The Dialectica models of type theory

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Contents

Introduction

This thesis is an attempt to understand how a family of models of type theory can be indexed by a base model of type theory. By 'type theory' we mean *intensional Martin-Löf type theory*, usually with dependent sums and dependent products. The notion of 'family of objects indexed by an object of the same kind' is widespread in mathematics, for us the most notable example of this phenomenon is the notion of *indexed category*, which can be modelled by that of *Grothendieck fibration*, introduced in [17]. We are more specifically interested in the derivation of a *total* model of type theory from the *indexed* or *fibred* one, and we will use this construction to build examples of *Dialectica models of type theory*. In building the Dialectica models we are continuing a programme that begins in de Paiva's thesis [11]. The idea is to find semantic manifestations of the ideas in various functional interpretations, starting with Gödel's original 'Dialectica' interpretation [16].

The original Dialectica interpretation was introduced by Gödel to prove the consistency of Heyting arithmetic relative to System T, a finite-type extension of primitive recursive arithmetic. Every formula of arithmetic is given an interpretation as a quantifier-free formula of System T together with two sets of fresh variables. A formula A is mapped to

$$A^D = \langle \vec{x}.\vec{y}.A_D(\vec{x};\vec{y})\rangle$$

which is understood to mean

$$\exists\vec{x}.\forall\vec{y}.A_D(\vec{x};\vec{y})$$

in a sense made precise by the following theorem (see, for instance, [28] or [7]).

Theorem. *Any proof of the formula A in Heyting arithmetic gives rises to a sequence of closed terms* $\vec{t}$ *and a proof of* $A_D(\vec{t};\vec{y})$ *in System T.*

We will not give details of the interpretation here other than to note the following. The question of the interpretation of function types in Dialectica can be thought of as asking for a reduction of a formula of the form

$$(\exists u^U.\forall x^X.A(u;x)) \to (\exists v^V.\forall y^Y.B(v;y)) \tag{1}$$

(where A and B are quantifier-free formulas and the superscript annotation indicates the type of each variable) to one in $\exists\forall$ form. For a suitable constructive interpretation of the implication, we may consider (1) equivalent to

$$\exists f^{U\to V}, F^{U\times Y\to X}.\forall u^U, y^Y.A(u;F(u,y)) \to B(f(u);y). \tag{2}$$

The reasoning goes thus: a proof of (1) should, *inter alia*, be a map of potential witnesses u of $\exists u.\forall x.A(u;x)$ to potential witnesses $v = f(u)$ of $\exists v.\forall y.B(v;y)$ with the property that, when u validates the former, $f(u)$ validates the latter. So, in particular, we get a function $f : U \to V$. Allowing for the fact that $f(u)$ might not be a witness v of $\exists v.\forall y.B(v;y)$ on account of that formula being false, a proof of (1) should also provide a means to map potential counterexamples y to $\forall y.B(f(u);y)$ to potential counterexamples x to $\forall x.A(u;x)$, in such a way that if y really is a counterexample then so is its associated x. Moreover, this x is allowed to depend on both u and y, so we may write it as $x = F(u,y)$, i.e. we have a function $F : U \times Y \to X$.

The import of the Diller-Nahm variant of the Dialectica interpretation relates to the extraction of a witness function F validating (2) from a proof of (1). It might be easy to produce a finite set of possible values for $F(u,y)$ from u and y but awkward to effectively decide which of them is actually a counterexample as required. This is the case when interpreting the contraction axiom $A \to A \wedge A$. This problem is resolved by extending the type system of System T to include a type constructor for the finite multisets in some type, written $A^\bullet$ for a type A, and extending the language to include quantification over a finite multiset. Then the analogue of (2) is

$$\exists f^{U\to V}, F^{U\times Y\to X^\bullet}.\forall u^U, y^Y.[\forall x \in F(u,y).A(u;x)] \to B(f(u);y). \tag{3}$$

The Dialectica category [11] of a cartesian closed category $\mathbb{C}$ is a category whose objects are Dialectica propositions and whose morphisms are of the form (2). More precisely, an object is a triple (U, X, α) where $U, X \in \mathbb{C}$ and $\alpha : [\alpha] \rightarrowtail U \times X$ is a subobject of $U \times X$. An arrow $(U, X, \alpha) \to (V, Y, \beta)$ consists of a morphism $f : U \to Y$ together with a morphism $F : U \times Y \to X$ such that, in the subobject lattice of $U \times Y$,

$$(\pi_U, F)^*\alpha \leq (f \times 1_Y)^*\beta.$$

This category is symmetric monoidal closed. The Diller-Nahm category, which we cover in detail in Chapter 4, is a variant where the maps are of the form (3), and it turns out to be *cartesian* closed. Its construction requires the existence of a monad on $\mathbb{C}$ which behaves like the finite multisets monad.

By finding type-theoretic versions of this construction, our work follows on directly from [46], in which von Glehn constructed a model of type theory based on a restricted version of Gödel's Dialectica. The literature on Dialectica categories and semantic versions of Dialectica interpretations includes [21], [39], [20], and the compilation [5] of whose constituent articles [4] is the most relevant to the present work.

Outline

In Chapter 1 we cover the basic background in category theory and categorical logic we need. This includes the fundamental concept of Grothendieck fibration and two notions of model of type theory: display map categories and comprehension categories. We consider it self-evident that these two notions capture roughly the same intuitions about type theory, and that many arguments about one can be applied to the other. Thus we usually only give arguments in whichever setting is convenient and do not trouble ourselves to prove every result or give every definition for both settings. We leave aside the problem of formally stating an equivalence between the two approaches. This chapter is nearly all review of literature and folklore, though we make a small contribution to the understanding of finite sum types.

One of the fundamental type constructors we will consider is the dependent product. There are many structures which model type theory but do not quite model dependent products in the usual sense: they satisfy the β- but not the η-rule. Hence in Chapter 2 we introduce the first of the main constructions of this thesis: the idempotent splitting construction. This generalizes the well-known construction from ordinary category theory of the *Karoubi envelope* [26] or *Cauchy completion* [30]. We say what it means for a model of type theory to have weak dependent products, then we set out the definition of the idempotent splitting construction in the context of split comprehension categories. Finally we consider when the resulting structure models Martin-Löf type theory. The most difficult type constructor to derive is the identity type. To do so, we need to make a modest assumption about the identity types in the starting model: we require that they *preserve idempotents*.

We review the notion of *additive monad* in Chapter 3, which is what we will need to construct the Diller-Nahm category and model of type theory. Our approach is to consider conditions on a monad which imply the existence of biproducts in the Kleisli category. We study properties of these biproducts which we will need later, and take a detour to study the commutative monoid structure induced by an algebra structure for an additive monad, the culmination of which is the result that an additive monad also admits biproducts in its Eilenberg-Moore category.

Chapter 4 is our interpretation of the constructions of [21] and [4] viewed from a fibrational point of view. We review the necessary input data to construct the Diller-Nahm fibration, and we introduce the *Kleisli fibrations* to show that the Diller-Nahm category is obtained by taking the total category after applying some basic operations to a composable pair of fibrations. We give a notion of *quasifibred exponential* which we use to simplify the calculation that the Diller-Nahm category is cartesian closed.

In Chapter 5, we detail the setting for our second main construction: the gluing construction. We introduce the notion of *fibred model of type theory*, in the context of both comprehension categories and display map categories. The advantage of the approach with comprehension categories is that it allows us to formulate and prove the result that dependent sums, one of the fundamental type constructors of Martin-Löf type theory, really are fundamental in that they are equivalent to context extension in the type theory given by working over a fixed context.

We give the gluing construction itself in Chapter 6. Acting on the plain models with no thought for type constructors, this is a generalization of the gluing construction in [38] and essentially the same as the fibrewise-to-total construction in [43]. In the display map case we detail how to derive the required type constructors in the gluing model. For dependent products, our approach is more general than that in [38] and [43] and it abstracts the proof in [46] that the polynomial model has dependent products. We also give a more general construction of a universe in the gluing model.

Chapter 7 is an examples chapter. As all the examples are models of type theory related to various Dialectica-style functional interpretations, we call them *Dialectica models of type theory.* We reconstruct the polynomial model of [46] in our framework using the gluing construction, showing that it models a universe and the correct notion of finite sum type to allow iteration. We also give an extension of the polynomial model to include an additional layer of predicates, bringing it more closely in line with the Dialectica category. An account of a Diller-Nahm variant of the polynomial model is given, and we sketch how this might be extended with an additional layer of predicates. We give a Dialectica model based on the exception monad, which only models weak dependent products. Finally, we combine this example with the idempotent completion construction to get another model with full dependent products.

The last chapter outlines some possible directions for future work, including the properties of the Dialectica models and their iterations, and the possibility of a general ‘model theory for type theory’, in which our gluing construction might have a universal property.

Chapter 1

Preliminaries

In this chapter we will review some of the fundamental background that we need. The central notion is that of *(Grothendieck) fibration*, introduced by Grothendieck in [17]. The importance of fibrations for logic is well-known, see for instance [24]. For us, a fibration allows us to speak of a system of contexts for type theory, where each context is equipped with a good notion of type in that context.

Next, we will set out several notions of 'model of type theory', and what it means for these models to admit various type constructors. The notions of model which are most important for us are that of *display map category* and that of *full comprehension category*. We take the view that these two notions encompass essentially the same intuition about type theory, so rather than formulating any sort of equivalence between them we simply use whichever notion is most convenient for the immediate task. It is usually clear how one would express the same argument in terms of the other language. There are other notions, which we do not consider here, such as that of *contextual category* (see [9]) and *category with families* (see [12]). In the investigation of the η-rule in Chapter 2 we will need the *split* version of full comprehension category, so we also cover that notion. It is apparent that many of the definitions and results apply to comprehension categories which are not necessarily full, and though it may be less apparent it is possible that some useful rôle will eventually be found for these more general structures, hence we give some parts of the exposition in terms of 'non-full comprehension categories'.

