DISSECTIONS, Hom-COMPLEXES AND THE CAYLEY TRICK

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Abstract. We show that certain canonical realizations of the complexes $\mathsf{Hom}(G, H)$ and $\mathsf{Hom}_+(G, H)$ of (partial) graph homomorphisms studied by Babson and Kozlov are in fact instances of the polyhedral Cayley trick. For G a complete graph, we then characterize when a canonical projection of these complexes is itself again a complex, and exhibit several well-known objects that arise as cells or subcomplexes of such projected Hom-complexes: the dissections of a convex polygon into k-gons, Postnikov's generalized permutohedra, staircase triangulations, the complex dual to the lower faces of a cyclic polytope, and the graph of weak compositions of an integer into a fixed number of summands.

1. Introduction

A *homomorphism* from a graph G to a graph H is a map $\varphi : V(G) \to V(H)$ between their vertex sets such that $(\varphi(x), \varphi(y))$ is an edge of H whenever (x, y) is an edge of G. The *Hom-complex* $\mathsf{Hom}(G, H)$ is a polytopal complex associated to the set of all homomorphisms from G to H that, intuitively, collects "compatible" homomorphisms into polytopal cells.

Recently, the study of Hom-complexes of graphs has led to a number of successes in topological combinatorics. One example is the recent proof of the *Lovász Conjecture* by Babson and Kozlov [1] (with simplications and extensions by Schultz [11]; see also the excellent survey article [7]). This result provides a lower bound for the chromatic number of a graph G in terms of a topological property (the connectivity) of the associated Hom-complex $\mathsf{Hom}(C, G)$, where C is an odd cycle. In the course of the original proof of the Lovász Conjecture, Babson and Kozlov define a certain simplicial complex $\mathsf{Hom}_+(G, H)$, which is related to the set of all "partial homomorphisms" from G to H, i.e., homomorphisms from an induced subgraph of G to the graph H. The definitions of $\mathsf{Hom}(G, H)$ and $\mathsf{Hom}_+(G, H)$ are purely combinatorial.

One of the first goals of this paper is to show that certain canonical realizations of the complexes $\mathsf{Hom}(G, H)$ and $\mathsf{Hom}_+(G, H)$ in Euclidean space are related via a (by now) rather famous geometric construction, namely the *polyhedral Cayley Trick* due to Sturmfels, Huber, Rambau and Santos [12] [4] [10]. This is done in Theorem 2.7.

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Next, we use the canonical geometric embedding of these complexes to project them, again canonically, to a lower-dimensional subspace. In general, this projection $\pi\mathsf{Hom}(G,H)$ is not itself a polytopal complex because the projected cells need not intersect in common faces. However, we can characterize the shape of these projected cells (Theorem 2.11): They are exactly the *generalized permutohedra* found by Postnikov [8].

In view of our application to dissection complexes, we then concentrate on the special case $G = K_g$. (Note that in the literature on topological methods in graph coloring, one is usually interested in the case $H = K_h$.) In this case, we can characterize when the projected Hom-complexes are themselves polytopal or simplicial complexes (Theorem 3.6): This happens if and only if $\omega(H) = g$, which means that the number of vertices in a largest clique of H is g. Along the way, we define two more complexes associated to Hom-complexes, namely *transversal* complexes $\mathsf{Hom}^t(G,H)$ and *induced* ones, $\mathsf{IHom}(G,H)$; moreover, we show that for any graph H, the 1-skeleton of the projection $\pi\mathsf{Hom}(K_g,H)$ is a subcomplex of the 1-skeleton of a hypersimplex (Proposition 3.5).

We are now ready to apply these tools to *dissection complexes*. For this, consider the set of dissections of a convex $(m(k-2)+2)$-gon into m convex k-gons.[1] We denote by $\delta(k,m)$ the set of all diagonals that can arise in such a dissection, and by $I(k,m)$ the *independence graph* on the vertex set $\delta(k,m)$, i.e., we connect two diagonals by an edge if the relative interiors of the diagonals do not intersect. In Proposition 4.4, we find some old acquaintances inside the projected complexes $\mathsf{D}(k,m) = \pi\mathsf{Hom}\big(K_{m-1}, I(k,m)\big)$ and $\mathsf{D}_+(k,m) = \pi\mathsf{Hom}_+\big(K_{m-1}, I(k,m)\big)$. Namely, the simplicial complex induced on the set of transversal $(m-2)$-dimensional faces of $\mathsf{D}_+(k,m)$ is a simplicial complex $\mathsf{T}(k,m)$ already considered by Tzanaki [13], and the 1-skeleton of $\mathsf{D}(k,m)$ is the *flip graph* on the dissections considered in [5].

Finally, in Section 5 we prove interesting isomorphisms between a certain polytopal complex $\mathcal{C}(r,s)$ whose graph is the graph of all *weak compositions* of the positive integer r into s non-negative summands, a certain induced subcomplex of a *polar-to-cyclic* polytope, and the *staircase triangulation* $\Sigma(r,s)$ of the product of simplices $\Delta^{r-1}\times\Delta^{s-1}$ — of course, here the polyhedral Cayley trick again plays a key role. As our last result, we show that $\mathcal{C}(r,s)$ and $\Sigma(r,s)$ are basic building blocks of $\mathsf{D}(k,m)$, respectively $\mathsf{D}_+(k,m)$.

2. The Cayley trick and Hom-complexes

2.1. The polyhedral Cayley trick.

Let $e_1,\dots,e_a$ be a linear basis of $\mathbb{R}^a$.

Proposition 2.1. [12] [4] [10] *Fix real $\lambda_1,\dots,\lambda_n$ such that each $\lambda_i > 0$ and $\sum_{i=1}^n \lambda_i = 1$. Then, for any polytopes $P_1,\dots,P_n \subset \mathbb{R}^d$, there is an isomorphism between the posets of polyhedral subdivisions of the* Cayley embedding

$$\mathcal{C}(P_1,\dots,P_n) \ = \ \operatorname{conv}\bigcup_{i=1}^{n} P_i\times e_i \ \subset \ \mathbb{R}^d\times\mathbb{R}^n$$

[1] As a historical aside, the interest in these objects goes back at least to 1791, when they were studied by Euler's assistant and student Nikolaus Fuss [3] in St. Petersburg; cf. also [9]

of the P_i's and the poset of mixed subdivisions of the Minkowski sum $\sum_{i=1}^n \lambda_i P_i \subset \mathbb{R}^d$, both ordered by refinement. The bijection between two corresponding subdivisions is given by intersecting a polyhedral subdivision $\mathcal{P}$ of $\mathcal{C}(P_1,\dots,P_n)$ with the d-dimensional plane $L = \mathbb{R}^d \times (\lambda_1,\dots,\lambda_n)$. This intersection produces from each cell $\operatorname{conv}\bigcup_{i=1}^n Q_i \times e_i$ of $\mathcal{P}$ the weighted Minkowski sum $(\sum_{i=1}^n \lambda_i Q_i) \times (\lambda_1,\dots,\lambda_n)$, where $Q_i \subset P_i$ are subpolytopes. □

2.2. **Simultaneous instances of the Cayley trick, related by joins and projections.** To paraphrase Proposition 2.1, the Cayley trick relates a "Cayley object" — namely a cell $Q = \operatorname{conv}\bigcup_{i=1}^n Q_i \times e_i$ of a polyhedral subdivision $\mathcal{P}$ of the Cayley embedding of the polytopes P_1, …, P_n — to its corresponding "Minkowski object", namely the (weighted and embedded) Minkowski sum of the subpolytopes Q_1, …, Q_n of the P_i's. The agent that produces this correspondence is a "morphing plane" L that intersects the cells of the subdivision (and also determines the weights in the Minkowski sum; however, here we will mostly just need the case of equal weights $\lambda_1 = \cdots = \lambda_n = \frac{1}{n}$).

In this paper, we will in fact work with *two* simultaneous "horizontal" instances of the Cayley trick, which will be related to each other by a "vertical" projection called $\pi_\square$. The "bottom" instance of the Cayley trick will be much as we have just outlined, but the "top" instance will be rather special: The top Cayley objects will always be joins of simplices (labeled by "J"), and the top Minkowski objects will be products of polytopes (labeled by "Π"); similarly, we label the bottom Cayley objects by "C" and the bottom Minkowski objects by "M".[2] We summarize this situation in the commutative diagram

$$\begin{array}{ccc} {}^{J}Q & \overset{\iota_L}{\longmapsto} & {}^{\Pi}\big((\sum_{i=1}^n \lambda_i Q_i) \times \boldsymbol{\lambda}\big) \\ \downarrow \pi_\square & & \downarrow \pi_\square \\ {}^{C}(\pi_\square Q) & \overset{\iota_{\pi_\square(L)}}{\longmapsto} & {}^{M}\big((\sum_{i=1}^n \lambda_i \pi_\square(Q_i)) \times \boldsymbol{\lambda}\big), \end{array}$$

where for any union of polytopes $\mathcal{P} \subset \mathbb{R}^d$ and any affine subspace $K \subset \mathbb{R}^d$, we denote the intersection of $\mathcal{P}$ with K by $\iota_K(\mathcal{P}) = \mathcal{P} \cap K$, and $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n)$. The Hom-complexes central to this paper, and their various projections, fit roughly as follows into this diagram:

$$\begin{array}{ccc} {}^{J}(\mathsf{Hom}_+(G,H)) & \overset{\iota_L}{\longmapsto} & {}^{\Pi}(\mathsf{Hom}(G,H) \times \boldsymbol{\lambda}) \\ \downarrow \pi_\square & & \downarrow \pi_\square \\ {}^{C}\big(\pi_\square \mathsf{Hom}_+(G,H)\big) & \overset{\iota_{\pi_\square(L)}}{\longmapsto} & {}^{M}\big(\pi_\square \mathsf{Hom}(G,H) \times \boldsymbol{\lambda}\big) \\ \downarrow \pi_\Delta & & \downarrow \pi_\Delta \\ \mathsf{D}_+(k,m) & & \mathsf{D}(k,m)\,. \end{array}$$

To aid the intuition of the reader, we have also included the dissection complexes associated to the special case $G = K_{m-1}$, $H = I(k,m)$ into this sketch; the projections $\pi_\square$ and π_Δ will be defined in a minute.

[2]Thanks to one of the referees for suggesting this language.

2.2.1. *The top instance.* To explicitly define the objects participating in the "top" instance of the Cayley trick, we first assemble some notation. For sets A and B of respective cardinalities $a = |A|$ and $b = |B|$, denote by Δ_A the simplex $\operatorname{conv}\{e_i : i \in A\} \subset \mathbb{R}^{|A|}$ on the vertex set A, so that $\dim \Delta_A = |A| - 1$, and similarly for Δ_B. We will often not distinguish between a subset $\tau \subset B$ and a face τ of Δ_B.

A key observation is now that the abstract $(ab-1)$-dimensional simplex that arises as the join $\star_{x \in A} \Delta_B$ can be geometrically realized as the Cayley embedding of the polytopes $\mu_1(\Delta_B), \ldots, \mu_a(\Delta_B)$ into $\mathbb{R}^{ab} \times \mathbb{R}^a$, where $\mu_i : \mathbb{R}^b \hookrightarrow \mathbb{R}^{ab}$ is the inclusion of $\mathbb{R}^b$ into the i-th component of $\mathbb{R}^{ab} = \mathbb{R}^b \times \cdots \times \mathbb{R}^b$. We obtain

$$\mathop{\star}_{i \in A} \Delta_B \;=\; \mathcal{C}\big(\mu_1(\Delta_B), \ldots, \mu_a(\Delta_B)\big) \;=\; \operatorname{conv} \bigcup_{i=1}^{a} \mu_i(\Delta_B) \times e_i\,.$$

Observe that this Cayley embedding is indeed a simplex (and therefore equal to the join $\star_{x \in A} \Delta_B$), because the $\mu_i(\Delta_B)$ are affinely independent from each other. Moreover, all faces of $\star_{i \in A} \Delta_B$ are of the form

$$\sigma \;=\; \mathop{\star}_{i \in A} \sigma_i \;=\; \mathcal{C}\big(\mu_1(\sigma_1), \ldots, \mu_a(\sigma_a)\big)$$

for some collection of faces $(\sigma_i : i \in A)$ of Δ_B. In accordance with our earlier discussion, we will sometimes explicitly identify such a face $\sigma = {}^J\sigma$ as being of "join type".

