The characteristic numbers of the variety of nodal plane cubics of $\mathbb{P}^3$

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Abstract

In this note we obtain, phrased in present day geometric and computational frameworks, the characteristic numbers of the family U_{nod} of non–degenerate nodal plane cubics in $\mathbb{P}^3$, first obtained by Schubert in his *Kalkül der abzählenden Geometrie*. The main geometric contribution is a detailed study of a variety X_{nod}, which is a compactification of the family U_{nod}, including the boundary components (degenerations) and a generalization to $\mathbb{P}^3$ of a formula of Zeuthen for nodal cubics in $\mathbb{P}^2$. The computations have been carried out with the OmegaMath intersection theory module WIT.

Introduction

Given an irreducible n-dimensional family of plane curves in $\mathbb{P}^3$, we are interested in the number of curves in the family that satisfy n conditions and, in particular, in its *characteristic numbers*, namely, the number of curves that go through i given points, intersect k given lines and are tangent to $n-2i-k$ given planes. Concerning the family of nodal cubics in $\mathbb{P}^2$, the characteristic numbers (and many other intersection numbers) were calculated by Maillard (6), Zeuthen (13) and Schubert (11), and were verified, in different ways, by Sacchiero (10), Kleiman–Speiser (5), Aluffi (1) and Miret–Xambó (7).

In this paper we study the characteristic numbers of the variety of nodal plane cubics in $\mathbb{P}^3$ given by Schubert. We first construct a compactification X_{nod} of the variety U_{nod} of non-degenerate nodal plane cubics of $\mathbb{P}^3$ by means of the

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projectivization of a suitable vector bundle. From this we get that the Picard group $\mathsf{Pic}(X_{nod})$ is a rank 3 free group generated by the classes μ, b and ν of the closures in X_{nod} of the hypersurfaces of U_{nod} determined, respectively, by the conditions:

- μ, that the plane determined by the nodal cubic go through a point;
- b, that the node be on a plane and
- ν, that the nodal cubic intersect a line.

We show that the boundary $X_{nod} - U_{nod}$ consists of two irreducible components of codimension 1 and we prove a formula which express the condition

- ρ, that the nodal cubic be tangent to a plane,

in terms of the two degenerations and the condition μ. This formula is a generalization to $\mathbb{P}^3$ of a degeneration relation given by Zeuthen (13) for nodal cubics in the projective plane. We compute, on the basis of the intersection theory of X_{nod} and using the OmegaMath package WIT (see (12)), the intersection numbers of the form $\mu^i \nu^k \rho^{11-i-k}$ given by Schubert in (11). In particular, we get the number ν^{11} of plane nodal cubics that intersect 11 lines which was used (and verified) by Kleiman–Strømme–Xambó in (4). Finally, the computation of the characteristic numbers $P^i \nu^k \rho^{11-2i-k}$ of the family of nodal plane cubics in $\mathbb{P}^3$ follows from the incidence formula $P = \nu\mu - 3\mu^2$.

1 The variety X_{nod} of nodal plane cubics

In the sequel, $\mathbb{P}^3$ will denote the projective space associated to a 4–dimensional vector space over an algebraically closed ground field $\mathbf{k}$ of characteristic 0, and the term *variety* will be used to mean a quasi-projective $\mathbf{k}$-variety. Moreover, we will also write z to indicate the degree of a 0–cicle z, if the underlying variety can be understood from the context.

Let $\mathbb{U}$ denote the rank 3 tautological bundle over the grassmannian variety Γ of planes of $\mathbb{P}^3$. Therefore, the projective bundle $\mathbb{P}(\mathbb{U})$ is a non singular variety defined by $\mathbb{P}(\mathbb{U}) = \{(\pi, x) \in \Gamma \times \mathbb{P}^3 \mid x \in \pi\}$. Let $\mathbb{L}$ be the tautological line subbundle of the rank 3 bundle $\mathbb{U}|_{\mathbb{P}(\mathbb{U})}$ over $\mathbb{P}(\mathbb{U})$ and let $\mathbb{Q}$ be the tautological quotient bundle. We will denote by a the hyperplane class of $\mathbb{P}(\mathbb{U})$ and by μ the pullback to $\mathbb{P}(\mathbb{U})$ of $c_1(\mathcal{O}_\Gamma(1))$ under the natural projection $\mathbb{P}(\mathbb{U}) \to \Gamma$.

We define $\mathbb{E}_{nod}$ as the subbundle of $S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$ whose fiber over $(\pi, x) \in \mathbb{P}(\mathbb{U})$ is the linear subspace of forms $\varphi \in S^3\mathbb{U}^*$ defined over π that have multiplicity at least 2 at x. In fact, given a point $(\pi, x) \in \mathbb{P}(\mathbb{U})$ and taking projective coordinates x_0, x_1, x_2, x_3 so that $\pi = \{x_3 = 0\}$ and $x = [1, 0, 0, 0]$, we can express the elements φ of the fiber of $\mathbb{E}_{nod}$ over (π, x) as follows:

$$\varphi = b_1 x_0 x_1^2 + b_2 x_0 x_1 x_2 + b_3 x_0 x_2^2 + a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3, \qquad (1)$$

where b_1, b_2, b_3 and a_i for $i = 1, \ldots, 4$ are in $\mathbf{k}$. Thus, $\mathbb{E}_{nod}$ is a rank 7 subbundle of $S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$.

