Mathematical Interactive Content: What, Why and How

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Abstract: The real challenge of e-Learning is to produce content that brings a *general* improvement in the way students learn and teachers teach. For that, "intelligent" interactivity is the single most important feature that such content should have. But the design and production of content with this kind of interactivity has turned out to be harder than expected, the main reason being that it requires the convergence of many kinds of experts in parallel to the convergence of the several technologies involved. In the case of mathematics, the least amount of expertise asks for the presence of professional mathematicians, software engineers, publishers, and perhaps learning theorists, that can productively talk to each other. In this paper we want to examine afresh, with a view to the future, what is (or can be) meant by interactivity; the reasons why interactivity should play a role in optimizing general mathematical learning and in making possible, and viable, that mathematics teachers at large embrace a role that is congruent with their present day mission. Finally we propose a check list for desirable features that advanced mathematical learning systems should have, state some guiding principles for their design, and describe what is involved in their production.

1 Introduction

Computational power of current mathematical systems makes it possible to write mathematical content in which the mathematics is live in the sense that one can experiment with various mathematical concepts. One can plot, differentiate or integrate functions, compute limits and perform many other tasks by using suitable applets which then perform the actual computation on a server. Mathematical content using such features is an example of interactive materials. One can, however, go much further. Using the capabilities of present day systems, one can build content that is adaptive, provides the user appropriate feedback according to his or her performance, and keeps track on the students' performances. In a sense one can provide automatic private instruction with such intelligent interactive content, I+ content for short. Proper I+ content will also contribute towards industrialization of instruction, a desperately needed development.

Concept-building and problem-solving are considered two cornerstones in the learning of mathematics. Another key factor, which is growing in importance in our present day world, is the ability to express and communicate the results. Therefore, in order to improve the general learning of mathematics we have to improve, and if possible optimize, the general performance of the students on these three factors.

To address these problems, technology is instrumental. Even if there were enough qualified teachers to insure a small number of students per class, a situation that in principle might guarantee excellent learning outcomes without the use of technology (as it did sometimes in the past), today it seems clear that the learning experience of those students would not be as rich as it could be with the use of technology. In fact, technology has brought back an experimental side to the learning of mathematics that had been sadly lost in past decades. Moreover it has increased the variety of experiments possible and the depth at which they can be carried out: think of simulations, conjecture testing, generation of example cases, and all these explorative tasks can be performed in a straightforward manner.

What do we need of technology? The fundamental requirement is that it should support the production of content aimed at enhancing the learning experience by improving concept-building, problem-solving and communication skills. If we start from the end, the ability to express and communicate results, technology should allow students to express their solutions in a way that is both readable by teachers and automatically assessable. At present, this means a computer interface endowed with the required functionality. Such an interface, let us call it the *student interface*, would provide the most precious assets of interactive teaching, which are meaningful feedback and accurate records of the individual learning process.

To train problem solving skills is quite an involved task which reaches beyond the subject field of mathematics and influences the way students will eventually be able to analyze a real life situation, abstract the main features of the problem, formulate a question to be answered, review the knowledge at hand and derive whether the problem can be tackled or not. Since mathematics is often an applied science, problem solving touches all areas of science and engineering where one may need to employ specific techniques of the application field together with the mathematical skills in order to achieve a result. Here technology can help by providing a variety of software tools, from different fields, all usable with a mouse click.

The above is more easily said than done: students tend to become overwhelmed and frustrated with, for instance, syntactical details of one system as opposed to another, or for having to spend too much time on low level technicalities that unduly postpone the overall learning goal of solving a problem. The Calculus Lecture Notes developed by M. Seppälä at Florida State University during 1996 – 2001 used heavily Maple to motivate and to illustrate mathematical results. Techepxlorer was used to display mathematical content on the web. This solution turned many students away. Even though a commercial version of techexplorer was provided to students for free by a site license, students did not use it. Hence they were unable to view the materials published on the web to support their learning. The use of Maple confused man students. They were suddenly faced with the challenge of learning Maple in addition to the mathematical content. This combination did not work, and was replaced by notes viewable to students without them having to install any additional software [Mika Seppälä, 2004]. This work was carried out by the Helsinki Learning System activities at the University of Helsinki. The Maple illustrations were replaced by static versions, which did not require nor offer any interactivity. This turned out to be successful. The moral of this account is that one should use caution when using interactive mathematical illustrations and other similar components in instruction. They may be very valuable but they also may be counter productive.

