

PARTICULAR SOLUTIONS OF THE MANY-BODY PROBLEM

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1. The planar n -body problem.

2. Central configurations. General setting: definition, properties, motivation and some results.

3. Two particular examples:
Example 1: Planar coreographies. Regular n -gon.
Example 2: the $1 + n$ -problem.

1. The n -body problem

The ODE of the n -body problem:

$$m_i \mathbf{r}_i'' = \sum_{j=1, j \neq i}^n G m_i m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3} = \frac{\partial U}{\partial \mathbf{r}_i} \quad i = 1, \dots, n$$

where $\mathbf{r}_i \in R^2$ is the vector position of the punctual mass m_k in an inertial system, $r_{ij} = \|\mathbf{r}_i - \mathbf{r}_j\|$, G is the gravitational constant and

$$U = \sum_{1 \leq i < j \leq n} \frac{G m_i m_j}{r_{ij}}$$

Since the center of mass of the system satisfies

$$\frac{\sum_{k=1}^n m_k \mathbf{r}_k}{m_1 + \dots + m_n} = \mathbf{a}t + \mathbf{b}$$

where \mathbf{a} and \mathbf{b} are constant vectors,

we can consider the motion relative to the center of mass, i.e. from now on we work in an inertial barycentric system with the center of mass fixed at the origin:

$$\sum_{k=1}^n m_k \mathbf{r}_k = 0$$

Eqs:

$$m_i \mathbf{r}_i'' = \frac{\partial U}{\partial \mathbf{r}_i} \quad i = 1, \dots, n, \quad U = \sum_{1 \leq i < j \leq n} \frac{Gm_i m_j}{r_{ij}}$$

- First integrals:

the energy $h = T - U$, $T = \frac{1}{2} \sum_k m_k v_k^2$ and the angular momentum $\mathbf{c} = \sum_k m_k (\mathbf{r}_k \times \mathbf{v}_k)$.

- For $n = 2$, Kepler problem, well known.

- For $n \geq 3$, no general solution.

- No equilibrium points:

$$\frac{\partial U}{\partial \mathbf{r}_i} = 0$$

but

$$\sum_i \mathbf{r}_i \frac{\partial U}{\partial \mathbf{r}_i} = -U = 0 \quad \textit{absurd}.$$

- Search for other simple solutions: **central configurations**.

Def. **Central configuration** (cc).

We consider the configuration space

$$E = \{\mathbf{r} \in R^{2n} : \sum_{i=1}^n m_i \mathbf{r}_i = 0, \mathbf{r}_i \neq \mathbf{r}_j, \text{ for } i \neq j\}$$

and

$$M\mathbf{r}'' = \frac{\partial U}{\partial \mathbf{r}}$$

where $M = \text{diag}(m_1, m_1, \dots, m_n, m_n)$, $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n)$

The configuration of n particles given by the vectors $\mathbf{r} = \mathbf{r}_1, \dots, \mathbf{r}_n$ is a central configuration:

\mathbf{r} is a cc \iff there exists $\lambda^2 > 0$ such that

$$M^{-1} \frac{\partial U}{\partial \mathbf{r}} = -\lambda^2 \mathbf{r}$$

Properties.

1. The set of cc is invariant with respect to homothetic transformations and rotations in R^2 .

2. \mathbf{r} is a cc iff $\frac{1}{m_i} \frac{\partial U}{\partial \mathbf{r}_i} = -\lambda^2 \mathbf{r}_i$ iff $\frac{\partial U}{\partial \mathbf{r}_i} + \lambda^2 m_i \mathbf{r}_i = 0$ iff

$$\frac{\partial U}{\partial \mathbf{r}_i} + \lambda^2 \frac{\partial I}{\partial \mathbf{r}_i} = 0$$

with $I = \frac{1}{2} \sum_{i=1}^n m_i \|\mathbf{r}_i\|^2$,

i.e. the cc is a critical point of the potential U restricted to a constant moment of inertia manifold $I = I_0 = \text{constant}$.