1.1 Fibrations

This section contains only review material, a good reference for which is [40]. We begin by recalling the definition of fibration.

Definition 1.1.1. Let $p : \mathbb{E} \to \mathbb{B}$ be a functor between categories. An arrow $\phi : Y \to X$ in $\mathbb{E}$ is *p-cartesian* (or just *cartesian*) if, for every arrow $\psi : Z \to X$ in $\mathbb{E}$, the function

$$\begin{gathered}\mathbb{E}/X(\psi, \phi) \to \mathbb{B}/p(X)(p(\psi), p(\phi)) \\ (\chi : Z \to Y) \mapsto (p(\chi) : p(Z) \to p(Y))\end{gathered}$$

between homsets of slice categories induced by p is a bijection. Given an object $X \in \mathbb{E}$ and an arrow $f : B \to p(X)$ in $\mathbb{B}$, a *cartesian lift* of f with codomain X is a cartesian arrow $\phi : Y \to X$ with $p(\phi) = f$. The functor p is a *fibration* if, for every $X \in \mathbb{E}$ and $f : B \to p(X)$ in $\mathbb{B}$, there exists a cartesian lift of f with codomain X.

Example 1.1.2. Let $\mathbb{B}$ be a category. Then there is a functor $\mathrm{cod} : \mathbb{B}^{\to} \to \mathbb{B}$ from the category of arrows of $\mathbb{B}$ to $\mathbb{B}$ which sends an arrow to its codomain. Then cod is a fibration if and only if $\mathbb{B}$ has all pullbacks. Indeed, an arrow in $\mathbb{B}^{\to}$ is cartesian if and only if it is a pullback when considered as a square in $\mathbb{B}$.

Definition 1.1.3. A *cartesian functor* (or *morphism of fibrations*) from a fibration $p_1 : \mathbb{E}_1 \to \mathbb{B}_1$ to a fibration $p_2 : \mathbb{E}_2 \to \mathbb{B}_2$ is a pair of functors $F_0 : \mathbb{B}_1 \to \mathbb{B}_2$ and $F_1 : \mathbb{E}_1 \to \mathbb{E}_2$ such that the square of functors

$$\begin{array}{ccc} \mathbb{E}_1 & \xrightarrow{F_1} & \mathbb{E}_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{B}_1 & \xrightarrow[F_0]{} & \mathbb{B}_2 \end{array}$$

commute and such that p_1-cartesian arrows in $\mathbb{E}_1$ are sent by F_1 to p_2-cartesian arrows in $\mathbb{E}_2$. We will often say cartesian functor to mean just such a functor F_1 when the functor F_0 between base categories is an identity functor.

The following definition contains two important variations on the notion of fibration.

Definition 1.1.4. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration. A *cleavage* for p is a choice of cartesian lift for every $X \in \mathbb{E}$ and $f : B \to p(X)$. The functor p is a *cloven fibration* if it is equipped with a cleavage. A cleavage is a *splitting* if the chosen cartesian lift of an identity arrow is again an identity, and the chosen cartesian lift of a composite of arrows is the composite of the chosen cartesian lifts of the two individuals. A *split fibration* is a fibration equipped with a splitting.

Definition 1.1.5. A *morphism of cloven fibrations* between cloven fibrations is a cartesian functor which preserves the cleavage. A *morphism of split fibrations* between split fibrations is the same as a morphism of cloven fibrations.

Example 1.1.6. Let $\mathbb{B}$ be a category equipped with the structure of chosen finite products. For an object $I \in \mathbb{B}$, the *simple slice category* $\mathbb{B}_I$ is the co-Kleisli category of the comonad $I \times (-)$. The simple slice categories assemble into the *simple slice fibration* $P_{\mathbb{B}} : \mathbb{B}_{(-)} \to \mathbb{B}$ whose fibre category over I is $\mathbb{B}_I$. The objects of $\mathbb{B}_{(-)}$ are pairs (I, X), and an arrow $(f, F) : (I, X) \to (J, Y)$ is given by an arrow $f : I \to J$ together with an arrow $F : I \times X \to Y$. Given $f : I \to J$ in $\mathbb{B}$ and (J, X) in $\mathbb{B}_{(J)}$, the arrow $(I, X) \to (J, X)$ given by the arrow f paired with the product projection $I \times X \to X$ is a cartesian lift of f with codomain (J, X). This choice of lifts exhibits $P_{\mathbb{B}}$ as a split fibration. It is possible to reformulate the definition so as to avoid reliance on *chosen* finite products in $\mathbb{B}$, but we omit the details.

Remark 1.1.7. The key intuition behind the notion of fibration is that it provides a notion of 'family of categories indexed by a category'. However, this is really only true of *cloven* fibrations. Assuming a sufficiently strong choice principle, every fibration has a cleavage. However, we may avoid limiting ourselves to cloven fibrations by making finitely many ad hoc choices of cartesian lifts for a given purpose. We will omit the details when the translation from the language of cloven fibrations to that of arbitrary fibrations is easy to give.

Notation 1.1.8. When $p : \mathbb{E} \to \mathbb{B}$ is a cloven fibration and $f : B \to A$ is a morphism in $\mathbb{B}$ and X an object of $\mathbb{E}$ with $p(X) = A$, we write $f^X : Y \to X$ for the chosen cartesian lift of f with codomain X. We will sometimes write f^X even when p is not a cloven fibration, to mean any choice of cartesian lift held constant throughout a given calculation.

Definition 1.1.9. Let $p : \mathbb{E} \to \mathbb{B}$. A *p-vertical* (or just *vertical*) morphism is a morphism $\phi : X \to Y$ in $\mathbb{E}$ with $p(\phi)$ an identity morphism. For an object $A \in \mathbb{B}$, the *fibre category* of p over A is the subcategory $\mathbb{E}(A)$ of $\mathbb{E}$ on those objects X with $p(X) = A$ together with all vertical morphisms between them.

Definition 1.1.10. Let $p : \mathbb{E} \to \mathbb{B}$ be a cloven fibration and let $f : B \to A$ be a morphism in $\mathbb{B}$. Then *reindexing along* f is a functor $f^* : \mathbb{E}(A) \to \mathbb{E}(B)$ which sends an object $X \in \mathbb{E}(A)$ to $\operatorname{dom} f^X$. The action of f^* on morphisms sends

$\chi : X \to Y$ to the unique factorization $f^*(\chi)$ of $\chi \circ f^X$ through f^Y.

$$\begin{array}{ccc} f^*X & \xrightarrow{f^X} & X \\ {\scriptstyle f^*(\chi)}\ \vdots & & \downarrow\ {\scriptstyle \chi} \\ f^*Y & \xrightarrow[f^Y]{} & Y \end{array}$$

The fibre categories and reindexing functors turn a cloven fibration p into a pseudofunctor

$$\mathbb{B}^{\mathrm{op}} \to \mathsf{Cat},$$

or in the case of a split fibration, a strict 2-functor. In fact, the notion of cloven fibration is essentially equivalent to the notion of pseudofunctor into Cat. Thus, pseudofunctors form an alternative to fibrations as a notion of indexed category, though we will not be concerned with pseudofunctors here.

We will still refer to the reindexing functors f^* even when p is not given as a cloven fibration. One can either take the view that we can always cleave any fibration using the axiom of choice, or as in Remark 1.1.7 we can understand that we are writing a shorthand for an elementary argument which really only requires finitely many choices of cartesian lift when written out.

The following is a useful construction on fibrations which we will use extensively in Chapters 4 and 7. It transforms a fibration into another whose fibre categories are the opposite categories of the fibre categories we started with.

Definition 1.1.11. For a fibration $p : \mathbb{E} \to \mathbb{B}$ its *opposite fibration* $p^{\mathrm{op}} : \mathbb{E}^{p,\mathrm{op}} \to \mathbb{B}$ is defined as follows. The category $\mathbb{E}^{p,\mathrm{op}}$ is the category whose objects are those of $\mathbb{E}$ and whose arrows $X \to Y$ are triples (Z, F, ϕ) where $Z \in \mathbb{E}$, $F : Z \to Y$ is a p-cartesian arrow in $\mathbb{E}$, and $\phi : Z \to X$. The triples are taken up to equivalence: (Z, F, ϕ) and (Z', F', ϕ') are equivalent as arrows $X \to Y$ if there exists a vertical isomorphism $\psi : Z \to Z'$ with $F' \circ \psi = F$ and $\phi' \circ \psi = \phi$. We omit the details of composition, which are obvious. It is easy to check that this is indeed a fibration, where the cartesian arrows are precisely those representable by a triple (Z, F, ϕ) where ϕ is an isomorphism.

An important advantage of the fibrational approach to 'indexed categories' over the pseudofunctorial approach is the relative ease with which we can handle the notion of 'family of fibrations indexed by a category'. This will allow us to speak about not just types definable in a context, but also the notion of dependent context in some context and types in that dependent context. Here dependent context means what it does in [15].

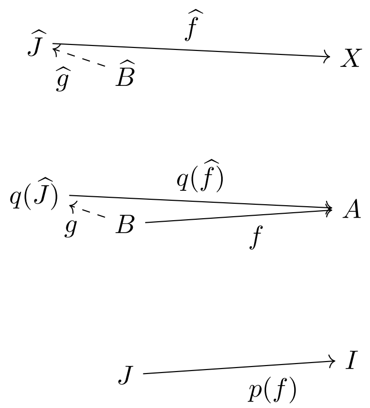
The following is Theorem 4.1 in the set of lecture notes [40]. See also Theorem 4.16 in [19].