Similarly, the Minkowski object $\frac{1}{a}(\mu_1(\sigma_1) + \cdots + \mu_a(\sigma_a))$ corresponding to ${}^J\sigma$ is in fact a cartesian product $\frac{1}{a}(\mu_1(\sigma_1) \times \cdots \times \mu_a(\sigma_a))$, because the $\mu_i(\sigma_i)$ lie in mutually skew subspaces by construction; hence we will refer to this Minkowski object as being of "product type" Π.

2.2.2. *The projections.* Next, we define the two projections

$$\begin{aligned} \pi_\Box \;&:\; \mathbb{R}^{ab} \times \mathbb{R}^a \;\to\; \mathbb{R}^b \times \mathbb{R}^a, \\ \pi_\Delta \;&:\; \mathbb{R}^b \;\times \mathbb{R}^a \;\to\; \mathbb{R}^b, \end{aligned}$$

as follows. The map π_Δ is just the projection onto the first factor; its purpose is to eliminate the extraneous factor "$\times\boldsymbol{\lambda}$". As for $\pi_\Box$, on the one hand we want it to leave the last factor $\mathbb{R}^a$ (and in particular the point $\boldsymbol{\lambda}$) invariant; on the other, for reasons that will become clear below, we would like it to superimpose all a copies of $\mathbb{R}^b$ in the factor $\mathbb{R}^{ab}$ onto each other. Therefore, we choose the matrix of $\pi_\Box$ to be

$$\begin{pmatrix} \mathbb{1}_b & \cdots & \mathbb{1}_b & 0 \\ 0 & \cdots & 0 & \mathbb{1}_a \end{pmatrix},$$

where the $\mathbb{1}_k$ denote $k \times k$ unit matrices, and the zeros stand for null matrices of the appropriate size. Note that, loosely speaking, each μ_i is a section of $\pi_\Box$, in the sense that $\pi_\Box|_{\mathbb{R}^{ab}} \circ \mu_i = \mathrm{id}_{\mathbb{R}^b}$ for all i. In particular,

$$\pi_\Box(\sigma) \;=\; \pi_\Box\, \mathcal{C}\big(\mu_1(\sigma_1), \ldots, \mu_a(\sigma_a)\big) \;=\; \mathcal{C}(\sigma_1, \ldots, \sigma_a) \tag{1}$$

for any face $\sigma = \star_{i \in A} \sigma_i$ of $\star_{i \in A} \Delta_B$.

Finally, let us fix our "morphing plane" L once and for all as the ab-dimensional plane $L \subset \mathbb{R}^{ab} \times \mathbb{R}^{a}$ defined by

$$L \;=\; \mathbb{R}^{ab} \times \tfrac{\mathbf{1}}{\boldsymbol{a}}\,, \tag{2}$$

where here and throughout we set $\frac{\mathbf{1}}{\boldsymbol{a}} = (\frac{1}{a}, \dots, \frac{1}{a}) \in \mathbb{R}^{a}$, so that $\pi_{\square}(L) = \mathbb{R}^{b} \times \frac{\mathbf{1}}{\boldsymbol{a}}$.

We summarize our discussion in the following proposition.

Proposition 2.2. *Let $\sigma = {}^{J}\sigma = \star_{i\in A}\sigma_i = \mathcal{C}\big(\mu_1(\sigma_1), \dots, \mu_a(\sigma_a)\big)$ be a face of $\star_{i\in A}\Delta_B$. Then the following diagram commutes:*

$$\begin{array}{ccccccc}
\underset{i\in A}{\star}\Delta_B & \supset & \sigma_1 \star \cdots \star \sigma_a & \xmapsto{\iota_L} & \frac{1}{a}\big(\mu_1(\sigma_1) \times \cdots \times \mu_a(\sigma_a)\big) \times \frac{\mathbf{1}}{\boldsymbol{a}} & \subset & \Delta_{A\times B} \times \frac{\mathbf{1}}{\boldsymbol{a}} \\
 & & \downarrow \pi_{\square} & & \downarrow \pi_{\square} & & \\
\Delta_B \times \Delta_A & \supset & \mathcal{C}(\sigma_1, \dots, \sigma_a) & \xmapsto{\iota_{\pi_{\square}(L)}} & \frac{1}{a}\big(\sigma_1 + \cdots + \sigma_a\big) \times \frac{\mathbf{1}}{\boldsymbol{a}} & \subset & \Delta_B \times \frac{\mathbf{1}}{\boldsymbol{a}} \\
 & & \downarrow \pi_{\Delta} & & \downarrow \pi_{\Delta} & & \\
\Delta_B & \supset & \operatorname{conv} \bigcup_{i=1}^{a} \sigma_i & & \frac{1}{a}\big(\sigma_1 + \cdots + \sigma_a\big) & \subset & \Delta_B
\end{array}$$

The (reverse) inclusions on the left-hand side of the diagram map vertices to vertices. This is generally not the case for the inclusions on the right-hand side.

Proof. It suffices to check the top middle square of the diagram. The horizontal maps are well-defined because they are just applications of the polyhedral Cayley trick, and the vertical maps are well-defined by (1) and the linearity of $\pi_{\square}$. Taken together, this also proves commutativity. The rest of the diagram follows by checking the definitions. □

Observation 2.3. If Δ_B is a join $\Delta_B = \star_{j\in C}\Delta_D$ and each σ_i resides in a different copy of Δ_D (which in particular implies $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$), we can glue the first row of another copy of this diagram onto the last row of this one, and in particular fill in the missing map in the last row.

Proof. Include $\Delta_B \subset \mathbb{R}^{b} = \mathbb{R}^{cd}$ into $\mathbb{R}^{cd} \times \mathbb{R}^{c}$, where $c = |C|$ and $d = |D|$, by the map that sends the j-th block of variables, $(x_{(j-1)d+1}, \dots, x_{jd})$, of $\mathbb{R}^{cd}$ to the block $(x_{(j-1)d+1}, \dots, x_{jd}, 1 - x_{(j-1)d+1} - \cdots - x_{jd})$ of $\mathbb{R}^{cd} \times \mathbb{R}^{c}$, for $1 \le j \le c$. This brings $\star_{j\in C}\Delta_D$ into the required canonical form. □

See Example 2.12 below for a detailed calculation with coordinates; here we first present a more conceptual illustration.

Example 2.4. Let $A = \{1, 2\}$, $B = \{3, 4, 5\}$, $\sigma_1 = \{3, 4\}$ and $\sigma_2 = \{4, 5\}$, and let us evaluate the middle row (i.e., the "lower instance of the Cayley trick") of the preceding diagram. We see that $\Delta_B \times \Delta_A$ is a triangular prism with vertex set $B \times A$ (but embedded into $\mathbb{R}^3 \times \mathbb{R}^2$), the Cayley embedding $\mathcal{C}(\sigma_1, \sigma_2)$ is the tetrahedron $T = \operatorname{conv}\{31, 41, 42, 52\}$, and the corresponding cell of the subdivision is the quadrilateral that results from slicing T with $\pi_{\square}(L) = \mathbb{R}^3 \times (\frac{1}{2}, \frac{1}{2})$. When we apply π_{Δ}, on the bottom row of the diagram we obtain on the left-hand side $\operatorname{conv}(\sigma_1 \cup \sigma_2) = \Delta_B$, and on the right-hand side $\frac{1}{2}(\sigma_1 + \sigma_2)$, a quadrilateral in Δ_B that is the Minkowski sum of two edges scaled by $\frac{1}{2}$; see the left-hand side of Figure 1.

On the other hand, if we choose $\sigma_1 = \{3,4\}$ and $\sigma_2 = \{5\}$ to be disjoint, we obtain on the left-hand side of the bottom row again $\text{conv}\{3,4,5\} = \Delta_B$, but on the right-hand side the segment $s = \frac{1}{2}(34+5)$. Observe that in this situation, $\Delta_B = \sigma_1 \star \sigma_2$, and that s arises by applying the Cayley trick to this join. See the right-hand side of Figure 1.

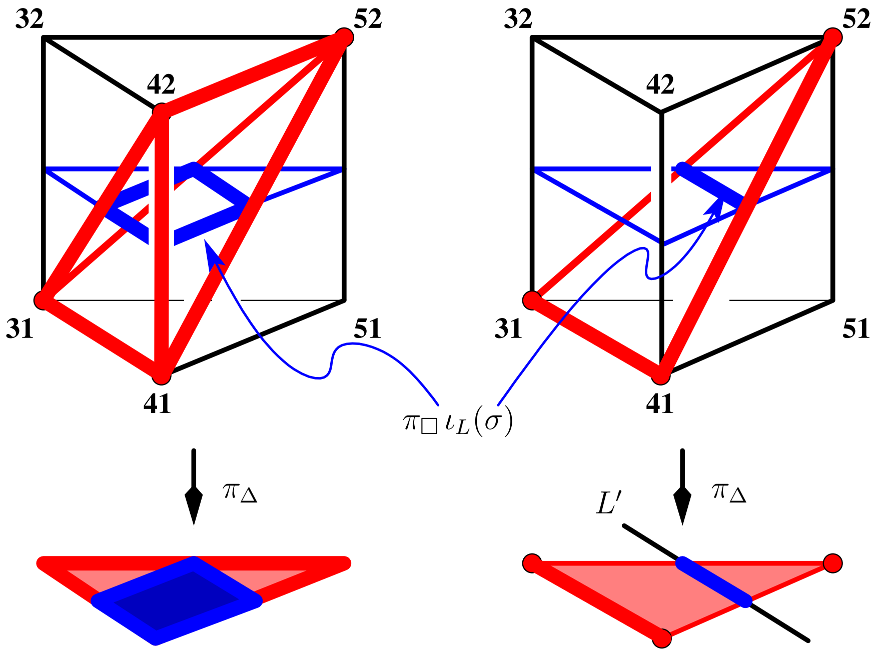

FIGURE 1. Two instances of the projection π_Δ from Proposition 2.2, applied to faces $\mathcal{C}(\sigma_1, \sigma_2) \subset \Delta_{\{3,4,5\}} \times \Delta_{\{1,2\}}$. *Left:* $\sigma_1 = \{3,4\}$ and $\sigma_2 = \{4,5\}$, so that $\sigma_1 \cap \sigma_2 \neq \emptyset$. *Right:* $\sigma_1 = \{3,4\}$ and $\sigma_2 = \{5\}$, so that $\sigma_1 \cap \sigma_2 = \emptyset$. In this case, $\Delta_B = \Delta_{\{3,4,5\}} = \sigma_1 \star \sigma_2$, and we obtain $\pi_\Delta \pi_\square\, \iota_L(\sigma) = \frac{1}{2}(\sigma_1 + \sigma_2) \subset \Delta_B$ via the polyhedral Cayley trick, by intersecting $\pi_\Delta \pi_\square(\sigma) = \text{conv}(\sigma_1 \cup \sigma_2) \subset \Delta_B$ with the affine subspace L' on the bottom right.

2.3. **Hom-complexes.** Let G and H be graphs on $g = |V(G)|$ and $h = |V(H)|$ vertices. When convenient, we will identify $V(G)$ and $V(H)$ with $[g] = \{1, 2, \dots, g\}$, respectively $[h]$. A *homomorphism* from G to H is a map $\varphi : V(G) \to V(H)$ such that for any edge (x, y) of G, $(\varphi(x), \varphi(y))$ is an edge of H.

Recall from [7] the following material related to the set of all homomorphisms between G and H. Let $\times_{x \in V(G)} \Delta_{V(H)}$ denote the cartesian product of g copies of the simplex $\Delta_{V(H)}$, so that the copies of $\Delta_{V(H)}$ are labeled by the vertices of G. Similarly, $\star_{x \in V(G)} \Delta_{V(H)}$ is the join of g labeled copies of $\Delta_{V(H)}$. Note that $\times_{x \in V(G)} \Delta_{V(H)}$ is a $g(h-1)$-dimensional polytope that is a product of simplices, while $\star_{x \in V(G)} \Delta_{V(H)}$ is a simplex of dimension $gh-1$. We will always think of $\star_{x \in V(G)} \Delta_{V(H)}$ as being embedded in $\mathbb{R}^{gh} \times \mathbb{R}^g$ as in Proposition 2.2.