3. $\lambda^2 = \frac{U(\mathbf{r})}{2I}$

Reasons why central configurations are of interest in celestial mechanics

1. They allow to compute all the homographic solutions:

A homographic solution of the n -body problem is a solution such that the configuration of the n particles at the instant t (with respect to an inertial barycentric system) remains similar to itself as t varies.

Two configurations are similar if we can go from one to the other by means of a dilatation and/or a rotation.

The first three homographic solutions were found by Euler in 1767 for the three-body problem. For these solutions the configuration of the three bodies is collinear.

In 1772 Lagrange found two additional homographic solutions in the 3-body problem. Now, the configuration formed by the 3 bodies is an equilateral triangle.

The Lagrangian equilateral triangle solutions

Theorem 1 ($n = 3$). For any values of the masses, there is one and only one noncollinear central configuration for the 3-body problem, namely, the three particles are at the vertices of an equilateral triangle.

The Euler-Moulton collinear solutions

Theorem 2 (n -body). There are exactly $n!/2$ collinear central configurations in the n -body problem, one for each ordering of the masses on the line.

More reasons

2. If the n bodies are going to a simultaneous collision, then the particles tend to a central configuration.

Sundman (1907), Hagihara (1970), Saari (1980), Chenciner (1998).

3. If the n bodies are going simultaneously to infinity in parabolic motion (i.e. the radial velocity of each particle tends to zero as the particle tends to infinity), then the particles tend to a central configuration.

Saari and Hilkower (1981).

More reasons

4. Central configurations play a key role when studying the integral manifolds I_{hc} , h being the energy and c the angular momentum. CC are the cause on the change of topology of these manifolds.

Smale (1970), Marchal, Saari (1975), Chenciner and Albouy (1993, 1998), McCord, Meyer and Wang (1998).

5. **Other results:** Stability of cc (Moeckel (1990)), n-dimensional cc (Palmore (1980), Cedo and Llibre (1989), Albouy and llibre (2000)), particular shapes of cc for a fixed n (Kemplerer (1962), Simó (1978), Elmabsout (1988), Hall (1988), Salo and Yoder (1988), Xia (1991)...

Last five years: Corbera, Cors, Delgado, Hampton, Kotsireas and Lazard, Lindstrom, LLibre, Ollé,...

Two obvious questions:

How many central configurations (modulus homothetic transformations and rotations) are there?

How do they look like?

Wintner's conjecture (1941): Is the number of classes of central configurations finite, in the n -body problem for any choice of the masses m_1, \dots, m_n ?

This conjecture also appears in the Smale's list on the open mathematical problems for the XXI century.

- For $n = 3$, known.
- For $n \geq 4$, open problem.

Remark. From now on we are interested in planar cc which are not collinear.

Example 1: Planar coreographies. n -gon

The ODE of the n -body problem:

$$r_i'' = \sum_{j=1, j \neq i}^n m_j \frac{r_j - q_i}{r_{ij}^3}$$

where $q_i \in R^2$, $r_{ij} = \|q_i - q_j\|$, $G = 1$ and the center of mass located at the origin.

For $n \geq 3$, a few solutions are known analytically, related to central configurations: relative equilibrium solutions (all the bodies rotate around the center of mass, with constant angular velocity, keeping the mutual distances constant).

- For $n = 3$, the simplest case:
 - Lagrange equilateral solutions
 - they exist for any value of the masses m_1, m_2, m_3
- For $n \geq 4$, the regular n -gon with equal masses.

Interesting property of the n -gon:

All bodies move periodically tracing the same curve on the plane; there is a time shift in the position to pass from the position of a body to the position of the next one, i.e., there exists a solution $q(t)$ such that the position of body k , $k = 1, \dots, n$ is given by:

$$q_k(t) = q(t - kT/n)$$

Natural question:

Are there other planar solutions of the n -body problem that move along the same path with a time shift T/n ?

