



UNIVERSITAT POLITÈCNICA
DE CATALUNYA

Modeling and control of electromechanical systems

Carles Batlle, Arnau Dòria Cerezo

IOC-DT-P-2005-11
Juliol 2005

**Institut d'Organització i Control
de Sistemes Industrials**



MODELING AND CONTROL OF ELECTROMECHANICAL SYSTEMS

Carles Batlle^(1,2,3) and Arnau Dòria^(1,2,4)

(carles.batlle)(arnau.doria)@upc.edu

(1) Escola Politècnica Superior d'Enginyeria de Vilanova i la Geltrú

(2) Institut d'Organització i Control de Sistemes Industrials

(3) Departament de Matemàtica Aplicada IV

(4) Departament d'Enginyeria Elèctrica

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Lectures for the II EURON/GEOPLEX Summer School on

Modeling and Control of Complex Dynamical Systems

Bertinoro, FC, Italy, July 18-22 2005

Contents

1	Preamble	1
2	Electromechanical energy conversion	2
2.1	Network description of systems	2
2.2	Energy storing elements	3
2.3	Electric capacitor	5
2.4	Magnetic stationary system	6
2.5	Elementary electromagnet	8
2.6	DC motor	9
2.7	Co-energy	11
3	Port Hamiltonian models of electromechanical systems	12
3.1	Dirac structures and port Hamiltonian dissipative systems: basic results . .	12
3.2	General electromechanical systems	13
4	Power converters	15
4.1	Variable structure systems	16
4.2	Second order power converters	19
4.3	SSA and GSSA for second order power converters	21
5	A system for storing excess energy in an electrical vehicle transportation network	24
6	Energy based control	28
6.1	Control as interconnection	32
6.2	Casimir functions and the dissipation obstacle	33
7	IDA control method: basic ideas and examples	35
7.1	Magnetic levitation system	36
7.2	Boost converter	39
7.3	DC motor	40
A	Solving quasilinear PDEs	41
B	Coordinate transformations for ac machines	44

1 Preamble

These lectures are devoted to modeling and control of electromechanical systems in the framework of port Hamiltonian dissipative systems (PHDS), which in the simplest, explicit version are of the form

$$\dot{x} = (\mathcal{J}(x) - \mathcal{R}(x))(\nabla H(x))^T + g(x)u,$$

where $x \in \mathbb{R}^n$, \mathcal{J} is antisymmetric, \mathcal{R} is symmetric and positive semi-definite and $u \in \mathbb{R}^m$ is the control. The function $H(x)$ is the Hamiltonian, or energy, of the system. The natural outputs in this formulation are

$$y = g^T(x)(\nabla H(x))^T.$$

Port Hamiltonian dissipative systems enjoy nice properties when interconnected, and yield themselves to passive control strategies quite naturally.

The material presented in these notes covers the sessions *Modeling of electromechanical systems*, *Passive control theory I* and *Passive control theory II* of the II EURON/GEOPLEX Summer School on Modeling and Control of Complex Dynamical Systems. A basic knowledge of port based modeling of physical systems, namely bond graph theory, and of the port Hamiltonian formulation, as presented in Breedveld's and Maschke's lectures in this summer school, is assumed. Nevertheless, some background material is included when needed. Throughout these notes, symbols like $\partial_x H$ or ∇H represent the gradient of a scalar function H , and whether they are to be taken as column or row vectors should be clear from the particular context.

We start with a general description of what an electromechanical system is from a network modeling point of view. Assuming no dissipation, the energy function is introduced and Maxwell's reciprocity relations are deduced from the principle of conservation of energy. Dissipation is introduced by ending some of the open ports with resistive elements. As examples, we consider an electrical capacitor with moving plates, an elementary electromagnet, a solenoid transducer and a simple DC motor. Next, a general formulation in terms of PHDS is introduced (with a slight modification for the DC motor), and some of the previous electromechanical systems are rewritten in this formalism.

Control of electromechanical systems, notably electrical machines, generally requires a device capable of delivering a variable voltage (or current, in some cases). This is the realm of power electronics, which uses solid state devices (for instance, sets of IGBT — Insulated Gate Bipolar Transistor), to switch between several topologies of special electric circuits, known as *power converters*. A series of topology changes performs the trick of temporally storing the energy into the converter and then delivering it in the required form to the load. Power converters can also be given a PHDS form; they are *variable structure systems* (VSS), and this is reflected in the fact that, generally, both \mathcal{J} and g , and even \mathcal{R} in some cases, depend on a set of discrete variables $\{S_i\}$, which describe which switches are open or closed. In feedback control, these discrete variables are made dependent on the state of the system to accomplish the power conversion goal. It turns out that averaged models of VSS are useful for simulation and control design purposes, and they inherit the PHDS form from the original VSS. For completeness, and also for later use in the last example, we present a PHDS description of the lowest order power converters, both in VSS and averaged forms.

We conclude the modeling part of these lectures with a rather complex example, showing the interconnection of subsystems from several domains, namely an arrangement to temporally store the surplus energy in a section of a metropolitan transportation system based on dc motor vehicles, using either arrays of supercapacitors or an electric powered flywheel.

The second part of the lectures addresses control of PHD systems. We first present the idea of control as power connection of a plant and a controller, establish the energy balance equation, introduce Casimir variables and present the dissipation obstacle which precludes the application of the simplest version of passivity based control for PHDS to many electrical and electromechanical systems. Next we discuss how to circumvent this obstacle and present the basic ideas of Interconnection and Damping Assignment (IDA) passivity-based control of PHD systems. We apply the method to a magnetic levitation system, a power converter and a dc motor.

Appendix A reviews the basic ideas of the *method of characteristics*, useful for solving some of the partial differential equations that appear when implementing the IDA control scheme, while appendix B contains some technical discussions related to alternating current machines.

Some proposed exercises are scattered around the text. Most of them are straightforward but some may require consulting the cited literature for hints. The reader is encouraged to model the systems and controls proposed with `20sim`.

2 Electromechanical energy conversion

2.1 Network description of systems

The network description of systems is based on the separation of a system into two parts: the *energy storing elements* (and eventually, energy dissipating elements) and the *network*, which just shuffles energy around. External ports can also be included, in order to deliver or extract energy to the system and connect it to other systems. Figure 1 displays all these elements. Energy is exchanged through *power ports* connected by *bonds*. A couple of variables, an *effort* e and a *flow* f , are assigned to each bond. Their product has dimensions of power.

The essential characteristic of the network is that it is power continuous:

$$\sum_{i=1}^N e_i f^i = 0, \quad N = n + m, \quad (1)$$

while the energy-storing elements impose *constitutive relations* between efforts and flows. A first formalization of the network description idea is provided by *bond graph theory*, also discussed in detail elsewhere in this school.

A trivial example of network description is lumped circuit theory. The efforts e are voltage drops, while the flows f are currents. The network imposes Kirchoff laws, which are power continuous (Tellegen's theorem).

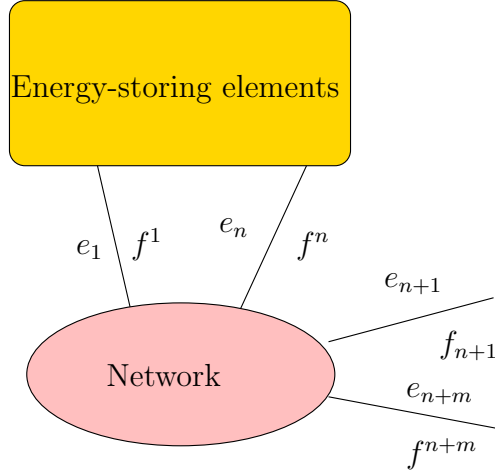


Figure 1: Conceptual elements in a network-based description of physical systems.

2.2 Energy storing elements

Consider a system with m power ports. The energy supplied by the ports between t_0 and t is given by the *field energy* increase

$$W_f(t) - W_f(t_0) = \int_{t_0}^t \sum_{i=1}^m e_i(\tau) f_i(\tau) d\tau. \quad (2)$$

We assume that the state of the system can be described by n state variables x_i , which, using vector notation, evolve, with the efforts as input signals, according to¹

$$\dot{x} = G(x) + g(x)e, \quad (3)$$

where $g(x) \in \mathbb{R}^{n \times m}$ indicates how the inputs connect to the variation of the state variables and G describes the *internal dynamics* of the system. We also suppose that the flows can be computed from the state as

$$f = g^T(x)\phi(x). \quad (4)$$

In compact form, we have

$$\begin{aligned} W_f(t) - W_f(t_0) &= \int_{t_0}^t \langle e(\tau), f(\tau) \rangle d\tau = \int_{t_0}^t \langle e(\tau), g^T(x(\tau))\phi(x(\tau)) \rangle d\tau \\ &= \int_{t_0}^t \langle g(x(\tau))e(\tau), \phi(x(\tau)) \rangle d\tau = \int_{t_0}^t \langle \dot{x}(\tau) - G(x(\tau)), \phi(x(\tau)) \rangle d\tau. \end{aligned}$$

Now we assume that the term $G(x)$ is actually of the form $G(x) = J(x)\phi_f(x)$, with $J^T = -J$. This can be justified from the general theory of Dirac structures and port Hamiltonian systems, but we can make plausible its introduction by noticing that it yields a term of the form

$$-\langle J(x(\tau))\phi(x(\tau)), \phi(x(\tau)) \rangle$$

¹Notice that, from a basic bond graph point of view, some of the inputs may actually be flows; however, to reduce notation, we do not distinguish here between I-like and C-like ports. In fact, by the use of gyrators, only one of the two classes of ports are actually needed. The final result we want to deduce is independent of this distinction.

which vanishes due to the skew-symmetry of J and hence does not contribute to the total energy variation of the system, as it fits to a shuffling around of energy among the internal degrees of freedom. We get thus

$$W_f(t) - W_f(t_0) = \int_{t_0}^t \langle \dot{x}(\tau), \phi(x(\tau)) \rangle d\tau = \int_{\gamma(x_0, x)} \phi(z) dz \quad (5)$$

where $\gamma(x_0, x)$ is any curve connecting $x_0 = x(t_0)$ and $x = x(t)$ in state space.

We will now assume that this system is *conservative*. This means that the variation of energy due to the power ports input only depends on the initial and final states, x_0 and x , and not on the particular curve in state space used to go from the former to the later. Otherwise, it would be possible to go from x_0 to x using a curve requiring less energy input and return from x to x_0 on a (reversed) curve with larger energy variation, yielding a system in the same initial state and a net surplus of energy, thus violating the First Principle of Thermodynamics. As it is well known, independence with respect to the curve means that the 1-form $\phi(x) dx$ is *exact*, *i.e.* in components

$$\phi_i(x) = \frac{\partial \Phi(x)}{\partial x_i}, \quad i = 1, \dots, n, \quad (6)$$

where Φ is a state space function. Actually, from (5) it follows immediately that then W_f is also a state space function and

$$W_f(x) - W_f(x_0) = \Phi(x).$$

In the sequel we will set $t_0 = 0$, $x_0 = 0$ and $W_f(0) = 0$ (there is no problem in doing so for time-invariant systems) and hence

$$W_f(x) = \int_{\gamma(x)} \phi(z) dz, \quad (7)$$

where $\gamma(x)$ is now *any* curve connecting the state space origin (or any other selected point) with x . The function $W_f(x)$ is thus the energy necessary to bring the system to state x , and it is the same no matter how the power is injected through the ports. It is thus *the* energy of the system in state x ; it coincides with the Hamiltonian function $H(x)$, and we will use the two notations indistinctively.

From (6) it follows, assuming continuity of the second order derivatives of Φ , that

$$\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x), \quad i, j = 1, \dots, n. \quad (8)$$

These are known as *Maxwell's reciprocity relations* for the constitutive equations of a conservative field. It allows, using derivation with respect to x_i followed by integration with respect to x_j , to obtain $\phi_i(x)$ if $\phi_j(x)$ is known. Also, from (6), one obtains that all the constitutive relations can be computed if the energy is known:

$$\phi_i(x) = \frac{\partial W_f(x)}{\partial x_i}. \quad (9)$$

Notice that we are adopting an *all-input* power convention, *i.e.* power is assumed to flow into the system if it is positive. In the electrical engineering literature, mechanical ports are treated with an output convention, so that mechanical efforts (force or torque) are defined as those done *by* the system. They are then computed from W_f by taking *minus* derivative with respect to the corresponding displacement.

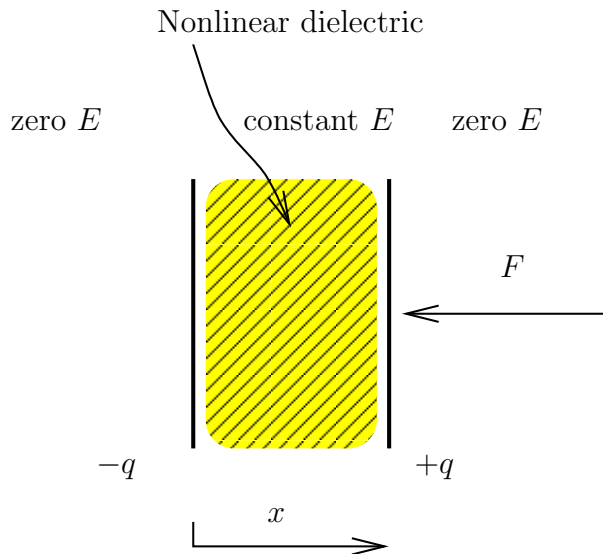


Figure 2: A nonlinear capacitor.

2.3 Electric capacitor

Figure 2 shows a general plane capacitor, where the dielectric has a nonlinear constitutive relation. As usual, we assume that the transversal dimensions of the plates are much bigger than the plate's separation x , so that the electric field is zero outside the capacitor and constant and perpendicular to the plates inside of it. In the vacuum or in linear isotropic dielectrics, it can be shown that the voltages of a set of conductors depend linearly on the charges. Here we assume a more general situation where the voltage difference depends nonlinearly on q . To make contact with the linear case, we introduce a capacitance-like function $C(x, q)$ so that

$$v(x, q) = \frac{q}{C(x, q)}.$$

This is a two-port system, one mechanical and the other electrical. We are given $v(x, q)$ and want to compute $F(x, q)$. According to our general discussion, Maxwell reciprocity condition tells us that

$$\frac{\partial F}{\partial q} = \frac{\partial v}{\partial x} = q \frac{\partial}{\partial x} \left(\frac{1}{C(x, q)} \right) = -\frac{q}{C^2(x, q)} \partial_x C(x, q).$$

Now we have to integrate this with respect to q . An integration by parts helps to make contact with the linear case:

$$\begin{aligned} F(x, q) &= \int_0^q \xi \frac{\partial}{\partial x} \left(\frac{1}{C(x, \xi)} \right) d\xi \\ &= \frac{q^2}{2} \frac{\partial}{\partial x} \left(\frac{1}{C(x, q)} \right) - \int_0^q \frac{\xi^2}{2} \frac{\partial^2}{\partial \xi \partial x} \left(\frac{1}{C(x, \xi)} \right) d\xi. \end{aligned}$$

For a linear system, $C(x, q) = C(x)$, and

$$F(x, q) = q \frac{E(x, q)}{2}$$

where

$$E(q, x) = q \frac{\partial}{\partial x} \left(\frac{1}{C(x, q)} \right) = \frac{\partial}{\partial x} \left(\frac{q}{C(x, q)} \right) = \frac{\partial}{\partial x} v(x, q)$$

is the electric field seen by the +plate when its charge is q and the separation is x . Furthermore, the field energy is given by

$$W_f(x, q) = \int_0^q \frac{\xi}{C(x)} d\xi = \frac{1}{2} \frac{q^2}{C(x)},$$

which is the well-known elementary result.

2.4 Magnetic stationary system

Figure 3 shows two coils coupled by the magnetic field they generate. Although this is not an electromechanical system (there are no moving parts) it allows us to introduce several concepts which will be used in the next example.

The current of any of the coils produces a magnetic field which goes through the other coil. Variations in any of the currents change the flux of the magnetic induction field and this, according to Lenz's law, originates an induced voltage in both coils. The iron core is introduced to "bend" the magnetic field lines so that the coupling is tighter. The coils have N_1 and N_2 turns, respectively. Any turn of any of the two coils has three contributions to the magnetic flux that goes through it:

- the flux due to the lines of the induction magnetic field generated by the current of the same coil and which close through the iron core,
- the flux due to the lines of the induction magnetic field generated by the current of the same coil and which do not close through the iron core, and
- the flux due to the lines of the induction magnetic field generated by the current of the other coil and which close through the iron core.

This can be written as

$$\begin{aligned} \Phi_1 &= \Phi_{l1} + \Phi_{m1} + \Phi_{m2} \\ \Phi_2 &= \Phi_{l2} + \Phi_{m2} + \Phi_{m1} \end{aligned}$$

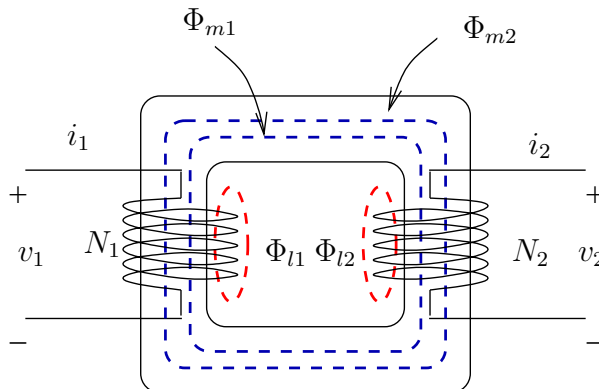


Figure 3: Magnetically coupled coils.

