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# Structural stability of $(C, A)$-marked and observable subspaces

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## Context

- Deformation of $f$ keeping the subspace invariant
- Deformation of invariant subspace keeping $f$ fixed
- Deformation of pairs of $f$ and invariant subspace

- $f \sim A$
- $f \sim (C,A)$

**Aim**

F.Puerta-X.Puerta-S.Tarragona obtain the implicit form of a miniversal deformation of a $(C,A)$-invariant subspace with regard to the usual equivalence relation

We obtain for a $(C,A)$-marked subspace:

- The explicit form of its miniversal deformation
- The dimension of its orbit
- The stable $(C,A)$-marked subspaces

**Notation**

$\begin{pmatrix} A \\ C \end{pmatrix}$ observable linear map defined in a subspace with Brunovsky numbers $k=(k_1,k_2,\ldots,k_r)$

$\mathrm{Inv}(k,h)$ manifold of the $(C,A)$-invariant subspaces with Brunovsky numbers of the restriction $h=(h_1,h_2,\ldots,h_s)$ with regard to the block equivalence

## Differentiable structure of **Inv(*k*,*h*)**

- $\mathrm{M}(k,h)$ set of matrices of full rank with regard to the partition $(k,h)$ formed by the matrices with constant diagonals depending on $k_i - h_j + 1$ parameters

$$X = \left[\begin{array}{cccc|c} x_{i,j,1} & 0 & \cdots & 0 & \\ x_{i,j,2} & x_{i,j,1} & \cdots & 0 & \\ \cdots & x_{i,j,2} & \cdots & x_{i,j,1} & \ddots \\ x_{i,j,k_i-h_j+1} & \cdots & \cdots & x_{i,j,2} & \\ 0 & x_{i,j,k_i-h_j+1} & \cdots & \cdots & \\ 0 & 0 & \cdots & x_{i,j,k_i-h_j+1} & \\ \hline & & \ddots & & \ddots \end{array}\right] \in \mathrm{M}(k,h)$$

- Differentiable structure as an orbit space

$$\boxed{\mathrm{Inv}(k,h) \cong \mathrm{M}(k,h)/\mathrm{M}(h,h)}$$

(J.Ferrer-F.Puerta-X.Puerta)

## Deformations

Let $M$ be a manifold and $G$ a Lie group acting on it:

$$\begin{aligned} G \times M &\to M \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

The orbit of the matrix $x \in M$

$$\mathcal{O}_x = \{g \cdot x \, , \, g \in G\}$$

is its equivalence class associated to this action

- Deformation

**Definition** An $l$-parametrized deformation of $x \in M$ is a differentiable map

$$\phi : U \to M$$

where

- $U$ is a neighbourhood of the origin
- $\phi(0) = x$

## Deformations

- Versal / Miniversal deformation

**Definition** $\phi: U \to M$ is called versal if for any other deformation

$$\psi: V \to M \quad , V \subset \mathbb{C}^k$$

there is:

- a neighbourhood of the origin $V' \subset V \;\; , \;\; 0 \in V'$
- a differentiable map $\alpha: V' \to U \;\; , \;\; \alpha(0) = 0$
- a deformation $\beta: V' \to G$ of $I \in G$

such that

$$\boxed{\psi(v) = \beta(v) \cdot \phi(\alpha(v)) \;\; , \forall v \in V'}$$

It is miniversal if the number of parameters is the minimum among all the versal deformations

## Deformations

- Transversality

**Theorem (Arnold)** A submanifold $N \subset M$ is a versal deformation of $x \in N$ if and only if $N$ is transversal to $\mathcal{O}_x$ at $x$:

$$\boxed{N \text{ versal } \Leftrightarrow T_x(N) + T_x\mathcal{O}_x = T_x(M)}$$

$$\boxed{N \text{ miniversal } \Leftrightarrow T_x(N) \oplus T_x\mathcal{O}_x = T_x(M)}$$

If $N$ is a miniversal deformation of $x \in M$, then

$$\boxed{\dim \mathcal{O}_x = \dim M - \dim N}$$

**Corollary** A supplementary subspace of $T_x\mathcal{O}_x$ determines a miniversal deformation of $x \in N$

## Miniversal deformation of (*C*,*A*)-invariant subspaces

- (*C*,*A*)-invariant subspace

**Definition** $A(W \cap \mathrm{Ker}\, C) \subset W$

- Implicit form deformation

**Theorem (F.Puerta-X.Puerta-S.Tarragona)**

$$X \in \mathrm{M}(k,h),\ V \in \mathrm{Sp}(X)$$

A miniversal deformation of $V$ in Inv($k$,$h$) with regard to the group action $\mathrm{M}(k,k)$ is

$$\mathrm{Sp}(X + W)$$

such that $W \in \overline{\mathrm{M}}(k,h)$ and

$$\boxed{\mathrm{tr}(PXW^*) = \mathrm{tr}(XQW^*) = 0}$$

$$\forall P \in \overline{\mathrm{M}}(k,k),\ \forall Q \in \overline{\mathrm{M}}(h,h)$$

## (*C*,*A*)-marked subspaces

• (*C*,*A*)-marked subspace

**Definition** A (*C*,*A*)-invariant subspace is marked if there is a Brunovsky basis of the restriction extendible to a Brunovsky basis of the whole space

$$\begin{pmatrix} A \\ C \end{pmatrix} \sim \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \\ \hline C_1 & C_3 \\ 0 & C_2 \end{pmatrix}$$