Finally let us consider the concept-building need. Before trying to solve a problem, a student should understand all the concepts that are involved. Hence the need that the concepts appearing in a piece of content be linked to whatever means are available that can contribute to their understanding. This may involve the widely used (and sometimes even abused) hypertext links or more recently, as we like to say, *hypermathematics* links, by which we mean links that supply the result of a mathematical computation in a form which can be embedded into the content, and hence used as any other item therein.

From the side of content design and production, the main actors are the authors. This leads to the need of an *author interface*, with all the required functionalities to produce content that is coherent with the learning outcomes and with the needs of students. This means that the students interface has to allow not only reading the content, but also editing a local copy of it. In other words, the reading and writing functionalities of content is shared by students and authors, the only difference being that the student does not have rights to modify server content. In addition, since the author need not coincide with the teacher or instructor, in principle we have to assume also an *instructor interface*, with enough functionality to meet the teacher's needs, like course design, assignments, assessments or record keeping.

I+ content. The thread binding together all the requirements above is, for want of a better qualifier, "intelligent" interactivity (*I*+), or just enhanced interactivity as defined above. Whatever this is, it is involved in the hypermathematical connections and in making viable the required interfaces (student, author and teacher interfaces). But judging for what we see, in comparison to promises made along the years, it can be safely said that the design and production of content with the sorts of interactivity that are required to optimize learning has turned out to be a difficult task. The main reason we see is that progress along these lines requires the cooperation of several kinds of experts in correspondence with the convergence (for the first time in history!) of the technologies involved. In the case of mathematics, the least amount of expertise asks for the presence of professional mathematicians, software engineers, publishers, and perhaps learning theorists, that can productively talk to each other.

Organization of this paper. In the following sections we first survey the present state of interactivity, up to what is possible in a limited amount of time for an exponentially growing area. Then we elaborate on the reasons why I+ is the road to follow and the implications that this has for the functionalities to be built in the different interfaces. Here we also outline the WebALT approach and how it may evolve in the near future. Next we consider a bottleneck that has to be circumvented in order to be successful, namely, that for authors it should be simple to transform their ideas into design patterns of (re)usable content. We end with a section about the sort of tools that authors will need to accomplish this, including the transfer of content to content providers, and a brief list of conclusions.

2 Interactivity today

We will examine interactivity from the point of view of the three basic aspects, namely concept building, problem solving and communication skills, that concur in promoting the learning mathematics.

Concept building. In principle most people would agree that today's Internet can be used as a concept-building tool. Unfortunately this is not so, as a little thought and inquiry reveal. Assuming that a student already knows what has to be conceptualized, it is very easy that he or she goes astray if unaided by the teacher (as supplier of the right links). This is due to the diffuse nature of internet, and, more seriously, to the lack of some sort of certification of the quality of what is being offered. Here goes a compelling illustration (just the first tried while writing this section): prompting Google with “rigid motion”, the first link that appears is http://mathworld.wolfram.com/RigidMotion.html and it supplies the following definition:

“A transformation consisting of rotations and translations which leaves a given arrangement unchanged.”

This definition is certainly not optimal, as it does not explain what is a “given arrangement”, nor what are the criteria for “unchanged”. Moreover, it differs its understanding to the special cases of “rotations” and “translations” (it is also silent about the rigid transformations that reverse the orientation). The accepted definition is that a rigid motion (say in the Euclidean plane or space) is “a transformation that leaves unchanged the distance between any pair of points”. Then, in the examples that good content should always supply, we would find the definition of translation and of rotation about of a given amplitude (about a point in the plane, about a line in space), among those that preserve orientation; and of symmetry with respect to a line (in the case of the plane) and with respect to a plane (in the case of space), among those reversing the orientation. These might be accompanied by suitable links to “live” or “interactive” geometry that could reinforce the intuitive and experimental feeling for the concepts (for an illustration of this, see the description of ICGS in next section). Finally we would refer to the result (actually a classification theorem, as in [Sebastian Xambó, 2001], for example) that any orientation preserving rigid motion of the plane (respectively in space) is either a translation or a rotation about a point (respectively either a translation or a rotation about a line followed by a translation that is parallel to the rotation axis) and that an orientation reversing rigid transformation of the plane (respectively of space) is the symmetry about a line (respectively a symmetry about a plane followed by a translation parallel to that plane). Let us pay attention here to A. Einstein ditto that *things should be done as simple as possible, but not simpler*.

