$, for $1 \leq j \leq d + n$, where $I_e \subset \{1, \ldots, 2^m\}$ indexes the non-zero μ_i^e. Note the constraint $\sum_{i \in I_e} \mu_i^e \leq \frac{1}{2}$ implied by the "0-th" column of this linear combination, which in turn implies $|M_j^e| \leq \frac{1}{2}$.

From columns $1 \leq j \leq d$ of (5), we learn that $\sum_{i \in I_e} \mu_i^e(-\delta_{ij}^e) + \nu_j^e = e_j$, so that in fact we know the coefficients $\nu_j^e = e_j + M_j^e$. With this information, we obtain from columns $d + 1 \leq j \leq d + n$ that $\sum_{i=1}^d (e_i + M_i^e) a_{i,j-d} = e_j + M_j^e$. Expressed in matrix notation, we have arrived at the following result:

Theorem 5. *Finding a cs-transform of a k-neighborly cs d-polytope with $2m = 2(n + d)$ vertices is equivalent to finding a $(d \times n)$-matrix A and a $(2^k \binom{m}{k} \times m)$-matrix M' that satisfy the following requirements:*

(a) *All entries of M' are bounded in absolute value by $\frac{1}{2}$.*

(b) *Let $E' = (E|F)$ be a $(2^k \binom{m}{k} \times m)$-matrix whose rows are the vertices of $E(m, k)$ in some order, and decompose it into a matrix E with d columns and a matrix F with n columns. Similarly, decompose $M' = (M|N)$ into a matrix M with d columns and a matrix N with n columns. Then the matrices A, E, F, M and N must satisfy*

$$(E + M)\, A \;=\; F + N. \tag{6}$$

(c) *The rows of M' must be expressible as linear combinations as in* (5).

As an aside, it is clear that any expression of $(1, e)$ as a linear combination as in (5) immediately yields an expression of $(1, -e)$ as a similiar linear combination, by reversing the signs of the relevant δ_{ij}^e and ν_i. It would therefore be enough to consider only one member of each pair of antipodal vertices of $E(m, k)$, and consequently only keep those rows of the $\{0, \pm 1\}$-matrix E' whose first non-zero entry is positive. However, to keep the symmetry of the problem we choose not to do this.

We do, however, partially order the rows of E'. Namely, we partition E' into $k+1$ blocks $E'_l = (E_l|F_{k-l})$ with $0 \leq l \leq k$, such that each row of E_l has exactly l non-zero entries (and consequently each row of F_{k-l} has $k-l$ of them). The order inside each such block is immaterial for our purposes. Note that with this partial ordering, the number of rows of both E_l and F_{k-l} is

$$2^l\binom{d}{l}\cdot 2^{k-l}\binom{n}{k-l} \;=\; 2^k\binom{d}{l}\binom{n}{k-l}.$$

By decomposing M and N into blocks labeled M_{k-l}, N_l with the same number of rows as E_l and F_{k-l}, equation (6) above decomposes into the $k+1$ equations

$$(E_l + M_{k-l})\,A \;=\; F_{k-l} + N_l\,, \qquad \text{for } \; 0 \le l \le k. \tag{7}$$

3. Hadamard matrices

In the diagonal case $n=d$, we will exhibit a very special solution of (7). Namely, we find matrices A, M_{k-l}, N_l such that

$$\begin{aligned} E_l A \;&=\; N_l, \\ M_{k-l} A \;&=\; F_{k-l}, \end{aligned}$$

in the following way:

Let d be such that there exists a Hadamard matrix of order d, i.e., a $(d\times d)$ matrix H_d with entries ± 1 such that $H_d^T H_d = d\,\mathrm{I}_d$, and set

$$A \;=\; \alpha H_d \;=\; \alpha \begin{pmatrix} | & & | \\ v_1 & \cdots & v_d \\ | & & | \end{pmatrix} \;=\; \alpha \begin{pmatrix} \text{---} & w_1 & \text{---} \\ & \vdots & \\ \text{---} & w_d & \text{---} \end{pmatrix},$$

for some real constant $\alpha > 0$ to be determined later. Here the (± 1)-vector v_i denotes the i-th column, and respectively w_j the j-th row of H_d; this implies that $v_i^T v_i = d$ and $v_i^T v_j = 0$ for $j \ne i$, and similarly for the w's. Moreover, set

$$N_l \;=\; E_l A$$

and

$$M_{k-l} \;=\; \beta F_{k-l} H_d^T \;=\; \beta \begin{pmatrix} v_1^T + \cdots + v_{k-l}^T \\ v_1^T + \cdots - v_{k-l}^T \\ \cdots \\ -v_{d-k+l+1}^T - \cdots - v_d^T \end{pmatrix},$$

for $\beta > 0$ another real constant. The displayed pattern of signs and indices in M_{k-l} reflects the one in F_{k-l}, and thus corresponds to a fixed but arbitrary ordering of the rows of F_{k-l}.