The following two complexes have proved to be useful in topological combinatorics; see [7] for a survey.

Definition 2.5. [7]
(a) $\mathsf{Hom}(G,H)$ is the polytopal subcomplex of $\times_{x\in V(G)}\Delta_{V(H)}$ of all cells $\times_{x\in V(G)}\sigma_x$ such that if $(x,y)\in E(G)$, then (σ_x,σ_y) is a complete bipartite subgraph of H.
(b) $\mathsf{Hom}_+(G,H)$ is the simplicial subcomplex of $\star_{x\in V(G)}\Delta_{V(H)}$ of all simplices $\star_{x\in V(G)}\sigma_x$ such that if $(x,y)\in E(G)$ and both σ_x and σ_y are nonempty, then (σ_x,σ_y) is a complete bipartite subgraph of H.

Note that the bipartite subgraphs are not required to be induced. Moreover, by definition all cells of $\mathsf{Hom}(G,H)$ are products of simplices. We will sometimes identify faces $\sigma={}^{\Pi}\sigma=\times_{x\in V(G)}\sigma_x$ of $\mathsf{Hom}(G,H)$, respectively faces $\sigma={}^{J}\sigma=\star_{x\in V(G)}\sigma_x$ of $\mathsf{Hom}_+(G,H)$, with the ordered list of (non-empty) labels $(\lambda_1,\dots,\lambda_g)$, where $V(G)=[g]$ and $\lambda_i\subset V(H)$ is the vertex set of the simplex σ_i. Moreover, define $L=\mathbb{R}^{gh}\times\frac{\mathbf{1}}{\boldsymbol{g}}$ as in (2).

Definition 2.6. A face $\star_{x\in V(G)}\sigma_x$ is *transversal* if $|\sigma_x|>0$ for all $x\in V(G)$. The simplicial complex $\mathsf{Hom}^t_+(G,H)$ is the subcomplex of $\mathsf{Hom}_+(G,H)$ induced by the set of all transversal faces.

Theorem 2.7. (i) $\iota_L\mathsf{Hom}^t_+(G,H)=\iota_L\mathsf{Hom}_+(G,H)=\mathsf{Hom}(G,H)\times\frac{\mathbf{1}}{\boldsymbol{g}}$. *In particular, we obtain a canonical embedding of all these complexes into the same Euclidean space.*
(ii) *The following diagram commutes:*

$$\begin{array}{ccccccc}
\underset{x\in V(G)}{\star}\Delta_{V(H)} & \supset & \mathsf{Hom}_+(G,H) & \xrightarrow{\iota_L} & \mathsf{Hom}(G,H)\times\frac{\mathbf{1}}{\boldsymbol{g}} & \subset & \Delta_{V(G)\times V(H)}\times\frac{\mathbf{1}}{\boldsymbol{g}}\\
 & & \downarrow\pi_\Box & & \downarrow\pi_\Box & & \\
\Delta_{V(H)}\times\Delta_{V(G)} & \supset & \pi_\Box\mathsf{Hom}_+(G,H) & \xrightarrow{\iota_{\pi_\Box(L)}} & \pi_\Box\mathsf{Hom}(G,H)\times\frac{\mathbf{1}}{\boldsymbol{g}} & \subset & \Delta_{V(H)}\times\frac{\mathbf{1}}{\boldsymbol{g}}
\end{array}$$

In particular, the image $\pi_\Box(\sigma)$ of any face σ of $\mathsf{Hom}_+(G,H)$ is the convex hull of some vertices of $\Delta_{V(H)}\times\Delta_{V(G)}$, and $\pi_\Box\mathsf{Hom}(G,H)=\iota_{\pi_\Box(L)}\pi_\Box\mathsf{Hom}_+(G,H)$.
(iii) *The same statements hold with $\mathsf{Hom}_+(G,H)$ replaced by $\mathsf{Hom}^t_+(G,H)$.*

Proof. (i) For the first equality, let $\sigma={}^{J}\sigma=\star_{x\in V(G)}\sigma_x$ be a simplex of $\mathsf{Hom}_+(G,H)$. Since $\star_{x\in G}\Delta_{V(H)}$ is embedded in $\mathbb{R}^{gh}\times\mathbb{R}^g$, any point $z\in\sigma$ can be written as the convex combination

$$z\;=\;\sum_{i=1}^{g}\lambda_i\sum_{v\in\operatorname{vert}\sigma_i}\lambda'_{iv}\mu_i(v)\times e_i\;\in\;\mathbb{R}^{gh}\times\mathbb{R}^g,\tag{3}$$

where $\sum_{i=1}^g\lambda_i=1$ and $\sum_{v\in\operatorname{vert}\sigma_i}\lambda'_{iv}=1$ for all $i=1,2,\dots,g$. Now look at the i-th entry of the $\mathbb{R}^g$-component of z. If $|\sigma_i|>0$, it is $\lambda_i\sum_{v\in\operatorname{vert}\sigma_i}\lambda'_{iv}=\lambda_i$, but otherwise it vanishes because the inner sum in (3) is empty. Therefore, no simplex of $\mathsf{Hom}_+(G,H)\setminus\mathsf{Hom}^t_+(G,H)$ intersects L. (However, notice that not all faces of $\mathsf{Hom}^t_+(G,H)$ intersect L.)

The second equality follows from the top row of the diagram in Proposition 2.2 with $a = g$ and $b = h$: Observe that $\iota_L({}^J\sigma) = {}^{\Pi}(\mu_1(\sigma_1) + \cdots + \mu_g(\sigma_g)) = {}^{\Pi}(\mu_1(\sigma_1) \times \cdots \times \mu_g(\sigma_g))$ for any simplices $\sigma_1, \ldots, \sigma_g \subset \Delta_{V(H)}$, because the $\mu_i(\sigma_i)$ lie in skew subspaces by construction.

(ii) Let $\sigma = \star_{i\in V(G)}\sigma_i$ be a face of $\mathsf{Hom}_+(G,H)$, where $\sigma_i \subset \Delta_{V(H)}$ for all $i \in V(G)$. The image under $\pi_\Box$ of ${}^J\sigma = \operatorname{conv} \bigcup_{i=1}^g \mu_i(\sigma_i) \times e_i$ is ${}^C(\pi_\Box(\sigma)) = \operatorname{conv} \bigcup_{i=1}^g \sigma_i \times e_i$, which is the convex hull of some vertices of $\Delta_{V(H)} \times \Delta_{V(G)}$. The well-definedness and commutativity of the diagram now follows from the fact that $\iota_L \mathsf{Hom}_+(G,H) = \mathsf{Hom}(G,H)$ and Proposition 2.2, after passing to faces.

Finally, (iii) follows from (i). ☐

2.4. **Projections of Hom-complexes and generalized permutohedra.** Suppose that $\mathsf{Hom}(G,H)$ is embedded into $\mathbb{R}^{gh} \times \mathbb{R}^g$ as in our previous discussion, and denote the symmetry groups of the graphs G and H by $S_G = \operatorname{Aut}(G)$, respectively $S_H = \operatorname{Aut}(H)$. By [7], the product group $S_G \times S_H$ acts on $\mathsf{Hom}(G,H)$.

Here and in the following we will use the notation $\pi = \pi_\Delta \pi_\Box : \mathbb{R}^{gh} \times \mathbb{R}^g \to \mathbb{R}^h$.

Proposition 2.8. *$\pi\big(\gamma(\sigma)\big) = \pi(\sigma)$ for any $\gamma \in S_G$ and any cell σ of $\mathsf{Hom}(G,H)$.*

Proof. Let $\sigma = {}^{\Pi}\sigma = (\times_{i\in V(G)}\sigma_i) \times \frac{\mathbf{1}}{\boldsymbol{g}}$ be a cell of $\mathsf{Hom}(G,H) \times \frac{\mathbf{1}}{\boldsymbol{g}}$ of "product type", and let $\sigma' = {}^{\Pi}(\sigma') = \big(\times_{i\in V(G)} \sigma_{\gamma(i)}\big) \times \frac{\mathbf{1}}{\boldsymbol{g}}$ be its image under $\gamma \in S_G$. We will chase σ' around the diagram of Theorem 2.7 to verify that $\pi_\Box(\sigma) = \pi_\Box(\sigma')$ in $\pi_\Box\mathsf{Hom}(G,H)$; since π_Δ restricted to $\pi_\Box\mathsf{Hom}(G,H)$ is an isomorphism, this implies that their images in $\pi\mathsf{Hom}(G,H)$ coincide.

First, let $\tilde\sigma$ and $\tilde\sigma'$ be the unique simplicial faces of $\mathsf{Hom}_+(G,H)$ such that $\iota_L(\tilde\sigma) = \sigma$ and $\iota_L(\tilde\sigma') = \sigma'$. We can write any point $z \in \pi_\Box(\tilde\sigma)$ as a convex combination

$$z \;=\; \sum_{i\in V(G)} \lambda_i \; x_i \times e_i$$

of points $x_i = \sum_{v\in \operatorname{vert}\sigma_i} \lambda_{iv} v \in \sigma_i$, so that

$$z' \;=\; \sum_{i\in V(G)} \lambda_i \; x_{\gamma(i)} \times e_i$$

is the corresponding point in $\pi_\Box(\tilde\sigma')$ under the action of γ. Now note that the h-plane $\pi_\Box(L) = \mathbb{R}^h \times \frac{\mathbf{1}}{\boldsymbol{g}}$ only intersects those cells $\pi_\Box(\tilde\sigma)$ with all σ_i non-empty; the reason (as in the proof of Theorem 2.7) is that $\sigma_j = \emptyset$ forces the j-th component of the $\mathbb{R}^g$-part of $z \in \mathbb{R}^h \times \mathbb{R}^g$ to be zero. Therefore, intersecting σ and $\tilde\sigma$ with $\pi_\Box(L)$ forces $\lambda_i = \frac{1}{g}$ for all $1 \le i \le g$, so that the images of z and z' under the map $\iota_{\pi_\Box(L)} \circ \pi_\Box$ agree, which is what we wanted to show. ☐

Observation 2.9. If $\operatorname{Aut}(G) \subsetneq S_{|G|}$ is not the full symmetric group on $|G|$ letters, then $\pi(\sigma) = \pi(\tau)$ may hold for faces σ, τ of $\mathsf{Hom}(G,H)$, even though τ is not of the form $\gamma(\sigma)$ for any $\gamma \in \operatorname{Aut}(G)$. Because of this, it would not be correct to say that each cell of $\pi\mathsf{Hom}(G,H)$ represents an S_G-equivalence class of faces of the polytopal complex $\mathsf{Hom}(G,H)$.

Definition 2.10. $\pi\mathsf{Hom}(G,H) = \{\pi(\sigma) : \sigma \text{ is a cell of } \mathsf{Hom}(G,H)\}$.

This is not in general a polytopal complex, because cells need not intersect in common faces. However, the next theorem identifies the faces of $\pi\mathsf{Hom}(G,H)$ as "generalized permutohedra", introduced by Postnikov [8].

For this, let $\Gamma \subset K_{m,n}$ be a bipartite graph with m "left" and n "right" vertices. We agree to denote its edges by (i,j), where i is in the "left" part and j in the "right" part of Γ.

For any such Γ, Postnikov defined the *generalized permutohedron* $P_\Gamma(\lambda_1,\dots,\lambda_m)$ to be the weighted Minkowski sum of simplices $P_\Gamma = \lambda_1\Delta_{I_1} + \dots + \lambda_m\Delta_{I_m}$, where $\lambda_i > 0$ and $I_i = I_i(\Gamma) = \{j : (i,j) \in \Gamma\} \subset [n]$ for $i = 1,2,\dots,m$. These may be obtained via the polyhedral Cayley Trick from the *root polytopes* $Q_\Gamma = \mathcal{C}(\Delta_{I_1},\dots,\Delta_{I_m}) \subset \Delta_{[n]} \times \Delta_{[m]}$, by intersecting with the subspace $\mathbb{R}^n \times (\frac{\lambda_1}{\lambda},\dots,\frac{\lambda_m}{\lambda})$, where $\lambda = \sum_{i=1}^m \lambda_i$ [8, Corollary 14.6].