Φ_{l1} and Φ_{l2} are known as the *leakage* fluxes, while Φ_{m1} and Φ_{m2} are the *magnetizing* fluxes. The total flux through the coils, which is the quantity which enters the electrical equations, is then

$$\begin{aligned}\lambda_1 &= N_1\Phi_1, \\ \lambda_2 &= N_2\Phi_2.\end{aligned}$$

For a linear magnetic system, fluxes can be expressed in terms of path reluctances, number of turns and currents as

$$\Phi_{l1} = \frac{N_1 i_1}{\mathcal{R}_{l1}} \equiv L_{l1} i_1, \quad \Phi_{m1} = \frac{N_1 i_1}{\mathcal{R}_m} \equiv L_{m1} i_1, \quad \Phi_{l2} = \frac{N_2 i_2}{\mathcal{R}_{l2}} \equiv L_{l2} i_2, \quad \Phi_{m2} = \frac{N_2 i_2}{\mathcal{R}_m} \equiv L_{m2} i_2,$$

and one finally gets the well-known linear relation in terms of inductances:

$$\begin{aligned}\lambda_1 &= L_{11} i_1 + L_{12} i_2, \\ \lambda_2 &= L_{21} i_1 + L_{22} i_2,\end{aligned}$$

where

$$L_{11} = N_1(L_{l1} + L_{m1}), \quad L_{22} = N_2(L_{l2} + L_{m2}), \quad L_{12} = N_1 L_{m2} = \frac{N_1 N_2}{\mathcal{R}_m} = N_2 L_{m1} = L_{21}.$$

For a system with a variable geometry, the mutual inductance $L_{12} = L_{21}$ depends on the geometrical parameters. If the system is nonlinear, the inductances also depend on the various currents. Usually, only the core experiences nonlinear effects (*i.e.* saturation), and hence only the mutual inductances are current-dependent.

The dynamical equations of the system are now given by

$$v_1 = r_1 i_1 + \frac{d\lambda_1}{dt}, \quad v_2 = r_2 i_2 + \frac{d\lambda_2}{dt}$$

where r_1, r_2 are the electric resistances of the respective windings.

Assume now that $L_{l1} = L_{l2} = 0$. It is easy to obtain then

$$\begin{aligned}\lambda_1 &= \frac{N_1 N_2}{\mathcal{R}_m} \left(\frac{N_1}{N_2} i_1 + i_2 \right), \\ \lambda_2 &= \frac{N_1 N_2}{\mathcal{R}_m} \left(i_1 + \frac{N_2}{N_1} i_2 \right),\end{aligned}$$

so that

$$\lambda_1 \frac{N_2}{N_1} = \lambda_2.$$

If, furthermore, $r_1 = r_2 = 0$, it follows from the dynamical equations that

$$v_1 \frac{N_2}{N_1} = v_2. \tag{10}$$

Now let

$$\frac{N_1 N_2}{\mathcal{R}_m}$$

become very large; from the expression of λ_1 (or λ_2), if one wants to keep fluxes finite, it is necessary that

$$i_2 = -\frac{N_1}{N_2} i_1. \tag{11}$$

Equations (10) and (11) constitute the relations of an ideal transformer, with an all-input power convention.

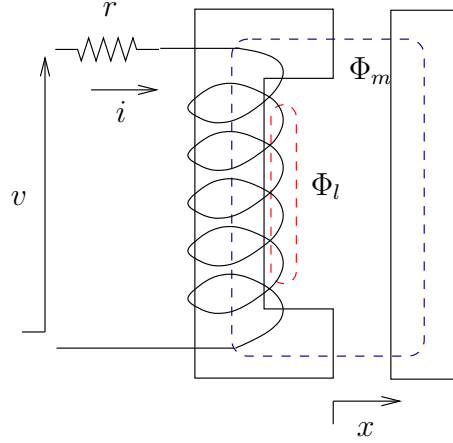


Figure 4: An elementary electromagnet: a magnetic system with a moving part.

2.5 Elementary electromagnet

Figure 4 shows an elementary electromagnet, a magnetic system with a moving part the flux linkage λ through the coil depends on a geometry variable, the “air gap” x .

The electrical equation of motion is

$$v = ri + \frac{d\lambda}{dt},$$

where the flux linkage can be computed from the number of turns, N , and the magnetic induction flux, Φ , as

$$\lambda = N\Phi.$$

In turn, Φ has a leakage, Φ_l , and a magnetizing, Φ_m , parts, $\Phi = \Phi_l + \Phi_m$, which can be computed in terms of the reluctances of the respective paths:

$$\Phi_l = \frac{Ni}{\mathcal{R}_l}, \quad \Phi_m = \frac{Ni}{\mathcal{R}_m}.$$

The reluctance of the magnetizing path has a fixed contribution, the part of the iron path, and a variable one, the part of the air gap:

$$\mathcal{R}_m = \frac{l_i}{\mu_{ri}\mu_0 A_i} + \frac{2x}{\mu_0 A_g},$$

where μ_{ri} , the relative magnetic permeability of the iron core, is of the order of 10^3 . Assuming that the sections of the iron and air gap paths are the same, $A_i = A_g = A$, one gets

$$\mathcal{R}_m = \frac{1}{\mu_0 A} \left(\frac{l_i}{\mu_{ri}} + 2x \right).$$

The relation between the current and the flux linkage can finally be written as

$$\lambda = \left(\frac{N^2}{\mathcal{R}_l} + \frac{N^2}{\mathcal{R}_m} \right) i = (L_l + L_m)i$$

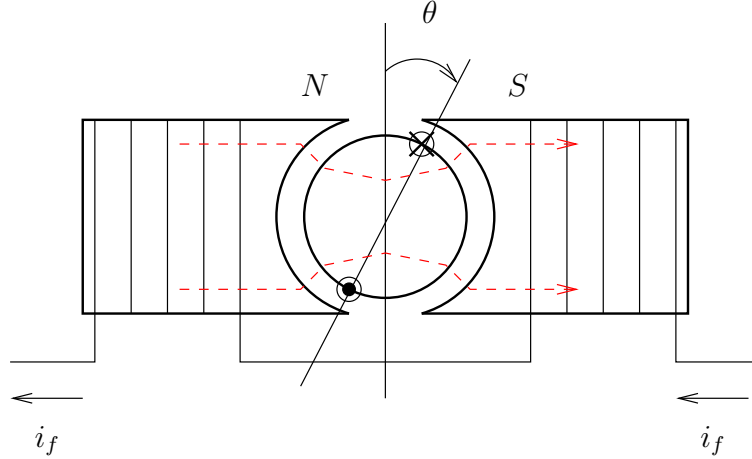


Figure 5: Transversal section of a dc motor with a single rotor loop.

with

$$L_m = \frac{N^2}{\mathcal{R}_m} = \frac{N^2 \mu_0 A}{\frac{l_i}{\mu_{ri}} + 2x} \equiv \frac{b}{c + x}.$$

EXERCISE Compute the electromechanical energy W_f and the force necessary to move x .

2.6 DC motor

Direct-current (dc) motors are somewhat singular from the point of view of energy modeling in that, in the simplest case and to first order, induced currents due to a variable geometry or a varying magnetic field are not present. Torque production is due to the direct action of the magnetic field on the rotor (or armature) currents. The magnetic field is produced either by a permanent magnet or by currents, called field currents, wrapped around an iron core.

Figure 5 shows a transversal section of a dc motor. Only a couple of magnetic lines are shown, but the geometry is such that an approximately radial constant magnetic field is produced in the air gap between the rotor and the external iron core. Only a single rotor current loop is shown; the current goes into the page in the upper leg of the loop and out of the page in the lower leg. To maintain this particular arrangement of currents as the rotor rotates, a mechanism, called *commutation* and consisting of slip rings and brushes, must be devised; see [9] for details.

From an elementary use of the law giving the force of a magnetic field \vec{B} on a current element $d\vec{l}$, it follows that the torques on the two current loops add each other in the clockwise direction, and the total torque is given by

$$\tau = l_1 l_2 B i_a \quad (12)$$

where i_a is the current in the rotor loop, l_2 is the radius of the rotor and l_1 is the longitudinal length of the rotor. Also, using Faraday's law applied to a semicylindrical surface having as base the rotor loop and closing through the air gap, it can be shown that a back electromotive force given

$$v_{emf} = -l_1 l_2 B \dot{\theta} \quad (13)$$

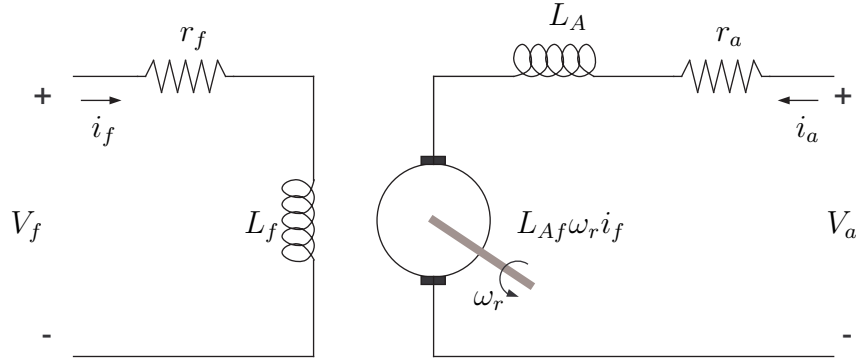


Figure 6: Total circuit of a dc machine.

is also produced. Essentially, this is due to the fact that, as it rotates, the portion of the cylindrical surface facing the north pole does vary.

EXERCISE Prove (12) and (13).

If the magnetic field is due to the field current, B is proportional, assuming no magnetic saturation or hysteresis, to the field current:

$$B = \frac{L_{Af}}{l_1 l_2} i_f, \quad (14)$$

and hence

$$\tau = L_{Af} i_f i_a \quad (15)$$

$$v_{emf} = -L_{Af} i_f \omega_r, \quad (16)$$

where $\omega_r = \dot{\theta}$ is the rotor angular speed. Here L_{Af} is a parameter depending on the exact geometry and the characteristics of the materials; it has dimensions of inductance but, despite the notation, it is not the mutual inductance between armature and field.

Putting everything together and, as displayed in Figure 6, assuming parasitic resistances in both rotor and field loops, and neglecting any mutual inductance effect between both loops, the equations of motion for the variables $\lambda_f = L_f i_f$ (the field flux), $\lambda_a = L_A i_a$ (the armature flux), and $p_m = J_m \omega_r$ (the mechanical angular momentum of the rotor) are

$$\begin{aligned} \dot{\lambda}_f &= -r_f i_f + V_f, \\ \dot{\lambda}_a &= -L_{Af} i_f \omega_r - r_a i_a + V_a, \\ \dot{p}_m &= L_{Af} i_f i_a - B_r \omega_r - T_L, \end{aligned} \quad (17)$$

where $-T_L$ is the external applied mechanical torque.

The bond graph corresponding to this system is shown in Figure 7. Notice the presence of the activated bond, and the fact that, under the stated conditions, armature and field do not exchange energy. The armature and field circuits can be fed in several configurations from the same source, giving what are known as *series* or *shunt* dc machines.

EXERCISE Show that, neglecting the field port, in the limit when L_A and J_m go to zero (define this formally!), the DC motor becomes a pure gyrator.

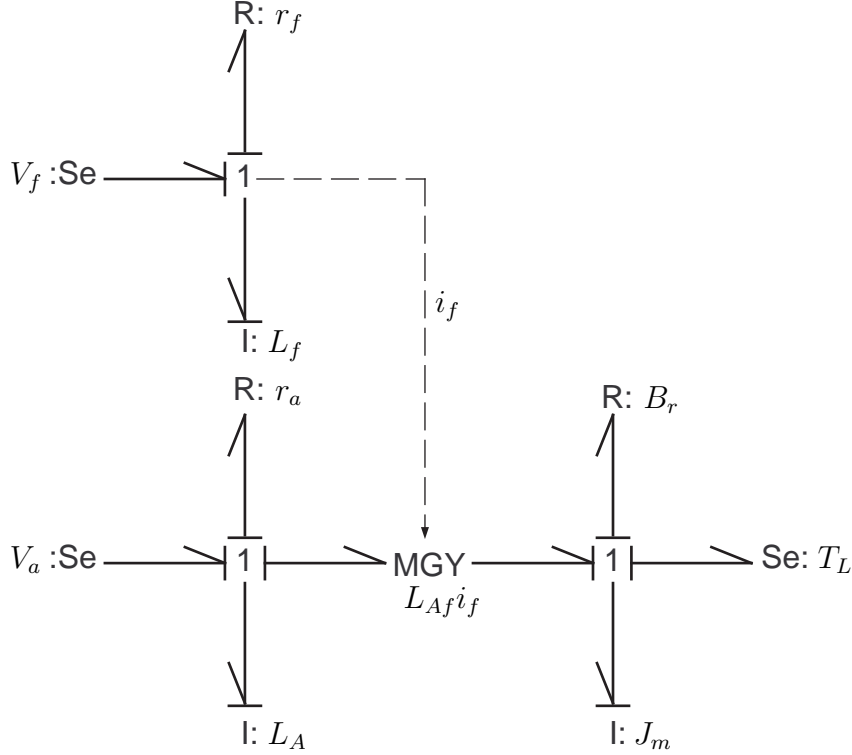


Figure 7: Bond Graph of a dc machine.

2.7 Co-energy

Assume now that $n = m$ and $g = \mathbb{I}$. We have, from (4) and (9), that

$$f_i = \frac{\partial W_f}{\partial x_i}(x), \quad i = 1, \dots, n. \quad (18)$$

One can then define the co-energy W_c as

$$W_c(f, x) = \sum_{i=1}^n f_i x_i - W_f(x). \quad (19)$$

However,

$$\frac{\partial W_c}{\partial x_i} = f_i - \frac{\partial W_f}{\partial x_i}(x) = f_i - f_i = 0, \quad i = 1, \dots, n.$$

Hence W_c is actually a function only of the (generalized) flows, $W_c = W_c(f)$, called the *co-energy* of the conservative system. The construction of W_c from W_f is a quite general operation, known as *Legendre transformation*, widely used in analytical mechanics and thermodynamics. While variables x are known as *energy* or *Hamiltonian* variables, transformed variables f are the *co-energy* or *Lagrangian* variables.

For linear electromagnetic systems, *i.e.* $q = C(x)v$, $\lambda = L(x)i$, energy equals co-energy:

$$W_c(v, i, x) = W_f(q, \lambda, x)|_{q=q(v,x), \lambda=\lambda(i,x)}. \quad (20)$$

In particular, for linear magnetic systems,

$$W_f(\lambda, x) = \frac{1}{2} \lambda^T L^{-1}(x) \lambda, \quad W_c(i, x) = \frac{1}{2} i^T L(x) i.$$

EXERCISE Prove (20).

EXERCISE Consider an electromechanical system with a nonlinear magnetic material such that

$$\lambda = (a + bx^2)i^2,$$

where a and b are constants and x is a variable geometric parameter. Compute W_f , W_c and f , and check all the relevant relations.

3 Port Hamiltonian models of electromechanical systems

3.1 Dirac structures and port Hamiltonian dissipative systems: basic results

This subsection is included for self-completeness; much more detailed results are exposed elsewhere in this summer school.

In order to further formalize the ideas of the network description, and also to generalize them to encompass, for instance, distributed parameter systems, the idea of *Dirac structure* must be introduced. In the simplest case (finite dimensional systems), flows corresponding to the open ports are arranged in an m -dimensional vector space \mathcal{V} , $f \in \mathcal{V}$, while the associated efforts are viewed as elements of its dual \mathcal{V}^* , $e \in \mathcal{V}^*$. The dual pairing between vectors and forms provides then the product which yields the power. One also needs an *state space* \mathcal{X} , with local coordinates $x \in \mathbb{R}^n$ and corresponding tangent and co-tangent spaces $T\mathcal{X}$ and $T^*\mathcal{X}$. These will eventually be associated to the bonds corresponding to the energy storing elements.