We now adjust k, α and β to make these matrices compatible with the conditions in Theorem 5. For this, first note that each row of $N_l = E_l A$ is of the form $\alpha \sum_{i\in I} \sigma_i w_i$, for some index set $I \in \binom{[d]}{l}$ and signs $\sigma_i \in \{\pm 1\}$. In particular, the absolute value of each entry of N_l is bounded by $\alpha l \le \alpha k$, so that α is constrained by

$$\alpha k \;\le\; \frac{1}{2}. \tag{8}$$

Similarly, each element of M_{k-l} is bounded in absolute value by $\beta(k-l) \leq \beta k$, so we also need

$$\beta k \ \leq \ \frac{1}{2}. \tag{9}$$

Because $A = \alpha H_d$, we obtain $M_{k-l}A = \alpha\beta F_{k-l}H_d^T H_d = \alpha\beta d F_{k-l}$, so we must set $\beta = \frac{1}{\alpha d}$ in order to fulfill (7), and thus condition (b) of Theorem 5. Now (8) and (9) taken together say that k, α and d must satisfy

$$\frac{2k}{d} \ \leq \ \alpha \ \leq \ \frac{1}{2k}, \tag{10}$$

so that we arrive at the bound $k \leq \frac{1}{2}\sqrt{d}$ for the cs-neighborliness of our cs-polytope. In fact, for $d \geq 4$, the choices $k := \lfloor \frac{1}{2}\sqrt{d} \rfloor$, $\alpha := \frac{1}{2k}$ and $\beta := \frac{1}{\alpha d} = \frac{2k}{d}$ satisfy (8) and (9), and thus we have found a matrix $M' = (M|N)$ that satisfies conditions (a) and (b) of Theorem 5.

It only remains to check condition (c), i.e., that the rows of M' can in fact be expressed as linear combinations as in (5). For this, note that by the definitions of M_{k-l} and N_l, each row of M' is a sum of row vectors of the form

$$\Big(\alpha \sum_{i\in I} \sigma_i v_i^T \,\Big|\, \mathbf{0}\Big) + \Big(\mathbf{0} \,\Big|\, \beta \sum_{j\in J} \sigma_j w_j\Big)$$

for index sets $I \in \binom{[d]}{k-l}$ and $J \in \binom{[d]}{l}$ and signs $\sigma_i, \sigma_j = \pm 1$, where $\mathbf{0}$ denotes the zero row vector of length d and $0 \leq l \leq k$. We now represent

$$\Big(\alpha \sum_{i\in I} \sigma_i v_i^T \Big| \mathbf{0}\Big) \ = \ \sum_{i\in I} \Big(\tfrac{\alpha}{2}\big(\sigma_i v_i^T \big| \mathbf{1}\big) + \tfrac{\alpha}{2}\big(\sigma_i v_i^T \big| {-\mathbf{1}}\big)\Big), \tag{11}$$

$$\Big(\mathbf{0} \Big| \beta \sum_{j\in J} \sigma_j w_j\Big) \ = \ \sum_{j\in J} \Big(\tfrac{\beta}{2}\big(\mathbf{1} \big| \sigma_j w_j\big) + \tfrac{\beta}{2}\big({-\mathbf{1}} \big| \sigma_j w_j\big)\Big) \tag{12}$$

as linear combinations of $2(k-l)$, respectively $2l$, vectors of length $2d$ with entries ± 1, where $\mathbf{1}$ represents the all-ones vector of length d. The sum over all coefficients in this linear combination is then

$$\begin{aligned}
\sum_{i\in I}\left(\frac{\alpha}{2} + \frac{\alpha}{2}\right) + \sum_{j\in J}\left(\frac{\beta}{2} + \frac{\beta}{2}\right) \ &= \ (k-l)\frac{1}{2k} + l\,\frac{2k}{d} \\
&= \ \frac{1}{2} - l\left(\frac{1}{2k} - \frac{2k}{d}\right) \\
&\leq \ \frac{1}{2}
\end{aligned}$$

by (10), as required. This concludes the proof of Theorem 1.

4. Discussion

We are plainly still quite far away from an explicit construction of $\Theta(d)$-neighborly d-dimensional cs-polytopes with $4d$ vertices. This situation is all too familiar: Linial & Novik find the linear subspace L defined by our matrix A using a probabilistic construction due to Kašin, and remark on the difficulty of explicitly finding such subspaces.

In the light of the discussion in Ball [1, p. 24], our explicit construction of $\Theta(\sqrt{d})$-neighborly cs polytopes using Hadamard matrices is what can reasonably be expected in this context, and it may not be realistic to hope for more: *"There are some good reasons, related to Ramsey theory, for believing that one cannot expect to find genuinely explicit matrices of any kind that would give the right estimates"*.

We close the present note by briefly mentioning some variations and alternatives.

4.1. **Special Hadamard matrices.** The bound $k = O(\sqrt{d})$ arises via (8), (9) from (10) because m is our best a priori upper bound for the largest absolute value of an entry of the sum of m rows (or columns) of H_d. If this largest absolute value could instead be taken of order $O(\sqrt{m})$ for $m = O(d)$, we would reach our goal of a cs-neighborliness of $k = \Theta(d)$.