If we insist Google with "rigid motion wikipedia", the first item we get is http://en.wikipedia.org/wiki/Kinematics, which is not what we expect in a geometry class (rather the opposite would be desirable: that the mathematical analysis of rigid motions be used to understand the kinematics of rigid bodies, as for example in [Sebastian Xambó, 2005]). Actually we have to go way down of the second page of the Google answer to find anything related to geometry (here is the link (*http://infosanctum.wikinerds.org/index.php/Geometry*)), then we see "Chapter 22 - Rigid motion", which leads to an empty wiki page (a "page doesn't exist yet" message).

So we have to be cautious and say that unless students can rely on some trustful guide, the reality they will encounter in the Web for the purpose of concept-building is not unlike a silent thunder, an oxymoron, in that it can only be helpful to those that already know. If much is done to stuff web pages without a true regard to aspects other then cosmetic, it will be of great added value to depart from that trend and publish only content that is a reference for its high quality. Even if we had been much unlucky in our trial searches reported in this section, it is clear that only an extended sense of the teachers' responsibilities will protect students from wasting a lot of time on Internet in trying to come to grips with this or that concept.

Problem solving. By *problem solving*, we mean here the ability to creatively come to the solution of a problem, whether of real life or theoretical nature. Interactivity in the form of feedback and hints has to be designed so that the students perception is that of an expert tutor guidance. Examples of technology which can support this type interaction is the LeActiveMath platform [E. Melis and J. Siekmann, 2005], or also more specifically the Problem Tree approach first implemented in mathematics in the Helsinki Learning System [Mika Seppälä, 2002] and now also in the MathDox problem solution tree [A. M. Cohen, H. Cuypers, D. Jibetean and M. Spanbroek, 2005]. As mentioned in the introduction, one has to be careful in the design of this highly-sophisticated interactive content to avoid alienating the student by imposing requirements that are too strict, for instance on the input syntax. This implies a careful balance between what is inferred automatically by the system and what is required from the student as key input: if the system is too clever, and forgives or "understands" too much, the student might end up not doing any thinking. On the other hand, if the system is unforgiving or totally ignorant, then the student is burdened by details which do not contribute to the actual learning goal but are merely programming details of the underlying computational system.

Communication skills. Interactive CourseWare is defined on the dictionary of computing as *a training program controlled by a computer that relies on trainee input to determine the order and pace of instruction delivery. The trainee advances through the sequence of instructional events by making decisions and selections. The instruction branches according to the trainee's responses.* This definition makes it very clear that interactivity is strongly linked to *communication*. In the case of communication of mathematics on a computer, we need to come to terms with the way mathematics is being represented, and is being written using today's interfaces. A representation of mathematics, by means of semantic markup, that is totally unambiguous and understandable by a computational program is still very cumbersome to produce. For students trained in using any formal software system, be it computer algebra or theorem proving, using the strict syntax of the system is a way to input a precise "decodable" mathematical expression. Any other way, unless the system is able to make heuristic guesses as to what the student means by considerably constraining the user input, is not guaranteed to produce a language-independent representation of mathematics. Designing I+ courseware based on an unambiguous representation of mathematics has many advantages since it becomes comparable to programming and is therefore suitable for being processed in a variety of ways including transformations that take care of notational and cultural differences in the presentation of mathematics, generation of natural language mathematical jargon from the abstract conceptual representation that produces several idioms and symbolic manipulation of the mathematical objects that analyses correctness of the answers or produces meaningful feedback.

3 Why interactivity will make the difference nevertheless

As said before, general improvement of mathematics learning cannot be expected unless we come up with technology solutions that:

(a) Allow students to be in continuous touch with reliable concept-building facilities, both for formal definitions and for intuitive perceptions derived from experimenting with interactive materials.

(b) Promote continuous improvement of the problem-solving abilities by judicious regular hand-outs on the part of the teacher, possibly customized to the student, by providing access to computational tools, by allowing the student to compose and hand-in the solutions (records in content form), and by getting meaningful feedback all along, be it from the teacher or automatic from the system (for example as described in (c)).

(c) Endow authors with powerful tools for composing content, and for designing and producing a set of (algorithmic) exercises subordinated to a goal problem be it for interactively helping the student toward its solution or as a recursive make-up mechanism in case the student's work gets stuck.