In the next proposition we give a resolution of the vector bundle $\mathbb{E}_{nod}$ over $\mathbb{P}(\mathbb{U})$. To do this, we consider the natural inclusion map $i : \mathbb{Q}^* \to \mathbb{U}^*$, the product map $\kappa : \mathbb{Q}^* \otimes S^2\mathbb{Q}^* \to S^3\mathbb{Q}^*$, and the maps

$$h : \mathbb{U}^* \otimes S^2\mathbb{Q}^* \to S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})} \quad \text{and} \quad j : S^3\mathbb{Q}^* \to S^3\mathbb{U}^*|_{\mathbb{P}(\mathbb{U})}$$

whose images are clearly contained in $\mathbb{E}_{nod}$.

Proposition 1.1 *The sequence*

$$0 \longrightarrow \mathbb{Q}^* \otimes S^2\mathbb{Q}^* \stackrel{\alpha}{\longrightarrow} (\mathbb{U}^* \otimes S^2\mathbb{Q}^*) \oplus S^3\mathbb{Q}^* \stackrel{\beta}{\longrightarrow} \mathbb{E}_{nod} \longrightarrow 0, \qquad (2)$$

where $\alpha = \binom{i\otimes 1}{-\kappa}$ *and* $\beta = h + j$*, is an exact sequence of vector bundles over* $\mathbb{P}(\mathbb{U})$*.*

Proof. From the definition of $\mathbb{E}_{nod}$ it follows that β is a surjective map and, since $i \otimes 1$ is injective, we get that α is also injective. Moreover, from the definitions of α and β it follows that $\beta\alpha = 0$. Now, to complete the proof it is enough to see, since $\mathsf{Im}\ \alpha \subseteq \mathsf{Ker}\ \beta$, that $\mathsf{rank}(\mathsf{Im}\ \alpha) = \mathsf{rank}(\mathsf{Ker}\ \beta)$. But this can be easily checked by simple computations. □

Let X_{nod} be the projective bundle $\mathbb{P}(\mathbb{E}_{nod})$ over $\mathbb{P}(\mathbb{U})$. Then, X_{nod} is a non singular variety of dimension 11 whose points are pairs $(f, (\pi, x)) \in \mathbb{P}(S^3\mathbb{U}^*) \times_\Gamma \mathbb{P}(\mathbb{U})$ such that the nodal cubic f is contained in the plane π and has a node at x.

We will denote by b the pullback to $\mathsf{Pic}(X_{nod})$ of the class a in $\mathsf{Pic}(\mathbb{P}(\mathbb{U}))$ under the natural projection $X_{nod} \to \mathbb{P}(\mathbb{U})$. Since this projection is flat, b is the class of the hypersurfaces of X_{nod} whose points $(f, (\pi, x))$ satisfy that x is on a given plane. Furthermore, the relation $\zeta = \nu - 3\mu$ holds in $\mathsf{Pic}(X_{nod})$, being ζ the hyperplane class of X_{nod} and being ν the class of the hypersurface of X_{nod} whose points $(f, (\pi, x))$ satisfy that f intersects a given line.

Proposition 1.2 *The intersection ring* $A^*(X_{nod})$ *is isomorphic to the quotient of the polynomial ring* $\mathbb{Z}[\mu, b, \nu]$ *by the ideal*

$$\langle \mu^4, b^3 - \mu b^2 + \mu^2 b - \mu^3, \nu^7 - 6b\nu^6 + 24b^2\nu^5 \rangle.$$

In particular, $\mathsf{Pic}(X_{nod})$ *is a rank 3 free group generated by* μ*,* b *and* ν*.*

Proof. Since $\zeta = \nu - 3\mu$, the intersection ring $A^*(X_{nod})$ is (see (2), ex. 8.3.4) isomorphic to $A^*(P(\mathbb{U}))[\nu]/\sum \overline{\pi}^* c_i(\mathbb{E}_{nod} \otimes \mathcal{O}_\Gamma(-3))\nu^{5-i}$, where $\overline{\pi} : \mathbb{E}_{nod} \to \mathbb{P}(\mathbb{U})$ is the natural projection. Now, using Proposition 1.1 we get the result taking into account the intersection ring of $\mathbb{P}(\mathbb{U})$. □

Thus, using the projection formula, we have

$$\int_{X_{nod}} \mu^i b^j \nu^{11-i-j} = \int_{\mathbb{P}(\mathbb{U})} \mu^i a^j s_{5-i-j}(\mathbb{E}_{nod} \otimes \mathcal{O}_\Gamma(-3)), \tag{3}$$

where the t–th Segre class $s_t(\mathbb{E}_{nod} \otimes \mathcal{O}_\Gamma(-3))$ can be calculated from the resolution (2). This allow us to compute all the intersection numbers of X_{nod} in the conditions μ, b and ν, that is to say

$$\begin{array}{llll} \mu^3\nu^8 = 12, & \mu^2\nu^9 = 216, & \mu\nu^{10} = 2040, & \nu^{11} = 12960 \\ \mu^3 b\nu^7 = 6, & \mu^2 b\nu^8 = 100, & \mu b\nu^9 = 872, & b\nu^{10} = 5040 \\ \mu^3 b^2\nu^6 = 1, & \mu^2 b^2\nu^7 = 18, & \mu b^2\nu^8 = 160, & b^2\nu^9 = 904 \\ & \mu^2 b^3\nu^6 = 1, & \mu b^3\nu^7 = 12, & b^3\nu^8 = 72 \end{array} \tag{4}$$

These numbers have been computed using the intersection theory package WIT (12) of the symbolic calculator OmegaMath in the following way:

```
variety(PU,5);
PU(monomial_values_)=
  {
    {m^3*b^2,   1},
    {m^2*b^3,   1}
  };
Ud=sheaf(3,[m,m^2,m^3],PU);
Qd=quotient(Ud,o(b));
Qq=quotient(Ud,Qd);
Enod=osum(tensor(Qq,symm(2,Qd)),symm(3,Qd));
Dnod=tensor(dual(Enod),o(3*m));
variety(Nodal,11);
Nodal(monomial_values_)={
    {
      m^i*b^j*n^(11-i-j),integral(PU,m^i*b^j*segre(5-i-j,Dnod))
    } with (i,j) in (0..3,0..min(3,5-i))
   };
```

We denote by ρ the class of the hypersurface of X_{nod} whose points $(f, (\pi, x))$ satisfy that f is tangent to a given plane. Notice that the dual f^* of an irreducible nodal cubic is a quartic curve. Furthermore, the indeterminacy locus of the map $f \mapsto f^*$ is the 4–codimensional closed set of X_{nod} consisting of points such that f degenerates to a double line and a simple line.