Proposition 1.1.12. *Suppose we have functors $p : \mathbb{C} \to \mathbb{B}$ and $q : \mathbb{D} \to \mathbb{C}$, and suppose that p is a fibration. Then q is a fibration if and only if*

- *the composite $p \circ q$ is a fibration;*
- *q is a cartesian functor from $p \circ q$ to p;*
- *each of the restrictions $q_I : \mathbb{D}(I) \to \mathbb{C}(I)$ is a fibration; and*
- *q_I-cartesian arrows are sent to q_J-cartesian arrows by $p \circ q$-reindexing along any map $f : J \to I$ in $\mathbb{C}$.*

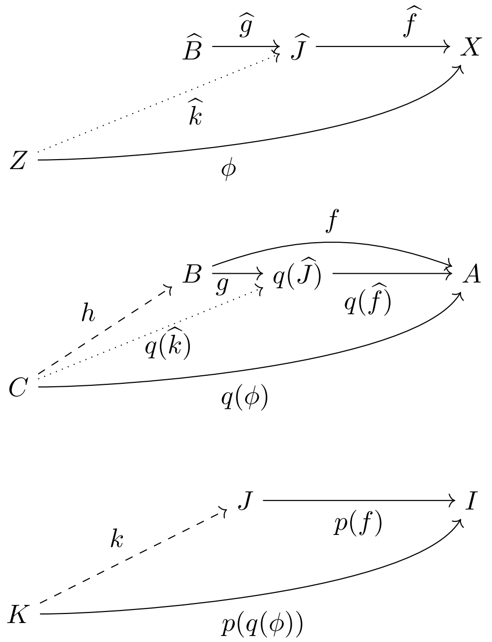
Proof. Suppose that q is a fibration. It is easy to check that fibrations are closed under composition, with cartesian lifts in the composite being given by successive cartesian lifts in the factors, hence $p \circ q$ is a fibration and q is a cartesian functor. Moreover, since fibrations are stable under change of base, each of the restrictions $q_I : \mathbb{D}(I) \to \mathbb{C}(I)$, which is the change of base of q along the inclusion $\mathbb{C}(I) \hookrightarrow \mathbb{C}$, is a fibration. Finally, q_I-cartesian arrows are just q-cartesian arrows in $\mathbb{D}(I)$, and q-cartesian arrows are preserved by $p \circ q$-reindexing (which is really just q-reindexing along p-cartesian arrows).

Now let us prove the converse. Suppose we have an object $X \in \mathbb{D}$ and an arrow $f : B \to A = q(X)$ in $\mathbb{C}$ of which we wish to find a q-cartesian lift. We construct a candidate by first taking a $p \circ q$-cartesian lift $\widehat{f} : \widehat{J} \to X$ of $p(f) : J = p(B) \to I = p(A)$, and then taking a q_J-cartesian lift $\widehat{g} : \widehat{B} \to \widehat{J}$ of the induced factorization $g : B \to q(\widehat{J})$, using that fact that $q(\widehat{f})$ is q-cartesian (as q is a cartesian functor).



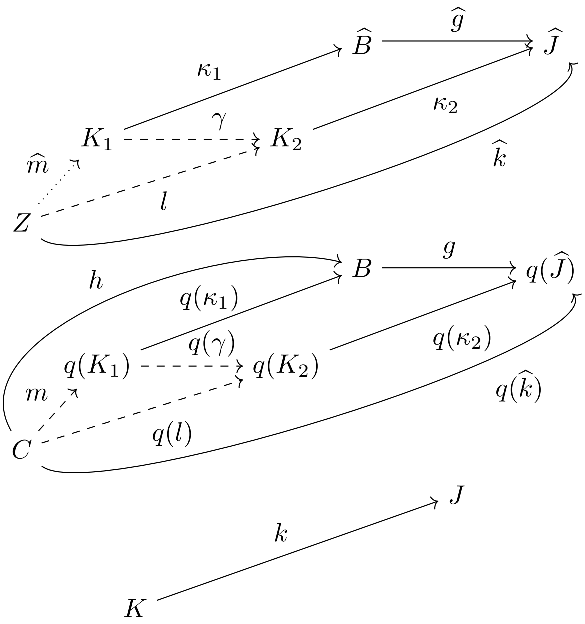
It remains to show that $\widehat{f} \circ \widehat{g}$ is indeed q-cartesian (over f). Suppose we have some map $\phi : Z \to X$ in $\mathbb{D}$ for which $q(\phi) : C = q(Z) \to A$ factorizes through f

via $h : C \to B$. Then since $\widehat{f}$ is $p \circ q$-cartesian, there is a map $\widehat{k} : Z \to \widehat{J}$ lifting $k = p(h) : K = p(C) \to J$. Moreover, $q(\widehat{k}) : C \to q(\widehat{J}) = g \circ h$, since both maps agree after composition with the p-cartesian arrow $q(\hat{f})$.



So we have reduced the problem to finding a factorization over h of $\widehat{k}$ through $\widehat{g}$. To do so, we choose $p \circ q$-cartesian lifts $\kappa_1 : \widehat{K}_1 \to \widehat{B}$ and $\kappa_2 : \widehat{K}_2 \to \widehat{J}$ of k and observe that the induced q-cartesian map $\gamma : K_1 \to K_2$, and that $\widehat{k}$ factorizes through κ_2 via some map $l : Z \to K_2$, say. Now $q(\kappa_1)$ and $q(\kappa_2)$ are p-cartesian, hence: $q(\gamma)$ is the p-reindexing along k of g, and h factorizes through $q(\kappa_1)$ via some map $m : X \to q(K_1)$, say, and $q(\gamma) \circ m = q(l)$. Thus we use the q-cartesianness of γ to lift m to a factorization $\widehat{m} : Z \to K_1$ over m

of l through γ. Hence $\kappa_1 \circ \widehat{m}$ is the required factorization of $\widehat{k}$ through $\widehat{g}$.



□

In the setting above, with two composable fibrations $p : \mathbb{C} \to \mathbb{B}$ and $q : \mathbb{D} \to \mathbb{C}$, we may think of the category $\mathbb{B}$ as having actions on both the categories $\mathbb{C}$ and $\mathbb{D}$, the category $\mathbb{C}$ as having an action on $\mathbb{D}$, and moreover the action of $\mathbb{C}$ on $\mathbb{D}$ being compatible with the two actions of $\mathbb{B}$. For instance, given $f : J \to I$ in $\mathbb{B}$, if $A \in \mathbb{C}(I)$ and $X \in \mathbb{D}(A)$, then we have an object $f^*(X)$ in $\mathbb{D}(f^*(A))$, where $f^*(A) \in \mathbb{C}(J)$. If now we have $F : B \to A$ in the fibre category $\mathbb{C}(I)$, then we can act on X to get $F^*(X) \in \mathbb{D}(B)$. If we then act by f, we get an object $f^*(F^*(X)) \in \mathbb{D}(f^*(B))$, which is canonically isomorphic to $(f^*(F))^*(f^*(X))$.

In the situation where both q and p are split fibrations, then the reindexing functors can be chosen in such a way that they are strictly functorial in f and F, and such that the connecting isomorphism is an identity.

Definition 1.1.13. A *fibred fibration* is a composable pair of (Grothendieck) fibrations. Given two fibred fibrations $\mathbb{E}_2 \xrightarrow{p_1} \mathbb{E}_1 \xrightarrow{p_0} \mathbb{E}_0$ and $\mathbb{F}_2 \xrightarrow{q_1} \mathbb{F}_1 \xrightarrow{q_0} \mathbb{F}_0$, a *morphism of fibred fibrations* is a triple of functors $f_i : \mathbb{E}_i \to \mathbb{F}_i$ $(i = 0, 1, 2)$ satisfying $f_i \circ p_i = q_i \circ f_{i+1}$ for $i = 0, 1$ such that (f_0, f_1) and (f_1, f_2) are morphisms of fibrations $p_0 \to q_0$ and $p_1 \to q_1$ respectively.

We conclude this section on fibrations with some review of completeness properties, which will be important in formulating the definition of type constructors in models of type theory.

Definition 1.1.14. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration and let $\mathcal{D}$ be a class of finite categories. Then p is said to have *fibred $\mathcal{D}$-limits* if each of the fibre categories has limits of shape $\mathbb{A}$ for each $\mathbb{A} \in \mathcal{D}$ and moreover, for each $f : I \to J$ in $\mathbb{B}$, the reindexing functor $f^* : \mathbb{E}(J) \to \mathbb{E}(I)$ preserves these limits.

Dually, p has *fibred $\mathcal{D}$-colimits* if each of the fibre categories has colimits of shape $\mathbb{A}$ for $\mathbb{A} \in \mathcal{D}$ which are preserved by reindexing, or, equivalently, if p^{op} has fibred $\mathcal{D}$-limits.

We will make use of the following folklore result.

Proposition 1.1.15. *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration over a category $\mathbb{B}$ with finite products. Then p has fibred finite products if and only if $\mathbb{E}$ has finite products and p preserves them.*

The following definition abstracts the idea of taking the product of some indexed family of objects. In this situation, the indexing objects come from the base of the fibration.