Let $\mathcal{B}(x) = T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{V} \times \mathcal{V}^*$. On $\mathcal{B}(x)$ one can define a symmetric bilinear form

$$\langle (v_1, v_1^*, f_1, e_1), (v_2, v_2^*, f_2, e_2) \rangle_+ = (v_1^*, v_2) + (v_2^*, v_1) + (e_1, f_2) + (e_2, f_1).$$

A *Dirac structure* on $\mathcal{B} = \cup_{x \in \mathcal{X}} \mathcal{B}(x)$ is a smooth subbundle $\mathcal{D} \in \mathcal{B}$ such that, for each x ,

$$\mathcal{D}(x) = \mathcal{D}^\perp(x),$$

$$\mathcal{D}^\perp(x) = \{(v_1, v_1^*, f_1, e_1) \mid \langle (v_1, v_1^*, f_1, e_1), (v_2, v_2^*, f_2, e_2) \rangle_+ = 0, \forall (v_2, v_2^*, f_2, e_2) \in \mathcal{D}(x)\}.$$

Dirac structures have the following important properties:

1. $\dim \mathcal{D}(x) = n + m$.
2. If $(v, v^*, f, e) \in \mathcal{D}(x)$, then $(v^*, v) + (e, f) = 0$.
3. In local coordinates, a Dirac structure can be characterized by $(n + m)$ -dimensional square matrices $E(x)$, $F(x)$, satisfying $F(x)E^T(x) + E(x)F^T(x) = 0$, as follows:

$$(v, v^*, f, e) \in \mathcal{D}(x) \Leftrightarrow F(x) \begin{pmatrix} v \\ f \end{pmatrix} + E(x) \begin{pmatrix} v^* \\ e \end{pmatrix} = 0.$$

Let \mathcal{H} be a smooth function on \mathcal{X} . The port controlled Hamiltonian system corresponding to $(\mathcal{X}, \mathcal{V}, \mathcal{D}, \mathcal{H})$ is defined by

$$(-\dot{x}, \partial_x \mathcal{H}(x), f, e) \in \mathcal{D}(x).$$

It follows from the self-duality of the Dirac structure that $\dot{\mathcal{H}}(x) = (e, f)$, which expresses the energy balance. Assuming that $F(x)$ is invertible for all x , one has

$$\begin{pmatrix} \dot{x} \\ -f \end{pmatrix} = F^{-1}(x)E(x) \begin{pmatrix} \partial_x \mathcal{H}(x) \\ e \end{pmatrix}.$$

It can be seen from $FE^T + EF^T = 0$ that $F^{-1}E$ must be skew-symmetric. Hence

$$\begin{pmatrix} \dot{x} \\ -f \end{pmatrix} = \begin{pmatrix} J(x) & g(x) \\ -g^T(x) & J_f(x) \end{pmatrix} \begin{pmatrix} \partial_x \mathcal{H}(x) \\ e \end{pmatrix}.$$

where $J(x)$, $J_f(x)$ are skew-symmetric.

EXERCISE Show that $FE^T + EF^T = 0$ implies that $F^{-1}E$ is skew-symmetric.

Dissipation may be included by terminating some of the open ports. Replacing $m \rightarrow m + m_r$ and setting $e_r = -R(x)f_r$ with $R^T(x) = R(x) \geq 0$, one gets

$$\dot{\mathcal{H}}(x) = (e, f) + (e_r, f_r) = (e, f) - f_r^T R(x) f_r \leq (e, f).$$

Again, in local coordinates the system can be expressed as

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\partial_x \mathcal{H}(x) + g(x)e \\ f &= g^T(x)\partial_x \mathcal{H}(x) - J_f(x)e. \end{aligned}$$

This is an explicit port Hamiltonian system with dissipation (PHD). In general, one can interconnect several PHD with open ports using Dirac structures, and the result is again a PHD, although not necessarily in explicit form.

As said before, all this can be generalized [35] to distributed parameter systems, described by partial differential equations, for which numerical spatial discretization schemes have also been developed [17].

3.2 General electromechanical systems

As explained before, an electromechanical system exchanges energy between a mechanical and an electrical part by means of geometry variations. For a wide class of systems, the total energy can be written as

$$H(\lambda, p, \theta) = \frac{1}{2}\lambda^T L^{-1}(\theta)\lambda + \frac{1}{2}p^T J_m^{-1}p, \quad (21)$$

we λ are the generalized electrical energy variables (they may be charges or magnetic fluxes), p are the generalized mechanical momenta (linear or angular, or associated to any other generalized coordinate), and θ are the generalized geometric coordinates. In a minimal formulation, the generalized inertia matrix J_m may be assumed to be independent of θ ; if not, the computation of the torque would yield non-electrical terms coming from

non-inertial effects, which are better accounted for by attaching the corresponding varying geometry mechanical subsystem to the mechanical port.

One has

$$\begin{aligned}\dot{\lambda} + R_e i &= Bv, \\ \dot{p} &= -R_m \omega - T_e(\lambda, \theta) + T_m, \\ \dot{\theta} &= J_m^{-1} p\end{aligned}\quad (22)$$

where B is a matrix indicating how the input voltages are connected to the electrical devices, T_e is the electrical torque, T_m is the external applied mechanical torque and

$$i = L^{-1}(\theta)\lambda = \partial_\lambda H, \quad \omega = J_m^{-1} p = \partial_p H \quad (23)$$

are the electrical currents and mechanical velocities $\omega = \dot{\theta}$. We have, according to our discussion on electromechanical energy conversion,

$$T_e = \frac{\partial H}{\partial \theta} = -\frac{1}{2} \lambda^T L^{-1} \partial_\theta L L^{-1} \lambda, \quad (24)$$

where $\partial_\theta L^{-1} = -L^{-1} \partial_\theta L L^{-1}$ has been used. Using (23) and (24), equations (22) can be written in an explicit port Hamiltonian form as

$$\dot{x} = \begin{pmatrix} -R_e & 0 & 0 \\ 0 & -R_m & -1 \\ 0 & 1 & 0 \end{pmatrix} \partial_x H + \begin{pmatrix} B & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ T_m \end{pmatrix} \quad (25)$$

where $x = (\lambda \ p \ \theta)^T$. This general model includes many of the classical electrical machines, as well as linear motors and levitating systems.

EXERCISE Write the port Hamiltonian model (25) for the electromagnet and for the variable geometry capacitor.

For the dc motor, the above formulation has to be modified slightly due to the different role played by the geometric coordinates (the angle in this case). Direct inspection of equations (17) shows that they can be given also a PHDS form, with Hamiltonian variables

$$x^T = (\lambda_f \quad \lambda_a \quad J_m \omega_r), \quad (26)$$

Hamiltonian function

$$H(x) = \frac{1}{2} \lambda L^{-1} \lambda^T + \frac{1}{2 J_m} \omega_r^2, \quad L = \begin{pmatrix} L_f & 0 \\ 0 & L_A \end{pmatrix}, \quad (27)$$

interconnection and damping matrices

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{Af} i_f \\ 0 & L_{Af} i_f & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} r_f & 0 & 0 \\ 0 & r_a & 0 \\ 0 & 0 & B_r \end{pmatrix}, \quad (28)$$

and port matrix $g = \mathbb{I}_3$, with inputs

$$u^T = (V_f \quad V_a \quad -T_L). \quad (29)$$

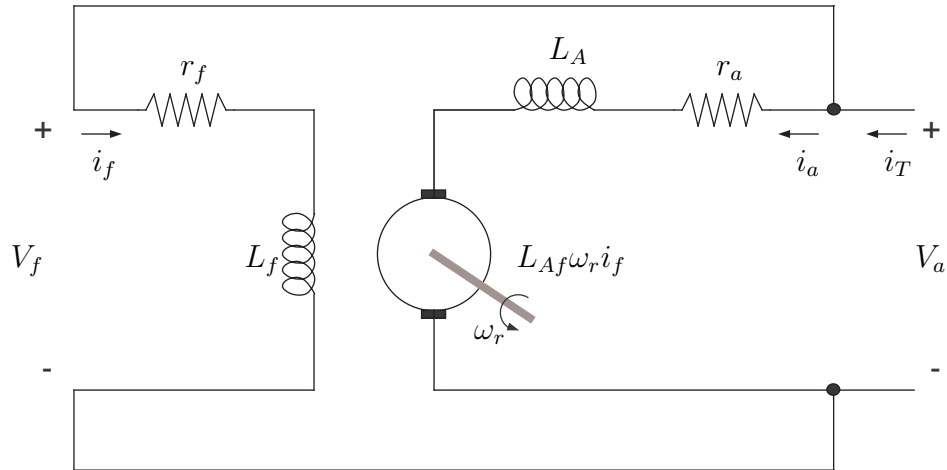


Figure 8: Shunt-connected dc machine.

The input voltages V_f and V_a can be obtained from the same source in several ways. Figure (8) shows a shunt-connected dc machine, for which $V_a = V_f$ and $i_T = i_a + i_f$. Figure (9) displays a series-connected dc machine, for which one has $i_a = i_f$ and $V_T = V_a + V_f$.

EXERCISE Obtain the port Hamiltonian models of the shunt-connected and of the series-connected dc machines. The later is in fact an implicit port Hamiltonian model. Check this by writing the corresponding bond graph and identifying the differential causality assignment.

Alternating current machines can also be written in PHDS form. However, several coordinate transformations are used in the electrical engineering literature to simplify the complex, geometry dependent constitutive relations involved in most of the cases. It turns out that, after carrying out those transformations, the system is still in PHDS form, with different, and nontrivial, interconnection matrices. We will not pursue this here. The interested reader is referred to [3]. Appendix B contains a brief account of Park's transformation from the point of view of bond graph theory and PHDS theory.

4 Power converters

Power converters are of great value in many growing application areas such as the control of electromechanical devices, portable battery-operated equipment or uninterruptible power sources. In particular, dc-to-dc power converters are used to reduce or elevate a given dc voltage, storing the energy in intermediate ac elements and delivering it as needed. The essential trick is the use of switches, which are operated in a periodic manner and which make the system to alternate between several dynamics. Generally, the individual dynamics are linear and can be solved analytically; the action of the switches yields a nonlinear dynamics which can display a rich behavior (see [14] and references therein). As power converters are variable structure systems, we present first a general theory for the later, and will retake concrete examples of the former after that.

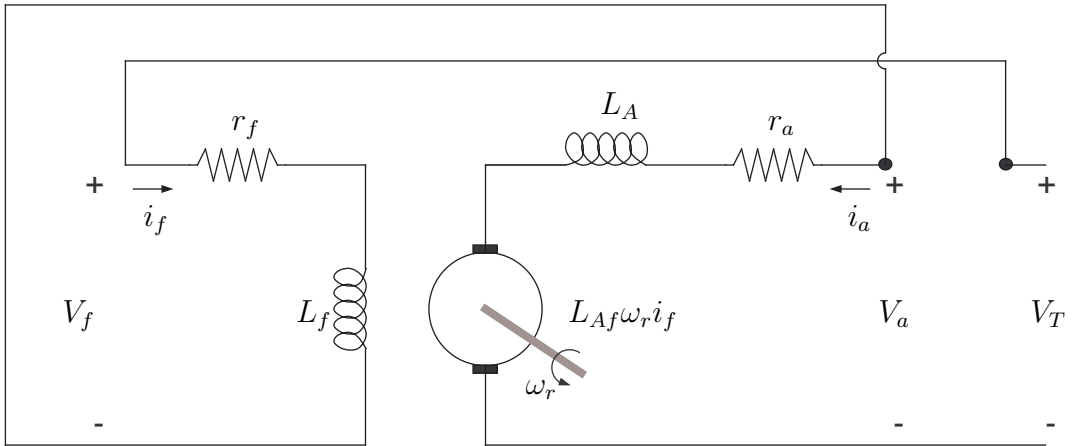


Figure 9: Series-connected dc machine.

4.1 Variable structure systems

Assume a VSS system such that the change in the state variables is small over the time length of an structure change, or such that one is not interested about the fine details of the variation. Then one may try to formulate a dynamical system for the time average of the state variables (*state space averaging*, or SSA)

$$\langle x \rangle(t) = \frac{1}{T} \int_{t-T}^t x(\tau) d\tau, \quad (30)$$

where T is the period, assumed constant, of a cycle of structure variations.

Let our VSS system be described in explicit port Hamiltonian form

$$\dot{x} = [\mathcal{J}(S, x) - \mathcal{R}(S, x)] (\nabla H(x))^T + g(S, x)u, \quad (31)$$

where S is a (multi)-index, with values on a finite, discrete set, enumerating the different structure topologies. For notational simplicity, we will assume from now on that we have a single index (corresponding to a single switch, or a set of switches with a single degree of freedom) and that $S \in \{0, 1\}$. Hence, we have two possible dynamics, which we denote as

$$\begin{aligned} S = 0 &\Rightarrow \dot{x} = (\mathcal{J}_0(x) - \mathcal{R}_0(x))(\nabla H(x))^T + g_0(x)u, \\ S = 1 &\Rightarrow \dot{x} = (\mathcal{J}_1(x) - \mathcal{R}_1(x))(\nabla H(x))^T + g_1(x)u. \end{aligned} \quad (32)$$

Note that controlling the system means choosing the value of S as a function of the state variables, and that u is, in most cases, just a constant external input.

From (30) we have

$$\frac{d}{dt} \langle x \rangle(t) = \frac{x(t) - x(t-T)}{T}. \quad (33)$$

Now the central assumption of the SSA approximation method is that for a given structure we can substitute $x(t)$ by $\langle x \rangle(t)$ in the right-hand side of the dynamical equations,

so that (32) become

$$\begin{aligned} S = 0 &\Rightarrow \dot{x} \approx (\mathcal{J}_0(\langle x \rangle) - \mathcal{R}_0(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_0(\langle x \rangle)u, \\ S = 1 &\Rightarrow \dot{x} \approx (\mathcal{J}_1(\langle x \rangle) - \mathcal{R}_1(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_1(\langle x \rangle)u. \end{aligned} \quad (34)$$

The rationale behind this approximation is that $\langle x \rangle$ does not have time to change too much during a cycle of structure changes. We assume also that the length of time in a given cycle when the system is in a given topology is determined by a function of the state variables or, in our approximation, a function of the averages, $t_0(\langle x \rangle)$, $t_1(\langle x \rangle)$, with $t_0 + t_1 = T$. Since we are considering the right-hand sides in (34) constant over the time scale of T , we can integrate the equations to get²

$$\begin{aligned} x(t) = x(t - T) &+ t_0(\langle x \rangle) [(\mathcal{J}_0(\langle x \rangle) - \mathcal{R}_0(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_0(\langle x \rangle)u] \\ &+ t_1(\langle x \rangle) [(\mathcal{J}_1(\langle x \rangle) - \mathcal{R}_1(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_1(\langle x \rangle)u]. \end{aligned}$$

Using (33) we get the SSA equations for the variable $\langle x \rangle$:

$$\begin{aligned} \frac{d}{dt}\langle x \rangle &= d_0(\langle x \rangle) [(\mathcal{J}_0(\langle x \rangle) - \mathcal{R}_0(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_0(\langle x \rangle)u] \\ &+ d_1(\langle x \rangle) [(\mathcal{J}_1(\langle x \rangle) - \mathcal{R}_1(\langle x \rangle))(\nabla H(\langle x \rangle))^T + g_1(\langle x \rangle)u], \end{aligned} \quad (35)$$

where

$$d_{0,1}(\langle x \rangle) = \frac{t_{0,1}(\langle x \rangle)}{T}, \quad (36)$$

with $d_0 + d_1 = 1$. In the power converter literature d_1 (or d_0 , depending on the switch configuration) is referred to as the duty cycle.

One can expect the SSA approximation to give poor results, as compared with the exact VSS model, for cases where T is not small with respect to the time scale of the changes of the state variables that we want to take into account. The GSSA approximation tries to solve this, and capture the fine detail of the state evolution, by considering a full Fourier series, and eventually truncating it, instead of just the “dc” term which appears in (30). Thus, one defines

$$\langle x \rangle_k(t) = \frac{1}{T} \int_{t-T}^t x(\tau) e^{-jk\omega\tau} d\tau, \quad (37)$$

with $\omega = 2\pi/T$ and $k \in \mathbb{Z}$.

The time functions $\langle x \rangle_k$ are known as index- k averages or k -phasors. Notice that $\langle x \rangle_0$ is just $\langle x \rangle$.

Under standard assumptions about $x(t)$, one gets, for $\tau \in [t - T, t]$ with t fixed,

$$x(\tau) = \sum_{k=-\infty}^{+\infty} \langle x \rangle_k(t) e^{jk\omega\tau}. \quad (38)$$

If the $\langle x \rangle_k(t)$ are computed with (37) for a given t , then (38) just reproduces $x(\tau)$ periodically outside $[t - T, t]$, so it does not yield x outside of $[t - T, t]$ if x is not T -periodic.

²We also assume that u does not vary over this time scale; in fact u is constant in many applications.

However, the idea of GSSA is to let t vary in (37) so that we really have a kind of “moving” Fourier series:

$$x(\tau) = \sum_{k=-\infty}^{+\infty} \langle x \rangle_k(t) e^{jk\omega\tau}, \quad \forall \tau. \quad (39)$$

A more mathematically advanced discussion is presented in [37].

In order to obtain a dynamical GSSA model we need the following two essential properties:

- **Derivation.** Writing (37) as

$$\langle x \rangle_k(t) = \frac{1}{T} \int_0^T x(\tau + t - T) e^{-jk\omega(\tau+t-T)} d\tau, \quad (40)$$

one immediately gets

$$\frac{d}{dt} \langle x \rangle_k(t) = \left\langle \frac{dx}{dt} \right\rangle_k(t) - jk\omega \langle x \rangle_k(t). \quad (41)$$

- **Convolution.** If x and y are two signals, then

$$\langle xy \rangle_k = \sum_{l=-\infty}^{+\infty} \langle x \rangle_{k-l} \langle y \rangle_l. \quad (42)$$

In particular, considering a first-harmonic GSSA approximation, one gets

$$\begin{aligned} \langle xy \rangle_0 &= \langle x \rangle_0 \langle y \rangle_0 + \langle x \rangle_{-1} \langle y \rangle_1 + \langle x \rangle_1 \langle y \rangle_{-1}, \\ \langle xy \rangle_1 &= \langle x \rangle_0 \langle y \rangle_1 + \langle x \rangle_1 \langle y \rangle_0, \\ \langle xy \rangle_{-1} &= \langle x \rangle_0 \langle y \rangle_{-1} + \langle x \rangle_{-1} \langle y \rangle_0. \end{aligned} \quad (43)$$

Using (41) and (31) one gets

$$\begin{aligned} \frac{d}{dt} \langle x \rangle_k &= \left\langle \frac{dx}{dt} \right\rangle_k - jk\omega \langle x \rangle_k \\ &= \langle [\mathcal{J}(S, x) - \mathcal{R}(S, x)] (\nabla H(x))^T + g(S, x)u \rangle_k - jk\omega \langle x \rangle_k. \end{aligned} \quad (44)$$

Assuming that the structure matrices \mathcal{J} and \mathcal{R} , the Hamiltonian H , and the interconnection matrix g have a series expansion in their variables, the convolution formula (42) can be used and an (infinite) dimensional system for the $\langle x \rangle_k$ can be obtained. Notice that, if we restrict ourselves to the dc terms (and without taking into consideration the contributions of the higher order harmonics to the dc averages), then (44) boils down to (35) since, under these assumptions, the zero-order average of a product is the product of the zero-order averages.