Theorem 2.11. *Let G and H be graphs. Then any cell of $\pi\mathsf{Hom}(G,H)$ is a generalized permutohedron, and any generalized permutohedron occurs as a cell of some $\pi\mathsf{Hom}(G,H)$. Moreover, a cell $\pi(\sigma)$ of $\pi\mathsf{Hom}(G,H)$ is a product of simplices if and only if $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.*

Proof. We start with a Hom-complex $\mathsf{Hom}(G,H)$, where G and H are graphs with g, respectively h vertices. Let $\sigma = \sigma_1 \times \dots \times \sigma_g$ be a cell of $\mathsf{Hom}(G,H)$, embedded in $\Delta_{V(H)}$ as in Theorem 2.7. (We will temporarily forget about the extra factor "$\times \frac{\mathbf{1}}{\boldsymbol{g}}$" here.) Moreover, let $\tilde\sigma$ be the unique simplicial cell of $\mathsf{Hom}_+(G,H)$ such that $\sigma = \iota_L(\tilde\sigma)$, and denote by $W = \bigcup_{i=1}^g \sigma_i \subset V(H)$ the union of all σ_i, regarded as subsets of $V(H)$. Now define the bipartite graph $\Gamma \subset K_{g,|W|}$ by connecting a left vertex i to a right vertex j whenever $j \in \sigma_i$. This graph defines a root polytope $Q_\Gamma = Q_\Gamma(1,\dots,1) = \mathcal{C}(\sigma_1,\dots,\sigma_g)$, and we only have to check that $Q_\Gamma = \pi_\Box\tilde\sigma$. But this is clear by the diagram of Proposition 2.2.

In the other direction, let us first suppose that we are given a generalized permutohedron $P_\Gamma(1,\dots,1)$ for some bipartite graph $\Gamma \subset K_{m,n}$. Define $\sigma_i = \{j : (i,j) \in \Gamma\} \subset [n]$ for $i = 1,2\dots,m$, and set $G_\Gamma = \operatorname{Ind}(\sigma_1,\dots,\sigma_m)$, the *independence graph* that has the σ_i's as vertices, and in which σ_i is joined to σ_j by an edge precisely if $\sigma_i \cap \sigma_j = \emptyset$. Moreover, define a graph H_Γ on the vertex set $[n]$ by adding the edges of a complete bipartite graph (σ_i,σ_j) whenever σ_i and σ_j form an edge in G_Γ. Checking the definitions yields that $\pi\mathsf{Hom}(G_\Gamma,H_\Gamma)$ contains a cell of the form $P_\Gamma(1,\dots,1)$. The more general case of permutohedra of the form $P_\Gamma(\lambda_1,\dots,\lambda_m)$ follows by constructing $\mathsf{Hom}(G_\Gamma,H_\Gamma)$ as the slice $\mathsf{Hom}_+(G_\Gamma,H_\Gamma) \cap \big(\mathbb{R}^{|G_\Gamma|\cdot|H_\Gamma|} \times (\frac{\lambda_1}{\lambda},\dots,\frac{\lambda_m}{\lambda})\big)$, where $\lambda = \sum_{i=1}^m \lambda_i$. □

Example 2.12. Let $\Gamma \subset K_{3,3}$ be the bipartite graph with edge set $\{\bar11,\bar12,\bar21,\bar23,\bar32,\bar33\}$, where we write left vertices with bars. The proof of the preceding theorem yields graphs G_Γ, H_Γ such that $\pi\mathsf{Hom}(G_\Gamma,H_\Gamma)$ contains a hexagon $P_\Gamma(1,1,1)$. Namely, the proof calls for setting $G_\Gamma = H_\Gamma = E_3$, the graph with 3 vertices and no edges, so that $\mathsf{Hom}(G_\Gamma,H_\Gamma) = \Delta_{[3]} \times \Delta_{[3]}$ and $S_{G_\Gamma} = S_3$, the symmetric group on 3 letters. Moreover, $\sigma_{\bar1} = \{1,2\}$, $\sigma_{\bar2} = \{1,3\}$ and $\sigma_{\bar3} = \{2,3\}$, so that the vertices of the simplicial cell $\sigma = \operatorname{conv}\bigcup_{i=1}^3 \mu_i(\sigma_{\bar i}) \times e_i$

in $\mathsf{Hom}_+(G_\Gamma, H_\Gamma)$ are the rows of the following matrix:

$$\begin{matrix} 1&0&0 & 0&0&0 & 0&0&0 & 1&0&0\\ 0&1&0 & 0&0&0 & 0&0&0 & 1&0&0\\ 0&0&0 & 1&0&0 & 0&0&0 & 0&1&0\\ 0&0&0 & 0&0&1 & 0&0&0 & 0&1&0\\ 0&0&0 & 0&0&0 & 0&1&0 & 0&0&1\\ 0&0&0 & 0&0&0 & 0&0&1 & 0&0&1 \end{matrix}$$

Intersecting the convex hull of these points (a 5-dimensional simplex in $\mathbb{R}^{12}$) with the 9-dimensional plane $L = \mathbb{R}^{3\cdot 3} \times (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ yields the 3-cube cell $\sigma \cap L$ of $\mathsf{Hom}(G_\Gamma, H_\Gamma) \times \mathbf{\frac{1}{3}}$ whose vertices are $\frac{1}{3}$ times the row vectors

$$\begin{matrix} 1&0&0 & 1&0&0 & 0&0&1 & 1&1&1\\ 1&0&0 & 0&0&1 & 0&0&1 & 1&1&1\\ 0&1&0 & 1&0&0 & 0&0&1 & 1&1&1\\ 0&1&0 & 0&0&1 & 0&0&1 & 1&1&1\\ 1&0&0 & 1&0&0 & 0&1&0 & 1&1&1\\ 1&0&0 & 0&0&1 & 0&1&0 & 1&1&1\\ 0&1&0 & 1&0&0 & 0&1&0 & 1&1&1\\ 0&1&0 & 0&0&1 & 0&1&0 & 1&1&1. \end{matrix}$$

Projecting this 3-cube down to $\mathbb{R}^3 \times \mathbb{R}^3$ by $\pi_\square$ yields, in order, $\frac{1}{3}$ times the points

$$\begin{matrix} 2&0&1 & 1&1&1\\ 1&0&2 & 1&1&1\\ 1&1&1 & 1&1&1\\ 0&1&2 & 1&1&1\\ 2&1&0 & 1&1&1\\ 1&1&1 & 1&1&1\\ 1&2&0 & 1&1&1\\ 0&2&1 & 1&1&1, \end{matrix}$$

and applying π_Δ now has the effect of eliminating the last three coordinates. We have found the cell $\pi(\sigma \cap L)$ of $\pi\mathsf{Hom}(G_\Gamma, H_\Gamma)$, namely $\frac{1}{3}$ times the convex hull of the points

$$(2,0,1),\ (1,0,2),\ (1,1,1),\ (0,1,2),\ (2,1,0),\ (1,1,1),\ (1,2,0),\ (0,2,1),$$

which is indeed the hexagon $Q_\Gamma(1,1,1)$. Notice how two antipodal vertices of $\sigma \cap L$ are projected down to the same point $(1,1,1)$ in the interior of $Q_\Gamma(1,1,1)$, and thus play no role in the convex hull of $\pi(\sigma \cap L)$.

3. The case $G = K_g$

In this paper, we will focus especially on the case where G is the complete graph on g vertices, with $V(G) = [g]$. The case $H = K_h$ has been widely studied in connection with coloring problems on graphs; see [7] for a survey.

3.1. **Projections and orbits of the symmetry group.** Both complexes $\mathsf{Hom}_+(K_g, H)$ and $\mathsf{Hom}(K_g, H)$ admit an S_g-action by permuting the vertices of K_g, where S_g is the symmetric group on g letters. By Proposition 2.8, and in contrast to the situation of Observation 2.9, it is now the case that each cell of $\pi\mathsf{Hom}(K_g, H)$ represents an S_g-equivalence class of faces of the polytopal complex $\mathsf{Hom}(K_g, H)$, so that we can think of $\pi\mathsf{Hom}(K_g, H)$ as a "quotient" $\mathsf{Hom}(K_g, H)/S_g$. Any cell $\pi(\sigma)$ of $\pi\mathsf{Hom}(K_g, H)$ is a product of simplices, because $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$ by the definition of $\mathsf{Hom}(K_g, H)$ and because H is loopless. However, in general $\pi\mathsf{Hom}(K_g, H)$ is not a polytopal complex; we will give a characterization for when this happens in Theorem 3.6 below. Before this, we prove that $\pi\mathsf{Hom}_+(K_g, H)$ and $\pi\mathsf{Hom}^t_+(K_g, H)$ have analogous properties to $\pi\mathsf{Hom}(K_g, H)$:

Theorem 3.1.

(i) *$\pi_\Box\mathsf{Hom}_+(K_g, H)$ is a simplicial immersion of $\mathsf{Hom}_+(K_g, H)$ into $\Delta_{V(H)} \times \Delta_{[g]}$. This means that locally on each simplex of $\mathsf{Hom}_+(K_g, H)$, the projection $\pi_\Box$ is a bijection onto a simplex whose vertices are among the vertices of $\Delta_{V(H)} \times \Delta_{[g]}$, but the images of different faces may intersect. Put differently, $\mathsf{Hom}_+(K_g, H)$ is a "horizontal" complex, i.e., it has no faces in the kernel of $\pi_\Box$. In summary, the following diagram commutes:*

$$\begin{array}{ccccccc}
\underset{i\in[g]}{\star}\Delta_{V(H)} & \supset & \mathsf{Hom}_+(K_g,H) & \xrightarrow{\iota_L} & \mathsf{Hom}(K_g,H)\times\frac{\mathbf{1}}{\mathbf{g}} & \subset & \Delta_{[gh]}\times\frac{\mathbf{1}}{\mathbf{g}} \\
 & & \downarrow\pi_\Box & & \downarrow\pi_\Box & & \\
\Delta_{V(H)}\times\Delta_{[g]} & \supset & \pi_\Box\mathsf{Hom}_+(K_g,H) & \xrightarrow{\iota_{\pi_\Box L}} & \pi\mathsf{Hom}(K_g,H)\times\frac{\mathbf{1}}{\mathbf{g}} & \subset & \Delta_{V(H)}\times\frac{\mathbf{1}}{\mathbf{g}} \\
 & & \downarrow\pi_\Delta & & \downarrow\pi_\Delta & & \\
\Delta_{V(H)} & \supset & \pi\mathsf{Hom}_+(K_g,H) & & \pi\mathsf{Hom}(K_g,H) & \subset & \Delta_{V(H)}
\end{array}$$

(ii) *Each cell of $\pi\mathsf{Hom}_+(K_g, H)$ represents an S_g-equivalence class of faces of the simplicial complex $\mathsf{Hom}_+(K_g, H)$.*

(iii) *The same statements hold with $\mathsf{Hom}_+(K_g, H)$ replaced by $\mathsf{Hom}^t_+(K_g, H)$.*

Proof. (i) We first check that $\pi_\Box(\sigma)$ is a simplex. For this, note that $\sigma_i \cap \sigma_j = \emptyset$ for any $i \neq j \in [g]$ by definition of $\mathsf{Hom}_+(K_g, H)$, because K_g is complete and H is loopless. Therefore, all simplices $\sigma_i \times e_i$ lie in skew subspaces of $\mathbb{R}^h \times \mathbb{R}^g$, and their convex hull is a simplex. It remains to check that no face σ of $\mathsf{Hom}_+(K_g, H)$ lies in the kernel of $\pi_\Box$. For this, suppose that there is some edge σ in $\mathsf{Hom}_+(K_g, H)$ that gets mapped to a point w by $\pi_\Box$. Then $\sigma = \star_{i\in[g]}\sigma_i$ with $\sigma_j = \sigma_k = \{w\}$ for some $j \neq k$ and $\sigma_i = \emptyset$ for $i \neq j, k$, but this again contradicts the fact that $\sigma_i \cap \sigma_j = \emptyset$ for any $i \neq j$.