2 Degenerations of X_{nod}

Let U_{nod} be the subvariety of X_{nod} whose points are pairs $(f,(\pi,x)) \in X_{nod}$ such that f is an irreducible nodal cubic contained in the plane π, with a node at x. In fact, X_{nod} is a compactification of U_{nod} whose boundary $X_{nod} - U_{nod}$ consists of the following two codimension 1 irreducible components, called *degenerations* of first order of X_{nod} :

- X_{ncusp}, that parameterizes pairs $(f,(\pi,x)) \in X_{nod}$ such that f is a cuspidal cubic with cusp x.
- X_{consec} parameterizes pairs $(f,(\pi,x)) \in X_{nod}$ such that f is a cubic consisting of a conic f' and a line l which intersects with the conic at two points, being x one of them.

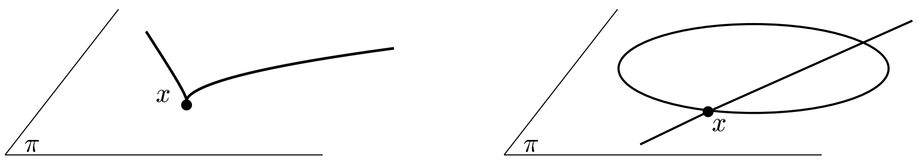

Figure 1. A closed point of X_{ncusp} and of X_{consec} .

We will denote the classes in $\mathsf{Pic}(X_{nod})$ of the degenerations X_{ncusp} and X_{consec} by γ and χ, respectively.

2.1 The variety X_{ncusp}

In (3) we introduced a compactification X_{cusp} of the variety of non–degenerate cuspidal plane cubics in $\mathbb{P}^3$ by means of the projectivization of a suitable vector bundle constructed over the flag variety $\mathbb{F} = \{(\pi,x,u) \mid x \in u,\ u \subset \pi\}$. Actually, X_{cusp} is the 10-dimensional subvariety of $\mathbb{P}(S^3\mathbb{U}^*|_{\mathbb{F}})$ whose points are pairs $(f,(\pi,x,u))$ such that f is a cuspidal cubic contained in the plane π, that has a cusp at x and u as the cuspidal tangent at x.

Moreover, we denote by μ and c the pullbacks to $\mathsf{Pic}(X_{cusp})$ of the hyperplane classes $\mu = c_1(\mathcal{O}_{\Gamma}(1))$ and $a = c_1(\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1))$, respectively, under the natural projections, so that μ is the class of the hypersurface of X_{cusp} such that π goes through a given point and c coincides with the class of the hypersurface of X_{cusp} such that x is on a given plane. In addition, let us denote by ν and

ρ the classes of the hypersurfaces of X_{cusp} consisting of the pairs $(f,(\pi,x,u))$ such that f intersect a given line and, respectively, that f is tangent to a given plane.

In (8) we verified and completed all the intersection numbers obtained by Schubert about cuspidal plane cubics in terms of the characteristic conditions and those relative to the singular triangle. In particular, we got:

$$\begin{array}{lcl}
\mu^3 & = & 24,\ 60,\ 114,\ 168,\ 168,\ 114,\ 60,\ 24 \\
\mu^2 & = & 384,\ 864,\ 1488,\ 2022,\ 2016,\ 1524,\ 924,\ 468,\ 192 \\
\mu & = & 3216,\ 6528,\ 10200,\ 12708,\ 12144,\ 9156,\ 5688,\ 3090,\ 1488,\ 624 \\
1 & = & 17760,\ 31968,\ 44304,\ 49008,\ 43104,\ 30960,\ 18888,\ 10284,\ 5088,\ 2304,\ 960 \\
\mu^3 c & = & 12,\ 42,\ 96,\ 168,\ 186,\ 132,\ 72 \\
\mu^2 c & = & 176,\ 536,\ 1082,\ 1688,\ 1844,\ 1496,\ 956,\ 512 \\
\mu c & = & 1344,\ 3576,\ 6388,\ 8852,\ 9108,\ 7264,\ 4706,\ 2688,\ 1392 \\
c & = & 6592,\ 14800,\ 22336,\ 25560,\ 22864,\ 16672,\ 10380,\ 5836,\ 3040,\ 1504 \\
\mu^3 c^2 & = & 2,\ 8,\ 20,\ 38,\ 44,\ 32 \\
\mu^2 c^2 & = & 32,\ 110,\ 240,\ 400,\ 452,\ 372,\ 240 \\
\mu c^2 & = & 248,\ 740,\ 1416,\ 2076,\ 2216,\ 1818,\ 1200,\ 696 \\
c^2 & = & 1168,\ 2896,\ 4592,\ 5408,\ 4952,\ 3708,\ 2376,\ 1392,\ 768
\end{array} \tag{5}$$

where the numbers listed to the right of a given $\mu^i c^j$ correspond to the intersection numbers $\mu^i c^j \nu^k \rho^{10-i-j-k}$, for $k = 10-i-j,\ldots,0$.