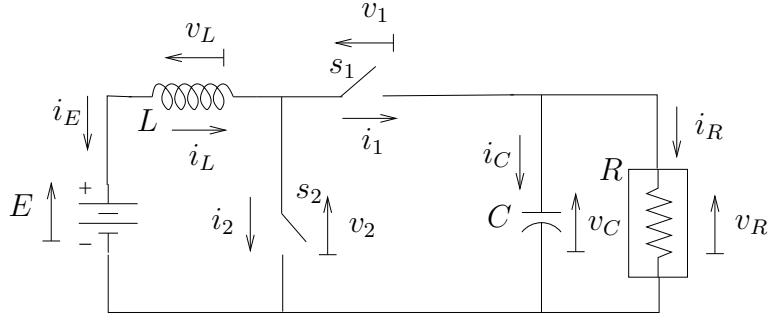


Figure 10: A functional description of the boost converter.

4.2 Second order power converters

Figure 10 shows a functional model of the boost (or elevator) converter (the detailed electronics of how the switches are implemented is not shown). The switches s_1 and s_2 are complementary: when s_1 is closed ($s_1 = 1$), s_2 is open ($s_2 = 0$), and viceversa. Thus, the different circuit topologies can be described with a single boolean variable $S = s_2$.

The port Hamiltonian modeling of electric circuits can be done in a systematic way using tools from graph theory [7], but since we are dealing here with a circuit of very small size we will adopt a more pedestrian approach and concentrate on the problems presented by the switches, using the ideas of [13]. A more in-deep conceptual analysis of the switches can be found in [16].

The Hamiltonian dynamical variables of the boost converter are the magnetic flux at the coil, ϕ_L , and the charge of the capacitor, q_C . Hence we have two one-dimensional Hamiltonian subsystems, with a global Hamiltonian $H = H_C + H_L$,

$$\frac{dq_C}{dt} = i_C, \quad v_C = \frac{\partial H}{\partial q_C}, \quad (45)$$

and

$$\frac{d\phi_L}{dt} = v_L, \quad i_L = \frac{\partial H}{\partial \phi_L}, \quad (46)$$

connected by Kirchoff's laws

$$\begin{aligned} i_L &= i_1 + i_2 \\ i_1 &= i_C + i_R \\ v_2 + v_L &= E \\ v_C + v_1 &= v_2 \\ v_C &= v_R \\ i_E + i_L &= 0 \end{aligned} \quad (47)$$

Here we treat the switches as ports, with their correspondent effort and flow variables. For the time being we do not terminate the resistive port, *i.e.* we do not use $v_R = Ri_R$.

EXERCISE Write the EF -representation [10] of the Dirac structure associated to (47).

Using (45) and (46), the first four equations of (47) can be written as

$$\begin{aligned}
\frac{\partial H}{\partial \phi_L} &= i_1 + i_2 \\
i_1 &= \frac{dq_C}{dt} + i_R \\
v_2 + \frac{d\phi_L}{dt} &= E \\
\frac{\partial H}{\partial q_C} + v_1 &= v_2
\end{aligned} \tag{48}$$

The second and third equations in (48) yield a Hamiltonian system with four inputs and $\mathcal{J} = \mathcal{R} = 0_{2 \times 2}$:

$$\frac{d}{dt} \begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = 0(\nabla H)^T + \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ v_2 \\ i_R \\ E \end{pmatrix}. \tag{49}$$

Next we will use the constraints imposed by the switches to absorb the ports s_1 and s_2 into the Hamiltonian structure:

- $S = 0 \Rightarrow s_1 = 1, s_2 = 0 \Rightarrow v_1 = 0, i_2 = 0,$
- $S = 1 \Rightarrow s_1 = 0, s_2 = 1 \Rightarrow i_1 = 0, v_2 = 0.$

Hence, when $S = 1$ we already have the values of the port variables i_1, v_2 in (49), while if $S = 0$, using the first and fourth equations in (48),

$$i_1 = \frac{\partial H}{\partial \phi_L}, \quad v_2 = \frac{\partial H}{\partial q_C}.$$

We can put together both results as

$$\begin{aligned}
i_1 &= (1 - S) \frac{\partial H}{\partial \phi_L}, \\
v_2 &= (1 - S) \frac{\partial H}{\partial q_C}.
\end{aligned} \tag{50}$$

Now

$$\begin{aligned}
\frac{d}{dt} \begin{pmatrix} q_C \\ \phi_L \end{pmatrix} &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} (1 - S) \frac{\partial H}{\partial \phi_L} \\ (1 - S) \frac{\partial H}{\partial q_C} \\ i_R \\ E \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 - S \\ -(1 - S) & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_C} \\ \frac{\partial H}{\partial \phi_L} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i_R \\ E \end{pmatrix}, \tag{51}
\end{aligned}$$

which is a port Hamiltonian system with outputs

$$y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} \frac{\partial H}{\partial q_C} \\ \frac{\partial H}{\partial \phi_L} \end{pmatrix} = \begin{pmatrix} -v_C \\ i_L \end{pmatrix} = \begin{pmatrix} -v_R \\ -i_E \end{pmatrix}.$$

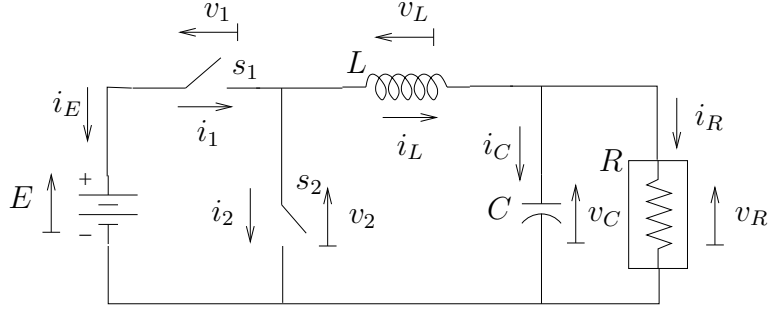


Figure 11: The buck converter.

Finally, we terminate the resistive port using

$$i_R = \frac{v_R}{R} = \frac{v_C}{R} = \frac{1}{R} \frac{\partial H}{\partial q_C}$$

and get our final port Hamiltonian representation of the boost converter

$$\frac{d}{dt} \begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \left[\begin{pmatrix} 0 & 1-S \\ -(1-S) & 0 \end{pmatrix} - \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \partial H / \partial q_C \\ \partial H / \partial \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} E, \quad (52)$$

with natural output

$$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} \partial H / \partial q_C \\ \partial H / \partial \phi_L \end{pmatrix} = i_L = -i_E.$$

Notice that the interconnection structure \mathcal{J} is modulated by the boolean variable S . Designing a control for this system means choosing S as a function of the state variables.

EXERCISE Figure 11 shows an scheme of the buck (or step-down) power converter. Show that the final port Hamiltonian structure is

$$\frac{d}{dt} \begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} \partial H / \partial q_C \\ \partial H / \partial \phi_L \end{pmatrix} + \begin{pmatrix} 0 \\ 1-S \end{pmatrix} E.$$

4.3 SSA and GSSA for second order power converters

Figure 12 shows the functional scheme of the buck-boost, which is the remaining elemental second order dc-dc converter.

The VSS port Hamiltonian models for the three converters presented, with the resistive port left open, can be written in an unified way as

$$\dot{x} = \begin{pmatrix} 0 & \alpha - \beta S \\ -(\alpha - \beta S) & 0 \end{pmatrix} (\nabla H(x))^T + \begin{pmatrix} -1 & 0 \\ 0 & 1 - \gamma S \end{pmatrix} \begin{pmatrix} i_R \\ E \end{pmatrix}, \quad (53)$$

where the state variables are $x = (q_C \ \phi_L)^T$ and the Hamiltonian function is

$$H(q_C, \phi_L) = \frac{1}{2C} q_C^2 + \frac{1}{2L} \phi_L^2, \quad (54)$$

with $i_L = \partial_{\phi_L} H$, $v_C = \partial_{q_C} H$. The parameters corresponding to the different converters are given in Table 1. The variable $S = 0, 1$ represents the state of switch 2 (0 closed and

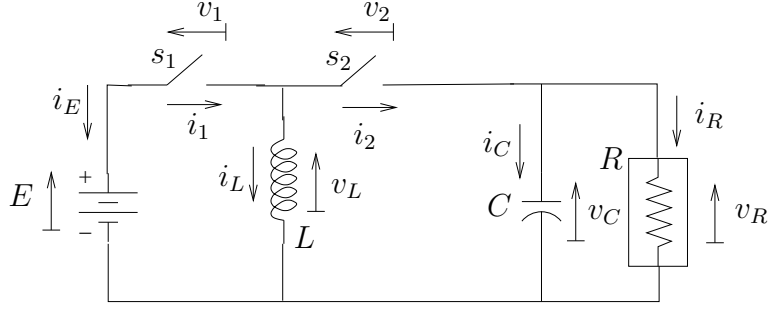


Figure 12: The buck-boost converter.

Converter	α	β	γ
buck	1	0	1
boost	1	1	0
buck-boost	0	1	1

Table 1: Parameter values for the unified description of the second-order power converters.

1 open), and switch 1 is complementary to switch 2. The natural Hamiltonian outputs of the system are

$$y = \begin{pmatrix} -1 & 0 \\ 0 & 1 - \gamma S \end{pmatrix}^T (\nabla H(x))^T = \begin{pmatrix} -v_C \\ (1 - \gamma S)i_L \end{pmatrix} = \begin{pmatrix} -v_R \\ -(1 - \gamma S)i_E \end{pmatrix}. \quad (55)$$

The resistive port can be terminated (*i.e.* one can use $v_R = Ri_R$) and then the structure matrix in (53) gets a dissipative part:

$$\dot{x} = \begin{pmatrix} -1/R & \alpha - \beta S \\ -(\alpha - \beta S) & 0 \end{pmatrix} (\nabla H(x))^T + \begin{pmatrix} 0 \\ 1 - \gamma S \end{pmatrix} E. \quad (56)$$

Applying (35) to (56), one immediately gets

$$\begin{aligned} \frac{d}{dt}\langle x \rangle &= d_0(\langle x \rangle) \left[\begin{pmatrix} -1/R & \alpha \\ -\alpha & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 \end{pmatrix} E \right] \\ &+ d_1(\langle x \rangle) \left[\begin{pmatrix} -1/R & \alpha - \beta \\ -(\alpha - \beta) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma \end{pmatrix} E \right] \\ &= \begin{pmatrix} -1/R & \alpha - \beta d_1(\langle x \rangle) \\ -(\alpha - \beta d_1(\langle x \rangle)) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma d_1(\langle x \rangle) \end{pmatrix} E \end{aligned} \quad (57)$$

where $d_0 + d_1 = 1$ has been used. Since

$$\langle S \rangle = 0 \cdot d_0 + 1 \cdot d_1,$$

one finally gets

$$\frac{d}{dt}\langle x \rangle = \begin{pmatrix} -1/R & \alpha - \beta \langle S \rangle \\ -(\alpha - \beta \langle S \rangle) & 0 \end{pmatrix} (\nabla H(\langle x \rangle))^T + \begin{pmatrix} 0 \\ 1 - \gamma \langle S \rangle \end{pmatrix} E, \quad (58)$$

which is well-known in the literature in terms of co-energy variables.

Let us turn now to a GSSA truncated approximation, considering the fundamental terms and the first harmonics. Using the Hamiltonian (54), we have

$$\frac{d}{dt} \begin{pmatrix} q_C \\ \phi_L \end{pmatrix} = \begin{pmatrix} -1/R & \alpha - \beta S \\ -(\alpha - \beta S) & 0 \end{pmatrix} \begin{pmatrix} \frac{q_C}{C} \\ \frac{\phi_L}{L} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 - \gamma S \end{pmatrix}. \quad (59)$$

We define real variables x_i , $i = 1, \dots, 6$ by means of

$$\begin{aligned} \langle \phi_L \rangle_1 &= x_1 + jx_2, \\ \langle \phi_L \rangle_{-1} &= x_1 - jx_2, \\ \langle q_C \rangle_1 &= x_3 + jx_4, \\ \langle q_C \rangle_{-1} &= x_3 - jx_4, \\ \langle \phi_L \rangle_0 &= x_5, \\ \langle q_C \rangle_0 &= x_6, \end{aligned}$$

so that, using (41),

$$\begin{aligned} \dot{x}_1 &= \omega x_2 + \frac{1}{2} (\langle \dot{\phi}_L \rangle_1 + \langle \dot{\phi}_L \rangle_{-1}), \\ \dot{x}_2 &= -\omega x_1 + \frac{1}{2j} (\langle \dot{\phi}_L \rangle_1 - \langle \dot{\phi}_L \rangle_{-1}), \\ \dot{x}_3 &= \omega x_4 + \frac{1}{2} (\langle \dot{q}_C \rangle_1 + \langle \dot{q}_C \rangle_{-1}), \\ \dot{x}_4 &= -\omega x_3 + \frac{1}{2j} (\langle \dot{q}_C \rangle_1 - \langle \dot{q}_C \rangle_{-1}), \\ \dot{x}_5 &= \langle \dot{\phi}_L \rangle_0, \\ \dot{x}_6 &= \langle \dot{q}_C \rangle_0. \end{aligned} \quad (60)$$

Next we compute the right-hand sides of (60) using (59) and the convolution property, and get

$$\begin{aligned} \dot{x}_1 &= \omega x_2 - (\alpha - \beta \langle S \rangle_0) \frac{x_3}{C} + \beta \langle S \rangle_{1R} \frac{x_6}{C} - \gamma E \langle S \rangle_{1R}, \\ \dot{x}_2 &= -\omega x_1 - (\alpha - \beta \langle S \rangle_0) \frac{x_4}{C} + \beta \langle S \rangle_{1I} \frac{x_6}{C} - \gamma E \langle S \rangle_{1I}, \\ \dot{x}_3 &= (\alpha - \beta \langle S \rangle_0) \frac{x_1}{L} - \frac{1}{R} \frac{x_3}{C} + \omega x_4 - \beta \langle S \rangle_{1R} \frac{x_5}{L}, \\ \dot{x}_4 &= (\alpha - \beta \langle S \rangle_0) \frac{x_2}{L} - \omega x_3 - \frac{1}{R} \frac{x_4}{C} - \beta \langle S \rangle_{1I} \frac{x_5}{L}, \\ \dot{x}_5 &= 2\beta \langle S \rangle_{1R} \frac{x_3}{C} + 2\beta \langle S \rangle_{1I} \frac{x_4}{C} - (\alpha - \beta \langle S \rangle_0) \frac{x_6}{C} + E(1 - \gamma \langle S \rangle_0), \\ \dot{x}_6 &= -2\beta \langle S \rangle_{1R} \frac{x_1}{L} - 2\beta \langle S \rangle_{1I} \frac{x_2}{L} + (\alpha - \beta \langle S \rangle_0) \frac{x_5}{L} - \frac{1}{R} \frac{x_6}{C}, \end{aligned}$$

where $\langle S \rangle_{1R,1I}$ are the real and imaginary parts of $\langle S \rangle_1$. As shown in [4], using the Hamiltonian

$$H_{PH} = \frac{1}{2L} (x_1^2 + x_2^2 + \frac{1}{2} x_5^2) + \frac{1}{2C} (x_3^2 + x_4^2 + \frac{1}{2} x_6^2), \quad (61)$$

this can be cast into Hamiltonian form

$$\dot{x} = (\mathcal{J}_{PH} - \mathcal{R}_{PH})(\nabla H_{PH}(x))^T + g_{PH}E$$

with

$$\mathcal{J}_{PH} = \begin{pmatrix} 0 & \omega L & -\alpha + \beta\langle S \rangle_0 & 0 & 0 & 2\beta\langle S \rangle_{1R} \\ -\omega L & 0 & 0 & -\alpha + \beta\langle S \rangle_0 & 0 & 2\beta\langle S \rangle_{1I} \\ \alpha - \beta\langle S \rangle_0 & 0 & 0 & \omega C & -2\beta\langle S \rangle_{1R} & 0 \\ 0 & \alpha - \beta\langle S \rangle_0 & -\omega C & 0 & -2\beta\langle S \rangle_{1I} & 0 \\ 0 & 0 & 2\beta\langle S \rangle_{1R} & 2\beta\langle S \rangle_{1I} & 0 & -2(\alpha - \beta\langle S \rangle_0) \\ -2\beta\langle S \rangle_{1R} & -2\beta\langle S \rangle_{1I} & 0 & 0 & 2(\alpha - \beta\langle S \rangle_0) & 0 \end{pmatrix}, \quad (62)$$

$$\mathcal{R}_{PH} = \text{diag} \left(0, 0, \frac{1}{R}, \frac{1}{R}, 0, \frac{2}{R} \right), \quad (63)$$

and

$$g_{PH} = \begin{pmatrix} -\gamma\langle S \rangle_{1R} \\ -\gamma\langle S \rangle_{1I} \\ 0 \\ 0 \\ 1 - \gamma\langle S \rangle_0 \\ 0 \end{pmatrix} \quad (64)$$

5 A system for storing excess energy in an electrical vehicle transportation network

Many metropolitan electrical-based vehicles use dc-motors which draw their power from segmented dc power lines. In general, the vehicles are able to brake in a regenerative way, returning power to the line. However, this power can be reused only if another vehicle is accelerating in the same segment of the power line; if not, this power is dissipated in special resistors and lost.