Therefore, the restriction of $\pi_\Box$ to any face of $\mathsf{Hom}_+(K_g, H)$ is a bijection onto some simplex that is the convex hull of vertices of $\Delta_{V(H)} \times \Delta_{[g]}$, but the images of different faces of $\mathsf{Hom}_+(K_g, H)$ may in general intersect.

Part (ii) is now an easy consequence of noting that $\pi\mathsf{Hom}_+(K_g, H)$ is indeed a simplicial subcomplex of $\Delta_{V(H)}$, and that we obtain all $g!$ faces in the S_g-orbit of a given face $\sigma \cong \pi_\Box(\sigma)$ by lifting each simplex $\sigma_i \subset \Delta_{V(H)}$ to $\sigma_i \times e_{\pi(i)}$, for all $\pi \in S_g$.

Finally, (iii) follows from Theorem 2.7 (i). $\Box$

In view of this theorem, $\pi\mathsf{Hom}_+^t(K_g, H)$ is the simplicial subcomplex of $\Delta_{V(H)}$ that is induced by the family of simplices

$$\{\Delta_V : V \text{ is the vertex set of a complete } g\text{-partite subgraph of } H\}.$$

The analogous construction $\pi\mathsf{Hom}_+(K_g, H)$ is not very interesting: since $\mathsf{Hom}_+(K_g, H)$ contains faces of the form $(\Delta_{V(H)}, \emptyset, \dots, \emptyset)$, we just get the whole simplex $\Delta_{V(H)}$.

Remark 3.2. At the level of faces, the left column of the diagram of Theorem 3.1 reads

$$\begin{array}{ccccccc}
\sigma & = & \operatorname{conv} \bigcup_{i=1}^g \mu_i(\sigma_i) \times e_i & \subset & \mathsf{Hom}_+^{(t)}(K_g, H) & \subset & \mathop{\star}\limits_{i\in[g]} \Delta_{V(H)} \\
 & & \downarrow \pi_\Box & & \downarrow \pi_\Box & & \downarrow \pi_\Box \\
\pi_\Box(\sigma) & = & \operatorname{conv} \bigcup_{i=1}^g \sigma_i \times e_i & \subset & \pi_\Box\mathsf{Hom}_+^{(t)}(K_g, H) & \subset & \Delta_{V(H)} \times \Delta_{[g]} \\
 & & \downarrow \pi_\Delta & & \downarrow \pi_\Delta & & \downarrow \pi_\Delta \\
\pi(\sigma) & = & \operatorname{conv} \bigcup_{i=1}^g \sigma_i & \subset & \pi\mathsf{Hom}_+^{(t)}(K_g, H) & \subset & \Delta_{V(H)}
\end{array}$$

3.2. **When is $\pi\mathsf{Hom}(K_g, H)$ a polytopal complex?** To answer this question, we introduce some special subcomplexes of Hom-complexes:

Definition 3.3. Let G, H be loopless graphs. The *induced* Hom-*complexes* $\mathsf{IHom}(G, H)$ and $\mathsf{IHom}_+(G, H)$ are the subcomplexes of $\mathsf{Hom}(G, H)$, respectively $\mathsf{Hom}_+(G, H)$, obtained by considering only induced complete bipartite subgraphs.

Recall that the *clique number* $\omega(H)$ of H is the number of vertices in a largest clique in H.

Proposition 3.4. *Let H be a loopless graph with $\omega(H) \geq g \geq 1$. Then* $\mathsf{IHom}(K_g, H) = \mathsf{Hom}(K_g, H)$ *and* $\mathsf{IHom}_+(K_g, H) = \mathsf{Hom}_+(K_g, H)$ *if and only if* $\omega(H) = g$.

Proof. A complete g-partite subgraph $\sigma = \sigma_1 \cup \dots \cup \sigma_g$ of H is not induced if and only if there exists an edge of H connecting two vertices x, y in the same part of σ, which we may assume to be σ_1; but then we can find a clique of size $g+1$ by taking x, y, and one vertex each from $\sigma_2, \dots, \sigma_g$. Conversely, any clique $\{v_1, \dots, v_{g+1}\}$ of size $g+1$ yields a non-induced complete g-partite graph $\sigma = \{v_1, v_2\} \cup \{v_3\} \cup \dots \cup \{v_{g+1}\}$. (Finally, note that $\mathsf{Hom}(K_g, H) = \emptyset$ if $\omega(H) < g$.) $\square$

Another ingredient to our answer is the *hypersimplex* $\Delta(h, g) \subset \mathbb{R}^h$, the convex hull of the set of all 0/1-vectors of length h with exactly g ones. It is an $(h-1)$-dimensional polytope that may be thought of as the slice of the h-dimensional 0/1-cube with the plane $\{\boldsymbol{x} \in \mathbb{R}^h : \sum_{i=1}^h x_i = g\}$. All faces of a hypersimplex are again hypersimplices, and its edges are spanned by pairs of vertices whose coordinates differ in exactly two locations.

Proposition 3.5. *For any (loopless) graph H, the* 1*-skeleton of* $\pi\mathsf{Hom}(K_g, H)$ *is a subcomplex of the* 1*-skeleton of the hypersimplex* $\Delta(h, g)$.

Proof. By abuse of notation, we replace $\pi\mathsf{Hom}(K_g, H)$ by a g times inflated copy but keep the same name. Let σ be a face of $\mathsf{Hom}(K_g, H)$, so that $\sigma_i \cap \sigma_j = \emptyset$ for all $1 \le i \ne j \le g$. As in the proof of Proposition 2.8, we can write any point $z \in \pi(\sigma)$ as

$$z \;=\; \sum_{i \in V(G)} \sum_{v \in \operatorname{vert} \sigma_i} \lambda_{iv} v \,.$$

The vertices of $\pi(\sigma)$ are 0/1-vectors of length h (because the vertices of the σ_i's are vertices of $\Delta_H \subset \mathbb{R}^h$) with exactly g ones (there are g mutually disjoint simplices σ_i, and to get a vertex of σ, exactly one λ_{iv} must be 1, for each i); in other words, they are vertices of $\Delta(h, g)$. The statement now follows because the sets σ_i of any edge σ of $\mathsf{Hom}(K_g, H)$ all have size 1, except for exactly one set of size 2. Therefore, the coordinates of the vertices of $\pi(\sigma)$ differ in exactly two places. ☐

We are now in a position to answer the question posed at the beginning of this section:

Theorem 3.6. *$\pi\mathsf{Hom}(K_g, H)$ is a non-empty polytopal complex, and $\pi\mathsf{Hom}_+(K_g, H)$ a non-empty simplicial complex, if and only if $\omega(H) = g$.*

Proof. Let supp : $\mathbb{R}^h \to [h]$ denote the map that assigns to any vector $v \in \mathbb{R}^h$ the set of all indices $i \in [h]$ such that $v_i \ne 0$. Moreover, for any face $\sigma = (\sigma_1, \dots, \sigma_g)$ of $\mathsf{Hom}(K_g, H)$, let G_σ denote the complete g-partite subgraph of H on the vertex set $V_\sigma = \sigma_1 \cup \dots \cup \sigma_g \subset [h]$ that is associated to σ by the definition of faces of Hom-complexes.

We want to rule out that two faces $\pi(\sigma)$ and $\pi(\tau)$ of $\pi\mathsf{Hom}(K_g, H)$ intersect badly, where σ, τ are faces of $\mathsf{Hom}(K_g, H)$. If they do, there exists a circuit (i.e., a minimal affine dependency) $C = C_+ \cup C_-$ among the vertices of $\Delta(h, g)$, such that $C_+ \subset \operatorname{vert} \pi(\sigma)$, $C_- \subset \operatorname{vert} \pi(\tau)$ and $C_+ \cap C_- = \emptyset$. In this case, we can find sets of positive real numbers $\{\lambda_v : v \in C_+\}$ and $\{\mu_w : w \in C_-\}$ with $\sum_{v \in C_+} \lambda_v = 1$ and $\sum_{w \in C_-} \lambda_w = 1$, such that $\sum_{v \in C_+} \lambda_v v = \sum_{w \in C_-} \lambda_w w$. In particular,

$$\operatorname{supp} \sum_{v \in C_+} v \;=\; \operatorname{supp} \sum_{w \in C_-} w \;=:\; V \subset [h] \,. \tag{4}$$

We claim that V is the vertex set of an — at this point not necessarily unique — complete g-partite subgraph of H of the form G_ρ for some face ρ of $\mathsf{Hom}(K_g, H)$. Indeed, each vertex in C yields a g-clique in H contained in the vertex set V. Because $C_+ \subset \operatorname{vert} \pi(\sigma)$, the vertices in C_+ yield a complete (but not necessarily induced) g-partite graph G_{ρ_+} on the vertex set V that corresponds to a face ρ_+ of σ. The same happens for $C_- \subset \operatorname{vert} \pi(\tau)$, and we obtain another complete g-partite graph G_{ρ_-} on the *same* vertex set V.

If all complete g-partite subgraphs of H are induced, then $G_{\rho_+} = G_{\rho_-}$; otherwise, some edge of $G_{\rho,-}$, say, would join two vertices belonging to the same part of $G_{\rho,+}$, and the latter graph would not be induced. Therefore, the graph $G_\rho := G_{\rho_+} = G_{\rho_-}$ is unique, and with it $V := V_\rho$ and the face $\rho := \rho_+ = \rho_-$ of $\mathsf{Hom}(K_g, H)$. But then ρ is a common face of both σ and τ, and (4) says that the circuit C is supported on the common face $\pi(\rho)$ of $\pi(\sigma)$ and $\pi(\tau)$ (which is a product of simplices by Theorem 2.11). In consequence, $\pi(\sigma)$ and $\pi(\tau)$ do not intersect badly after all.

Conversely, if $G_{\rho_+} \neq G_{\rho_-}$ are different complete g-partite graphs on the common vertex set V given by (4), then the face ρ_- of τ is not a face of σ, and the projections $\pi(\sigma)$ and $\pi(\tau)$ do have a bad intersection. The theorem now follows from Proposition 3.4. □

Example 3.7. $\pi\mathsf{Hom}(K_2, K_4)$ is not a polytopal complex, as predicted by Theorem 3.6: not all complete bipartite graphs in K_4 are induced. The complex $\mathsf{Hom}(K_2, K_4)$ is isomorphic to the boundary complex of the 3-dimensional cuboctahedron, and by Proposition 3.5, the 1-skeleton of $\pi\mathsf{Hom}(K_2, K_4)$ is contained in the 1-skeleton of the octahedron $\Delta(4, 2)$ (they actually coincide in this case). However, the six square faces of $\mathsf{Hom}(K_2, K_4)$ get mapped to three square faces in $\pi\mathsf{Hom}(K_2, K_4)$, namely the three "internal squares" of the octahedron $\Delta(4, 2)$, and the relative interiors of any two of these intersect.

Remark 3.8. The vertices of $\pi\mathsf{Hom}(K_g, H)$ are those vertices of the hypersimplex $\Delta(h, g)$ that correspond to cliques in H, and by Proposition 3.5 the 1-skeleton of each cell $\pi(\sigma)$ is entirely contained in the 1-skeleton of $\Delta(h, g)$. Thus, each cell $\pi(\sigma)$ is a *matroid polytope* [6], corresponding to the associated *clique matroid* M_σ. The elements of M_σ are the vertices of the complete g-partite subgraph σ of H, and its bases the g-cliques of σ. These matters, as well as the connections to tropical geometry, are however beyond the scope of this article and will be pursued in a future publication.