Answer: Planar coreographies.

-For $n = 3$, Figure eight curve: Chenciner and Montgomery (2000). **Equal masses.**

-For $n \geq 3$, Simó (2001). **Equal masses.**

Open question and motivation for our problem:

Existence of coreographies with unequal masses and unequal time spacings between two bodies.

Chenciner's open question in HAMSYS2001 (Guanajuato, Mexico):

Is the regular n -gon with equal masses the unique central configuration such that

- 1) all the bodies lie on a circle,
- 2) the center of mass coincides with the center of the circle?

OUR AIM

Circular coreographies as central configurations:

Which are the circular central configurations?

n particles located at a circumference of radius one and center the origin satisfying the cc definition.

Eqs:

$$Mq'' = U_q$$

where $M = \text{diag}(m_1, m_1, \dots, m_n, m_n)$, $q = (q_1, \dots, q_n)$ and

$$U(q_1, \dots, q_n) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\|q_i - q_j\|}$$

Given M , q is a cc \iff there exists $\lambda^2 > 0$ such that

$$M^{-1}U_q = -\lambda^2 q$$

- For $n = 2$: m_1 and m_2 diametrically opposite $\implies m_1 = m_2$

- Useful equations for $n \geq 3$

Proposition.

$$cc \iff \sum_{k \neq i, j} m_k(i, j, k) \left(\frac{1}{r_{ik}^3} - \frac{1}{r_{j,k}^3} \right) = 0$$

where

$$(i, j, k) = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_k & y_k & 1 \end{vmatrix}$$

is the oriented area of the triangle with vertices r_i , r_j and r_k .

- For $n = 3$: m_1 , m_2 and m_3 in an equilateral triangle: $m_1 (1, 0)$, $m_2 (-1/2, \sqrt{3}/2)$, $m_3 (-1/2, -\sqrt{3}/2)$ $\implies m_1 = m_2 = m_3$.

- For $n = 4$

$$\begin{aligned}
m_3(1, 2, 3) \left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} \right) + m_4(1, 2, 4) \left(\frac{1}{r_{14}^3} - \frac{1}{r_{24}^3} \right) &= 0 \\
m_2(1, 3, 2) \left(\frac{1}{r_{12}^3} - \frac{1}{r_{23}^3} \right) + m_4(1, 2, 4) \left(\frac{1}{r_{14}^3} - \frac{1}{r_{34}^3} \right) &= 0 \\
m_2(1, 4, 2) \left(\frac{1}{r_{12}^3} - \frac{1}{r_{24}^3} \right) + m_3(1, 4, 3) \left(\frac{1}{r_{13}^3} - \frac{1}{r_{34}^3} \right) &= 0 \\
m_1(2, 3, 1) \left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3} \right) + m_4(2, 3, 4) \left(\frac{1}{r_{24}^3} - \frac{1}{r_{34}^3} \right) &= 0 \\
m_1(2, 4, 1) \left(\frac{1}{r_{12}^3} - \frac{1}{r_{14}^3} \right) + m_3(2, 4, 3) \left(\frac{1}{r_{23}^3} - \frac{1}{r_{34}^3} \right) &= 0 \\
m_1(3, 4, 1) \left(\frac{1}{r_{13}^3} - \frac{1}{r_{14}^3} \right) + m_2(3, 4, 2) \left(\frac{1}{r_{23}^3} - \frac{1}{r_{24}^3} \right) &= 0
\end{aligned}$$

Let $D_1 = (2, 3, 4)$, $D_2 = (3, 4, 1)$, $D_3 = (4, 1, 2)$, $D_4 = (1, 2, 3)$.

Proposition. (a) Given a triangle T of sides a, b, c in a circle of radius one, $\text{Area}(T) = \frac{abc}{4}$.