Now, we will see that there exists a birational isomorphism between the variety X_{cusp} and the degeneration X_{ncusp} of X_{nod}. Notice that the dual of a $(f,(\pi,x)) \in X_{ncusp}$, where f is a non–degenerate cuspidal cubic, consists of the dual cuspidal cubic together with the cusp as a simple focus.

Proposition 2.1 *The map $\psi_{cusp} : X_{cusp} \to X_{nod}$ that assigns $(f,(\pi,x))$ to $(f,(\pi,x,u))$ is a birational isomorphism between X_{cusp} and $X_{ncusp} \subseteq X_{nod}$. Moreover, we have that $\psi^*_{cusp}(\mu) = \mu$, $\psi^*_{cusp}(b) = c$, $\psi^*_{cusp}(\nu) = \nu$ and $\psi^*_{cusp}(\rho) = \rho + c$.*

Proof. Since u is the tangent line of f at x (f a non–degenerate cuspidal cubic over π with cusp x), it is clear that ψ_{cusp} induced a birational isomorphism. On the other hand, the relation $\psi^*_{cusp}(\rho) = \rho + c$ can be proved considering the commutative diagram:

$$\begin{array}{ccc}
X_{cusp} & \xrightarrow{\ \psi_{cusp}\ } & X_{nod} \\
\Big\downarrow{\scriptstyle(\varphi_{cusp},p)} & & \Big\downarrow{\scriptstyle\varphi_{nod}} \\
\mathbb{P}(S^3\mathbb{U}) \times_\Gamma \mathbb{P}(\mathbb{U}) & \xrightarrow{\ \kappa\ } & \mathbb{P}(S^4\mathbb{U})
\end{array}$$

where p is the natural projection, φ_{nod} and φ_{cusp} are the birational maps over X_{nod} and X_{cusp} that assign $f \mapsto f^*$, and κ is the map that assigns $((f^*, \pi), (\pi, x)) \mapsto (f^* \cdot x^*, \pi)$, where x^* is the pencil of planes that go through x (its focus). From this, we have that

$$\begin{aligned}\psi^*_{cusp}(\rho) &= \psi^*_{cusp}\varphi^*_{nod}(c_1\mathcal{O}_{\mathbb{P}(S^4\mathbb{U})}(1)) = (\varphi_{cusp}, p)^*\kappa^*(c_1\mathcal{O}_{\mathbb{P}(S^4\mathbb{U})}(1)) \\ &= (\varphi_{cusp}, p)^*(c_1\mathcal{O}_{\mathbb{P}(S^3\mathbb{U})}(1),\ c_1\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1)) = \rho + c.\end{aligned}$$

The remaining relations can be proved in a similar way. □

Now, from this proposition and from the intersection numbers (5) of X_{cusp}, we can compute the intersection numbers of the degeneration X_{ncusp} in this way:

$$\int_{X_{nod}} \mu^i b^j \nu^k \rho^t \gamma = \int_{X_{cusp}} \mu^i c^j \nu^k (\rho + c)^t.$$

Proposition 2.2 *In $A^*(X_{nod})$ we have:*

$$\begin{aligned}
\mu^3\gamma &= 24,\ 72,\ 200,\ 480,\ 960,\ 1424,\ 1512,\ 1200 \\
\mu^2\gamma &= 384,\ 1040,\ 2592,\ 5600,\ 10240,\ 14944,\ 17440,\ 16512,\ 12800 \\
\mu\gamma &= 3216,\ 7872,\ 17600,\ 34112,\ 56320,\ 76896,\ 87152,\ 83520,\ 70032,\ 52320 \\
\gamma &= 17760,\ 38560,\ 75072,\ 124800,\ 173952,\ 203840,\ 204320,\ 179712,\ 142720,\ 105312,\ 75520
\end{aligned}$$

where the numbers listed to the right of a given $\mu^i\gamma$ correspond to the intersection numbers $\mu^i\nu^k\rho^{10-i-k}\gamma$, for $k = 10 - i, \ldots, 0$.

It is worth while to notice that the value we find in the *Tabelle von Zahlen* γ, page 154 of (11), for $\mu^2\nu^3\rho^6\gamma$ is 14744 instead of 14944. This looks like a misprint, rather than a mistake, since the remaining numbers do coincide.

Corollary 2.1 *The following relation holds in* $\mathsf{Pic}(X_{nod})$*:*

$$\gamma = -4\mu + 2\nu.$$

Proof. From Proposition 1.2 we know that $\gamma = \alpha_1\mu + \alpha_2\nu + \alpha_3 b$, with $\alpha_i \in \mathbb{Z}$, holds in $\mathsf{Pic}(X_{nod})$. By substituting this expression in to the numbers $\gamma\mu^3\nu^5 b^2 = 2$, $\gamma\mu^3\nu^6 b = 12$ and $\gamma\mu^2\nu^6 b^2 = 32$ we obtain the desired formula. □

2.2 The variety X_{consec}

In this section we introduce a birational model of the variety $X_{consec} \subseteq X_{nod}$. To do this, we consider the variety $\mathbb{G} = \mathbb{F} \times_{\mathbb{P}(\mathbb{U}^*)} \mathbb{F}$ consisting of the points (π, x_a, x_b, u_l) such that $(\pi, x_a, u_l) \in \mathbb{F}$ and $(\pi, x_b, u_l) \in \mathbb{F}$. The pullback to $\mathbb{G}$ of the classes μ, ℓ of $\mathbb{P}(\mathbb{U}^*)$ will be denoted by the same notations and so will do a and b of $\mathbb{F}$.