Definition 1.1.16. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, where $\mathbb{B}$ has finite products. Then p has *simple products* if for any objects $I, J \in \mathbb{B}$, the reindexing functor

$$\pi_I^* : \mathbb{E}(I) \to \mathbb{E}(I \times J)$$

admits a right adjoint

$$\Pi_J : \mathbb{E}(I \times J) \to \mathbb{E}(I)$$

naturally in I. The naturality requirement, known as the *Beck-Chevalley condition*, means that, for any morphism $f : K \to I$ in $\mathbb{B}$, the canonical natural transformation $f^* \Pi_J \Rightarrow \Pi_J \circ (f \times 1_J)^*$ is an isomorphism.

The reader should take note that, while π_I is the projection from $I \times J$ *onto* I, the right adjoint to π_I^* is Π_J because it is a J*-indexed* product.

Remark 1.1.17. A more transparent way to state the Beck-Chevalley condition is in terms of the counit of the adjunction. Recall that a $\pi_I^* : \mathbb{E}(I) \to \mathbb{E}(I \times J)$ admits a right adjoint if and only if every object $X \in \mathbb{E}(I \times J)$ admits a *coreflection along* π_I^*, i.e. an object $PX \in \mathbb{E}(I)$ together with a map $\epsilon_X : \pi_I^*(PX) \to X$ which is universal. Universality means that for any $Y \in \mathbb{E}(I)$ and $\phi : \pi_I^*(Y) \to X$ there exists a unique $\psi : Y \to PX$ such that $\phi = \epsilon_X \circ \pi_I^*(\psi)$. The Beck-Chevalley condition states that for any $f : K \to I$ and $X \in \mathbb{E}(I \times J)$, the object $f^*(PX) \in \mathbb{E}(K)$ together with the map

$$\pi_I^*(f^*(PX)) \cong (f \times 1_J)^* \pi_I^*(PX) \xrightarrow{(f \times 1_J)^*(\epsilon_X)} (f \times 1_J)^*(X)$$

is a coreflection of $(f \times 1_J)^*(X)$ along $\pi_K^* : \mathbb{E}(K) \to \mathbb{E}(I \times K)$.

Dual to simple products we have simple sums, which abstract the idea of taking the coproduct of some indexed family of objects.

Definition 1.1.18. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration, where $\mathbb{B}$ has finite products. Then p has *simple sums* if for any objects $I, J \in \mathbb{B}$, the reindexing functor

$$\pi_I^* : \mathbb{E}(I) \to \mathbb{E}(I \times J)$$

admits a left adjoint

$$\Sigma_J : \mathbb{E}(I \times J) \to \mathbb{E}(I)$$

naturally in I (satisfying the Beck-Chevalley condition).

Similarly to the situation for simple products, the Beck-Chevalley condition for simple sums can be formulated in terms of a canonical natural transformation $\Sigma_J(f \times 1_J)^* \Rightarrow f^* \Sigma_J$ or, equally, in terms of *reflections along* π_I^*.

Remark 1.1.19. The existence of a left adjoint $\Sigma_J : \mathbb{E}(I \times J) \to \mathbb{E}(I)$ is equivalent to the existence of, for each $X \in \mathbb{E}(I \times J)$, a cocartesian arrow $X \to \Sigma_J X$ lying over the product projection $\pi_I : I \times J \to I$. Given such a left adjoint, the cocartesian arrow can be recovered as the composite

$$X \xrightarrow{\epsilon_X} \pi_I^*(\Sigma_J X) \xrightarrow{\pi_I^{\Sigma_J X}} \Sigma_J X$$

where $\pi_I^{\Sigma_J X}$ is a cartesian lift of $\pi_I : I \times J \to I$ with codomain $\Sigma_J X$ and ϵ_X is the counit of the adjunction $\Sigma_J \dashv \pi_I^*$. In this reformulation, the Beck-Chevalley condition corresponds to the stability of such cocartesian arrows under reindexing.

1.2 The semantics of type theory

Let us set out the notions of 'model of type theory' with which we will be concerned. The first of these appears in [41]. The idea is that the domain of a display map $f : X \twoheadrightarrow A$ is thought of as disjoint union $\coprod_{a \in A} X_a$ of a family of types $(X_a \mid a \in A)$ indexed by the codomain of that display map.

Definition 1.2.1. A *display map category* is a category $\mathbb{B}$ together with a class $\mathcal{F}$ of morphisms, called *display maps*, containing all isomorphisms, such that pullbacks of display maps along arbitrary maps exist and are again display maps. A display map category $(\mathbb{B}, \mathcal{F})$ is *well-rooted* if it has a terminal object $\top$ and for every object X the unique morphism $!_X : X \to \top$ is a display map.

We are typically only interested in well-rooted display map categories, but it is useful to be able to talk about more general ones.

Example 1.2.2. Let $\mathbb{B}$ be a category with all finite products. Then $\mathbb{B}$ is a well-rooted display map category when equipped with the class $\mathcal{F}$ defined by $f \in \mathcal{F}$ precisely when f is (isomorphic to) a product projection.

Observe that the pullback condition on $\mathcal{F}$ is equivalent to the following, which should be compared with 1.1.2: the full subcategory $\mathcal{F}$ of $\mathbb{B}^{\to}$ whose objects are the display maps is a subfibration of $\mathrm{cod} : \mathbb{B}^{\to} \to \mathbb{B}$. The following definition, which is found in [23], may be seen as taking this alternative view as primary.

Definition 1.2.3. A *comprehension category* consists of a *base* category $\mathbb{B}$ and a fibration $p : \mathbb{E} \to \mathbb{B}$ over $\mathbb{B}$ equipped with a functor $\chi : \mathbb{E} \to \mathbb{B}^{\to}$ (a *comprehension*) such that $\mathrm{cod} \circ \chi = p$ and χ sends p-cartesian arrows to pullback squares.

Since, as stated in 1.1.2, pullback squares are precisely the cod-cartesian arrows in $\mathbb{B}^{\to}$, and the notion of cartesian functor makes sense no matter whether the target functor is a fibration or not, we may abridge this definition to: 'a fibration $p : \mathbb{E} \to \mathbb{B}$ together with a cartesian functor $\chi : p \to \mathrm{cod}_{\mathbb{B}}$'.

Notation 1.2.4. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then for $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, we will often use the notation $\{A\}$ as shorthand for $\mathrm{dom}\, \chi(A)$. We use double-headed arrow notation as in $\{A\} \twoheadrightarrow I$ to emphasize that a displayed arrow is in the image of χ. We will also use the notation

$$\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$$

for the functor sending $A \in \mathbb{E}(I)$ to $\chi(A) : \{A\} \twoheadrightarrow I$. This functor is the factorization of the restriction $\chi|_{\mathbb{E}(I)} : \mathbb{E}(I) \to \mathbb{B}^{\to}$ through the inclusion $\mathbb{B}/I \hookrightarrow \mathbb{B}^{\to}$.

Example 1.2.5. We can give a version of Example 1.2.2 in terms of comprehension categories. Let $\mathbb{B}$ be a category with finite products. Then let $p : \mathbb{E} \to \mathbb{B}$ be the simple slice fibration of Example 1.1.6. We can equip p with a comprehension which sends the pair (I, X) to the product projection $I \times X \to I$.

Remark 1.2.6. It is possible to discuss well-rootedness for comprehension categories, but we shall not do this. We would like to say that well-rootedness makes no essential difference to the type-theoretic content of a display map category or a comprehension category, but this is not quite true. A more serious defect in its own right, which would resolve this issue, is the absence from Definitions 1.2.1 and 1.2.3 of what we call *dependent sums*, (see 1.6.2 and 1.6.1). This is because dependent sums are not just a type-forming operation, but semantically they play the rôle of context extension, and thus they allow us to consider 'type theory over some context'. We remark further on this for display map categories in 1.6.3 and formalize this idea for comprehension categories in Theorem 5.6.4.

Definition 1.2.7. A *split comprehension category* is a comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ for which p is a split fibration.

The significance of Definition 1.2.7 is primarily for using the syntax of type theory as a language for reasoning about comprehension categories. It is not directly possible to do this for general comprehension categories, because reindexing in the fibration p is in general only pseudofunctorial in the base whereas its syntactic counterpart, substitution, is a strictly associative operation. Hence we can only reason directly about comprehension categories where the reindexing is strictly functorial in the base, i.e. about split comprehension categories. If one wishes to reason about more general comprehension categories, one would apply a coherence theorem such as the one in [32] to find an equivalent split comprehension category.

Example 1.2.8. The comprehension category of Example 1.2.5 is a split comprehension category.

While we will not formally prove any sort of equivalence between display map categories and comprehension categories here, it is worth noting at least how to convert between the two. It is easy to check that the following constructions are well-defined.

Definition 1.2.9. Let $(\mathbb{B}, \mathcal{F})$ be a display map category. Then $\mathcal{F}$ induces a comprehension category structure on $\mathbb{B}$ whose underlying fibration is $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$ and whose comprehension is the inclusion $\mathcal{F} \subseteq \mathbb{B}^{\to}$.

Definition 1.2.10. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then p and χ induce a display map category structure on $\mathbb{B}$ whose display maps are the objects of the essential image of $\chi : \mathbb{E} \to \mathbb{B}^{\to}$. Equivalently, the display maps with codomain I are all those of the form $\chi(A) \circ s$, where $A \in \mathbb{E}(I)$ and s is an isomorphism.

Notation 1.2.11. For a display map category $(\mathbb{B}, \mathcal{F})$, we denote by $\mathcal{F}/I$ the full subcategory of the slice $\mathbb{B}/I$ whose objects are $\mathcal{F}$-maps (with codomain I). Observe that these restricted slices are the fibre categories of the underlying fibration of the induced comprehension category. In particular, every morphism $f : J \to I$ induces a pullback functor

$$f^* : \mathcal{F}/I \to \mathcal{F}/J$$

between restricted slice categories.