(b) $D_1 = \frac{r_{23}r_{24}r_{34}}{4}$, $D_2 = \frac{r_{13}r_{14}r_{34}}{4}$, $D_3 = \frac{r_{12}r_{14}r_{24}}{4}$, $D_4 = \frac{r_{12}r_{13}r_{23}}{4}$

So we have equations

$$e_1 = 0, \dots, e_6 = 0$$

only in variables r_{ij} and masses.

Proposition. For 4 bodies in a circle (radius 1 and center at 0), we have

$$m_2 r_{12}^2 + m_3 r_{13}^2 + m_4 r_{14}^2 = m_1 r_{12}^2 + m_3 r_{23}^2 + m_4 r_{34}^2$$

$n = 4$, we assume equal masses

- Let $c_{ij} = r_{ij}^2$, then,

$$d_1 = c_{13} - c_{23} + c_{14} - c_{24} = 0$$

$$d_2 = c_{12} - c_{13} + c_{24} - c_{34} = 0$$

$$d_3 = c_{13} + c_{23} - c_{14} - c_{24} = 0$$

$$d_4 = -c_{12} - c_{13} + c_{24} + c_{34} = 0$$

$d_1 + d_3 = 2(c_{13} - c_{24}) = 0$, $d_1 - d_3 = -2(c_{23} - c_{14}) = 0$,
 $d_2 - d_4 = 2(c_{12} - c_{34}) = 0 \implies r_{13} = r_{24}$, $r_{23} = r_{14}$,
 $r_{34} = r_{12}$, i.e. rectangle.

- 2 remaining variables: r_{12} , r_{14} ; $r_{13} = \sqrt{r_{12}^2 + r_{14}^2}$.

- We impose the 6 remaining equations: $e_1, \dots, e_6 = 0$,
with $a = r_{12}$ and $x = r_{14}$,

$$e_2 = -e_5 = \frac{(a-x)\sqrt{a^2+x^2}(a^2+ax+x^2)}{2a^2x^2} = 0 \implies a = x$$

therefore 4-gon.

$n = 5$, equal masses

Ingredients:

1. Eqs:

$$\begin{aligned}
e_1 &= D_{45} \left(\frac{1}{r_{13}^3} - \frac{1}{r_{23}^3} \right) + D_{35} \left(\frac{1}{r_{14}^3} - \frac{1}{r_{24}^3} \right) + D_{34} \left(\frac{1}{r_{15}^3} - \frac{1}{r_{25}^3} \right) = 0 \\
e_2 &= -D_{45} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{23}^3} \right) + D_{25} \left(\frac{1}{r_{14}^3} - \frac{1}{r_{34}^3} \right) + D_{24} \left(\frac{1}{r_{15}^3} - \frac{1}{r_{35}^3} \right) = 0 \\
e_3 &= -D_{35} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{24}^3} \right) - D_{25} \left(\frac{1}{r_{13}^3} - \frac{1}{r_{34}^3} \right) + D_{23} \left(\frac{1}{r_{15}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_4 &= -D_{34} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{25}^3} \right) - D_{24} \left(\frac{1}{r_{13}^3} - \frac{1}{r_{35}^3} \right) - D_{23} \left(\frac{1}{r_{14}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_5 &= D_{45} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{13}^3} \right) + D_{15} \left(\frac{1}{r_{24}^3} - \frac{1}{r_{34}^3} \right) + D_{14} \left(\frac{1}{r_{25}^3} - \frac{1}{r_{35}^3} \right) = 0 \\
e_6 &= D_{35} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{14}^3} \right) - D_{15} \left(\frac{1}{r_{23}^3} - \frac{1}{r_{34}^3} \right) + D_{13} \left(\frac{1}{r_{25}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_7 &= D_{34} \left(\frac{1}{r_{12}^3} - \frac{1}{r_{15}^3} \right) - D_{14} \left(\frac{1}{r_{23}^3} - \frac{1}{r_{35}^3} \right) - D_{13} \left(\frac{1}{r_{24}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_8 &= D_{25} \left(\frac{1}{r_{13}^3} - \frac{1}{r_{14}^3} \right) + D_{15} \left(\frac{1}{r_{23}^3} - \frac{1}{r_{24}^3} \right) + D_{12} \left(\frac{1}{r_{35}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_9 &= D_{24} \left(\frac{1}{r_{13}^3} - \frac{1}{r_{15}^3} \right) + D_{14} \left(\frac{1}{r_{23}^3} - \frac{1}{r_{25}^3} \right) - D_{12} \left(\frac{1}{r_{34}^3} - \frac{1}{r_{45}^3} \right) = 0 \\
e_{10} &= D_{23} \left(\frac{1}{r_{14}^3} - \frac{1}{r_{15}^3} \right) + D_{13} \left(\frac{1}{r_{24}^3} - \frac{1}{r_{25}^3} \right) + D_{12} \left(\frac{1}{r_{34}^3} - \frac{1}{r_{35}^3} \right) = 0
\end{aligned}$$