We will denote by $\mathbb{E}_{consec}$ the rank 4 subbundle of $S^2\mathbb{U}^*|_{\mathbb{G}}$ whose fiber over a point $(\pi, x_a, x_b, u_l) \in \mathbb{G}$ is the linear subspace of forms $\varphi \in S^2\mathbb{U}^*$ that vanish at x_a and x_b. Next statement provides a resolution of $\mathbb{E}_{consec}$. We use the following notations:

- $\mathbb{Q}_a^*$, respectively $\mathbb{Q}_b^*$, for the pullback of $\mathbb{Q}^*$ to $\mathbb{G}$ under the projection $\mathbb{G} \to \mathbb{F}$ which assigns (π, x_a, u_l) to (π, x_a, x_b, u_l), respectively (π, x_a, u_l) to (π, x_a, x_b, u_l);
- $\mathcal{O}_{\mathbb{G}}(-1)$, for the pullback to $\mathbb{G}$ of the tautological line subbundle of $\mathbb{P}(\mathbb{U}^*)$.

Lemma 2.1 *The sequence*

$$0 \to \mathcal{O}_{\mathbb{G}(-2)} \to \mathcal{O}_{\mathbb{G}(-1)} \otimes \mathbb{Q}_a^* \otimes \mathbb{Q}_b^* \to (\mathbb{U}^* \otimes \mathcal{O}_{\mathbb{G}(-1)}) \oplus (\mathbb{Q}_a^* \otimes \mathbb{Q}_b^*) \to \mathbb{E}_{consec} \to 0$$

is an exact sequence of vector bundles over $\mathbb{G}$.

Proof. It is similar to the proof given in Proposition 1.1. □

Thus, $\mathbb{P}(\mathbb{E}_{consec})$ is the 10-dimensional subvariety of $\mathbb{P}(S^2\mathbb{U}^*|_{\mathbb{G}})$ whose points are pairs $(f', (\pi, x_a, x_b, u_l))$ such that f' is a conic contained in the plane π that goes through the points x_a and x_b.

Furthermore, we denote by μ, a, b and ℓ the pullbacks to $\mathsf{Pic}(\mathbb{P}(\mathbb{E}_{consec}))$ of the homonymous classes of $\mathsf{Pic}(\mathbb{G})$ under the natural projections. In addition, let us denote by ν' the class of the hypersurface of $\mathbb{P}(\mathbb{E}_{consec})$ consisting of the pairs $(f', (\pi, x_a, x_b, u_l))$ such that the conic f' intersect a given line.

Using again the projection formula, we have

$$\int_{\mathbb{P}(\mathbb{E}_{consec})} \mu^i a^j b^k \ell^h \nu'^{10-i-j-k-h} = \int_{\mathbb{G}} \mu^i a^j b^k \ell^h s_{7-i-j-k-h}(\mathbb{E}_{consec} \otimes \mathcal{O}_{\Gamma}(-3)),$$

where $s_t(\mathbb{E}_{consec} \otimes \mathcal{O}_{\Gamma}(-3))$ can be calculated from the resolution given in Lemma 2.1. This allows us to compute all the intersection numbers of $\mathbb{P}(\mathbb{E}_{consec})$ in the conditions μ, a b, ℓ and ν'. In particular, we have:

$\mu^3 a\ell\nu'^5 = 2,$	$\mu^2 a\ell\nu'^6 = 16,$	$\mu a\ell\nu'^7 = 68,$	$a\ell\nu'^8 = 184$
$\mu^3 \ell^2\nu'^5 = 2,$	$\mu^2 \ell^2\nu'^6 = 16,$	$\mu \ell^2\nu'^7 = 68,$	$\ell^2\nu'^8 = 184$
$\mu^3 a^2\ell\nu'^4 = 1,$	$\mu^2 a^2\ell\nu'^5 = 8,$	$\mu a^2\ell\nu'^6 = 34,$	$a^2\ell\nu'^8 = 92$
$\mu^3 a\ell^2\nu'^4 = 1,$	$\mu^2 a\ell^2\nu'^5 = 10,$	$\mu a\ell^2\nu'^6 = 50,$	$a\ell^2\nu'^8 = 160$
	$\mu^2 \ell^3\nu'^5 = 4,$	$\mu \ell^3\nu'^6 = 32,$	$\ell^3\nu'^7 = 136$
	$\mu^2 a^3\ell\nu'^4 = 1,$	$\mu a^3\ell\nu'^5 = 6,$	$a^3\ell\nu'^6 = 18$
	$\mu^2 a^2\ell^2\nu'^4 = 2,$	$\mu a^2\ell^2\nu'^5 = 14,$	$a^2\ell^2\nu'^6 = 52$
	$\mu^2 a\ell^3\nu'^4 = 2,$	$\mu a\ell^3\nu'^5 = 16,$	$a\ell^3\nu'^6 = 68$
		$\mu \ell^4\nu'^5 = 4,$	$\ell^4\nu'^6 = 32$
		$\mu a^3\ell^2\nu'^4 = 1$	$a^3\ell^2\nu'^5 = 6$
		$\mu a^2\ell^3\nu'^4 = 2$	$a^2\ell^3\nu'^5 = 12$
		$\mu a\ell^4\nu'^4 = 2$	$a\ell^4\nu'^5 = 12$