In order to be able to take advantage of this excess power, and also to be able to accelerate a given vehicle if no other one is braking nearby, one may think of installing in each power segment some device to store the energy. Several kinds of devices may in principle be used, but the special characteristics of the system under consideration, namely large power peaks and huge values of the total energy which must be stored or delivered, electrochemical batteries, due to their low power, and normal electrical capacitors, due to their low energy storage capacity, must be discarded. Two kinds of storage devices apt for the task at hand are supercapacitors, which have a high cost but are in general very efficient, and electromechanical devices, either dc or ac powered massive flywheels, which are cheaper but require much more space, have friction losses during normal operation and require much more maintenance.

In either case, a system to absorb the required amount of power during regenerative braking, and to return it at the desired rate and voltage to the line, must be introduced. This can be done with power electronic devices and Figure 13 shows a topology that can do the job. A supercapacitor is used in this case, but one can as well replace the supercapacitor with a normal one and connect to it a dc-motor, to which a mechanical flywheel can then be attached.

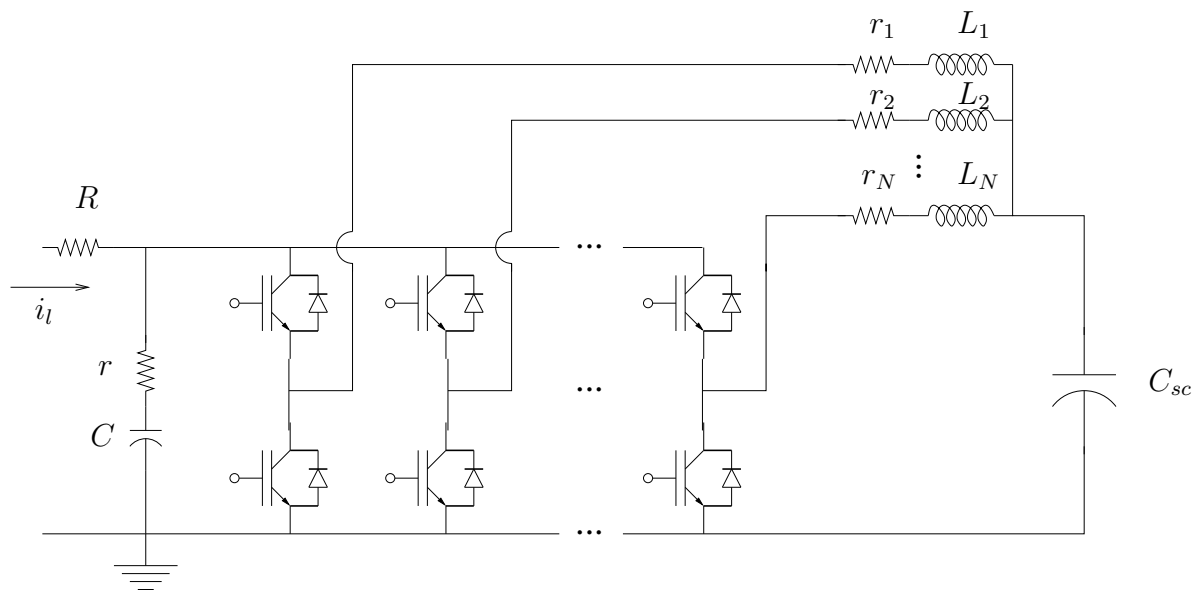


Figure 13: Topology for storing and returning line energy.

The control objective of this system would be to keep the line voltage constant, whatever the line current i_l might be.

Notice that several identical branches are used to feed the (super)capacitor. This is done for economical reasons, since less current must flow through each branch and cheaper power electronic components can then be used, and also for technical reasons, having to do with the dynamics of the system when it is returning power to the line; the circuit acts in that situation as a boost (the line voltage is higher ($\sim 1200\text{V}$) than the maximum (super)capacitor voltage ($\sim 600\text{V}$)) and the control algorithms cease to work if the current through a given branch exceeds a certain threshold, due to parasitic resistances and unmodeled dynamics of the IGBTs. The maximum power that can be returned is thus higher if several branches are used. This also has the advantage of scalability, which, as we will see, translates also to the modeling approach based on energy. Each branch contains an inductor, a resistor and a pair of IGBT+diode sets. Thus, this is a variable structure system; however, we are going to deduce its zeroth order averaged model, and see that we can replace each pair of IGBT+diode by a modulated transformer.

EXERCISE Denote each IGBT+diode set by $S_i, T_i, i = 1, \dots, N$. Show that the bond graph of the proposed system is the one given in Figure 14.

In Figure 14 we have not indicated the causality for the bonds attached to the S_i and T_i elements. This is due to the fact that each position of the switches defines a different causality. In fact, for each set $\{S_i, T_i\}$, only two causality assignments are possible, as shown in Figure 15. However, both possible internal configurations define the same external causality.

When an ideal switch (IGBT+diode) is closed, it acts as an ideal zero-voltage source, while it is an ideal zero-current source when it is open. Using this, it is easy to see from the above bond graphs that the pair of complementary switches act as an ideal transformer,

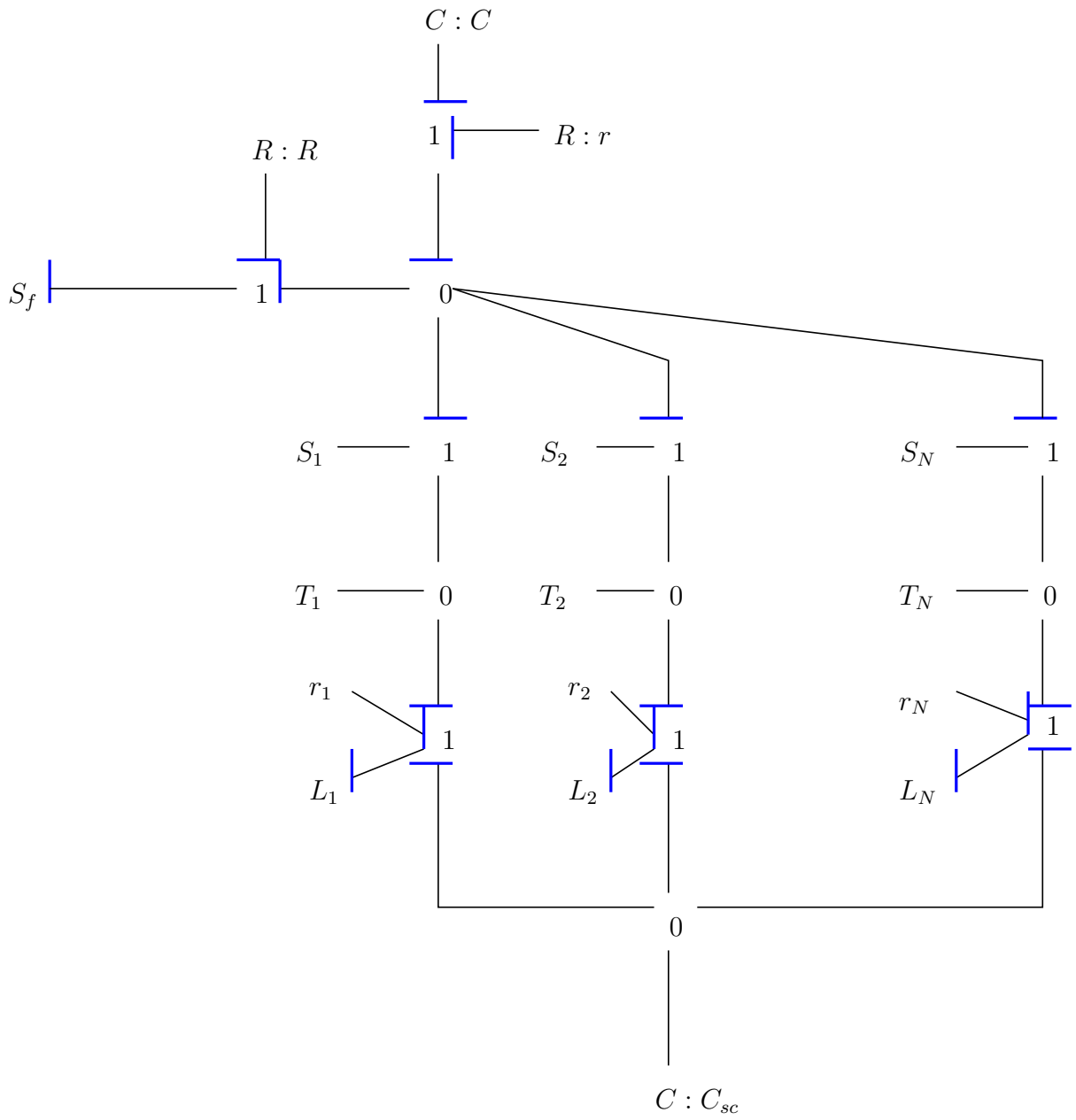


Figure 14: Bond graph of the system of Figure 13.

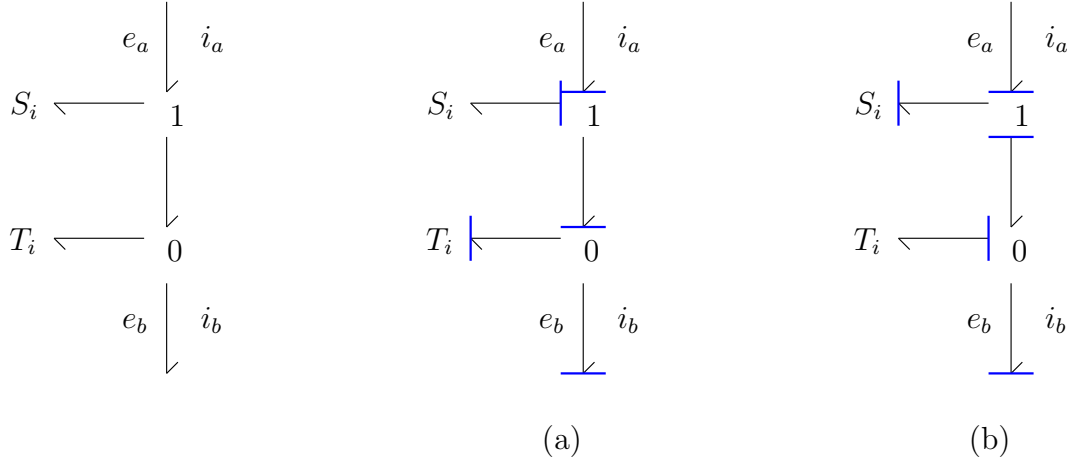


Figure 15: Possible causalities for a pair of complementary IGBT+diode. Case (a) $S_i = 1$: S_i closed and T_i open. Case (b) $S_i = 0$: S_i open and T_i closed.

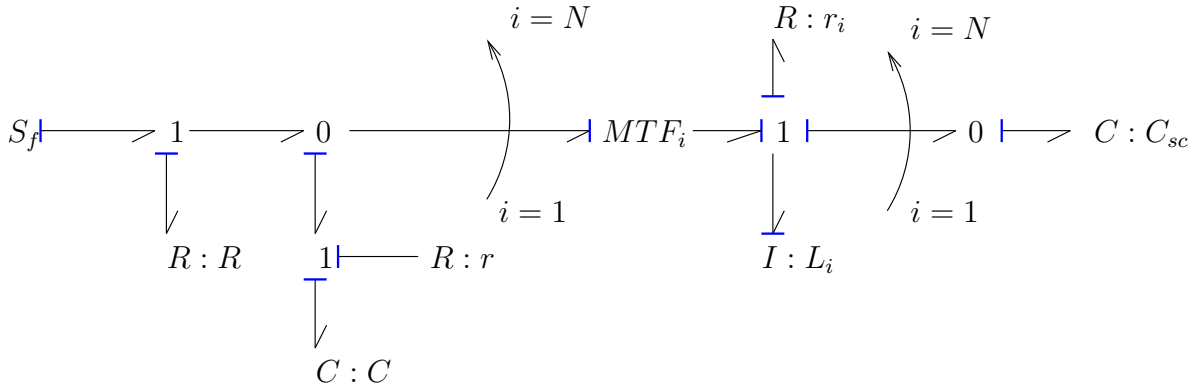


Figure 16: Bond graph with the equivalent transformers.

with relation

$$\begin{aligned} e_b &= S_i e_a, \\ i_a &= S_i i_b. \end{aligned} \quad (65)$$

Notice that S_i is not the standard transformer modulus as implemented in `20sim` (it would rather be S_i^{-1} , so the relations in the standard TF element must be changed to implement this. Using this, one can write the final bond graph as depicted in Figure 16, where a special notation has been used to denote the multiple branches and each MTF_i implements the relation in (65) with modulated signal S_i .

EXERCISE Assign labels to each bond in the bond graph of Figure 16 and compute the state space equations. If the state space variables are denoted by q (charge of C), Q (charge of C_{sc}) and λ_i (flux of L_i , $i = 1, \dots, N$), show that they can be given a port Hamiltonian form

$$\dot{x} = (\mathcal{J} - CR)\partial_x H + gi_l,$$

with

$$H(Q, q, \lambda_i) = \frac{1}{2C_{sc}}Q^2 + \frac{1}{2C}q^2 + \sum_{i=1}^N \frac{1}{2L_i}\lambda_i^2,$$

where all energy storage elements are taken as linear, the interconnection matrix is given by

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & -S_1 & \cdots & -S_N \\ -1 & S_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & S_N & 0 & \cdots & 0 \end{pmatrix}$$

and the dissipation matrix by

$$\mathcal{R} = \begin{pmatrix} 0_{2 \times 2} & 0_{2 \times N} \\ 0_{N \times 2} & \delta_{ij}R_j + rS_iS_j \end{pmatrix},$$

while the port matrix is

$$g = \begin{pmatrix} 0 \\ 1 \\ rS_1 \\ rS_2 \\ \vdots \\ rS_N \end{pmatrix}.$$

Notice that all the matrices, including the dissipation structure and the open ports matrix, depend on the states of the switches.

EXERCISE Write the `20sim` model for the above system, but replacing the supercapacitor by an ordinary one, and adding to it a dc motor with a flywheel attached. Assign parameters, put a variable (piecewise constant) i_i and do simulations. The `20sim` model should look like the one in Figurebg20sim, where 3 branches have been used and all the transformers are driven by the same signal. In fact, each S_i can be taken as constant, giving the duty cycle of a zeroth order averaged model.

Important remark: the switches that we have tried to model are, in fact, two-quadrant. They allow current in both ways but can sustain a non-zero voltage only in one way. This restriction has not been incorporated in our model (the switches in the model are four-quadrant) and hence the model is not competent to simulate realistically situations where the switches invert polarity. These kind of sign restrictions appear frequently in power electronics when one considers actual devices, giving rise to what are called *generalized discontinuous conduction modes*, which can be treated, among other options, in the framework of *complementarity dynamical systems*. [36][5]

6 Energy based control

Traditionally, control problems have been approached adopting a signal-processing viewpoint. This is very useful for linear time-invariant systems, where signals can be discriminated via filtering. However, for nonlinear systems, frequency mixing invalidates this approach due to the following reasons:

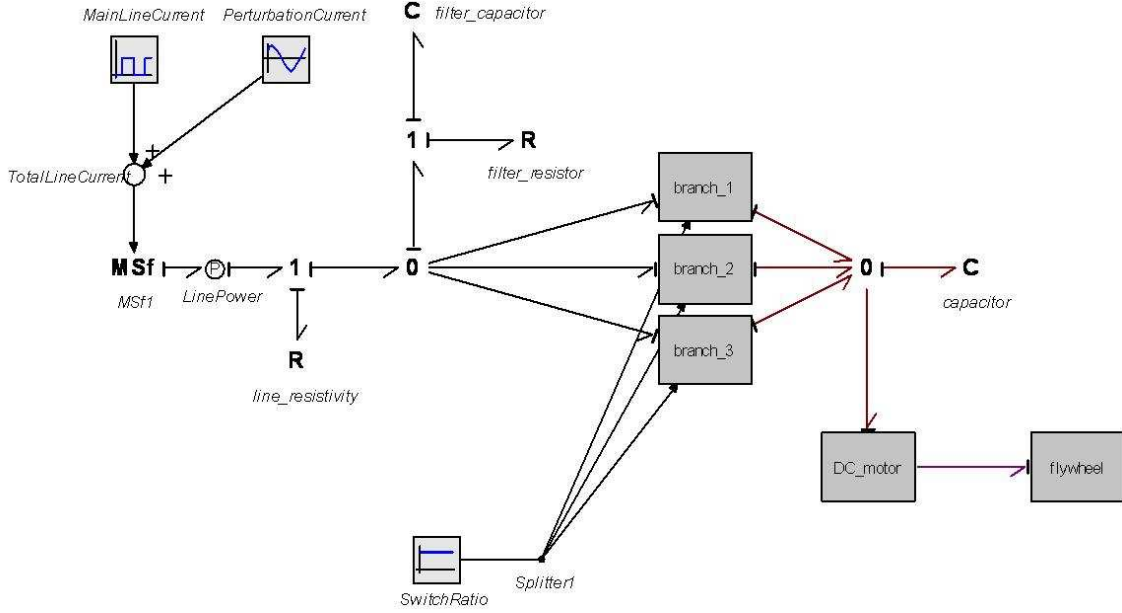


Figure 17: 20sim model with an electromechanical subsystem attached.