We propose the following notions of *morphism* between comprehension categories.

Definition 1.2.12. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ and $(q : \mathbb{F} \to \mathbb{C}, \psi)$ be comprehension categories. Then a *lax morphism of comprehension categories*

$$(F_0, F_1, \theta) : (p, \chi) \to (q, \psi)$$

consists of a morphism of fibrations $(F_0, F_1) : p \to q$ together with a natural transformation $\theta : \psi \circ F_1 \to (F_0)^{\to} \circ \chi$ with cod-vertical components.

Let us spell this out. A lax morphism includes a functor $F_0 : \mathbb{B} \to \mathbb{C}$ and a functor $F_1 : \mathbb{E} \to \mathbb{F}$ making the square

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{F_1} & \mathbb{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{B} & \xrightarrow[F_0]{} & \mathbb{C} \end{array}$$

commute, and moreover F_1 preserves cartesian arrows. For any $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, we have two ways to map the comprehension $\chi(A) : \{A\} \twoheadrightarrow I$ into $\mathbb{C}$. Firstly, we can apply F_0 directly to this comprehension arrow. Secondly, we can apply F_1 to A to get a type $F_1(A) \in \mathbb{F}(F_0(I))$ and apply the comprehension ψ in the target comprehension category. The final piece of data in the lax morphism is a comparison arrow θ_A in $\mathbb{C}$ making the diagram

$$\begin{array}{ccc} F_0(\{A\}) & \xrightarrow{\theta_A} & \{F_1(A)\} \\ & {\scriptstyle F_0(\chi(A))}\searrow \quad \swarrow{\scriptstyle \psi(F_1(A))} & \\ & F_0(I) & \end{array}$$

in $\mathbb{C}$ commute. Moreover, θ_A is natural in A.

Definition 1.2.13. A *strong morphism of comprehension categories* is a lax morphism of comprehension categories for which each of the arrows θ_A is an isomorphism. A *strict morphism of (split) comprehension categories* is a lax morphism of comprehension categories for which each of the arrows θ_A is an identity.

In 7.5.10 we will consider *split* strict morphisms, which are the natural notion of homomorphism for split comprehension categories considered as models of an essentially algebraic theory (for instance, see [32]). The (non-split) notion of strict morphism we give in 1.2.13 is the same as the notion of 'morphism' for non-split comprehension categories given in [32]. Our notion of strong morphism was suggested in that same article.

It is natural also to consider the dual notion to lax morphism.

Definition 1.2.14. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ and $(q : \mathbb{F} \to \mathbb{C}, \psi)$ be comprehension categories. Then a *colax morphism of comprehension categories*

$$(F_0, F_1, \zeta) : (p, \chi) \to (q, \psi)$$

consists of a morphism of fibrations $(F_0, F_1) : p \to q$ together with a natural transformation $\zeta : (F_0)^{\to} \circ \chi \to \psi \circ F_1$ with cod-vertical components.

The only difference to the notion of lax morphism is that the natural transformation ζ goes in the opposite direction to θ, so that we get a family of commutative triangles

$$\begin{array}{ccc} F_0(\{A\}) & \xleftarrow{\zeta_A} & \{F_1(A)\} \\ & {\scriptstyle F_0(\chi(A))} \searrow \quad \swarrow {\scriptstyle \psi(F_1(A))} & \\ & F_0(I) & \end{array}$$

in $\mathbb{C}$. Clearly we could just as well define strong and strict morphisms of comprehension categories in terms of colax morphisms, and it makes sense to ask whether a particular colax morphism is indeed a strong one. We will see some examples of morphisms of comprehension categories in chapters 5 and 6.

Remark 1.2.15. There is an evident notion of 2-cell between lax morphisms of comprehension categories, which gives us a 2-category of comprehension categories. We can instead use colax morphisms. There are also 2-categories given by restricting to a fixed base category.

Full comprehension categories are slightly closer to display map categories than general comprehension categories and typically we will mostly be concerned only with these.

Definition 1.2.16. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then (p, χ) is *full* if the functor $\chi : \mathbb{E} \to \mathbb{B}^{\to}$ is full and faithful.

A full comprehension category may be thought of as a display map category in which every display map is equipped with a set of 'structures' and for each display map with structure and morphism with the same codomain there is a specified pullback square together with a display map structure on the pullback of the display map. This is the idea behind the notion of 'cloven type-theoretic fibration categories' in [38].

The following is related to Lemma 2.5 in [23].

Lemma 1.2.17. *Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then (p, χ) is full if and only if for each $I \in \mathbb{B}$ the functor*

$$\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$$

is full and faithful.

Proof. Since χ_I is just a restriction of χ, the 'only if' part is obvious. For the converse, observe that for $f : I \to J$ and $X \in \mathbb{E}(I)$ and $Y \in \mathbb{E}(J)$, the set of maps $X \to Y$ in $\mathbb{E}$ lying over f is in bijection with the set of maps $X \to f^*(Y)$ in $\mathbb{E}(I)$, which via χ_I is in bijection with maps $\{X\} \to \{f^*(Y)\}$ over I in $\mathbb{B}$. Since the cartesian arrow $f^Y : f^*(Y) \to Y$ is sent to a pullback square in $\mathbb{B}$, this latter set is in bijection with the maps $\{X\} \to \{Y\}$ making the square

$$\begin{array}{ccc} \{X\} & \dashrightarrow & \{Y\} \\ {\scriptstyle \chi(X)}\downarrow & & \downarrow{\scriptstyle \chi(Y)} \\ I & \xrightarrow[f]{} & J \end{array}$$

commute. As the composite bijection is just the action of the functor χ, the result follows. $\square$

Proposition 1.2.18. *The comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ associated to a display map category $(\mathbb{B}, \mathcal{F})$ is full.*

Proof. Trivial since the fibre category $\mathbb{E}(I)$ is by definition $\mathcal{F}/I$. $\square$

1.3 Finite product types

Comprehension categories contain just the structure required to model the most basic notions of type theory: context, dependent type, term in context. Interesting type theories contain certain basic types and type constructors for building new types out of old. We spend most of the rest of this chapter detailing what it means for a model to admit various type constructors, starting with the most basic.

Definition 1.3.1. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then (p, χ) has *unit types* if the fibration p has fibred terminal objects and χ sends fibred terminal objects to isomorphisms. The latter condition says equivalently that each functor $\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$ preserves terminal objects. Also, (p, χ) has *binary product types* if the fibration P has fibred binary products and χ sends each product diagram

$$\begin{array}{ccc} & A \times_I B & \\ \swarrow & & \searrow \\ A & & B \end{array}$$

in $\mathbb{E}(I)$ to a pullback square

$$\begin{array}{ccccc} & & \{A \times_I B\} & & \\ & \swarrow & & \searrow & \\ \{A\} & & & & \{B\} \\ & \chi(A) \searrow & & \swarrow \chi(B) & \\ & & I & & \end{array}$$

in $\mathbb{B}$. Equivalent to this latter condition is the statement that each functor $\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$ preserves binary products. We say that (p, χ) has *finite product types* if it has both unit types and binary product types.

In view of the following, we need not define unit types for display map categories: every display map category already has them.

Proposition 1.3.2. *Let $(\mathbb{B}, \mathcal{F})$ be a display map category. Then its induced comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ has unit types*

Proof. For each $I \in \mathbb{B}$, the identity morphism $1_I : I \to I$ is a fibred terminal object in $\mathbb{E}(I)$, whose comprehension, 1_I, is an isomorphism. $\square$

It is almost the case that every display map category has binary product types too. All our examples have dependent sums (see 1.6.2), and we will show in 1.6.6 that binary products are a special case of dependent sums.

1.4 Fullness and Ehrhard comprehension

As well as the notion of fullness, there is another specialization of the notion comprehension category to make it align more closely with type theory. We briefly explore its connection to fullness and having unit types.

Definition 1.4.1. Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration. We say that p has *Ehrhard comprehension* if p has a full and faithful right adjoint $\top : \mathbb{B} \to \mathbb{E}$ which in turn has a right adjoint $\mathcal{P}_0 : \mathbb{E} \to \mathbb{B}$.

What we are calling fibrations with Ehrhard comprehension were called *D-categories* by Ehrhard in [13] and *comprehension categories with unit* by Jacobs in [23]. Note that having Ehrhard comprehension is merely a property of the fibration p, whereas being a comprehension category is a structure.

Proposition 1.4.2. *If a fibration $p : \mathbb{E} \to \mathbb{B}$ has Ehrhard comprehension, then we may choose $\top : \mathbb{B} \to \mathbb{E}$ such that $p \circ \top = 1_{\mathbb{B}}$ (i.e. $\top$ is a section of p) and the counit of the adjunction $p \vdash \top$ is an identity morphism. It follows that, for each $I \in \mathbb{B}$, the object $\top_I$ is a fibred terminal object in $\mathbb{E}(I)$ and that, for each $A \in \mathbb{E}(I)$, the unit at A is the unique vertical morphism $A \to \top_{p(A)} = \top_I$.*

Proof. This is Proposition 7 in [13]. □

Thus, we will assume that the right adjoint $\top : \mathbb{B} \to \mathbb{E}$ is indeed a section of p which picks out the fibred terminal objects of p. Since $\top$ is full and faithful, not only is the counit $\top \circ p \to 1$ an isomorphism, but so is the unit $1 \to \mathcal{P}_0 \circ \top$.