Finally, in order to compute intersection numbers involving the ρ condition, we will consider $\overline{\mathbb{P}}(\mathbb{E}_{consec})$, the closure of the graph in $\mathbb{P}(\mathbb{E}_{consec}) \times_{\mathbb{G}} \mathbb{P}(S^2\mathbb{U}|_{\mathbb{G}})$ of the rational map $\psi : \mathbb{P}(\mathbb{E}_{consec}) \longrightarrow \mathbb{P}(S^2\mathbb{U}|_{\mathbb{F}})$ that assigns the conic of tangents to a given conic of rank ≥ 2. Notice that the points of $\overline{\mathbb{P}}(\mathbb{E}_{consec})$ consist of triples $(f',\ f'^{*}, (\pi, x_a, x_b, u_l))$ where f'^{*} is the dual conic of f' over π, so that ψ is undefined precisely at a closed set D of codimension 2 of $\mathbb{P}(\mathbb{E}_{consec})$ which has two irreducible components:

- D_1 consisting of pairs $(f', (\pi, x_a, x_b, u_l))$ such that f' is a double line which coincides with the line u_l;
- D_2 consisting of pairs $(f', (\pi, x_a, x_b, u_l))$ such that $x_b = x_a$ and f' is a double line that goes through the point x_a.

Then, the projection map $h : \overline{\mathbb{P}}(\mathbb{E}_{consec}) \to \mathbb{P}(\mathbb{E}_{consec})$ is just the blow–up of $\mathbb{P}(\mathbb{E}_{consec})$ along D. The geometric description of the two irreducible components of the exceptional divisor $E = h^{-1}(D)$ is given below (see figure 2):

- E_1 parameterizes triples $(f', f'^{*}, (\pi, x_a, x_b, u_l))$ such that f' is a double line which coincides with u_l and the dual conic f'^{*} consists of two pencils whose foci lie on u_l;
- E_2 parameterizes triples $(f', f'^{*}, (\pi, x_a, x_b, u_l))$ such that $x_b = x_a$, f' is a double line over π that goes through x_a and the dual conic f'^{*} consists of a pair of pencils whose foci lie on this double line.

We will also write μ, a, b, ℓ and ν' to denote the pullbacks to $\mathsf{Pic}(\overline{\mathbb{P}}(\mathbb{E}_{consec}))$ of their homonymous classes in $\mathsf{Pic}(\mathbb{P}(\mathbb{E}_{consec}))$ under the blow–up $h : \overline{\mathbb{P}}(\mathbb{E}_{consec}) \to \mathbb{P}(\mathbb{E}_{consec})$. Then, μ, a, b, ℓ and ν' are the classes of the hypersurfaces of $\overline{\mathbb{P}}(\mathbb{E}_{consec})$ whose points $(f', f'^{*}, (\pi, x_a, x_b, u_l))$ satisfy that π goes through a given point, x_a is on a given plane, x_b is on a given plane, u_l intersects a line and f' intersects a line, respectively. Let ρ' be the class of the hypersurface of $\overline{\mathbb{P}}(\mathbb{E}_{consec})$ whose points $(f', f'^{*}, (\pi, x_a, x_b, u_l))$ satisfy that $\pi \cap \pi' \in f'^{*}$ for a

Figure 2. A closed point of the component E_1 and E_2 .

given plane π' (that is, f' is tangent to a given plane).

Lemma 2.2 *The following relation holds in* $\mathsf{Pic}(\overline{\mathbb{P}}(\mathbb{E}_{consec}))$*:*

$$\rho' = 2\nu' - 2\mu - 2\varepsilon_1 - 4\varepsilon_2.$$

Proof. Due to the properties of the blow–up, there exists a morphism $\overline{\psi}$: $\overline{\mathbb{P}}(\mathbb{E}_{consec}) \to \mathbb{P}(S^2\mathbb{U}|_{\mathbb{G}})$ which makes the following diagram commutative :

$$\begin{array}{ccc} \overline{\mathbb{P}}(\mathbb{E}_{consec}) & & \\ \downarrow h & \searrow^{\overline{\psi}} & \\ \mathbb{P}(\mathbb{E}_{consec}) & \xrightarrow{\psi} & \mathbb{P}(S^2\mathbb{U}|_{\mathbb{G}}), \end{array}$$

that is, $\overline{\psi}$ coincides, as a rational map, with $\psi \circ h$. Thus, as we know that ψ is univocally given by sections of the invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathbb{E}_{consec})}(2)$ over $\mathbb{P}(\mathbb{E}_{consec})$, we can conclude, see (9), that

$$\overline{\psi}^*(\mathcal{O}_{\mathbb{P}(S^2\mathbb{U}|_{\mathbb{F}})}(1)) = h^*(\mathcal{O}_{\mathbb{P}(\mathbb{E}_{consec})}(2)) \otimes \mathcal{O}_{\overline{\mathbb{P}}(\mathbb{E}_{consec})}(-E).$$

Taking Chern classes, we get $c_1(\overline{\psi}^*(\mathcal{O}_{\mathbb{P}(S^2\mathbb{U}|_{\mathbb{G}})}(1))) = c_1(h^*(\mathcal{O}_{\mathbb{P}(\mathbb{E}_{consec})}(2))) - (2\varepsilon_1 + 4\varepsilon_2)$. Finally, we know $c_1(h^*(\mathcal{O}_{\mathbb{P}(\mathbb{E}_{consec})}(1))) = \nu' - 2\mu$, and, by duality, $c_1(\overline{\psi}^*(\mathcal{O}_{\mathbb{P}(S^2\mathbb{U}|_{\mathbb{F}})}(1))) = \rho' - 2\mu$, so the formula follows. □

Now, in order to calculate the intersection numbers $\mu^i a^j b^k \ell^h \nu'^s \rho'^t$ with $t = 10-i-j-k-h-s$, and since the intersection numbers $\mu^i a^j b^k \ell^h \nu'^{9-i-j-k-h}$ are easily calculated using the resolution of $\mathbb{E}_{conic}$ given in Lemma 2.1, we only need to compute numbers over $\overline{\mathbb{P}}(\mathbb{E}_{consec})$ which involve any of the two components of the exceptional divisor, that is, numbers of the form $\mu^i a^j b^k \ell^h \nu'^s \rho'^{9-i-j-k-h-s} \varepsilon_1$ or $\mu^i a^j b^k \ell^h \nu'^s \rho'^{9-i-j-k-h-s} \varepsilon_2$, and then proceed down recursively by induction on the order of the ρ condition.