1. computations are far from obvious.
2. very complex controls are needed to quench the large set of undesirable signals, and the result is very inefficient, with a lot of energy being consumed and always on the verge of instability (a typical example is provided by bipedal walking machines; see Stramigioli's lectures in this summer school).

Most of the problem stem from the fact that no information about the structure of the system is used. A change of control paradigm is needed:

control systems as energy exchanging entities

We present the basic ideas involved in this energy-based approach to control. We follow the pedagogical account in [28]; complete proofs are presented in [27].

The map $u \mapsto y$ is *passive* if there exists a state function $H(x)$, bounded from below, and a nonnegative function $d(t) \geq 0$ such that

$$\underbrace{\int_0^t u^T(s)y(s) \, ds}_{\text{energy supplied to the system}} = \underbrace{H(x(t)) - H(x(0))}_{\text{stored energy}} + \underbrace{d(t)}_{\text{dissipated}} .$$

The simplest example of a passive system is probably the forced mass-spring-damper arrangement of Figure 18.

One has

$$\begin{aligned} \int_0^t F(s)v(s) \, ds &= \int_0^t (m\dot{v}(s) + kq(s) + \lambda v(s))v(s) \, ds \\ &= \left(\frac{1}{2}mv^2(s) + \frac{1}{2}kq^2(s) \right) \Big|_0^t + \lambda \int_0^t v^2(s) \, ds \\ &= H(x(t)) - H(x(0)) + \lambda \int_0^t v^2(s) \, ds. \end{aligned}$$

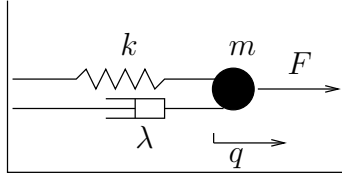


Figure 18: Example of a mechanical passive system.

If x^* is a global minimum of $H(x)$ and $d(t) > 0$, and we set $u = 0$, $H(x(t))$ will decrease in time and the system will reach x^* asymptotically. The rate of convergence can be increased if we actually extract energy from the system with

$$u = -K_{\text{di}} y$$

with $K_{\text{di}}^T = K_{\text{di}} > 0$. However, the minimum of the natural energy H of the system is not a very interesting point in most engineering problems.

The key idea of passivity based control (PBC) is as follows: use feedback

$$u(t) = \beta(x(t)) + v(t)$$

so that the closed-loop system is again a passive system, with energy function H_{d} , with respect to $v \mapsto y$, and such that H_{d} has the global minimum at the desired point. Passivity for the closed-loop system is far from obvious: physically, the controller is injecting energy into the system. PBC is robust with respect to unmodeled dissipation, and has built-in safety: even if we don't know H exactly, if passivity is preserved the system will stop somewhere instead of running away and finally blowing up.

If

$$-\int_0^t \beta^T(x(s))y(s) ds = H_{\text{a}}(x(t)) \quad (66)$$

then the closed-loop system has energy function $H_{\text{d}}(x) = H(x) + H_{\text{a}}(x)$. One has the following *energy balance equation* (EBE), which yields an interpretation to PBC:

$$\underbrace{H_{\text{d}}(x(t))}_{\text{closed-loop energy}} = \underbrace{H(x(t))}_{\text{stored energy}} - \underbrace{\int_0^t \beta^T(x(s))y(s) ds}_{\text{supplied energy}}.$$

For (f, g, h) systems, the EBE is equivalent to the PDE

$$-\beta^T(x)h(x) = \left(\frac{\partial H_{\text{a}}}{\partial x}(x) \right)^T (f(x) + g(x)\beta(x)). \quad (67)$$

As an example, consider the electrical system in Figure 19.

$$\begin{aligned} x &= \begin{pmatrix} q \\ \phi \end{pmatrix} \text{ state} \\ u &= V \text{ control} \\ y &= i = \frac{\phi}{L} \text{ output} \end{aligned}$$

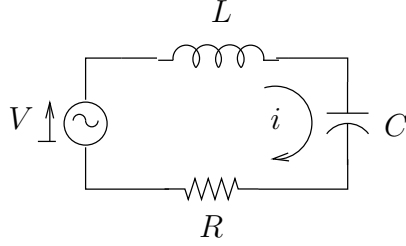


Figure 19: Example of electrical passive system.

$$\dot{x} = \begin{pmatrix} x_2/L \\ -x_1/C - x_2R/L \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} V(t)$$

The map $V \mapsto i$ is passive with energy function

$$H(x) = \frac{1}{2C}x_1^2 + \frac{1}{2L}x_2^2$$

and dissipation $d(t) = \int_0^t \frac{R}{L^2} \phi^2(s) ds$. Notice that the natural minimum is $(0, 0)$, but forced equilibrium points are of the form $(x_1^*, 0)$.

The PDE (67) is in this case

$$\frac{x_2}{L} \frac{\partial H_a}{\partial x_1} - \left(\frac{x_1}{C} + \frac{R}{L}x_2 - \beta(x) \right) \frac{\partial H_a}{\partial x_2} = -\frac{x_2}{L}\beta(x).$$

Since $x_2^* = 0$ is already a minimum of H , we only have to shape the energy in x_1 . Hence, we can take $H_a = H_a(x_1)$ and the above PDE boils down to

$$\beta(x_1) = -\frac{\partial H_a}{\partial x_1}(x_1)$$

i.e. it defines the closed-loop control. Then we are free to choose H_a so that H_d has the minimum at x_1^* . The simplest solution is

$$H_a(x_1) = \frac{1}{2C_a}x_1^2 - \left(\frac{1}{C} + \frac{1}{C_a} \right) x_1^*x_1 + K$$

where C_a is a design parameter. The closed-loop energy H_d can then be computed and it is seen that it has a minimum at $(x_1^*, 0)$ if $C_a > -C$. Finally, one can compute the control as

$$u = -\frac{\partial H_a}{\partial x_1}(x_1) = -\frac{x_1}{C_a} + \left(\frac{1}{C} + \frac{1}{C_a} \right) x_1^*.$$

This control is an energy-balancing PBC that stabilizes x^* under the stated parameter restrictions.

EXERCISE Show that the total energy supplied by the controller is finite.

Consider now the slightly different circuit of Figure 20. With the same states, energy, input and outputs than the preceding system, the equations of motion are now

$$\begin{aligned} \dot{x}_1 &= -\frac{1}{RC}x_1 + \frac{1}{L}x_2, \\ \dot{x}_2 &= -\frac{1}{C}x_1 + V(t). \end{aligned}$$

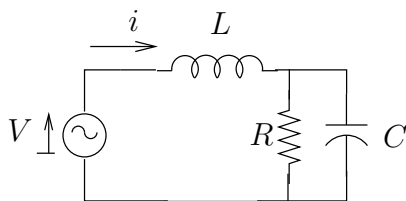


Figure 20: Another electrical passive system.

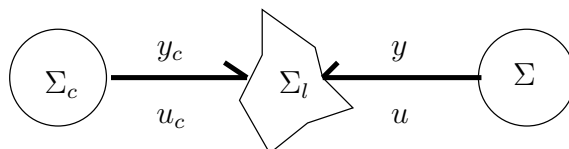


Figure 21: Network interpretation of control.

Only the dissipation structure has changed, but the admissible equilibria are of the form

$$x_1^* = CV^*, \quad x_2^* = \frac{L}{R}V^*$$

for any constant V^* . The power delivered by the source, $V \frac{x_2}{L}$, is nonzero at any equilibrium point except for the trivial one. Hence, the source has to provide an infinite amount of energy to keep any nontrivial equilibrium point, a task which is clearly not feasible. This situation will reappear later when we discuss invariant and Casimir functions. Notice that pure mechanical systems are free of this problem, since any equilibrium point has velocities equal to zero and hence no power is necessary to keep the system at the equilibrium point.

6.1 Control as interconnection

Now we would like to have a physical interpretation of PBC. To be precise, we would like to think of the controller as a *system exchanging energy with the plant*. Consider two systems, Σ and Σ_c , exchanging energy through an interconnection network given by Σ_l , as depicted in Figure 21.

The condition for the interconnection to be power continuous is

$$u_c^T(t)y_c(t) + u^T(t)y(t) = 0 \quad \forall t.$$

As an example, consider the typical negative feedback interconnection displayed in Figure 22. The interconnection is given by

$$\begin{aligned} u_c &= y \\ u &= -y_c \end{aligned}$$

and is clearly power continuous.

Suppose now that we add some extra inputs $u \rightarrow u + v$, $u_c \rightarrow u_c + v_c$ to the interconnected system. Then it is easy to show the following. Let Σ and Σ_c have state variables x and ξ . If Σ and Σ_c are passive with energy functions $H(x)$ and $H_c(\xi)$ and Σ_l is power preserving, then the map $(v, v_c) \mapsto (y, y_c)$ is passive for the interconnected system, with energy function $H_d(x, \xi) = H(x) + H_c(\xi)$. Or, in short,

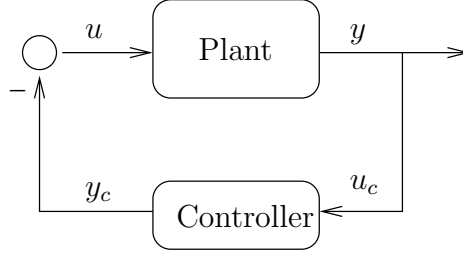


Figure 22: Typical negative feedback interconnection.

power continuous interconnection of passive systems yields passive systems.

Now we have a passive system with energy function $H_d(x, \xi) = H(x) + H_c(\xi)$, but this is not very useful unless we get an energy function depending only on x . To solve this, we restrict the dynamics to a submanifold of the (x, ξ) space parameterized by x :

$$\Omega_K = \{(x, \xi) ; \xi = F(x) + K\},$$

and dynamically invariant:

$$\left(\left(\frac{\partial F}{\partial x} \right)^T \dot{x} - \dot{\xi} \right)_{\xi=F(x)+K} = 0.$$

Instead of solving this in general, it is convenient to formulate the problem for PHD systems.

6.2 Casimir functions and the dissipation obstacle

Consider a PHD system in explicit form given by

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned}$$

with $J^T(x) = -J(x)$, and $R^T(x) = R(x) \geq 0$. A PHD is passive with respect to the pair (u, y) :

$$\int_0^t u^T y = H(x(t)) - H(x(0)) + \int_0^t \left(\frac{\partial H}{\partial x} \right)^T R(x) \frac{\partial H}{\partial x}.$$

More precise results about the possibility of obtaining invariant manifolds expressing the controller variables in terms of the variables of the system can be formulated if both system and controller are PHDS. Let thus

$$\Sigma : \begin{cases} \dot{x} &= (J(x) - R(x)) \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \end{cases}$$

define the plant and

$$\Sigma_c : \begin{cases} \dot{\xi} &= (J_c(\xi) - R_c(\xi)) \frac{\partial H_c}{\partial \xi}(\xi) + g_c(\xi)u_c \\ y_c &= g_c^T(\xi) \frac{\partial H_c}{\partial \xi}(\xi) \end{cases}$$

define the controller. With the power preserving, standard negative feedback interconnection $u = -y_c$, $u_c = y$, one gets

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & -g(x)g_c^T(\xi) \\ g_c(\xi)g^T(x) & J_c(\xi) - R_c(\xi) \end{pmatrix} \begin{pmatrix} \frac{\partial H_d}{\partial x} \\ \frac{\partial H_d}{\partial \xi} \end{pmatrix}$$

where $H_d(x, \xi) = H(x) + H_c(\xi)$. Let us look now for invariant manifolds of the form

$$C_K(x, \xi) = F(x) - \xi + K.$$

Condition $\dot{C}_k = 0$ yields

$$\left(\left(\frac{\partial F}{\partial x} \right)^T \mid -\mathbb{I} \right) \begin{pmatrix} J - R & -gg_c^T \\ g_c g^T & J_c - R_c \end{pmatrix} \begin{pmatrix} \frac{\partial H_d}{\partial x} \\ \frac{\partial H_d}{\partial \xi} \end{pmatrix} = 0.$$

Since we want to keep the freedom to choose H_c , we demand that the above equation is satisfied on C_K for every Hamiltonian, *i.e.* we impose on F the following system of PDEs:

$$\left(\left(\frac{\partial F}{\partial x} \right)^T \mid -\mathbb{I} \right) \begin{pmatrix} J - R & -gg_c^T \\ g_c g^T & J_c - R_c \end{pmatrix} = 0.$$

Functions $C_K(x, \xi)$ such that F satisfies the above PDE on $C_K = 0$ are called *Casimir functions*. They are invariants associated to the structure of the system (J, R, g, J_c, R_c, g_c) , independently of the Hamiltonian function.

One can show [27] that the PDE for F has solution iff, on $C_K = 0$,

1. $\left(\frac{\partial F}{\partial x} \right)^T J \frac{\partial F}{\partial x} = J_c$,
2. $R \frac{\partial F}{\partial x} = 0$,
3. $R_c = 0$,
4. $\left(\frac{\partial F}{\partial x} \right)^T J = g_c g^T$.

Conditions 2 and 3 are easy to understand: essentially, no Casimir functions exist in presence of dissipation. Given the structure of the PDE, $R_c = 0$ is unavoidable, but we can have an effective $R = 0$ just by demanding that the coordinates on which the Casimir depends do not have dissipation, and hence condition 2.

If the preceding conditions are fulfilled, an easy computation shows that the dynamics on C_K is given by

$$\dot{x} = (J(x) - R(x)) \frac{\partial H_d}{\partial x}$$

with $H_d(x) = H(x) + H_c(F(x) + K)$. Notice that, due to condition 2,

$$R(x) \frac{\partial H_c}{\partial x}(F(x) + K) = \underbrace{R(x) \frac{\partial F}{\partial x}}_{=0} \frac{\partial H_c}{\partial \xi}(F(x) + K) = 0,$$

so we can say that, in energy-balancing PBC,

dissipation is only admissible for those coordinates which do not require energy shaping.

For regulation problems in mechanical systems, where the state consists of positions and velocities, dissipation only appear associated to the later, while energy shaping is necessary only in the position part, since the kinetic energy already has the minimum where we want it (that is, at velocity equal to zero). Hence, the dissipation obstacle is always absent for mechanical regulation problems. For the first of the two simple *RLC* circuits considered previously, dissipation appears in a coordinate, x_2 , which already has the minimum at the desired point. For the second one, the minimum of the energy has to be moved for both coordinates, and hence the dissipation obstacle is unavoidable.

7 IDA control method: basic ideas and examples

The previous Section has exposed some shortcomings of the passivity based control of PHDS by means of control-as-interconnection:

- Nonlinear PDE for the Casimir function.
- Dissipation obstacle.

One can get a method with more freedom if not only the energy function is changed but also the interconnection (J) and the dissipation (R), *i.e.* one aims at a closed-loop system of the form

$$\dot{x} = (J_d(x) - R_d(x)) \frac{\partial H_d}{\partial x}(x),$$

where $J_d^T(x) = -J_d(x)$, $R_d^T(x) = R_d(x) > 0$, and x^* a minimum of $H_d(x)$.

One has the following fundamental result [27]:

If one can find a (vector) function $K(x)$, a function $\beta(x)$, an antisymmetric matrix $J_a(x)$, and a symmetric, semipositive definite matrix $R_a(x)$ such that

$$(J(x) + J_a(x) - R(x) - R_a(x))K(x) = -(J_a(x) - R_a(x)) \frac{\partial H}{\partial x}(x) + g(x)\beta(x), \quad (68)$$

with K the gradient of a scalar function, $K(x) = \frac{\partial H_a}{\partial x}(x)$, then the closed loop dynamics with $u = \beta(x)$ is a Hamiltonian system with $H_d = H + H_a$, $J_d = J + J_a$, $R_d = R + R_a$. One can then try to impose conditions on H_a (or on K) so that x^* is an asymptotically stable point of the dynamics. This is called IDA-PBC (from *Interconnection and Damping Assignment Passivity Based Control*). Notice that (68) comes from matching the original system in closed loop to the desired dynamics:

$$(J(x) - R(x))\partial_x H(x) + g(x)\beta(x) = (J_d(x) - R_d(x))\partial_x H_d(x). \quad (69)$$

This is called *the matching equation*. Notice that, using if necessary algebraic combinations, one can split the n row equations in (69) into those that contain $u = \beta(x)$ and those that not. The former are not, in fact, equations to be solved, as they give the desired controller; the remaining equations are the ones that have to be satisfied using the freedom in J_a , R_a and H_a .

Several techniques have been proposed in the literature [27, 29, 26] to solve (69). One idea is to fix $J_a(x)$ and $R_a(x)$. Pre-multiplying by a left annihilator of $g(x)$ yields a linear PDE for H_a . After solving it, $\beta(x)$ can be computed using $(g^T(x)g(x))^{-1}$. Alternatively,

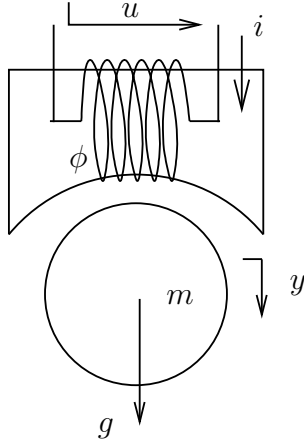


Figure 23: A magnetic levitation system.

one can fix the desired H_d and then solve the algebraic equations for J_a and R_a that are obtained. In between, one can fix parts of the interconnection and damping structures, and also part of the H_a dependence with respect to some of the variables, and solve the resulting mixture of algebraic and PDE equations.