To address this issue, our construction of Section 3 works for *any* Hadamard matrix, but these are in fact quite a varied and structured lot, cf. [3]. In particular, there exist so-called *regular* Hadamard matrices of order d, for which all the entries of the sum of *all* d rows (or columns) are precisely $\sqrt{d}$. However, this is not good enough for our purposes: it follows from elementary considerations that any row or column of a (conveniently normalized) regular Hadamard matrix contains exactly $\frac{1}{2}(d+\sqrt{d})$ entries '1' and $\frac{1}{2}(d-\sqrt{d})$ entries '-1'; therefore, there exist choices of $l = O(d)$ rows or columns such that the maximal entry of their sum will be $O(d)$ in absolute value, and via (10) this ruins our cs-neighborliness.

Another reason for doubting the efficacy of Hadamard matrices in this respect is that the fraction of the total number of vertices of the $2d$-dimensional cube involved in the concrete instances (11), (12) of the linear combination (5) is quite small.

4.2. **Pseudo-inverses.** Moving away from Hadamard matrices, one should really try to *find* the right matrix A in (6) or (7), instead of prescribing it. In this context, we recall the concept of *generalized inverses*, and refer to [2] for further discussion and notation.

A *Moore-Penrose* $\{1\}$*-inverse* of a real $(m \times n)$ matrix G is any real $(n \times m)$ matrix $G^{(1)}$ such that $GG^{(1)}G = G$; the set of all $\{1\}$-inverses of G is denoted $G\{1\}$. These are important for our purposes because by [2, Theorem 2.1], the matrix equation

$$GA \;=\; H$$

has a solution A if and only if there exists $G^{(1)} \in G\{1\}$ such that

$$(GG^{(1)} - I_m)H \;=\; 0.$$

To apply these notions to our context, we set $G = E + M$ and $H = F + N$, and remark that in the diagonal case $n = d$, an especially nice ordering of the rows of $E'_l = (E_l|F_{k-l})$ is the "doubly lexicographic" one:

Proposition 6. *If $n = d$, one can choose a total order on the rows of E' (that refines the partial order given above), in such a way that the matrices E_l and F_{k-l} satisfy*

$$\begin{aligned} E_l^T E_l &= 2^k \binom{d-1}{l-1} \binom{d}{k-l} I_d, \\ E_l^T F_{k-l} &= 0. \end{aligned}$$

Proof. Let $\tilde{C}_l$ be the matrix of size $\binom{d}{l} \times d$ whose rows are, in lexicographical order, all $0/1$-vectors of length d with exactly l entries '1', set $n(l) = 2^l \binom{d}{l}$, and let C_l be the matrix of size $n(l) \times d$ obtained from $\tilde{C}_l$ by replacing each row with the 2^l rows obtained by choosing all possible signs for the non-zero entries, again in lexicographical order. Thus, the non-zero entries of each column of C_l come in $\binom{d-1}{l-1}$ blocks of size 2^l each, so that the scalar product of each column with itself is $2^l \binom{d-1}{l-1}$. Moreover, it easily follows by induction that distinct columns of C_l are mutually orthogonal, so that

$$C_l^T C_l = 2^l \binom{d-1}{l-1} I_d.$$

Now denote the all-ones column vector of length i by $\mathbf{1}_i$. Then

$$\begin{aligned} E_l &= C_l \otimes \mathbf{1}_{n(k-l)}, \\ F_{k-l} &= \mathbf{1}_{n(l)} \otimes C_{k-l} \end{aligned}$$

combine in such a way that the matrix $(E_l | F_{k-l})$ is a valid representation of E'_l. (Recall that $A \otimes B$ is the matrix obtained from A by replacing each entry a_{ij} by the block matrix $a_{ij} B$, so that $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$.) Now, as claimed,

$$E_l^T E_l = (C_l^T \otimes \mathbf{1}_{n(k-l)}^T)(C_l \otimes \mathbf{1}_{n(k-l)}) = 2^k \binom{d-1}{l-1} \binom{d}{k-l} I_d,$$

and

$$E_l^T F_{k-l} = (C_l^T \otimes \mathbf{1}_{n(k-l)}^T)(\mathbf{1}_{n(l)} \otimes C_{k-l}) = (C_l^T \mathbf{1}_{n(l)}) \otimes (\mathbf{1}_{n(k-l)}^T C_{k-l}) = 0,$$

because, again by induction, the sum of all entries in any column of each C_i vanishes. □

Therefore, we can choose our matrices G and H to be

$$G = \begin{pmatrix} C_k & & + M_k \\ C_{k-1} \otimes \mathbf{1}_{n(1)} & & + M_{k-1} \\ & \ldots & \\ C_1 & \otimes \mathbf{1}_{n(k-l)} & + M_1 \\ & & M_0 \end{pmatrix}, \qquad H = \begin{pmatrix} & & N_0 \\ \mathbf{1}_{n(k-1)} \otimes C_1 & & + N_1 \\ & \ldots & \\ \mathbf{1}_{n(1)} & \otimes C_{k-1} & + N_{k-1} \\ & C_k & + N_k \end{pmatrix}.$$

The set of all $\{1\}$-inverses of G can be parametrized explicitly using the techniques in [9]; however, so far we have not succeeded in turning this to our advantage.

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