Proposition 1.4.3. *A fibration $p : \mathbb{E} \to \mathbb{B}$ with Ehrhard comprehension gives rise to a comprehension category structure on p which models unit types and for which $\chi : \mathbb{E} \to \mathbb{B}^{\to}$ sends a type $A \in \mathbb{E}(I)$ to the result of applying $\mathcal{P}_0$ to the unit $A \to \top_{p(A)}$ composed with the inverse of the unit $I \to \mathcal{P}_0(\top_I)$.*

Proof. Firstly, we must check that χ preserves cartesian arrows. Let $F : B \to A$ be a cartesian arrow lying over $f : J \to I$. In the diagram

$$\begin{array}{ccc} B & \xrightarrow{F} & A \\ {\scriptstyle \eta_B}\downarrow & & \downarrow{\scriptstyle \eta_A} \\ \top_J & \xrightarrow[\top_f]{} & \top_I \end{array}$$

the horizontally displayed arrows are p-cartesian and the vertically displayed arrows are p-vertical. Hence the square is a pullback in $\mathbb{E}$. Since $\mathcal{P}_0$ is a right adjoint it preserves this pullback square. Since the components of the unit $K \to \mathcal{P}_0(\top_K)$ are isomorphisms, every naturality square (of its inverse) is a pullback square. This proves that χ preserves cartesian arrows.

It remains to check that χ sends the fibred terminal objects to isomorphisms. This is immediate from the fact that, for any $I \in \mathbb{B}$, the map $\top_I \to \top_{p(\top_I)}$ which is the component of the unit at $\top_I$ is an isomorphism. $\square$

The following proposition helps to explain the connection between Ehrhard comprehension and the usual notion of comprehension.

Proposition 1.4.4. *Let $(p : \mathbb{E} \to \mathbb{B}, \top, \mathcal{P}_0)$ be a fibration with Ehrhard comprehension, and $\chi : \mathbb{E} \to \mathbb{B}^{\to}$ the induced comprehension. Let $f : I \to J$ be a morphism in $\mathbb{B}$ and let $A \in \mathbb{E}(J)$. Then the bijection of homsets*

$$\mathbb{E}(\top_I, A) \cong \mathbb{B}(I, \mathcal{P}_0(A))$$

restricts to a bijection

$$\mathbb{E}(I)(\top_I, f^*A) \cong \mathbb{B}/J(f, \chi(A))$$

Proof. For $f : I \to J$, if $F : \top_I \to A$ lies over f then it suffices to show that its corresponding map $G : I \to \mathcal{P}_0(A)$ satisfies $\chi(A) \circ G = f$. But $\chi(A)$ is given by applying $\mathcal{P}_0$ to the unit $A \to \top_J$ and postcomposing with the inverse of the unit $J \to \mathcal{P}_0(\top_J)$, whereas G is given by applying $\mathcal{P}_0$ to F and precomposing with the unit $I \to \mathcal{P}_0(\top_I)$. But F composed with the unit $A \to \top_J$ is the unique arrow $\top_I \to \top_J$ lying over f. The result follows by functoriality of $\mathcal{P}_0$ and naturality of the unit $K \to \mathcal{P}_0(\top_K)$ in K. $\square$

Proposition 1.4.5. *Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category with unit types. Then (p, χ) is induced by an Ehrhard comprehension for p if and only if, for any $f : I \to J$ and $A \in \mathbb{E}(J)$, the operation sending a map $t : \top_I \to A$ lying over f to a map $\widehat{t} : f \to \chi(A)$ in $\mathbb{B}/J$ is a bijection.*

Proof. We have a composite operation

$$\begin{aligned}
\mathbb{E}(\top_I, A) &\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{E}(I)(\top_I, f^*(A)) \\
&\to \{f \in \mathbb{B}(I, J)\} \times \mathbb{B}/I(\chi(\top_I), \chi(f^*(A))) \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{B}/I(1_I, \chi(f^*(A))) \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{B}/J(f, \chi(A)) \\
&\cong \mathbb{B}(I, \{A\})
\end{aligned}$$

which is a bijection by hypothesis. Hence the terminal objects functor has a right adjoint, and so p has Ehrhard comprehension. One checks easily that χ is indeed induced by this Ehrhard comprehension. $\square$

We finish by giving two ways to connect Ehrhard comprehension and fullness. In the first, the cocartesianness condition corresponds to the type-theoretic statement that $\Sigma_{x:A}1 \cong A$ (see 1.6.2).

Proposition 1.4.6. *Let $p : \mathbb{E} \to \mathbb{B}$ be a fibration with Ehrhard comprehension and χ its induced comprehension functor. Then (p, χ) is a full comprehension category if and only if, for any $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, the counit*

$$\top_{\{A\}} \to A,$$

which lies over $\{A\} \to I$, is cocartesian.

Proof. Suppose that (p, χ) is full. Let $A \in \mathbb{E}(I)$. Then, for any $B \in \mathbb{E}(J)$, we have a bijection

$$\begin{aligned}
\mathbb{E}(A, B) &\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{E}(I)(A, f^*(B)) \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{B}/I(\chi(A), \chi(f^*(B))) \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{E}(\{A\})(\top_{\{A\}}, \chi(A)^* f^*(B)), \\
&\cong \mathbb{E}(\top_{\{A\}}, B)
\end{aligned}$$

which one verifies easily is given by precomposing with the counit, as required. Conversely, suppose the counit $\top_{\{A\}} \to A$ is cocartesian. Then we have a bijection

$$\begin{aligned}
\mathbb{E}(A, B) &\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{E}(I)(A, f^*(B)) \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{E}(\{A\})(\top_{\{A\}}, \chi(A)^* f^*(B)), \\
&\cong \{f \in \mathbb{B}(I, J)\} \times \mathbb{B}/I(\chi(A), \chi(f^*(B))) \\
&\cong \mathbb{B}^{\to}(\chi(A), \chi(B))
\end{aligned}$$

which one verifies easily is given by the functor χ, as required. ☐

Proposition 1.4.7. *Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a full comprehension category. Then p has Ehrhard comprehension and χ is the induced comprehension functor if and only if (p, χ) has unit types.*

Proof. The forward direction is Proposition 1.4.3. For the converse, we define $\top : \mathbb{B} \to \mathbb{E}$ to be the functor sending I to the fibred terminal object $\top_I$ in $\mathbb{E}(I)$, which is easily seen to be a full and faithful right adjoint to p. We define

$\mathcal{P}_0 : \mathbb{E} \to \mathbb{B}$ to be $\mathrm{dom} \circ \chi$. The adjunction is given by, for $I \in \mathbb{B}$ and $A \in \mathbb{E}(J)$,

$$\begin{aligned}
\mathbb{B}(I, \mathcal{P}_0(A)) &\cong \{f \in \mathbb{B}(I,J)\} \times \mathbb{B}/J(f, \chi(A)) \\
&\cong \{f \in \mathbb{B}(I,J)\} \times \mathbb{B}/I(1_I, \chi(f^*(A))) \\
&\cong \{f \in \mathbb{B}(I,J)\} \times \mathbb{B}/I(\chi(\top_I), \chi(f^*(A))) \\
&\cong \{f \in \mathbb{B}(I,J)\} \times \mathbb{E}(I)(\top_I, f^*(A)) \\
&\cong \mathbb{E}(\top_I, A).
\end{aligned}$$

One checks easily that this bijection is indeed natural, and that the unit of the adjunction $\top \dashv \mathcal{P}_0$ is the inverse of the comprehension of $\top_I \in \mathbb{E}(I)$. Hence, the reconstructed comprehension functor $\chi' : \mathbb{E} \to \mathbb{B}^{\to}$, which is given on $A \in \mathbb{E}(I)$ by applying $\mathrm{dom} \circ \chi$ to the unit $A \to \top_I$ and then composing with the comprehension of $\top_I \in \mathbb{E}(I)$. This is clearly equal to $\chi(A)$. It is straightforward to check that χ' and χ also agree on morphisms. $\square$

1.5 Dependent products

The infinitary version of the product type is the *dependent product* or Π-*type*. This generalizes the notion of simple product (Definition 1.1.16).

Definition 1.5.1. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a comprehension category. Then (p, χ) has *dependent products* if for every $I \in \mathbb{B}$, $A \in \mathbb{E}(I)$, the reindexing functor

$$\chi(A)^* : \mathbb{E}(I) \to \mathbb{E}(\{A\})$$

admits a right adjoint

$$\Pi_A : \mathbb{E}(\{A\}) \to \mathbb{E}(I).$$

Moreover, these right adjoints satisfy the Beck-Chevalley condition: for every morphism $f : J \to I$ in $\mathbb{B}$, we require that the canonical natural transformation $f^* \Pi_A \Rightarrow \Pi_{f^*(A)} \{f^A\}^*$, which we obtain by considering the pullback square

$$\begin{array}{ccc}
\{f^*(A)\} & \xrightarrow{\{f^A\}} & \{A\} \\
{\scriptstyle \chi(f^*(A))} \big\downarrow & & \big\downarrow {\scriptstyle \chi(A)} \\
J & \xrightarrow[f]{} & I
\end{array}$$

in $\mathbb{B}$, is an isomorphism.

Remark 1.5.2. One can easily generalize Remark 1.1.17 to this situation, to express the Beck-Chevalley condition in terms of the stability under reindexing of coreflections along $\chi(A)^*$.