Proposition 2.3 *The map* $\psi_{consec} : \overline{\mathbb{P}}(\mathbb{E}_{consec}) \longrightarrow X_{nod}$ *that assigns to each* $(f', f'^*, (\pi, x_a, x_b, u_l))$ *the pair* $(f' \cdot u_l, (\pi, x_b))$ *is a birational isomorphism be-*

tween $\overline{\mathbb{P}}(\mathbb{E}_{consec})$ and $X_{consec} \subseteq X_{nod}$. Moreover, we have that $\psi^*_{consec}(\mu) = \mu$, $\psi^*_{consec}(b) = b$, $\psi^*_{consec}(\nu) = \nu' + \ell$ and $\psi^*_{consec}(\rho) = \rho' + 2a$.

Proof. Notice that if we take a system of projective coordinates $\{x_0, x_1, x_2, x_3\}$ of $\mathbb{P}^3$ such that $x = [1, 0, 0, 0]$ and $\pi = \{x_3 = 0\}$ then $\partial f/\partial x_0$ is the tangent cone of f at x over π. From this it is easy to see that ψ_{consec} induced a birational isomorphism. On the other hand, all the relations of the proposition can be proved in the same way and so we we will show how to prove that $\psi^*_{consec}(\nu) = \nu' + \ell$. It is enough to consider the commutative diagram:

$$\begin{array}{ccc}
\overline{\mathbb{P}}(\mathbb{E}_{consec}) & \xrightarrow{\psi_{consec}} & X_{nod} \\
{\scriptstyle (p_2,p_1)}\downarrow & & \downarrow{\scriptstyle p_3} \\
\mathbb{P}(S^2\mathbb{U}^*) \times_\Gamma \mathbb{P}(\mathbb{U}^*) & \xrightarrow{q} & \mathbb{P}(S^3\mathbb{U}^*)
\end{array}$$

where p_1, p_2 and p_3 are the natural projections and q is the map that assings $(f', u_l, \pi) \mapsto (f' \cdot u_l, \pi)$. From this, we have that

$$\begin{aligned}
\psi^*_{consec}(\nu) &= \psi^*_{consec} p_3^*(c_1\mathcal{O}_{\mathbb{P}(S^3\mathbb{U}^*)}(1)) = (p_2, p_1)^* q^*(c_1\mathcal{O}_{\mathbb{P}(S^2\mathbb{U}^*)}(1)) \\
&= (p_2, p_1)^*(c_1\mathcal{O}_{\mathbb{P}(S^2\mathbb{U}^*)}(1),\ c_1\mathcal{O}_{\mathbb{P}(\mathbb{U}^*)}(1)) = \nu' + \ell.
\end{aligned}$$

□

Then,

$$\int_{X_{nod}} \mu^i b^j \nu^k \rho^t \chi = \int_{\overline{\mathbb{P}}(\mathbb{E}_{consec})} \mu^i b^j (\nu' + \ell)^k (\rho' + 2a)^t,$$

and so we have the following table:

Proposition 2.4 *In $A^*(X_{nod})$ we have:*

$$\begin{aligned}
\mu^3\chi &= 42,\ 114,\ 260,\ 480,\ 588,\ 422,\ 144,\ 0 \\
\mu^2\chi &= 672,\ 1652,\ 3424,\ 5840,\ 7264,\ 6452,\ 3952,\ 1344,\ 0 \\
\mu\chi &= 5640,\ 12568,\ 23632,\ 36864,\ 44040,\ 39820,\ 26968,\ 13452,\ 4224,\ 0 \\
\chi &= 31320,\ 62160,\ 103328,\ 141792,\ 153984,\ 130960,\ 86560,\ 44088,\ 16072,\ 3984,\ 0
\end{aligned}$$

where the numbers listed to the right of a given $\mu^i\chi$ correspond to the intersection numbers $\mu^i\nu^k\rho^{10-i-k}\chi$, for $k = 10 - i, \ldots, 0$.

Corollary 2.2 *The following relation holds in $\mathsf{Pic}(X_{nod})$:*

$$\chi = 3\mu - 3b + 5\nu.$$

Proof. We obtain the expression of χ in terms of the basis $\{\mu, b, \nu\}$ of $\mathsf{Pic}(X_{nod})$ from table (4) and the former one, proceeding as in Corollary 2.1. □

3 Characteristic numbers of X_{nod}

In this section we express the condition $\rho \in \mathsf{Pic}(X_{nod})$, that the nodal cubic $(f,(\pi,x))$ is tangent to a given plane, in terms of the μ condition and the γ and χ degenerations, generalizing to $\mathbb{P}^3$ the degeneration formula for ρ given by Zeuthen in (13).