(i) $\begin{pmatrix} A_1 \\ C_1 \end{pmatrix}$ and $\begin{pmatrix} A_2 \\ C_2 \end{pmatrix}$ Brunovsky matrices

(ii) $\begin{pmatrix} A \\ C \end{pmatrix}$ similar by a permutation to a Brunovsky matrix

**Remark**

$h_1 \geq h_2 \geq \cdots \geq h_s > h_{s+1} = \cdots = h_r = 0$

If $V \in \mathrm{Inv}(k,h)$ is a (*C*,*A*)-marked subspace we can suppose: $k_i \geq h_i$ and $k_i \geq k_{i+1}$ if $h_i = h_{i+1}$

## (*C*,*A*)-marked subspaces

- Canonical matrix representation

**Definition**

$(C,A)$ observable pair with Brunovsky numbers $k$

$V \in \mathrm{Inv}(k,h)$ $(C,A)$-marked subspace, $X \in \mathrm{M}(k,h)$

$X$ such that $\mathrm{Sp}(X) \in \mathcal{O}(V)$ is a canonical matrix representation of $\mathcal{O}(V)$ if it is

$$X_{i,i} = \begin{pmatrix} 0 \\ I_{h_i} \end{pmatrix} \text{ for all } 1 \le i \le s$$
$$X_{i,j} = 0 \text{ in other case}$$

**Example**

$$k = (4,2),\ h = (3,1)$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

## Miniversal deformation of (*C*,*A*)-marked subspaces

- Explicit form deformation

**Proposition**

(*C*,*A*) observable pair with Brunovsky numbers *k*

$V = \mathrm{Sp}(X) \in \mathrm{Inv}(k,h)$ (*C*,*A*)-marked subspace

$X \in \mathrm{M}(k,h)$ canonical matrix representation of $\mathcal{O}(V)$

$W \in \overline{\mathrm{M}}(k,h)$ is a solution of

$$\mathrm{tr}(PXW^*) = 0 \quad \forall P \in \overline{\mathrm{M}}(k,k)$$

$$\mathrm{tr}(XQW^*) = 0 \quad \forall Q \in \overline{\mathrm{M}}(h,h)$$

if and only if

| | | |
|---|---|---|
| $w_{i,j,l} = 0$ | $k_j - h_j < l \le k_i - h_j + 1$ | $1 \le i \le r,\ 1 \le j \le s,\ k_i \ge k_j$ |
| $w_{i,j,l} = 0$ | $k_i - h_i < l \le k_i - h_j + 1$ | $1 \le i \le r,\ 1 \le j \le s,\ h_i \ge h_j$ |

## Miniversal deformation of (*C*,*A*)-marked subspaces

- Explicit form deformation

**Theorem**

(*C*,*A*) observable pair with Brunovsky numbers $k$

$V = \mathrm{Sp}(X) \in \mathrm{Inv}(k,h)$ (*C*,*A*)-marked subspace

$X \in \mathrm{M}(k,h)$ canonical matrix representation of $\mathcal{O}(V)$

A miniversal deformation of $V$ is

$$\mathrm{Sp}(X+W)$$

such that $W \in \overline{\mathrm{M}}(k,h)$ and

$$\boxed{\begin{array}{l} w_{i,j,l} = 0 \quad \text{if} \quad \min(k_i - h_i, k_j - h_j) < l \le k_i - h_j + 1, \\ \qquad\qquad k_i \ge k_j \ \text{or} \ h_i \ge h_j \end{array}}$$

## Miniversal deformation of (*C*,*A*)-marked subspaces

- Explicit form deformation

**Examples**

$k=(4,2)$ , $h=(3,1)$

$$\left[\begin{array}{ccc|c} w_1 & 0 & 0 & w_3 \\ 1 & w_1 & 0 & 0 \\ 0 & 1 & w_1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & w_2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$k=(4,2)$, $h=(2,1)$

$$\left[\begin{array}{cc|c} w_1 & 0 & w_3 \\ w_2 & w_1 & 0 \\ 1 & w_2 & 0 \\ 0 & 1 & 0 \\ \hline w_4 & 0 & w_5 \\ 0 & w_4 & 1 \end{array}\right]$$

## Miniversal deformation of (*C*,*A*)-marked subspaces

- Dimension of the orbit

**Corollary**

(*C*,*A*) observable pair with Brunovsky numbers $k$

$V \in \mathrm{Inv}(k,h)$ (*C*,*A*)-marked subspace

$$\operatorname{codim}\mathcal{O}(V) = \sum_{\substack{i\le r,\, j\le s \\ k_i \ge k_j \text{ or } h_i \ge h_j}} \min(k_i - h_i, k_j - h_j) + \sum_{\substack{i\le r,\, j\le s \\ k_j > k_i \ge h_j > h_i}} (k_i - h_j + 1)$$

**Miniversal deformation of (*C*,*A*)-marked subspaces**

- Structural stability

**Definition**

Characterization structurally stable elements:

$$\operatorname{codim} \mathcal{O}(V) = 0$$

**Corollary**

$(C,A)$ observable pair with Brunovsky numbers $k$

$V \in \operatorname{Inv}(k,h)$ $(C,A)$-marked subspace is structurally stable with regard to the considered equivalence relation if and only if

$$\boxed{k_j = h_j \quad \forall\, 1 \le j \le s}$$

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