$$\begin{aligned}
e_{11} &= 10 - r_{12}^2 - r_{13}^2 - r_{14}^2 - r_{15}^2 = 0 \\
e_{12} &= r_{12} \sqrt{1 - \frac{r_{12}^2}{4}} + r_{13} \sqrt{1 - \frac{r_{13}^2}{4}} \pm r_{14} \sqrt{1 - \frac{r_{14}^2}{4}} - r_{15} \sqrt{1 - \frac{r_{15}^2}{4}}
\end{aligned}$$

with $D_{45} = r_{12}r_{23}r_{13}/4$, $D_{35} = r_{12}r_{14}r_{24}/4$, etc.

2.

$$\begin{aligned}
d_1 &= r_{13}^2 - r_{23}^2 + r_{14}^2 - r_{24}^2 + r_{15}^2 - r_{25}^2 = 0 \\
d_2 &= r_{24}^2 - r_{34}^2 + r_{25}^2 - r_{35}^2 + r_{12}^2 - r_{13}^2 = 0 \\
d_3 &= r_{35}^2 - r_{45}^2 + r_{13}^2 - r_{14}^2 + r_{23}^2 - r_{24}^2 = 0 \\
d_4 &= r_{14}^2 - r_{15}^2 + r_{24}^2 - r_{25}^2 + r_{34}^2 - r_{35}^2 = 0 \\
d_5 &= 0, \dots, d_{10} = 0
\end{aligned}$$

3. Ptolomeo's theorem.

$$\begin{aligned}
p_1 &= r_{12}r_{34} + r_{14}r_{23} - r_{24}r_{13} = 0 \\
p_2 &= r_{12}r_{35} + r_{15}r_{23} - r_{25}r_{13} = 0 \\
p_3 &= r_{12}r_{45} + r_{15}r_{24} - r_{25}r_{14} = 0 \\
p_4 &= r_{13}r_{45} + r_{34}r_{15} - r_{14}r_{35} = 0 \\
p_5 &= r_{23}r_{45} + r_{34}r_{25} - r_{24}r_{35} = 0
\end{aligned}$$

$$r_{13}^2 r_{14}^2 r_{15}^3 r_{23}^2 r_{24}^2 + r_{13}^2 r_{14}^3 r_{15}^2 r_{23}^2 r_{25}^2 + r_{13}^3 r_{14}^2 r_{15}^2 r_{24}^2 r_{25}^2 - r_{14}^2 r_{15}^2 r_{23}^3 r_{24}^2 r_{25}^2 - r_{13}^2 r_{15}^2 r_{23}^2 r_{24}^3 r_{25}^2 - r_{12}^3 r_{14}^2 r_{15}^2 r_{34}^2 r_{35}^2 = 0$$

we have 10 eqs, just 5 variables.

Two approaches:

(i) Analytical: (?).

(ii) Numerical.

(ii) Numerical.