Let us formulate the definition for display map categories

Definition 1.5.3. A display map category $(\mathbb{B}, \mathcal{F})$ has *dependent products* if for every display map $f : B \twoheadrightarrow A$ the pullback functor

$$f^* : \mathcal{F}/A \to \mathcal{F}/B$$

admits a right adjoint

$$\Pi_f : \mathcal{F}/B \to \mathcal{F}/A.$$

Moreover, these right adjoints satisfy the Beck-Chevalley condition. For every morphism $h : C \to A$ in $\mathbb{B}$ and any pullback square

$$\begin{array}{ccc} D & \xrightarrow{\overline{h}} & B \\ {\scriptstyle \overline{f}} \downarrow & & \downarrow {\scriptstyle f} \\ C & \xrightarrow[h]{} & A \end{array}$$

the canonical transformation $h^* \Pi_f \Rightarrow \Pi_{\overline{f}} \overline{h}^*$ is an isomorphism.

Remark 1.5.4. As in the comprehension category case, the Beck-Chevalley condition can be formulated in terms of coreflections along f^* which are stable under pullback.

Actually, there is another way to formulate the Beck-Chevalley condition in this special case.

Proposition 1.5.5. *A display map category* $(\mathbb{B}, \mathcal{F})$ *has dependent products if and only if for every display map* $f : B \twoheadrightarrow A$*, the reindexing functor*

$$f^* : \mathbb{B}/A \to \mathbb{B}/B$$

between the whole slice categories admits a coreflection of every object in the subcategory $\mathcal{F}/B \hookrightarrow \mathbb{B}/B$ *and moreover each coreflection of such is in the subcategory* $\mathcal{F}/A$ *of* $\mathbb{B}/A$*.*

Proof. The latter condition says that for any display map $g : C \twoheadrightarrow B$, there exists a display map $p : \Pi_f\, g \twoheadrightarrow A$ such that for *arbitrary* maps $w : W \to A$

there is a natural bijection

$$\mathbb{B}/A(w,p) \cong \mathbb{B}/B(f^*(w),g)$$

rather than just for those w which are display maps. Suppose that $(\mathbb{B},\mathcal{F})$ has dependent products, and let $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow B$ be display maps and $w : W \to A$ an arbitrary map into A. Fix some pullback square

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\overline{f}} & W \\ {\scriptstyle \overline{w}}\downarrow & & \downarrow{\scriptstyle w} \\ B & \xrightarrow[f]{} & A \end{array}$$

in $\mathbb{B}$. Writing $p : \Pi_f\, g \twoheadrightarrow A$ for the dependent product of g along f and using the Beck-Chevalley condition to see that it is stable under pullback along w, we get a bijection

$$\mathbb{B}/A(w,p) \cong \mathcal{F}/W(1_w, w^*(p)) \cong \mathcal{F}/\overline{W}(1_{\overline{W}}, \overline{w}^*(g)) \cong \mathbb{B}/B(\overline{w},g)$$

which is easily seen to be natural in w, as required.

Let us prove the converse. Given $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow B$, our candidate for the dependent product is the reflection of g (considered just as a morphism) along $f^* : \mathbb{B}/A \to \mathbb{B}/B$, which we write $p : \Pi_f\, g \twoheadrightarrow A$. It remains to show that this choice of dependent product is stable under pullback. Let $w : W \to A$ be an arbitrary map, and fix some pullback of w and f, as above. Then for any display map $h : D \twoheadrightarrow W$ we fix a pullback square

$$\begin{array}{ccc} \overline{D} & \xrightarrow{\overline{\overline{f}}} & D \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h} \\ \overline{W} & \xrightarrow[\overline{f}]{} & W \end{array}$$

and we obtain a bijection

$$\mathcal{F}/W(h, w^*(p)) \cong \mathbb{B}/A(wh,p) \cong \mathbb{B}/B(\overline{w}\overline{h},g) \cong \mathcal{F}/\overline{W}(\overline{h}, \overline{w}^*(g))$$

which is clearly natural in h, as required. □

We also give the appropriate definition of dependent product for split comprehension categories.

Definition 1.5.6. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a split comprehension category. Then when (p, χ) has *strictly stable dependent products* it is equipped with, for every $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, a right adjoint

$$\Pi_{I,A} : \mathbb{E}(\{A\}) \to \mathbb{E}(I)$$

to the reindexing functor

$$\chi(A)^* : \mathbb{E}(I) \to \mathbb{E}(\{A\})$$

with counit

$$\mathsf{ev}_{I,A,B} : \chi(A)^*(\Pi_{I,A}(B)) \to B.$$

For any $h : J \to I$ in $\mathbb{B}$ the equations

$$\Pi_{J,h^*(A)}(\chi(h^A)^*(B)) = h^*(\Pi_{I,A}(B))$$

and

$$\mathsf{ev}_{J,h^*(A),\chi(h^A)^*(B)} = \chi(h^A)^*(\mathsf{ev}_{I,A,B})$$

hold.

1.6 Dependent sums

Whereas dependent products are modelled by right adjoints to reindexing functors, *dependent sums* (or Σ*-types*) are left adjoints. We can easily define what we would call *weak dependent sums* by analogy to Definitions 1.5.1 and 1.5.3 which generalize Definition 1.1.18, but we normally require a stronger condition. In type theory Σ-types usually have an eliminator where the target type is allowed to be a dependent type over the Σ-type itself. The following is Definition 5.8 in [23].

Definition 1.6.1. A comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ has *dependent sums* if for every $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, the weakening functor

$$\chi(A)^* : \mathbb{E}(I) \to \mathbb{E}(\{A\})$$

has a left adjoint

$$\Sigma_A : \mathbb{E}(\{A\}) \to \mathbb{E}(I)$$

such that, for any $B \in \mathbb{E}(\{A\})$, the canonical map $\{B\} \to \{\Sigma_A B\}$ is an isomorphism. Moreover, this family of left adjoints satisfies the Beck-Chevalley condition.

The canonical map $\{B\} \to \{\Sigma_A B\}$ is the comprehension of the composite

$$B \xrightarrow{\eta_{I,A,B}} \chi(A)^*(\Sigma_A B) \xrightarrow{\chi(A)^{\Sigma_A B}} \Sigma_A B$$

in $\mathbb{E}$, where $\eta_{I,A,B}$ is the unit of the dependent sum adjunction.

Definition 1.6.2. A display map category $(\mathbb{B}, \mathcal{F})$ has *dependent sums* if $\mathcal{F}$ is closed under composition.

Remark 1.6.3. As we remarked in 1.2.6, the notion of dependent sum is a fundamental one as it allows us to take localizations of display map categories. Given a display map category $(\mathbb{B}, \mathcal{F})$ with dependent sums and an object $I \in \mathbb{B}$, the *slice* at I is a display map category whose underlying category is the restricted slice $\mathcal{F}/I$. Using the fact that $\mathcal{F}$ is closed under composition, one checks easily that dom : $\mathcal{F}/I \to \mathbb{B}$ creates pullbacks of morphisms whose underlying $\mathbb{B}$-morphism is in $\mathcal{F}$. Hence the collection of such maps is a class of display maps in $\mathcal{F}/I$, which moreover models dependent sums. The fact that for any map $f : I \to J$ there is a pullback functor $f^* : \mathcal{F}/J \to \mathcal{F}/I$ allows us to view a display map category $(\mathbb{B}, \mathcal{F})$ with dependent sums which is not well-rooted as a family of well-rooted display map categories indexed by $\mathbb{B}$.

Lemma 1.6.4. *Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a full comprehension category. Then (p, χ) has dependent sums if and only if the induced display map category $(\mathbb{B}, \mathcal{F})$ has dependent sums.*

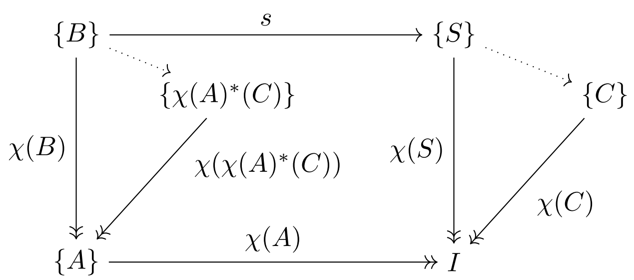
Proof. If (p, χ) has dependent sums then since $\mathcal{F}$ is always closed under composition with isomorphisms, it is easy to see that $\mathcal{F}$ is closed under composition.

Conversely, let $I \in \mathbb{B}$, $A \in \mathbb{E}(I)$ and $B \in \mathbb{E}(\{A\})$. Then the composite $\chi(A)\chi(B)$ is a display map, so by the description in 1.2.10 there is a commuting square

$$\begin{array}{ccc} \{B\} & \xrightarrow{s} & \{S\} \\ {\scriptstyle \chi(B)} \downarrow & & \downarrow {\scriptstyle \chi(S)} \\ \{A\} & \xrightarrow[\chi(A)]{} & I \end{array}$$

where S is some object of $\mathbb{E}(I)$ and s is an isomorphism. We will show that S

is a reflection of B along $\chi(A)^* : \mathbb{E}(I) \to \mathbb{E}(\{A\})$. Suppose $C \in \mathbb{E}(I)$.