The next example, this time related to concept-building and problem-solving, is the [S. Molina and Xambó S., 2006] Interactive Classical Geometry System (ICGS). The situation now prevailing in many countries is that high school geometry is a rather neglected field. This is very unfortunate, from the point of view of learning and education. Indeed, geometry is not only useful knowledge, but it is a proven and fabulous way of mastering reasoning from an early age. Being the ability to reason one of the key goals of education, high school students are thus deprived of a real treasure, and their plight is not easily remedied upon entrance at the university. Hopefully ICGS can contribute, to some extend, to this remedy. The content is based on chapter 0 of [Sebastian Xambó, 2001] which itself is the natural evolution of notes begun twelve years ago for similar purposes. The material, which consists of html files produced by means of the University of Helsinki Wiris, can be studied at two levels. The aim of the easiest level is just to understand the description of the different constructions and statements, roughly ordered by increasing order of difficulty, aided by the interactive graphics and by the links of the concepts to the definitions assembled in a glossary. The other level is trying to come up with solutions or proofs, perhaps by clicking for a hint. From the WebALT point of view it is not more than another experimental prototype for interactive content. In that role it may be useful to observe what happens with an earlier prototype that was made available, in poster and demo forms, to the Catalan high schools during the Spanish Science Week (7-13 Nov, 2005). For general features of systems like the UH Wiris, see [R. Eixarch, D. Marquès and S. Xambó, 2002].

4 Helping authors to design interactive content

The content used for lectures, independent study or private instruction usually consists of web pages representing typical printed content in an electronic form. In many cases this is not ideal for the new media: computer screens. A different way to present content needs to be used. From what was said above, it is also clear that too many gizmos might be counter-productive when the students become more wrapped into technology than into concepts.

When preparing for talks almost all persons will prepare slides. These may be produced by PowerPoint, or by some open source variant of it. Most mathematicians use LaTeX with a special class for presentations. Common feature to these systems is that they all produce end results that very much look like PowerPoint slide shows. Such presentations form an important part of the mathematical educational on-line content that we use. The PowerPoint slide shows are not, however, intelligent in the way defined above. They are nevertheless an important part of instruction and the lecturer may include links and animations or java applets to be replayed by the students when studying at home. To produce these slides, the major help available to authors, besides authoring tools, is the availability of searchable repositories of applets, animations and multimedia learning objects that are described by good metadata. An example of such a repository is for instance MERLOT, at http://www.merlot.org. The WebALT repository WALTER uses a custom-made taxonomy, expanding the LivingTaxonomy (http://livingtaxonomy.org/index.php/Math) on the subject areas of Calculus and Linear Algebra, for classifying mathematical learning objects with a finer accuracy.

Well written slide shows can make a big difference in normal contact instruction and even much bigger in instruction delivered over the web. In the production of such slide shows, authors should be encouraged to use the most advanced tools available. This is still relatively new, and a lot of attention is wasted to either insisting on using certain type of technical solutions or to avoiding some others. When reading newspapers, we do not think of the actual process how the newspaper was printed and delivered, but rather the content. Content rules in the printed press. So it will also in the new media.

Intelligent interactivity will go much beyond well written slide shows. I+ content is content which understand the readers' problems and responds accordingly. Such content can be produced by systems like the commercial Maple T.A., the open source systems LeActiveMath and STACK and by the forthcoming WebALT system. The most valuable part of such content consists of algorithmic problems, i.e., of problems which are really specialized programs. An algorithmic problem then produces a new version of the problem for the student to solve each time such a program is invoked. The algorithmic problems will also contain algorithmic hints and solutions. These really make a difference. Such content will help to industrialize instruction by providing automatic private instruction.

To author such content is still a challenging task for most instructors since it requires knowledge of the underlying programming system. Here again the best advice is to use the most advanced tools that exist. The aforementioned systems, Maple T.A., LeActiveMath, STACK and WebALT also necessarily provide the authors with an editing mode yet these editors have along way to go before they can be used by a novice with no outside expert guidance. The fact remains that, nevertheless, it is difficult to create good I+ content, and the best advice is simply to search for existing content, use it as such or modify it to fit the needs of the students who will use it. Good metadata will be instrumental and make it possible to locate such content. Gold and diamonds are not of much value unless they are first found and made available to people. Good content can be worth millions, but not unless it is made available.