Remark 3.9. We have seen that $\pi\mathsf{Hom}(K_g, H) \times \frac{\mathbf{1}}{\boldsymbol{g}} = \pi_\square\mathsf{Hom}_+(K_g, H) \cap (\mathbb{R}^h \times \frac{\mathbf{1}}{\boldsymbol{g}})$ is the "slice parallel to $\Delta_{[g]}$" given by the polyhedral Cayley trick. Therefore it seems natural to ask about $\Sigma_h := h \cdot \pi_\square\big(\mathsf{Hom}_+(K_g, H)\big) \cap (\frac{\mathbf{1}}{\boldsymbol{h}} \times \mathbb{R}^g)$, the "slice parallel to $\Delta_{V(H)}$". If $\sigma = \sigma_1 \star \cdots \star \sigma_g$ is a cell of $\mathsf{Hom}_+(K_g, H)$, then as before any point $x \in \pi_\square(\sigma) = \bigcup_{i=1}^g \sigma_i \times e_i$ can be written as a convex combination

$$x \;=\; \sum_{i=1}^{g} \lambda_i \sum_{v \in \operatorname{vert} \sigma_i} \lambda_{i,v} v \times e_i,$$

so that the intersection of $\pi_\square(\sigma)$ with $\frac{\mathbf{1}}{\boldsymbol{h}} \times \mathbb{R}^g$ is non-empty if and only if $\bigcup_{i=1}^g \sigma_i = [h]$, where we view the σ_i as subsets of $[h]$. In other words, every complete g'-partite subgraph of H supported on *all* h vertices of H contributes a cell to Σ_h, for all $1 \leq g' \leq g$. For instance, the cells of $\mathsf{Hom}_+(K_g, H)$ of the form $\Delta_{V(H)} \star \emptyset \star \ldots \star \emptyset$ each contribute a vertex of the form $b \times e_i$, where b is the barycenter of $\Delta_{V(H)}$.

4. Dissection complexes

For $k \geq 3$ and $m \geq 1$, consider the set of dissections of a convex N-gon into m convex k-gons. Note that for such a dissection to be possible, it is necessary and sufficient that $N = m(k-2) + 2$. We agree to label the vertices of the N-gon by 0, 1, ..., $N-1$. Let $\delta(k, m)$ be the set of *k-allowable* diagonals of the N-gon, i.e., those diagonals that can appear in a dissection into k-gons. It is easy to check that these are precisely the diagonals that connect a vertex x with one of the form $x + k - 1 + j(k-2) \bmod N$, for $0 \leq j \leq m-2$, and that $|\delta(k, m)| = (m-1)N/2$. Let $\mathrm{Cr}(\delta(k, m))$ and $I(k, m) = \mathrm{Ind}(\delta(k, m))$ be the *crossing graph* and *independence graph* of $\delta(k, m)$. These are complementary graphs on the vertex set $\delta(k, m)$, such that two vertices are joined by en edge in $\mathrm{Cr}(\delta(k, m))$ if the corresponding k-allowable diagonals intersect in their relative interior, while the same two vertices are joined in $I(k,m)$ if this is not the case. For a graph G, the simplicial complexes $\mathsf{Ind}(G)$ and $\mathsf{Cl}(G)$ are the *independence complex* and *clique complex* of G, whose simplices are the independent sets, respectively the cliques, of G. Thus, $I(k,m) = \mathrm{sk}^1 \mathsf{Ind}(\delta(k,m))$ and $\mathrm{Cr}(\delta(k,m)) = \mathrm{sk}^1 \mathsf{Cl}(\delta(k,m))$.

Proposition 4.1. $\mathsf{Hom}\big(K_{m-1}, I(k,m)\big)$ *is the polytopal complex on the vertex set* $\delta(k,m)$ *whose cells are the products* $\Delta^{C_1} \times \cdots \times \Delta^{C_{m-1}}$ *of* $m-1$ *simplices such that each* C_i *is a non-empty clique in* $\mathrm{Cr}(\delta(k,m))$*, and* C_i *and* C_j *are independent in* $\mathrm{Cr}(\delta(k,m))$ *for* $i \neq j$.

Proof. $\mathsf{Hom}\big(K_{m-1}, I(k,m)\big)$ arises by labelling the vertices of K_{m-1} with non-empty lists of elements in $\delta(k,m)$, such that any two diagonals of two lists λ, λ' on different vertices are joined by an edge in $I(k,m)$. But this means precisely that the diagonals in λ do not cross the diagonals in λ'. Moreover, no list can contain two or more independent diagonals, because on the one hand all $m-1$ lists must be non-empty, and on the other hand the maximal size of an independent set in $\delta(k,m)$ is $m-1$ by definition. □

Remark 4.2. That $\widetilde{\mathsf{D}}(k,m) = \mathsf{Hom}\big(K_{m-1}, I(k,m)\big)$ is a polytopal complex follows from its definition as Hom-complex, but can also be proven directly from Proposition 4.1: First note that any face of a cell of $\widetilde{\mathsf{D}}(k,m)$ is again a product of $m-1$ simplices, and therefore again a cell of $\widetilde{\mathsf{D}}(k,m)$. On the other hand, let $C = \prod_{i=1}^{m-1} \Delta^{C_i}$ and $D = \prod_{i=1}^{m-1} \Delta^{D_i}$ be two cells of $\widetilde{\mathsf{D}}(k,m)$. We must show that $C \cap D$ is either a common face of both C and D, or else does not index a face in the complex. For this, let $D_{\pi(1)}, \ldots, D_{\pi(m-1)}$ for $\pi \in S_{m-1}$ be a permutation of the D_i such that $C_i \cap D_{\pi(i)} \neq \emptyset$ for all $1 \leq i \leq m-1$. If such a permutation π does not exist, then the intersection $\mathrm{vert}\, C \cap \mathrm{vert}\, D$ is not a union of $m-1$ independent cliques, and so $C \cap D$ is not a face of $\widetilde{\mathsf{D}}(k,m)$. On the other hand, if such a permutation exists, then it is unique: Suppose that $C_i \cap D_{\pi(i)} = V_i \neq \emptyset$ and $C_i \cap D_{\sigma(i)} = W_i \neq \emptyset$ for some $1 \leq i \leq m-1$ and $\pi, \sigma \in S_{m-1}$. Then $V_i, W_i \subset C_i$, so that V_i and W_i are not independent; but then $V_i \subset D_{\pi(i)}$, $W_i \subset D_{\sigma(i)}$ forces $\pi(i) = \sigma(i)$. The cell of $\widetilde{\mathsf{D}}(k,m)$ corresponding to $C \cap D$ is then $\prod_{i=1}^{m-1} \Delta^{C_i \cap D_{\pi(i)}}$.

The number of dissections of a convex polygon into m convex k-gons was already determined in 1791 by Fuss [3]; see also the simplified proof in [9]. Two related complexes were considered somewhat later: In 2005, Tzanaki [13] proved that the simplicial complex $\mathsf{T}(k,m) = \mathsf{Ind}(\delta(k,m))$ is homotopy equivalent to a wedge of $\frac{1}{m}\binom{m(k-2)}{m-1}$ spheres of dimension $m-2$. Its dual graph is the *flip graph* $D(m,k)$, whose vertices are the dissections of the polygon (the facets of $\mathsf{T}(k,m)$), and in which two dissections are adjacent if they differ in the placement of exactly one interior diagonal [5].

Definition 4.3. We will use the following abbreviations:

$$\begin{aligned} \mathsf{D}(k,m) &= \pi\mathsf{Hom}\big(K_{m-1}, I(k,m)\big), \\ \mathsf{D}_+(k,m) &= \pi\mathsf{Hom}_+\big(K_{m-1}, I(k,m)\big), \\ \mathsf{D}_+^t(k,m) &= \pi\mathsf{Hom}_+^t\big(K_{m-1}, I(k,m)\big). \end{aligned}$$

Proposition 4.4. *With these notations,*
(a) $\mathsf{D}(k,m)$ *is a polytopal complex, and* $\mathsf{D}_+(k,m)$ *and* $\mathsf{D}^t_+(k,m)$ *are simplicial complexes. Moreover,* $\mathsf{D}(k,m)$ *arises as a linear section of* $\mathsf{D}_+(k,m)$ *and* $\mathsf{D}^t_+(k,m)$.
(b) $\mathsf{T}(k,m)$ *is the simplicial complex induced on the set of transversal* $(m-2)$*-dimensional faces of* $\mathsf{D}_+(k,m)$.
(c) $D(k,m) = \mathrm{sk}^1\, \mathsf{D}(k,m)$.

Proof. The first assertion of (a) follows from Theorem 3.6 and the definition of $I(k,m)$, the second one from Theorem 2.7, and the third is true by definition. For (b), note that the faces of a transversal $(m-2)$-dimensional simplex of $\mathsf{Hom}_+(K_{m-1}, I(k,m))$ are obtained by labelling each vertex of K_{m-1} with a list of diagonals of size 0 or 1. By the definition of $I(k,m)$ these diagonals are mutually non-crossing, so after dividing out by the S_{m-1}-symmetry we obtain exactly the simplices of $\mathsf{T}(k,m)$. The reasoning for (c) is similar. ☐

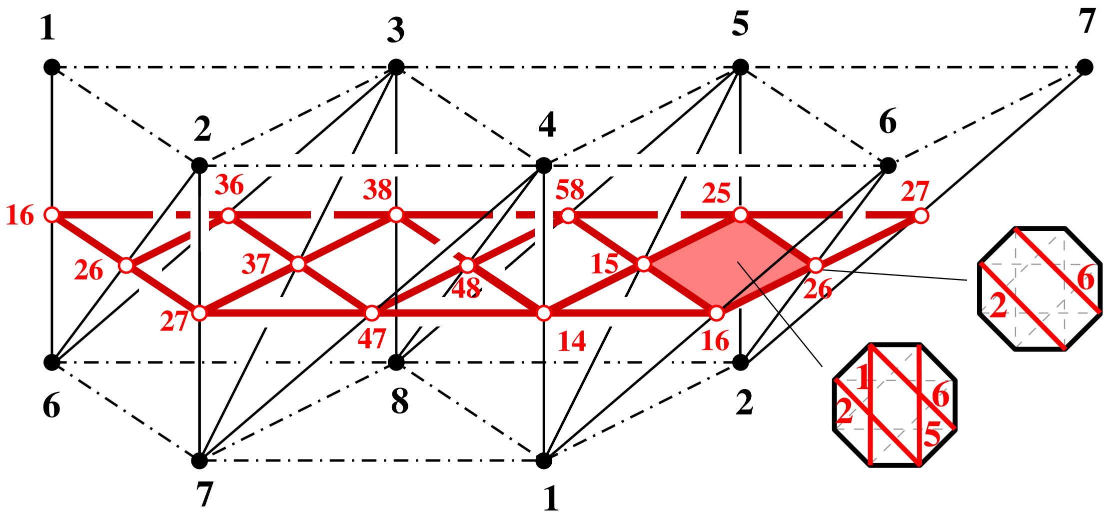

FIGURE 2. The projected simplicial complex $\mathsf{D}^t_+(4,3)$ with the polytopal complex $\mathsf{D}(4,3)$ arising as a section. Notice that $\mathsf{D}(4,3)$ is a Möbius band, while its double cover $\mathsf{Hom}(K_2, I(4,3))$ is homeomorphic to $S^1 \times [0,1]$. The solid edges of $\mathsf{D}^t_+(4,3)$ make up the complex $\mathsf{T}(4,3)$. The 4-allowable diagonals of the corresponding 8-gon are numbered cyclically from 1 to 8.

Remark 4.5. $\mathsf{D}_+(k,m)$ is not the only simplicial complex that contains $\mathsf{T}(k,m)$ as a subcomplex. For example, one can also consider the simplical complex $\mathsf{IC}_\Delta(k,m)$ on the vertex set $\delta(k,m)$ whose simplices are unions of independent cliques in $\mathrm{Cr}(\delta(k,m))$. This means that all diagonals in any of the cliques intersect in their relative interior, but any two diagonals from different cliques do not. $\mathsf{IC}_\Delta(k,m)$ is a strict subcomplex of $\mathsf{D}_+(k,m)$, because any tuple of mutually independent cliques is a tuple of mutually independent sets of diagonals, but not vice versa. Moreover, one can check that $\mathsf{IC}^t_\Delta(k,m) = \mathsf{D}^t_+(k,m)$, where $\mathsf{IC}^t_\Delta(k,m)$ is the induced simplicial complex on the set of unions of $m-1$ non-empty mutually independent cliques. Therefore, $\mathsf{D}(k,m)$ is also a linear section of $\mathsf{IC}_\Delta(k,m)$.