On a more theoretical note, it can be shown [27] that

- IDA-PBC generates all asymptotically stabilizing controllers for PCH systems.
- If $R\partial_x H_a = 0$, IDA-PBC is energy-balancing.
- If, additionally, $J_a = R_a = 0$, one can think of IDA-PBC in terms of a Casimir function $C(x, \xi) = -H_a(x) - \xi$, a feedback interconnection modulated by $\beta(x)$ and a controller with energy $H_c(\xi) = -\xi$.

Here we are going to illustrate some of the basic techniques with concrete examples. Appendix A contains a very simple account of the method of characteristics, which is the basic tool for solving the kind of PDE that arises from (68).

7.1 Magnetic levitation system

Figure 23 shows a very simplified model of a magnetic levitation system.

The flux lines generated by the current at the coil close through the air gap and the iron ball. Since the air gap has a variable reluctance, the system tries to close it, and this counteracts the gravity.

The equations of motion are

$$\begin{aligned}\dot{\phi} &= -Ri + u \\ \dot{y} &= v \\ m\dot{v} &= F_m + mg\end{aligned}$$

with $\phi = L(y)i$ the linkage flux, R the resistance of the coil, and F_m the magnetic force, given by

$$F_m = \frac{\partial W_c}{\partial y},$$

where the magnetic coenergy is (we assume a linear magnetic system)

$$W_c = \frac{1}{2} \frac{\partial L}{\partial y} i^2.$$

In general, L is a complicated function of the air gap, y . A classical approximation for L for this kind of systems for small y is

$$L(y) = \frac{k}{a + y}$$

with k , a constants, which is essentially the relation used for our model of the electro-magnet, without the leakage term.

Taking $x_1 = \phi$, $x_2 = y$, $x_3 = mv$, this can be written as a PCH

$$\dot{x} = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \frac{\partial H}{\partial x} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

with Hamiltonian

$$H(x) = \frac{1}{2k}(a + x_2)x_1^2 + \frac{1}{2m}x_3^2 - mgx_2.$$

Note that the gravity term could also have been included as an external mechanical force.

The gradient of the Hamiltonian is

$$(\nabla H)^T = \begin{pmatrix} x_1 \frac{a+x_2}{k} \\ \frac{1}{2k}x_1^2 - mg \\ \frac{x_3}{m} \end{pmatrix}.$$

Given a desired y^* , the equilibrium point is

$$x^* = \begin{pmatrix} \sqrt{2kmg} \\ y^* \\ 0 \end{pmatrix},$$

with an equilibrium control

$$u^* = \frac{R}{k}x_1^*(a + x_2^*).$$

We will first try to compute the IDA-PBC control without using the specific theory presented for general electromechanical systems.

Taking first $J_a = R_a = 0$, the IDA-PBC equation $(J - R)K(x) = g\beta(x)$ yields in this case

$$\begin{aligned} -RK_1(x) &= \beta(x) \\ K_3(x) &= 0 \\ -K_2(x) &= 0, \end{aligned}$$

and we see that $H_a(x) = H_a(x_1)$, which can be chosen so that

$$H_d(x) = H(x) + H_a(x_1)$$

has a critical point at $x = x^*$.

Unfortunately

$$\frac{\partial^2 H_d}{\partial x^2}(x) = \begin{pmatrix} \frac{1}{k}(a + x_2) + H_a''(x_1) & \frac{x_1}{k} & 0 \\ \frac{x_1}{k} & 0 & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}$$

has at least one negative eigenvalue no matter what H_a we choose, so x^* will not be asymptotically stable. The source of the problem is the lack of coupling between the mechanical and magnetic part in the interconnection matrix J . To solve this, we aim at

$$J_d = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 1 \\ \alpha & -1 & 0 \end{pmatrix}, \quad i.e. \quad J_a = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}.$$

Taking $R_a = 0$, the IDA-PBC equation now becomes

$$\begin{aligned} -\alpha K_3 - RK_1(x) &= \frac{\alpha}{m}x_3 + \beta(x) \\ K_3(x) &= 0 \\ \alpha K_1(x) - K_2(x) &= -\frac{\alpha}{k}(a + x_2)x_1, \end{aligned}$$

Now $H_a = H_a(x_1, x_2)$. Using the second equation, the first equation yields the control

$$u = \beta(x) = RK_1 - \alpha \frac{x_3}{m},$$

while the third equation is a PDE for $H_a(x_1, x_2)$:

$$\alpha \frac{\partial H_a}{\partial x_1} - \frac{\partial H_a}{\partial x_2} = -\alpha \frac{x_1(a + x_2)}{k}. \quad (70)$$

EXERCISE Solve (70) by the method of characteristics. It is better to give the initial condition as $(0, s, \Phi(s))$ instead of $(s, 0, \Phi(s))$.

The way in which (68) has been solved in this case seems to be quite model-dependent. In [29] a more general method for systems of the form (25) with $T_m = \partial_\theta V(\theta)$ is proposed. The central idea is to aim for a H_d given by

$$H_d(x) = \frac{1}{2}(\lambda - \mu_d(\theta, p))^T L^{-1}(\theta)(\lambda - \mu_d(\theta, p)) + V_d(\theta) + \frac{1}{2J}p^2,$$

where $\mu_d(\theta, p)$ is a kind of desired permanent magnet, and consider a Hamiltonian structure of the form

$$\mathcal{J}_d - \mathcal{R}_a = \begin{pmatrix} -R & \alpha(x) & \beta(x) \\ -\alpha^T(x) & 0 & 1 \\ -\beta^T(x) & -1 & -r_a(p) \end{pmatrix}.$$

It can be seen then that the method boils down to an algebraic equation for

$$i_d = L^{-1}(\theta)\mu_d(\theta, p),$$

namely

$$\frac{1}{2}i_d^T \partial_\theta L(\theta) i_d + \partial_\theta V_d(\theta) - \partial_\theta V(\theta) - (r_m - r_a(p)) \frac{p}{J} = 0, \quad (71)$$

and that α , β and u can be computed from it.

EXERCISE Try to derive (71). See [29] for details.

EXERCISE Solve (71) for the levitating ball system. Use $V_d(y) = K_p \frac{(y-y^*)^2}{\sqrt{1+(y-y^*)^2}}$, $r_a(p) = r_{a1} + r_{a2} \frac{p^2}{1+p^2}$, with $K_p > 0$, $r_{a1} > 0$, $r_{a2} > 0$.

7.2 Boost converter

We retake the boost converter, or rather, its averaged model, where $u = 1 - S$, instead of taking values in the set $\{0, 1\}$, varies over $[0, 1]$ (see [11] for a discussion of this averaging process; essentially, S is changed periodically with a frequency much higher than the highest natural frequency of the system).

We have

$$\mathcal{J}(u) = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} 1/R & 0 \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and, assuming a linear electromagnetic system,

$$H(x_1, x_2) = \frac{1}{2C} x_1^2 + \frac{1}{2L} x_2^2.$$

We set as control objective the regulation of the load voltage at V_d , so that the equilibrium point is $x^* = (CV_d, \frac{LV_d^2}{RE})$ and the equilibrium value of the control is

$$u^* = \frac{E}{V_d}.$$

Notice that this makes sense (*i.e.* $u^* \in [0, 1]$) since this is a boost converter and $V_d \geq E$.

To solve (68) we take $\mathcal{J}_a = \mathcal{R}_a = 0$ and get

$$\begin{pmatrix} -1/R & u \\ -u & 0 \end{pmatrix} \frac{\partial H_a}{\partial x} = gE,$$

i.e.

$$\begin{aligned} -u \frac{\partial H_a}{\partial x_1} &= E, \\ -\frac{1}{R} \frac{\partial H_a}{\partial x_1} + u \frac{\partial H_a}{\partial x_2} &= 0. \end{aligned}$$

The standard way to solve this system [27] is to solve for the derivatives of H_a and then impose the identity of the second order cross derivatives. One gets

$$\begin{aligned} \frac{\partial H_a}{\partial x_1} &= -\frac{E}{u}, \\ \frac{\partial H_a}{\partial x_2} &= -\frac{E}{R u^2}. \end{aligned}$$

Imposing the identity of the cross derivatives we arrive at the PDE

$$\frac{2}{R} \frac{\partial u}{\partial x_1} - u \frac{\partial u}{\partial x_2} = 0. \quad (72)$$

EXERCISE Solve (72). As initial condition for the characteristic's method, take $(0, k_1 s, a s + b)$, with k_1 , a and b constants. In [30, 31] it is discussed how to choose the values of k_1 , a and b to impose the stability of the equilibrium point, as well as how to improve the resulting controller so that it is robust with respect to E and R .

EXERCISE Repeat the above analysis for the buck converter. See the beginning of Section 3.2 of [27] for some hints.

EXERCISE Implement the buck and boost systems and controllers in 20-sim. To test them, use $L = 20$ mH, $C = 20$ μ F, $E = 15$ V and $R = 30$ Ω .

7.3 DC motor

Consider a permanent magnet dc motor, or either a field dc motor for which the field dynamics is neglected. The PHDS model with Hamiltonian variables $x \in \mathbb{R}^2$

$$x^T = (\lambda, p) \quad (73)$$

where $\lambda \in \mathbb{R}$ is the inductor flux and $p \in \mathbb{R}$ is the angular momentum, and Hamiltonian

$$H(x) = \frac{1}{2L}\lambda^2 + \frac{1}{2J}p^2. \quad (74)$$

yielding the co-energy variables

$$i = \frac{1}{L}\lambda, \quad \omega = \frac{1}{J}p, \quad (75)$$

where L is the inductance, i is the current, $J \in \mathbb{R}$ is the inertia of the motor, and ω is the mechanical speed, is given by

$$\dot{x} = (J - R)\partial_x H + g + g_u u. \quad (76)$$

where the interconnection, damping, and port matrices are (we have split the efforts according to whether we can control them or not when the machine works as a motor)

$$J = \begin{pmatrix} 0 & -K \\ K & 0 \end{pmatrix} \quad R = \begin{pmatrix} r & 0 \\ 0 & b \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ -\tau_L \end{pmatrix} \quad g_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (77)$$

where $K \in \mathbb{R}$ is the torque constant, $r \in \mathbb{R}$ is the resistance losses in the electrical subsystem, $b \in \mathbb{R}$ is the damping in the mechanical subsystem, $\tau_L \in \mathbb{R}$ is an incoming external torque and the control action $u \in \mathbb{R}$ is the voltage $V \in \mathbb{R}$.

Assume that the control objective is a desired speed ω_d . In terms of ω_d , the equilibrium values of i and u are

$$i^* = \frac{1}{K}(b\omega_d + \tau_L) \quad (78)$$

$$u^* = r i^* + K\omega_d. \quad (79)$$

To apply the IDA-PBC technique we fix a desired Hamiltonian function H_d as

$$H_d = \frac{1}{2L}(\lambda - \lambda^*)^2 + \frac{1}{2J}(p - p^*)^2, \quad (80)$$

which implies

$$\partial_x H_d = \begin{pmatrix} i - i^* \\ \omega - \omega_d \end{pmatrix}. \quad (81)$$

In order to solve the matching equation of the IDA-PBC method, we consider desired interconnection and damping matrices given by

$$J_d - R_d = \begin{pmatrix} -r_d & -j_d \\ j_d & -b_d \end{pmatrix}. \quad (82)$$

The first row of the matching equation will yield the desired control action, while the second row imposes

$$j_d(i - i^*) - b_d(\omega - \omega_d) = Ki - b\omega - \tau_L. \quad (83)$$

Setting $b_d = b$, and using the equilibrium point expression, j_d is computed as

$$j_d = K, \quad (84)$$

and r_d is still a free parameter to tune the controller. Finally, substituting into the first row of the matching equation,

$$u = -r_d(i - i^*) - ri + K\omega_d. \quad (85)$$

Notice that this is just a proportional + constant compensation controller. Although we have not discussed robustness issues, it is well known that adding an integral correction eliminates regulation point errors due to non-nominal parameters. Hence, we can consider

$$u_i = -r_d(i - i^*) - ri + K\omega_d - \int (\omega - \omega_d) dt. \quad (86)$$

and compare the performances of both controllers. Notice that this improved controller has not been deduced in the framework of the IDA-PBC method for PHDS, but some results in that direction are available.[2]

Figure 24 shows the system behavior with the two previous controller (with and without integral term). The motor parameters are $r = 2 \Omega$, $L = 2 \text{ mH}$, $K = 0.07 \text{ V s rad}^{-1}$, $b = 0.0004 \text{ N m s rad}^{-1}$, $J = 6 \cdot 10^{-5} \text{ Kg m}$, and the nominal torque is $\tau_L = 2 \text{ N m}$. The desired mechanical speed is fixed at $\omega_d = 250 \text{ rad s}^{-1}$, and the control parameter is $r_d = 0.1 \Omega$.

The system starts with the nominal torque τ_L and at $t = 1 \text{ s}$ the torque changes at $\tau_L = 1.75 \text{ N m}$. In Figure 24 the mechanical speed is depicted. Note that for both controllers the system goes to the desired speed ω_d , but when the torque changes the non-integral controller is unable to reach ω_d while the integral one drives again the system to ω_d .

A Solving quasilinear PDEs

Equations of the form

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \quad (87)$$

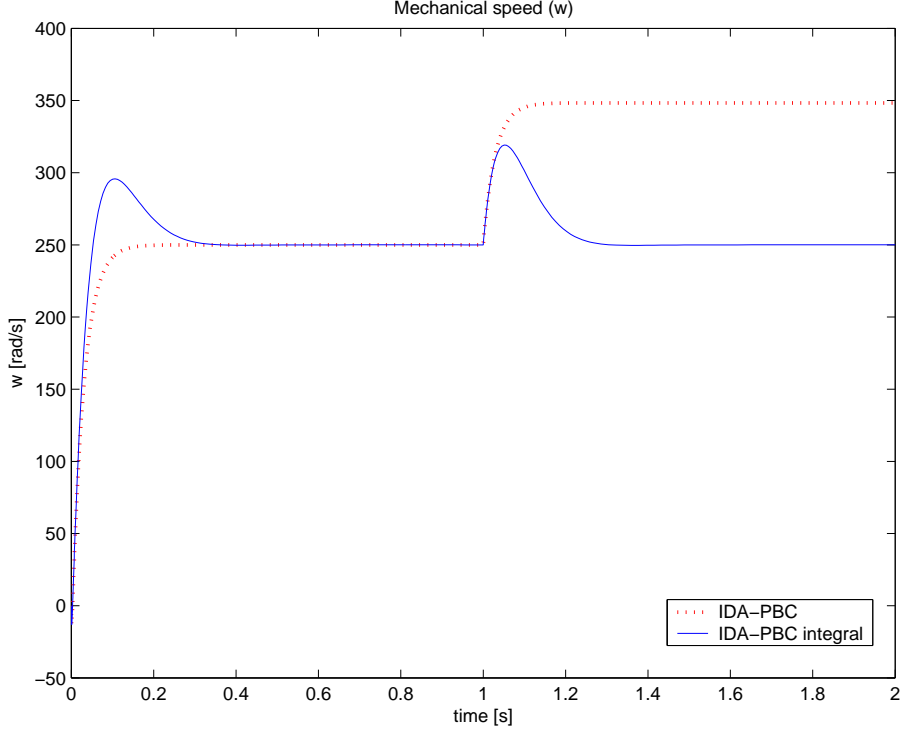


Figure 24: Mechanical speed ω with a classical IDA-PBC controller (dotted line) and with an IDA-PBC controller with integral action (continuous line).

where $u_x = \partial u / \partial x$, $u_y = \partial u / \partial y$, appear frequently in modern control theory, and in particular in the IDA-PBC scheme. Equation (87) is called a quasilinear PDE because the derivatives in u appear linearly, although in general the dependence on u is nonlinear.

Geometrically, the solution to (87) is a surface $u = u(x, y)$ whose normal $(u_x, u_y, -1)$ is constrained by (87). This simple fact allows the explicit construction of a solution for this low dimension case, although the resulting method, called characteristics's method in the literature, can be generalized to higher order cases as well as to fully nonlinear PDEs.

Let (x_0, y_0, u_0) be a point on a solution surface and let $(x(\tau), y(\tau), u(\tau))$ be a curve on the surface through it at $\tau = 0$. This means that the tangent vector $(x'(0), y'(0), u'(0))$ must be tangent to the surface at the point. Let us see how can we impose this condition.

Let $(p = u_x(x_0, y_0), q = u_y(x_0, y_0), -1)$ be a normal at the point. According to the preceding discussion, it must satisfy

$$c(x_0, y_0, u_0) = a(x_0, y_0, u_0)p + b(x_0, y_0, u_0)q. \quad (88)$$

The set of all planes through the point (x_0, y_0, u_0) is given by

$$u - u_0 = p(x - x_0) + q(y - y_0), \quad (89)$$

and p and q must satisfy (88) for this plane to be tangent to the surface. We can be sure that the tangent vector to the curve is indeed tangent to the surface if we impose that it belongs to the whole family of planes. Now it can be shown that the equation of the line common to all the planes in the set is

$$\frac{x - x_0}{a(x_0, y_0, u_0)} = \frac{y - y_0}{b(x_0, y_0, u_0)} = \frac{u - u_0}{c(x_0, y_0, u_0)}. \quad (90)$$

EXERCISE Show that (90) yields the equation of the line common to the family of planes (89).