Proposition 3.1 *The following relation holds in* $\mathsf{Pic}(X_{nod})$*:*

$$3\rho = 4\mu + \gamma + 2\chi.$$

Proof. From Proposition 1.2 and corollaries 2.1 and 2.2 we know that there exist rational numbers s_i such that $\rho = s_0\mu + s_1\gamma + s_2\chi$ holds in $\mathsf{Pic}(X_{nod})$. Taking into account the degeneration formula of Zeuthen verified by Kleiman-Speiser (5) we know that $s_1 = \frac{1}{3}$ and $s_2 = \frac{2}{3}$. In order to determine s_0 we compute the intersection number $\mu^2\nu^7\rho b$ in two different ways. First, we have $\mu^3\nu^7\rho = \frac{1}{3}\mu^3\nu^7\gamma + \frac{2}{3}\mu^3\nu^7\chi = 36$. Now, from Corollary 2.1, we get $\mu^2 b\nu^7\rho = 2\mu^3 b\nu^6\rho + \frac{1}{2}\mu^2 b\nu^6\rho\gamma = 2\cdot 22 + \frac{1}{2}\cdot 568 = 328$. Finally, by substituting the expression of ρ in the relation $\mu^2 b\nu^7\rho = 328$, we obtain $s_0 = \frac{4}{3}$. □

This proposition implies that the intersection numbers $\mu^i\nu^k\rho^{11-i-k}$ in X_{nod} can be obtained as $\mu^i\nu^k\rho^{11-i-k} = \frac{1}{3}(\mu^i\nu^k\rho^{10-i-k}(4\mu+\gamma+2\chi))$, because the unique degenerations of the 1–dimensional systems $\mu^i\nu^k\rho^{10-i-k}$ are the ones consisting of a cuspidal cubic or a degenerated conic with a secant line. Thus, from Proposition 2.2 and Proposition 2.4, we are now able to compute all the non–zero intersection numbers of the form $\mu^i\nu^k\rho^{11-i-k}$ in X_{nod}.

Proposition 3.2 *In* $A^*(X_{nod})$ *we have:*

μ^3 = 12, 36, 100, 240, 480, 712, 756, 600, 400

μ^2 = 216, 592, 1496, 3280, 6080, 8896, 10232, 9456, 7200, 4800

μ = 2040, 5120, 11792, 23616, 40320, 56240, 64040, 60672, 49416, 35760, 23840

$*$ = 12960, 29520, 61120, 109632, 167616, 214400, 230240, 211200, 170192, 124176, 85440, 56960

where the numbers listed to the right of a given μ^i *(* $*$ *for* μ^0*) correspond to the intersection numbers* $\mu^i\nu^k\rho^{11-i-k}$*, for* $k = 11-i,\dots,0$ *.*

Finally, from the formula $P = \mu\nu - 3\mu^2$ given by Schubert (see (3)), where P is the class of the subvariety of X_{nod} consisting of pairs $(f,(\pi,x))$ such that f goes through a given point, and from the table of Proposition 3.2, we get the characteristic numbers of nodal plane cubics in $\mathbb{P}^3$ that involve the P condition. Our results confirms the characteristic numbers computed by Schubert and listed in page 159 of (11).

Theorem 3.1 *The following results hold in $A^*(X_{nod})$:*

$$P^2 = 144,\ 376,\ 896,\ 1840,\ 3200,\ 4624,\ 5696,\ 5856$$
$$P = 1392,\ 3344,\ 7304,\ 13776,\ 22080,\ 29552,\ 33344,\ 32304,\ 27816,\ 21360$$

where the numbers listed to the right of a given P^i correspond to the characteristic numbers $P^i\nu^k\rho^{11-2i-k}$, for $k = 11-2i, \ldots, 0$.

References

[1] P. Aluffi. The enumerative geometry of plane cubics II. Nodal and cuspidal cubics. *Math. Ann.*, 289, 543-572, 1991.

[2] W. Fulton. *Intersection Theory.* Ergebnisse der Math. 3 Folge, Band 2, Springer-Verlag, 1984.

[3] X. Hernández and J. Miret. The characteristic numbers of cuspidal plane cubics in $\mathbb{P}^3$. *Bull. Belg. Math. Soc. Simon Stevin*, 10, 115-124, 2003.

[4] S. Kleiman, S. Strømme and Xambó. *Sketch of a verification of Schubert's number 5819539783680 of twisted cubics.* LNM 1266, 156-180, 1987.

[5] S. Kleiman and R. Speiser. Enumerative geometry of nodal plane cubics. LNM 1311, 156-196, 1988.

[6] S. Maillard. *Récherche des charactéristiques des systèmes élémentaires de courbes planes du troisième ordre.* PhD thesis, Paris, 1871. Publ. by Cusset.

[7] J. Miret and S. Xambó. On the geometry of nodal plane cubics: the condition *p*. *Contemporary Mathematics*, 123, 169-187, 1991.

[8] J. Miret, S. Xambó and X. Hernández. Completing H. Schubert's work on the enumerative geometry of cuspidal cubics in $\mathbb{P}^3$. *Comm. in Algebra*, 31(8), 4037-4068, 2003.

[9] J. Roberts and R. Speiser. Enumerative geometry of triangles I. *Comm. in Algebra*, 12:1213–1255, 1984.

[10] G. Sacchiero. *Numeri caratteristici delle cubiche piane nodale.* Preprint Univ. di Roma II, 1984.

[11] H. Schubert. *Kalkül der abzählenden Geometrie.* Teubner, 1879. Repr. 1979 by Springer.

[12] S. Xambó. WIT, a system for computations in intersection theory. It will appear in the Proc. of the ISSAC2002 Conference.

[13] H. Zeuthen. Détermination des charactéristiques des systèmes elémetaires des cubiques, I: Cubiques doueés d'un point double. *CR. Acad. Sc. Paris*, 74, 604-607, 1872.