4.1. On the dimension and homotopy type of the dissection complexes.

Proposition 4.6. *$\dim \mathsf{D}(k,m) = \lfloor \frac{m}{2} \rfloor (k-2)$, and $\dim \mathsf{D}_+(k,m) = \dim \mathsf{D}(k,m) + m - 2$.*

Proof. The second statement holds generally, because the dimension of a maximal cell $\sigma_1 \times \cdots \times \sigma_{m-1}$ is $|\sigma_1| + \cdots + |\sigma_{m-1}| - m + 1$, while $\dim \sigma_1 \star \cdots \star \sigma_{m-1} = |\sigma_1| + \cdots + |\sigma_{m-1}| - 1$.

To prove the first statement, we must find $m-1$ sets of mutually intersecting k-allowable diagonals of the N-gon in such a way that no two diagonals from distinct sets cross, and the total number of diagonals is maximized. For this, we fix diagonals $d_1, \ldots, d_{m-1}$ of the N-gon. To each d_i (with endpoints x_i and y_i) and some choice of $\delta_i^+, \delta_i^- \in \mathbb{N}$, we adjoin all k-allowable diagonals with one endpoint between $x_i - \delta_i^-$ and $x_i + \delta_i^+$, and the other endpoint between $y_i - \delta_i^+$ and $y_i + \delta_i^-$ (cf. Figure 3). Note that $0 \le \delta_i^+ + \delta_{i+1}^- \le \ell_i$ and

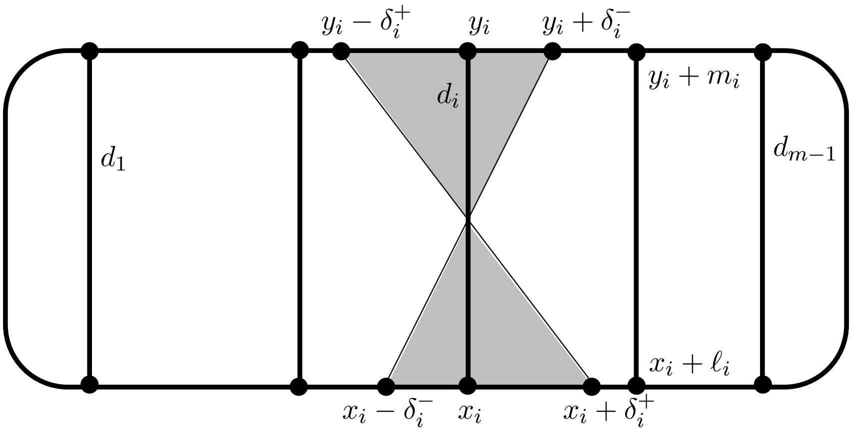

FIGURE 3. Finding a cell of $\mathsf{D}(k,m)$ of maximal dimension.

$0 \le \delta_i^- + \delta_{i+1}^+ \le m_i$ for $1 \le i \le m-2$, where $\ell_i = x_{i+1} - x_i$ and $m_i = y_{i+1} - y_i$, so that $\ell_i + m_i = k-2$. All calculations are to be considered modulo N. The dimension of the cell $\sigma = \sigma_1 \times \cdots \times \sigma_{m-1}$ obtained in this way is then

$$\begin{aligned} -m+1+\sum_{i=1}^{m-1} |\sigma_i| &= \sum_{i=1}^{m-1} \delta_i^- + \delta_i^+ \\ &\le \min \left\{ \delta_1^- + \delta_{m-1}^+ + \sum_{i=1}^{m-2} \ell_i, \quad \delta_1^+ + \delta_{m-1}^- + \sum_{i=1}^{m-2} (k-2-\ell_i) \right\}. \end{aligned}$$

For even m, we can maximize this value by choosing $\ell_{2j-1} = \lfloor \frac{k-2}{2} \rfloor$ and $\ell_{2j} = \lceil \frac{k-2}{2} \rceil$ for $1 \le j \le \frac{m-2}{2}$; moreover, it is readily verified that in this case δ_1^- and δ_{m-1}^+ can be chosen such that $\delta_1^- + \delta_{m-1}^+ = k-2$. Therefore,

$$\dim \sigma \;=\; k-2+(m-2)\frac{k-2}{2} \;=\; \frac{m}{2}(k-2),$$

as claimed. The proof for odd m is similar. □

The simplicial complex $\mathsf{D}_+(k,m)$ and the polytopal one $\mathsf{D}(k,m)$ are homotopy equivalent, because the former arises by taking as simplices the union of vertices of polytopal cells of the latter; cf. [2]. Since $\mathsf{D}_+(k,m)$ is obtained from $\mathsf{T}(k,m)$ by adjoining extra cells, and $\mathsf{T}(k,m)$ is homotopy equivalent to a wedge of "many" $(m-2)$-dimensional spheres, it seems reasonable to hope for the extra cells to simplify the topology.

Explicit computation in small cases reveals that this is indeed the case, but perhaps not to the greatest extent possible. Although the homology of $\mathsf{D}_+(k,m)$, and therefore of $\mathsf{D}(k,m)$, is substantially simpler than that of $\mathsf{T}(k,m)$, Table 1 below does not support the conjecture that these complexes are homotopy equivalent to a wedge of a "simple" number of spheres.

$k \setminus m$	3	4	5	6	7	8	9
3	2 (1) $r_1 = 1$	3–4 (1–2) $r_2 = 1$	4–5 (1–2) $r_3 = 1$	5–7 (1–3) $r_4 = 1$	7–8 (2–3) $r_5 = 1$	8–10 (2–4) $r_6 = 1$	9–11 (2–4)
4	3 (2) $r_1 = 1$	4–6 (2–4) $r_3 = 1$	5–7 (2–4) $r_3 = 4$ $r_4 = 4$	6–10 (2–6) $r_5 = 1$	7–11 (2–6) $r_5 = 17$ $r_6 = 20$	8–14	
5	4 (3) $r_1 = 1$	5–8 (3–6) $r_3 = 1$	6–9 (3–6) $r_3 = 1$	7–13 (3–9) $r_5 = 17$	8–14		
6	5 (4) $r_1 = 1$	6–10 (4–8) $r_3 = 1$	7–11 (4–8) $r_3 = 1$	8–16			
7	6 (5) $r_1 = 1$	7–12 (5–10) $r_3 = 1$	8–13				

Table 1. Dimensions of facets and nonzero integer homology ranks of $\mathsf{D}_+(k,m)$. The first line of each entry (k,m) lists the range of dimensions of the facets of $\mathsf{D}_+(k,m)$, respectively $\mathsf{D}(k,m)$, and the next lines the nonzero ranks r_i of their reduced integer homology groups, so that $\tilde{H}_i(\mathsf{D}_+(k,m),\mathbb{Z}) = \mathbb{Z}^{r_i}$. No homology groups in the table have torsion.

5. Staircase triangulations and cyclic polytopes

In Proposition 4.4, we found $\mathsf{T}(k,m)$ and $D(k,m)$ as subcomplexes of $\mathsf{D}_+(k,m)$, respectively of $\mathsf{D}(k,m)$. Next, we identify other "nice" subcomplexes of $\mathsf{D}(k,m)$ and its relatives:

Theorem 5.1. *Let r, s be integers such that $1 \le r \le \frac{(m-1)(k-2)+2}{k-1}$ and $1 \le s \le k-1$.*

(a) *The simplicial complex $\mathsf{D}_+(k,m)$ contains copies of the staircase triangulation $\Sigma(r,s)$ of the product of simplices $\Delta^{r-1} \times \Delta^{s-1}$.*

(b) *The polytopal complex $\mathsf{D}(k,m)$ contains copies of the polytopal complex $\mathcal{C}(r,s)$, where $d = 2s-2$ and $n = r+d$.*

In order to define the complexes mentioned in this theorem, we must first recall some facts about cyclic polytopes. A standard realization of the *cyclic polytope* $C_d(n) \subset \mathbb{R}^d$ is given by the convex hull of any n distinct points $\mu(t_1), \dots, \mu(t_n)$ on the *moment curve* $\mu : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \dots, t^d)$, where we assume $t_1 < \cdots < t_n$. Implicit in this definition is the fact that the combinatorial type of $C_d(n)$ does not depend on the concrete values of the t_i. A set $I \subset [n]$ indexes a face of $C_d(n)$ if I satisfies *Gale's evenness criterion*: For any $j, k \in [n] \setminus I$, the number of elements of I between j and k must be even. Henceforth, we will always identify faces of $C_d(n)$ with their index sets. A facet of $C_d(n)$ indexed by $I \subset [n]$ is a *lower facet* if the cardinality of the end-set of I is even, where the *end-set* of I consists of the last block of contiguous elements of I. We leave it to the reader to check (or consult in the literature) the fact that the last entry of the normal vector of any lower facet of a standard realization of $C_d(n)$ is negative.

5.1. **Weak compositions and cyclic polytopes** [5]**.** Let $r, s \geq 1$ be integers. A *(weak) composition*[3] *of r into s parts* is an ordered s-tuple $(a_1, a_2, \dots, a_s)$ of non-negative integers such that $a_1 + a_2 + \cdots + a_s = r$. We make the set $C(r, s)$ of all compositions of r into s parts into a graph by declaring two of them to be adjacent if they differ by one in exactly two positions that are connected by a (perhaps empty) sequence of 0's. For example, the composition $(1, 0, 2, 4, 0, 1)$ is adjacent to $(1, 0, 2, 3, 0, 2)$, but not to $(0, 0, 2, 4, 0, 2)$.

Definition 5.2. Let $n \geq d \geq 2$, let d be even and $C_d(n)$ be a d-dimensional cyclic polytope with n vertices. For any set $I = I(F) = \{i_1, i_1 + 1, \dots, i_{d/2}, i_{d/2} + 1\} \subset [n]$ that indexes a lower facet $F = F(I)$ of $C_d(n)$, the sequence $\chi(F) = \chi(I)$ records the sizes of the "holes" in I. More precisely, $\chi(F) := \big(i_{j+1} - i_j - 2 : 0 \leq j \leq d/2\big)$, where $i_0 := -1$ and $i_{d/2+1} := n + 1$.

For example, if $n = 6$, $d = 4$ and $I = I(F) = \{2, 3, 5, 6\}$, then $\chi(I) = (1, 1, 0)$.

Proposition 5.3. *Let $r, s \geq 1$ be integers and set $d = 2s - 2$ and $n = r + d$.*

(a) $\chi(F) \in \{0, 1, \dots, n-d\}^{\frac{d}{2}+1}$ *and* $\sum_{k=1}^{\frac{d}{2}+1} \chi(F)_k = n-d$ *for any lower facet F of $C_d(n)$.*
(b) *The map χ induces a bijection between the set of lower facets of $C_d(n)$ and the set of vertices of $C(r, s)$ that takes a facet F with $\chi(F) = (a_1, \dots, a_s)$ to the weak composition $r = a_1 + \cdots + a_s$.*

Proof. Part (a) and the forward direction of part (b) follow because any facet F leaves $n - d = r$ "holes" in $\{1, 2, \dots, n\}$. For the other direction of (b), Gale's evenness criterion uniquely reconstructs F from any weak composition $r = a_1 + \cdots + a_s$ by inserting a pair of indices between each pair of "holes" of sizes a_i and a_{i+1}. As a check, note that the number of lower facets of the cyclic polytope $C_d(r + d)$ is $\binom{r+d-\lceil d/2 \rceil}{\lfloor d/2 \rfloor} = \binom{r+s-1}{s-1} = |C(r, s)|$. □

5.2. **Cyclic polytopes and staircase triangulations.** Let $r, s \geq 1$ be integers. A *lattice path* in the grid $[r] \times [s]$ is a connected chain of horizontal and vertical line segments of unit length that connects $(1, 1)$ to (r, s) and is weakly monotone with respect to both

[3] This seems to be the standard name in the literature for ordered partitions.

coordinates. Thus, any lattice path has $r+s-1$ vertices. In this paper, we will always think of a lattice path as its set of vertices.