Using the vector along the line of (90), we impose

$$\begin{aligned}x'(0) &= a(x_0, y_0, u_0), \\y'(0) &= b(x_0, y_0, u_0), \\u'(0) &= c(x_0, y_0, u_0),\end{aligned}$$

or, taking into account that this must be valid for any point on the curve,

$$\begin{aligned}x'(\tau) &= a(x(\tau), y(\tau), u(\tau)), \\y'(\tau) &= b(x(\tau), y(\tau), u(\tau)), \\u'(\tau) &= c(x(\tau), y(\tau), u(\tau)).\end{aligned}\tag{91}$$

The solutions to this system of ODE are called *characteristic curves* of the PDE, while their projections on the plane $u = 0$ are simply called *characteristics*. To generate a solution surface, one must start with a curve of initial conditions $(x(0, s), y(0, s), u(0, s))$ and solve (91) for each point on the curve. This way one gets $(x(\tau, s), y(\tau, s), u(\tau, s))$. If the curve of initial conditions does not lie on a characteristic curve, it is possible to solve for τ and s in terms of x and y , and finally get $u(x, y)$.

As a (manifestly trivial) example, let us consider

$$3u_x + 5u_y = u,$$

with an initial curve $(s, 0, f(s))$ where f is arbitrary. We have to solve

$$\begin{aligned}x' &= 3, \\y' &= 5, \\u' &= u.\end{aligned}$$

The solution satisfying the initial conditions is

$$\begin{aligned}x &= 3\tau + s, \\y &= 5\tau, \\u &= f(s)e^\tau.\end{aligned}$$

From the first two equations we get $\tau = y/5$ and $s = x - 3y/5$, and the corresponding solution surface is

$$u(x, y) = f\left(x - \frac{3y}{5}\right)e^{\frac{y}{5}}.$$

EXERCISE For the above example, show that we cannot get the solution surface if the initial condition is given on $(3s, 5s, f(s))$.

Finally, let us remark that when several PDE for the same function are involved, some compatibility conditions must be met for the system to be solvable. See Chapter 2 of [25] for an account in terms of Fröbenius theorem and the integrability of a distribution of vector fields.

B Coordinate transformations for ac machines

Laboratory frame models of ac machines have complicate expressions for the constitutive relations, and Park transformation is commonly used to transform to rotating frames where equations simplify dramatically, at the expense of increased nonlinearity. See [6] for a more general presentation.

From any set of three-phase electrical variables (voltages, currents, fluxes...) y_{abc} we compute transformed variables $y_{\alpha\beta\gamma}$ by means of

$$y_{\alpha\beta\gamma} = T y_{abc} \quad (92)$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix},$$

where

$$T = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Notice that, since $T^T = T^{-1}$, this is a power-continuous transformation:

$$\langle i, v \rangle = \langle i_{abc}, v_{abc} \rangle.$$

If the three phases are electrically equilibrated, one has $y_a + y_b + y_c = 0$, and the transformation just presented allows to work only with the two first components (the so called $\alpha - \beta$ components), and neglect the third one (the homopolar, or γ , component) which is zero for any balanced set and which, in any case, it can be seen to be decoupled from the remaining dynamical equations.[19][20]

In most situations of interest, the state variables are sinusoidal functions of time, and this allows a further transformation which maps these periodic functions into fixed points. This transformation also eliminates the dependence of the equations on θ (the mechanical position of the rotor). From the $\alpha - \beta$ variables, one defines new variables y_{dq} via

$$\begin{pmatrix} y_{\alpha\beta s} \\ y_{\alpha\beta r} \end{pmatrix} = K(\theta, \delta) \begin{pmatrix} y_{dqs} \\ y_{dqr} \end{pmatrix} \quad (93)$$

where

$$K(\theta, \delta) = \begin{bmatrix} e^{J_2\delta} & O_2 \\ O_2 & e^{J_2(\delta-\theta)} \end{bmatrix}$$

and where δ is an arbitrary function of time (usually $\dot{\delta}$ is the stator frequency ω_s), subindex s and r denote stator and rotor respectively, and

$$e^{J_2\phi} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The dq-transformation can be seen, in bond graph terms, as a modulated transformation by two steps, as displayed in Figure (25). First the T transformation reduces, in an

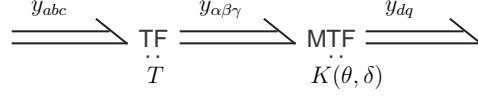


Figure 25: Bond graph of the dq-transformation.

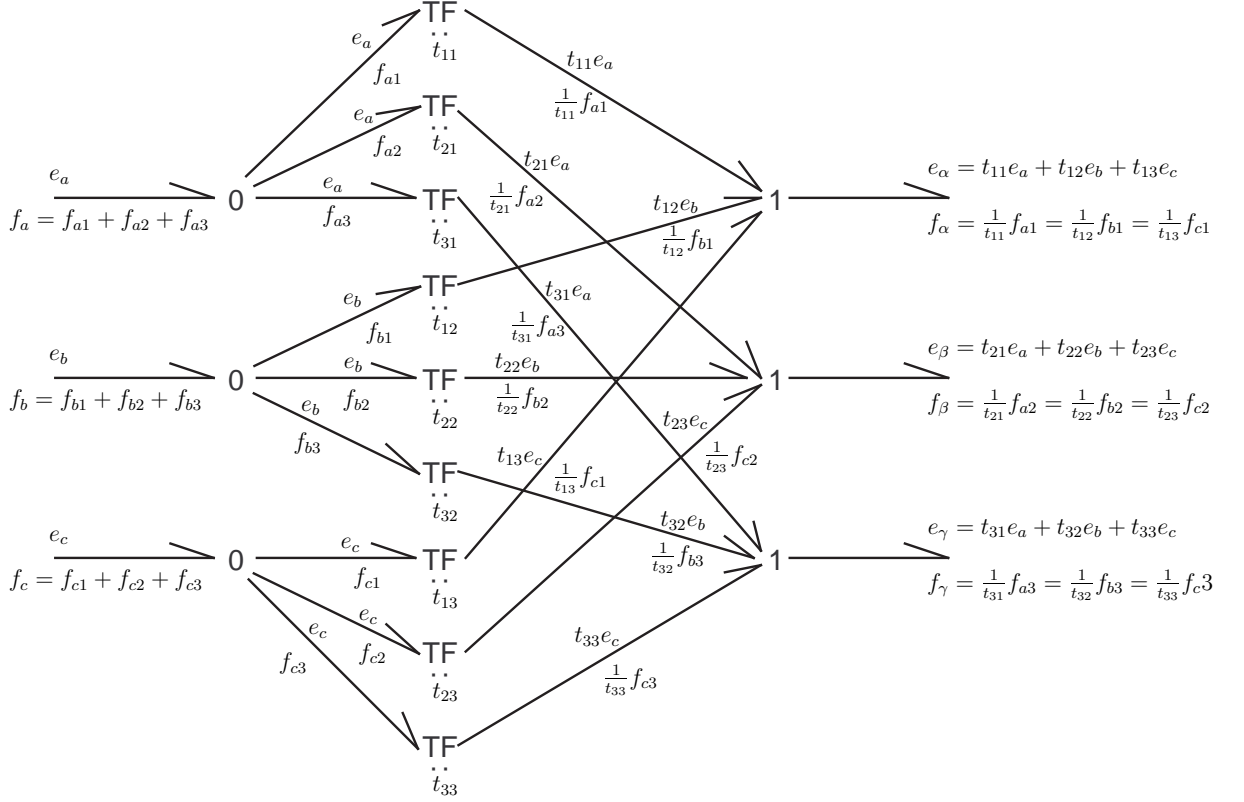


Figure 26: Detailed bond graph of the T transformation.

equilibrated case, from a 3-phase (abc) to a 2-phase ($\alpha\beta$) system. Then the $K(\delta, \theta)$ transformation, which is also power continuous, can be implemented by means of a modulated transformer.

Fig.(26) shows a completed bond graph of the first step of dq-transformation.

Note that for an equilibrated 3-phase system the output of the γ port is zero.

For the second part of the transformation, the bond graph is shown in Fig.(27). Using

$$e^{-J_2\phi} = (e^{J_2\phi})^T$$

one can readily check that the bond graph yields indeed the K transformation.

Finally, the whole dq-transformation is depicted in Fig.(28), with a 3-inputs port (abc) and a 2-output port(dq).

References

- [1] Banerjee, S., and G. Verghese (editors), *Nonlinear Phenomena in Power Electronics*, IEEE Press, 2001.

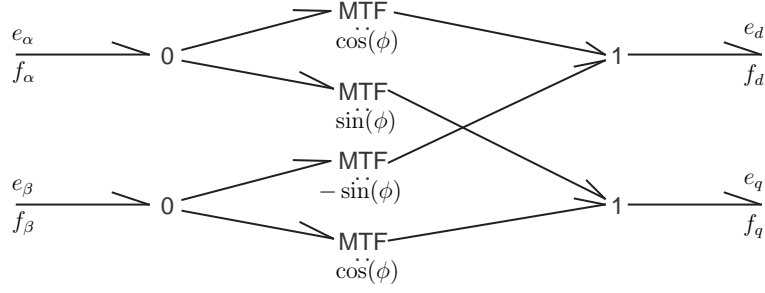


Figure 27: Bond graph of the modulated K transformation.

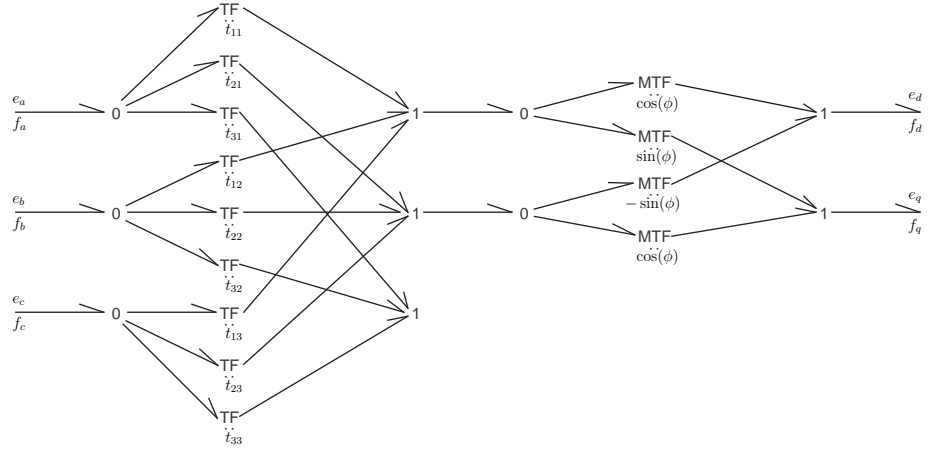


Figure 28: Complete bond graph of the dq-transformation for an equilibrated 3-phase system.

- [2] Batlle, C., A. Dòria, E. Fossas, and R. Ortega, Improved robustness for Hamiltonian passivity based control , in preparation.
- [3] Batlle, C., A. Dòria, and R. Ortega, Power flow control of a doubly-fed induction machine coupled to a flywheel (in press), *European Journal of Control*, 2005.
- [4] Batlle, C., E. Fossas, R. Griñó, and S. Martínez, Generalized state space averaging for port controlled Hamiltonian systems, In *Proc. 16th IFAC World Congress (IFAC2005)*, 2005.
- [5] Batlle, C., E. Fossas, I. Merillas, and A. Miralles, Generalized discontinuous conduction modes in the complementarity formalism (in press), *IEEE Trans. on Circuits and Systems — II*, 2005.
- [6] Breedveld, P., A generic dynamic model of multiphase electromechanical transduction in rotating machinery, Proceedings WESIC 2001, June 27-29, 2001, University of Twente, Enschede, The Netherlands, pp. 381-394, ISBN 90-365-16102, 2001.
- [7] Breedveld, P., *et al*, An intrinsic Hamiltonian formulation of the dynamics of LC -circuits, *IEEE Trans. on Circuits and Systems — I* **42**, pp. 73-82, 1995.

- [8] Caliskan, V.A., G.C. Verghese, and A.M. Stankovic, Multi-frequency averaging of dc/dc converters, *IEEE Transactions on Power Electronics*, **14**, pp. 124-133, 1999.
- [9] Chiasson, J., *Modeling and High Performance Control of Electric Machines*, John Wiley & Sons Inc, 2005.
- [10] Dalsmo, M., and A. van der Schaft, On representations and integrability of mathematical structures in energy-conserving physical systems, *SIAM J. Control Optim.* **37**, pp. 54-91, 1998.
- [11] Delgado, M., and H. Sira-Ramírez, Modeling and simulation of switch regulated dc-to-dc power converters of the boost type, *Proc. of the First IEEE International Caracas Conference on Devices, Circuits and Systems*, pp. 84-88, 1995.
- [12] Erickson, R., *Fundamentals of Power Electronics*, Kluwer, 1997.
- [13] Escobar, G., A.J. van der Schaft, and R. Ortega, A Hamiltonian viewpoint in the modeling of switching power converters, *Automatica* **35**, pp. 445-452, 1999.
- [14] Fossas, E., and G. Olivar, Study of chaos in the buck converter, *IEEE Trans. Circuit Systems-I* **43**, pp. 13-25, 1996.
- [15] Gaviria, C., E. Fossas, and R. Griñó, Robust controller for a full-bridge rectifier using the IDA-PBC approach and GSSA modelling, *IEEE Trans. Circuits and Systems* **52**, pp. 609-616, 2003.
- [16] Gawthrop, Peter H., *Hybrid bond graphs using switched I and C components*, Centre for Systems and Control report 97005, University of Glasgow, 1997.
- [17] Golo, G., V. Talasila, A. van der Schaft and B. Maschke, Hamiltonian discretization of boundary control systems, *Automatica* **40**, pp. 757-771, 2004.)
- [18] Khalil, H. K., *Nonlinear Systems*, Prentice Hall, 2 edition, 1996.
- [19] Krause, Paul C., *Analysis of Electric Machinery*, McGraw-Hill, 1986.
- [20] Krause, Paul C., and Oleg Wasynczuk, *Electromechanical Motion Devices*, McGraw-Hill, 1989.
- [21] Krein, P.T., J. Bentsman, R. M. Bass, and B. L. Lesieutre, On the use of averaging for the analysis of power electronic systems, *IEEE Transactions on Power Electronics* **5**, pp. 182-190, 1990.
- [22] Kugi, A., *Non-linear control based on physical models*, Springer, 2001.
- [23] Lyshevski, Sergey E., *Electromechanical systems, electric machines and applied mechatronics*, CRC Press LLC, 2000.
- [24] Mahdavi, J., A. Emaadi, M. D. Bellar, and M. Ehsani, Analysis of power electronic converters using the generalized state-space averaging approach, *IEEE Transactions on Circuits and Systems I* **44**, pp. 767-770, 1997.

- [25] Nijmeijer, H., and A. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, 1990.
- [26] Ortega, R., *et al*, Stabilization of a class of underactuated mechanical systems via interconnection and damping assignment, *IEEE Trans. on Automatic Control* **47**, pp. 1218-1233, 2002.
- [27] Ortega, R., A. van der Schaft, and B. Maschke, Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems, *Automatica* **38**, pp. 585-596, 2002.
- [28] Ortega, R., A. van der Schaft, I. Mareels, and B. Maschke, Putting energy back in control *IEEE Control Systems Magazine* **21**, pp. 18-33, 2001.
- [29] Rodríguez, H., and R. Ortega, Stabilization of electromechanical systems via interconnection and damping assignment, *Int. Journal of Robust and Nonlinear Control* **13**, pp. 1095-1111, 2003.
- [30] Rodríguez, H., R. Ortega, and G. Escobar, A new family of energy-based nonlinear controllers for switched power converters, Proc. of the ISIE 2001, Pusan, Korea, pp. 723-727, 2001.
- [31] Rodríguez, H., *et al*, A robustly stable output feedback saturated controller for the boost dc-to-dc converter, *Systems & Control Letters* **40**, pp. 1-8, 2000.
- [32] Sanders, Seth R., J. Mark Noworolski, Xiaojun Z. Liu, and G.C. Verghese. Generalized averaging method for power conversion circuits. *IEEE Transactions on Power Electronics* **6**, pp. 251-259, 1991.
- [33] van der Schaft, A., and B. Maschke, Port controlled Hamiltonian systems: modeling origins and system theoretic properties, In *Proc. 2nd IFAC Symp. on Nonlinear Control Systems Design (NOLCOS'92)*, pp. 282-288, 1992.
- [34] van der Schaft, A., *L₂-gain and passivity techniques in nonlinear control*, Springer-Verlag, 2000.
- [35] van der Schaft, A., and B. Maschke, Hamiltonian formulation of distributed-parameter systems with boundary energy flow, *Journal of Geometry and Physics* **42**, pp. 166-194, 2002.
- [36] van der Schaft, A., and J. Schumacher, Complementarity modeling of hybrid systems, *IEEE Trans. Automat. Contr.* **43**, pp. 483-490, 1998.
- [37] Tadmor, G., On approximate phasor models in dissipative bilinear systems. *IEEE Trans. on Circuits and Systems I* **49**, pp. 1167-1179, 2002.
- [38] Yu, X., and J.X. Xu, editors. *Advances in Variable Structure System, Analysis, Integration and Applications*. World Scientific, Singapur, 2001.