5 On the production of interactive content

We now like to make a list of desirable authoring features that would support the production of I+ content with respect to the learning goals to be achieved by the material. The ultimate intelligent editor should behave in much the same in which Google is able to suggest the right spelling when the results from the user query are not good enough. Such an editor while leaving the author the possibility to choose its own input language and syntax for the mathematical fragments, would be able to analyze the resulting document in term of its quality with respect to concept building, problem solving and communication. Can we design such a tool?

Technology solutions that support concept building, problem solving, and communication skills are now available by threading together different fields of expertise, and sometimes by improving the existing knowledge in them. This is the point of view adopted by the WebALT project. For example, in the case of the author's interface, one of the features to aim at is that the content produced by an author can be automatically translated into other languages (see [Sebastian Xambó and Jordi Saludes, 2006] for an analysis of this problem, and for an approach to its solution for a limited segment of the mathematical texts). This entails that the content must be a formal expression (actually a tree produced by some grammar) that captures the semantics of the mathematical text, including the semantics of the mathematical expressions that belong to the text. Consequently, the author's interface must have options for the author's language and for the user's language, and must be capable of constructing the content (as an abstract representation) in so friendly a manner that it could be confused with just a friendly editor. The interactivity in this case must be at a higher level, since in addition to features that are familiar in advanced editors, it has to offer enough meaningful choices at any moment for the author to be able to continue composing his mathematical text in a natural way.

The Helsinki Learning System [Mika Seppälä, 2002] provided a first version of author's interface that supports the construction of problem (solution) trees used to enhance the problem solving capabilities. Further development of HLS is being carried out by the WebALT Project. The solution here will rely on fetching related documents (that might be leaves of the solution tree) by proper metadata and topic/subtopic relations in it. A similar approach is taken by smart proof assistants which try to guess which lemma or theorem is needed next to carry on with the proof [Claudio Sacerdoti Coen and S.

Zacchiroli, 2003]. Whilst in such a situation, the metadata attached to the formalized mathematical library is produced by the system, in the generic situation of editing course material one has to rely on hand-crafted metadata and at best we can hope for a good way to describe the specific field of mathematics. This has motivated the refinement of the mathematics taxonomy given at LivingTaxonomy by the WebALT CCDs used for classification. Future direction may well use semantic web technology such as RDF and OWL to capture more structure and infer more relations from the metadata records [M. Nilsson, 2003], [N. Nilsson, W. Siberski and J. Tane, 2004].

Intellectual Property. Open Source Software forms an important and growing part of the current software market. Authors of various open source products have, in a way, given up the monetary exchange mechanism and are, instead, voluntarily giving their work to others in return of the right to use their colleagues work in the same way. This has developed to such a big part of economy that one should take it into account when computing GDPs and other parameters measuring the size of an economy. Open source products have turned out to be important and there are many examples of successful commercial products using open source components. I+ educational content are mainly a collection of end products. While I+ content can use open source components, the sustainability of these activities require commercialization of the development of I+ educational content. Publishers are busy doing this, but often only to protect their core business, printing and selling books and journals. The protection of intellectual property is an important problem in the business of I+ educational content. Here one can use the solutions provided by the main stream commercial companies to restrict the delivery and the editing of the content as well as to set expiry dates on the content.

6 Conclusions

I+ educational content will eventually make a big difference in the delivery and availability of education. Today education is still manual labor. Instructors work hard to produce high quality instruction that cannot easily be reproduced. The professors today are like jewelers. In best cases their students enjoy brilliant instruction, but the content disappears as soon as the lecture is over. Using the advanced tools available today, the professors have a possibility to preserve their knowledge and share it with their colleagues today and in the future. Such sharing is facilitated if most standard tools are being used. The content should be such that other people using it could edit it without extreme difficulties. In particular this means that one should avoid the use of highly specialized systems like LaTeX with many custom made macro packages which are not readily available to other users. The package devoted to producing proper italic versions of the letters ä, ö, å, etc. is an example of macros to avoid. While it is true that LaTeX did not place the accents correctly in the italic versions of the above mentioned Scandinavian letters, the use of such macros in LaTeX source rendered such source code virtually unusable for people not having access to these packages. Strive to perfectionism can lead to bad results. Pretty good is good enough, the motto of Ralph's Pretty Good Grocery ([John Mueller, 2001]). This applies also to I+ educational content.

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