Denote by $\mathcal{L}(r,s)$ the set of all *partial lattice paths* in $[r]\times[s]$, i.e., all subsets of lattice paths. By identifying any partial lattice path with its vertex set, we make $\mathcal{L}(r,s)$ into a simplicial complex. This simplicial complex also appears in the guise of the *staircase triangulation* $\Sigma(r,s)$ of the product of simplices $\Delta^{r-1}\times\Delta^{s-1}$: it is straightforward to check that each partial lattice path in fact indexes a simplex in this product polytope, and that all these simplices combine to a triangulation.

For any partial lattice path $\lambda\in\mathcal{L}(r,s)$, let $\lambda_j=\lambda\cap(\{j\}\times[s])$ be the j-th "vertical slice" of λ, for $j=1,\dots,r$, and let $\boldsymbol{a}(\lambda)=(a_1,\dots,a_s)$ be the vector whose i-th entry is $a_i=|\lambda\cap([r]\times\{i\})|$, the cardinality of the i-th "horizontal slice". Now let $\mathcal{L}^t(r,s)$ be the set of partial lattice paths λ such that $|\lambda_j|\ge 1$ for all $j=1,\dots,r$, and denote the set of inclusion-minimal members of $\mathcal{L}^t(r,s)$ — those with $|\lambda_j|=1$ for all j — by $M(r,s)$.

Define the $(s-1)$-dimensional polytopal complex $\mathcal{C}(r,s)=\Sigma(r,s)\cap L$ to be the intersection of the staircase triangulation $\Sigma(r,s)$ of $\Delta^{r-1}\times\Delta^{s-1}$ and the $(s-1)$-dimensional plane $L=(\frac{1}{r},\dots,\frac{1}{r})\times\mathbb{R}^{s-1}$. The complex $\mathcal{C}(r,s)$ is a mixed polyhedral subdivision of the interior of an r times inflated standard $(s-1)$-dimensional simplex $r\Delta^{s-1}$, such that the vertices of $\mathcal{C}(r,s)$ are precisely the lattice points of $r\Delta^{s-1}$. Finally, let $\Sigma^t(r,s)$ denote the set of *transversal* simplices of $\Sigma(r,s)$, i.e. those that have non-empty intersection with L.

Clearly, the simplices in $\Sigma^t(r,s)$ correspond on the one hand bijectively to the partial lattice paths in $\mathcal{L}^t(r,s)$, and on the other hand — via the polytopal Cayley trick — to the polytopal cells of $\mathcal{C}(r,s)$. Under this bijection, the set $M(r,s)$ of minimal partial lattice paths corresponds to the vertices of $\mathcal{C}(r,s)$.

Example 5.4. Figure 1 (left) illustrates this correspondence for $r=2$, $s=3$. In fact, the bottom part of that figure shows $C(2,3)$, the graph of (weak) compositions of 2 into 3 non-negative summands, embedded as the 1-skeleton of a polyhedral decomposition of the twice dilated standard simplex $2\Delta^2$. In the top part, we can see $C(2,3)$ as a section of the transversal part $\Sigma^t(2,3)$ of the staircase triangulation of $\Delta^2\times\Delta^1$.

Theorem 5.5. *Let $r,s\ge 1$ be integers, and set $d=2s-2$ and $n=r+d$. Then $\mathcal{C}(r,s)$ is isomorphic to the polytopal subcomplex of the polar-to-cyclic polytope $C_d(n)^\Delta$ induced on the vertices dual to lower facets, whose faces are of dimension at most $d/2$. Moreover, the 1-skeleton of each of these complexes is the composition graph $C(r,s)$.*

Example 5.6. Figure 4 illustrates Theorem 5.5 for $r=4$ and $s=3$ (so that $d=4$ and $n=8$). On the left, we see the $s-1=2$-dimensional polytopal complex $\mathcal{C}(4,3)$, whose graph is the composition graph $C(4,3)$ of weak compositions of 4 into 3 summands; cf. the labels to the lower right of each vertex. At the same time, this complex is the subcomplex of the polar-to-cyclic polytope $C_4(8)^\Delta$ that is induced on the vertices dual to lower facets of $C_4(8)$: each vertex is dual to the lower facet indexed by the union of the labels below and to the left of it.

On the right, we see a (full) lattice path in $\mathcal{L}(4,3)$ that is the union of the minimal transversal lattice paths $\{11,21,31,43\}$, $\{11,21,32,43\}$ and $\{11,21,33,43\}$ in $M(4,3)$.

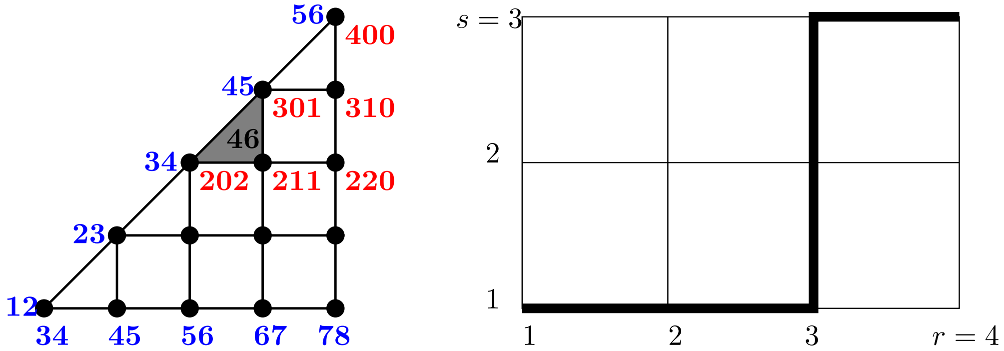

FIGURE 4. An example for the correspondences of Theorem 5.5.

Their $\boldsymbol{a}$-vectors are $(3,0,1)$, $(2,1,1)$ and $(2,0,2)$, respectively, and so they correspond to the lower facets 4567, 3467 and 3456 of $C_4(8)$. The entire lattice path thus corresponds to the face of $\mathcal{C}(4,3)$ dual to the intersection of these facets, namely the $d/2 = 2$-dimensional triangular face dual to the edge 46.

Proof of Theorem 5.5. The second statement follows from the first via the bijection of Proposition 5.3. To prove the first one, the remark before Example 5.4 yields an isomorphism between the face posets of $\mathcal{C}(r,s)$ and $\mathcal{L}^t(r,s)$. It therefore suffices to identify the latter complex as an interval of $C_d^{\downarrow}(n)$, the poset of lower faces of the cyclic polytope $C_d(n)$. This is done by the following inclusion-reversing maps:

$$\phi: \quad \mathcal{L}^t(r,s) \to C_d^{\downarrow}(n), \qquad \phi(\lambda) \;=\; \bigcap_{\mu\in M(r,s):\mu\subseteq\lambda} \chi^{-1}\big(\boldsymbol{a}(\mu)\big),$$

$$\psi: \quad C_d^{\downarrow}(n) \to 2^{[s]\times[r]}, \qquad \psi(G) \;=\; \bigcup_{F\in\mathcal{F}(G)} \boldsymbol{a}^{-1}\big(\chi(F)\big).$$

Here $\mathcal{F}(G)$ denotes the set of all those lower facets of $C_d(n)$ that contain the face G, and $\boldsymbol{a}^{-1} : \{0,1,\dots,r\}^s \to M(r,s)$ the bijection that maps "hole size vectors" to inclusion-minimal partial lattice paths. In particular, $\boldsymbol{a}^{-1}\chi$ and $\chi^{-1}\boldsymbol{a}$ are mutually inverse bijections between the set of lower facets of $C_d(n)$ and the set $M(r,s)$. Now note that

$$\phi\psi|_{\mathrm{Im}\phi}(G) \;=\; \phi\Big(\bigcup_{F\in\mathcal{F}(G)} \boldsymbol{a}^{-1}\chi(F)\Big) \;=\; \phi\Big(\bigcup_{\mu\in\widetilde{M}(G)} \mu\Big) \;=\; \bigcap_{\mu\in\widetilde{M}(G)} \chi^{-1}\boldsymbol{a}(\mu) \;=\; G,$$

where $\widetilde{M}(G) = \{\boldsymbol{a}^{-1}\chi(F) \in M(r,s) : F \supseteq G\} = \{\mu \in M(r,s) : \chi^{-1}\boldsymbol{a}(\mu) \supseteq G\}$, and that

$$\psi\phi(\lambda) \;=\; \psi\Big(\bigcap_{\mu\in M(r,s):\mu\subseteq\lambda} \chi^{-1}\boldsymbol{a}(\mu)\Big) \;=\; \psi\Big(\bigcap_{F:\boldsymbol{a}^{-1}\chi(F)\subseteq\lambda} F\Big) \;=\; \bigcup_{F:\boldsymbol{a}^{-1}\chi(F)\subseteq\lambda} \boldsymbol{a}^{-1}\chi(F) \;=\; \lambda.$$

It now follows that $\lambda \subset \lambda'$ in $\mathcal{L}^t(r,s)$ if and only if $\phi(\lambda') \subset \phi(\lambda)$ in $C_d^{\downarrow}(n)$. The forward direction is clear, as the intersection in the definition of $\phi(\lambda')$ is taken over a larger subset than in $\phi(\lambda)$, while the reverse direction follows from $\psi\phi(\lambda) = \lambda$ and the fact that ψ reverses inclusions. The statement about the dimension of the complex follows because any member of $M(r,s)$ has r elements, while the cardinality of a full lattice path is $r+s-1 = r+d/2$. □

5.3. **Proof of Theorem 5.1.** By the discussion leading up to the proof of Theorem 5.5, we identify the staircase triangulation $\Sigma(r,s)$ as $\mathsf{IHom}\big(K_r, S(r,s)\big)$, where $S(r,s)$ is the graph on the vertex set $[r]\times[s]$ in which an edge joins (i_1,j_1) to (i_2,j_2) exactly if $i_1 < i_2$ and $j_1 \le j_2$. If we can show that $S(r,s) \subset I(k,m)$ for some values of r,s, then by the definition of $\mathsf{IHom}\big(K_r, S(r,s)\big)$ and the functoriality of $\mathsf{Hom}(K_r,-)$ we obtain $\mathsf{IHom}\big(K_r, S(r,s)\big) \subset \mathsf{Hom}\big(K_r, S(r,s)\big) \subset \mathsf{Hom}(K_r, H)$. In fact, $\mathsf{IHom}\big(K_r, S(r,s)\big) = \mathsf{Hom}\big(K_r, S(r,s)\big)$ because $\omega(S(r,s)) = r$.

We now find r,s such that $S(r,s)\subset I(k,m)$. First, for any x and s with $0 \le x \le N$ and $1 \le s \le k-1$, let $\sigma_x(s) \subset \delta(k,m)$ be the set of s diagonals

$$\sigma_x(s) \;=\; \big\{(x+j, x+j+k-1) \bmod N : 0 \le j \le s-1\big\}.$$

Clearly, any two diagonals in each $\sigma_x(s)$ cross, so that $\sigma_x(s)$ is an independent set in $I(k,m)$. Now choose an integer r such that $1 \le r \le \frac{(m-1)(k-2)+2}{k-1}$ and put

$$S \;=\; \bigcup_{b=0}^{r-1} \sigma_{b(k-1)}(s).$$

By our choice of r, the set S is the vertex set of a copy of $S(r,s)$ inside $I(k,m)$, as desired (cf. Figure 5).

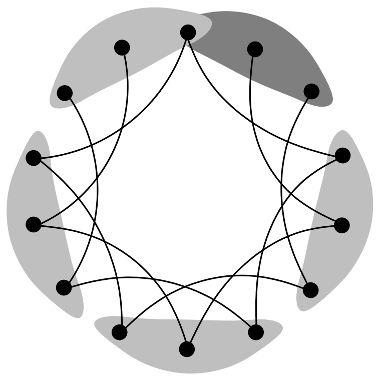

FIGURE 5. The construction for $k = 4$, $m = 6$, $r = 4$ and $s = 3$.

This proves part (a) of the theorem. Part (b) follows by combining Theorem 4.4 and Proposition 5.3 with part (a). The proof of Theorem 5.1 is now complete. □

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