- 12 eqs in variables x_2, x_3, x_4 since $1 + x_2 + x_3 + x_4 + x_5 = 0$) with some restrictions: fixed $x_2 \in (-1, 1)$,

- Two cases:

Case 1: $y_2 > 0, y_3 < 0, y_4 < 0, y_5 < 0,$
 $-1 \leq x_2 < 1, \quad -1 \leq x_3 < x_4 < x_5 < 1,$

Case 2: $y_2 > 0, y_3 > 0, y_4 < 0, y_5 < 0,$
 $-1 \leq x_3 < x_2 < 1, \quad -1 \leq x_4 < x_5 < 1$

Example 2:

cc of the planar $1 + n$ -problem with infinitesimal masses.

J. M. Cors, J. Llibre, M. Ollé, *Cel. mech. and dynam. astron.* 89(4), 319–342 (2004)

We consider $N = 1 + n$,
 $q(\epsilon) = (q_0(\epsilon), q_1(\epsilon), \dots, q_n(\epsilon)) \in \tilde{\mathcal{C}}$ be a central configuration of the planar $1 + n$ body problem with $m_0 = 1$, $m_i = \epsilon$, $i = 1, \dots, n$.

Def. $q = (q_0, q_1, \dots, q_N)$ is a *central configuration* of the planar $1 + n$ body problem if there exists $\lim_{\epsilon \rightarrow 0} q(\epsilon)$ and this limit is equal to q .

Proposition All central configurations of the planar $1 + n$ body problem lie on a circle centered at $q_0 = 0$.

Proposition Let $q = (q_0, \dots, q_n)$ be a non-collision central configuration of the planar $1 + n$ body problem. Denoting by α_i the angle defined by the position of the i -th infinitesimal mass on a circle centered at $q_0 = 0$, we have for $i = 1, \dots, n$

$$\sum_{j=1, j \neq i}^n \sin(\alpha_j - \alpha_i) \left(1 - \frac{1}{2\sqrt{2}\sqrt{(1 - \cos(\alpha_j - \alpha_i))^3}} \right) = 0, \quad (1)$$

We assume that the circle has radius 1 and that $\alpha_1 = 0$. We take as coordinates the angles

$$\theta_i = \alpha_{i+1} - \alpha_i, \quad i = 1, \dots, n - 1.$$

and

$$\theta_n = 2\pi - \sum_{i=1}^{n-1} \theta_i$$

Let

$$f(\theta) = \sin \theta \left(1 - \frac{1}{2\sqrt{2}\sqrt{(1 - \cos \theta)^3}} \right).$$

Central configurations iff

$$\begin{aligned} f(\theta_1) + f(\theta_1 + \theta_2) + \cdots + f(\theta_1 + \dots + \theta_{n-1}) &= 0, \\ f(\theta_2) + f(\theta_2 + \theta_3) + \cdots + f(\theta_2 + \dots + \theta_n) &= 0, \\ f(\theta_3) + f(\theta_3 + \theta_4) + \cdots + f(\theta_3 + \dots + \theta_n + \theta_1) &= 0, \\ &\dots \\ f(\theta_{n-1}) + f(\theta_{n-1} + \theta_n) + \cdots + \\ + f(\theta_{n-1} + \theta_n + \theta_1 + \cdots + \theta_{n-3}) &= 0, \\ \theta_1 + \cdots + \theta_n &= 2\pi. \end{aligned}$$

Proposition The regular n -gon is always a central configuration of the $1 + n$ problem.

– **Conjecture:** for $9 \leq n$, only one cc .

Conjecture proved for n large enough: there exists only one cc (the regular n -gon) if $n \approx \exp(73)$ (Casasayas, Llibre, Nunes, '94).

– **Conjecture:** All the central configurations of the $1 + n$ body problem are symmetric with respect to a straight line.

– Analytical results for small n :

$n = 2$, Euler and Lagrange

$n = 3$, Hall (unpublished paper)

$n = 4$, Cors, LLibre and Ollé (2004)