Then we have a bijection

$$\begin{aligned}\mathbb{E}(\{A\})(B, \chi(A)^*(C)) &\cong \mathbb{B}/\{A\}(\chi(B), \chi(\chi(A)^*(C))) \\ &\cong \mathbb{B}/I(\chi(A)\chi(B), \chi(C)) \\ &\cong \mathbb{B}/I(\chi(S), \chi(C)) \\ &\cong \mathbb{E}(I)(S, C)\end{aligned}$$

which is clearly natural in C, as required. It is straightforward to check that the canonical map $\{B\} \to \{S\}$ induced by this injection is indeed just s, so in particular it is an isomorphism. Finally, it is clear that S is stable under reindexing as a reflection of B along $\chi(A)$, since the isomorphism $s : \{B\} \to \{S\}$ always pulls back to an isomorphism. $\square$

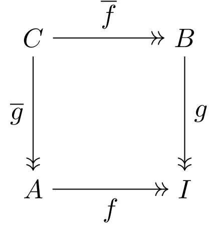
Lemma 1.6.5. *A display map category* $(\mathbb{B}, \mathcal{F})$ *has dependent sums if and only if its induced full comprehension category* $(p : \mathbb{E} \to \mathbb{B}, \chi)$ *has dependent sums.*

Proof. It is easy to check that $\mathcal{F}$ is the class of display maps in $\mathbb{B}$ induced by the comprehension category structure $(p : \mathbb{E} \to \mathbb{B}, \chi)$ on $\mathbb{B}$ which is induced by $\mathcal{F}$. Hence the result follows from Lemma 1.6.4, (p, χ) has dependent sums. $\square$

Dependent sums can be used to construct binary products.

Proposition 1.6.6. *Let* $(\mathbb{B}, \mathcal{F})$ *be a display map category with dependent sums. Then its induced comprehension category* $(\mathbb{B}, \mathcal{F})$ *has binary product types.*

Proof. Let $I \in \mathbb{B}$ and $f : A \twoheadrightarrow I$ and $g : B \twoheadrightarrow I$ be two objects in $\mathbb{E}(I)$. Let us form a pullback square



in $\mathbb{B}$. Then since $\mathcal{F}$ is closed under composition, $g\overline{f} = f\overline{g}$ is in $\mathcal{F}$. Moreover, since $\mathbb{E}(I) = \mathcal{F}/I$, we see that $g\overline{f}$ is a binary product of f and g in $\mathbb{E}(I)$, which is preserved by inclusion $\mathbb{E}(I) = \mathcal{F}/I \hookrightarrow \mathbb{B}^{\to}$. Finally, since pullback squares are stable under pullback, $g\overline{f}$ is reindexing-stable as a product of f and g. □

We finish our treatment of dependent sums by recalling the definition of the strictly stable version. For simplicity we restrict to the full case.

Definition 1.6.7. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a full split comprehension category. When (p, χ) has *strictly stable dependent sums* it is equipped with, for each triple $I \in \mathbb{B}$, $A \in \mathbb{E}(I)$ and $B \in \mathbb{E}(\{A\})$, an object $\Sigma_{I,A}\, B \in \mathbb{E}(I)$ together with an isomorphism

$$\theta_{I,A.B} : \{B\} \to \{\Sigma_{I,A}\, B\}$$

in $\mathbb{B}$ making the diagram

$$\begin{array}{ccc} \{B\} & \xrightarrow{\theta_{I,A,B}} & \{\Sigma_{I,A}\, B\} \\ {\scriptstyle \chi(B)}\big\downarrow & & \big\downarrow{\scriptstyle \chi(\Sigma_{I,A}\, B)} \\ \{A\} & \xrightarrow[\chi(A)]{} & I \end{array}$$

commute. For any $h : J \to I$, the equations

$$h^*(\Sigma_{I,A}\, B) = \Sigma_{J,h^*(A)}\{h^A\}^*(B) \in \mathbb{E}(J)$$
$$h^*(\theta_{I,A,B}) = \theta_{J,h^*(A),\{h^A\}^*(B)} \in \mathbb{B}$$

hold.

1.7 Finite sum types

We can define finite sum types in a way analogous to Definition 1.3.1. Neither display map categories nor comprehension categories have sum types automatically. In fact we will mostly be concerned with some much stronger formulations, hence the qualifier "weak" in our terminology below — even though much weaker versions of finite sums are possible. We give definitions of various notions of finite sum types in display map categories only, based on the notions of finite sum in [46].

Definition 1.7.1. A display map category $(\mathbb{B}, \mathcal{F})$ has *weak finite sum types* if the fibration $\mathcal{F} \to \mathbb{B}$ has fibred finite coproducts, and for each I the inclusion

$\mathcal{F}/I \to \mathbb{B}/I$ preserves finite coproducts.

Remark 1.7.2. Since, for any $I \in \mathbb{B}$, the forgetful functor $\mathrm{dom} : \mathbb{B}/I \to \mathbb{B}$ is a left adjoint (with right adjoint $A \mapsto (\pi_I : I \times A \to I)$) it preserves colimits. Hence, when $(\mathbb{B}, \mathcal{F})$ has weak finite sum types, the *empty type* (fibred initial object) in any context is an initial object (for the whole category $\mathbb{B}$) and a binary sum in any context is a binary coproduct (for the whole category $\mathbb{B}$).

Recall that a *strict initial object* is an initial object 0 such that every map $X \to 0$ is an isomorphism.

Lemma 1.7.3. *A well-rooted display map category $(\mathbb{B}, \mathcal{F})$ has weak finite sum types if and only if $\mathbb{B}$ has finite coproducts including a strict initial object such that copairing pullback-stably preserves display maps. This latter condition means that for any two display maps $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$, their copairing $[f, g] : A + B \to C$ is also a display map and, for any $h : D \to C$, the square*

$$\begin{array}{ccc} \overline{A} + \overline{B} & \xrightarrow{h_A + h_B} & A + B \\ {\scriptstyle [\overline{f}, \overline{g}]} \downarrow & & \downarrow {\scriptstyle [f, g]} \\ D & \xrightarrow[h]{} & C \end{array}$$

is a pullback, where $\overline{f}$, $\overline{g}$, h_A and h_B are given by the following pullback squares.

$$\begin{array}{ccc} \overline{A} & \xrightarrow{h_A} & A \\ {\scriptstyle \overline{f}} \downarrow & & \downarrow {\scriptstyle f} \\ D & \xrightarrow[h]{} & C \end{array} \qquad \begin{array}{ccc} \overline{B} & \xrightarrow{h_B} & B \\ {\scriptstyle \overline{g}} \downarrow & & \downarrow {\scriptstyle g} \\ D & \xrightarrow[h]{} & C \end{array}$$

Proof. Suppose that $(\mathbb{B}, \mathcal{F})$ has weak finite sum types. Then $\mathbb{B} \simeq \mathcal{F}/1$ has finite coproducts. Let 0 be the initial object. Since, for any $I \in \mathbb{B}$, the forgetful functor $\mathrm{dom} : \mathbb{B}/I \to \mathbb{B}$ is a left adjoint we see that the unique map $!_I : 0 \to I$ is a display map and is the initial object in $\mathcal{F}/I$. Moreover, display maps of this form are stable under pullback. Now suppose we have a map $f : X \to 0$. Then by the above the square

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow[f]{} & 0 \end{array}$$

is a pullback, hence $X \cong 0$. So 0 is indeed a strict initial object.

Now suppose that $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$ are display maps. Then their sum in $\mathcal{F}/I$ is preserved by the inclusion into $\mathbb{B}/I$ and by the functor $\mathrm{dom} : \mathbb{B}/I \to \mathbb{B}$. Hence their sum is the copairing $[f, g] : A + B \to C$. This shows that the copairing preserves display maps. Pullback-stability follows immediately from the pullback-stability of sums in $\mathcal{F}/I$.

For the converse, it is easy to see that if 0 is a strict initial object then all pullbacks of maps of the form $!_X : 0 \to X$ exist and maps of this form are stable under pullback. Since the unique map $0 \to \top$ is a display map, by pulling back we see that every map $0 \to X$ is a display map. Hence such maps define a fibred initial object in the fibration $\mathcal{F} \to \mathbb{B}$ and, for each I, the inclusion $\mathcal{F}/I \to \mathbb{B}/I$ preserves initial objects. If $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$ are two display maps, then the copairing $[f, g] : A + B \to C$ is a display map and is a binary coproduct of f and g in $\mathbb{B}/I$ and hence a binary coproduct in $\mathcal{F}/I$, which is pullback-stable. Hence $\mathcal{F} \to \mathbb{B}$ has fibred binary coproducts and the inclusion $\mathcal{F}/I \to \mathbb{B}/I$ preserves binary coproducts, for each I. □

We will need the following formulation of weak finite sums in a full split comprehension category.

Definition 1.7.4. A full split comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ has *strictly stable finite sum types* if for each I there is an operation sending $A, B \in \mathbb{E}(I)$ to a coproduct $A +_I B \in \mathbb{E}(I)$ which is strictly preserved by reindexing in I. These coproducts are preserved by the functors $\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$. Moreover, there is a strictly-reindexing-stable choice of initial object in each $\mathbb{E}(I)$, and these are also preserved by the functors $\chi_I : \mathbb{E}(I) \to \mathbb{B}/I$.

Given a display map $f : D \twoheadrightarrow A + B$ into a sum, we can pullback along each of the coproduct inclusions to get $f_A : D_A \twoheadrightarrow A$ and $g_B : D_B \twoheadrightarrow B$. Our first strengthening of the notion of finite sum types ensures that morphisms between types over a sum $A + B$ are equivalent to morphisms between their restrictions to A and B.

Definition 1.7.5. A display map category $(\mathbb{B}, \mathcal{F})$ has *strong finite sum types* if it has weak finite sum types (hence $\mathbb{B}$ has finite coproducts) and moreover, for any $A, B \in \mathbb{B}$, the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ given by pulling back along the coproduct inclusions is full and faithful.

We can give an alternative characterization of this property, based on the notion of extensive category, see [8].

Definition 1.7.6. A display map category $(\mathbb{B}, \mathcal{F})$ is *semi-extensive* if it has weak finite sum types and, in any diagram of the form

$$\begin{array}{ccccc} X & \longrightarrow & C & \longleftarrow & Y \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array} \tag{1.1}$$

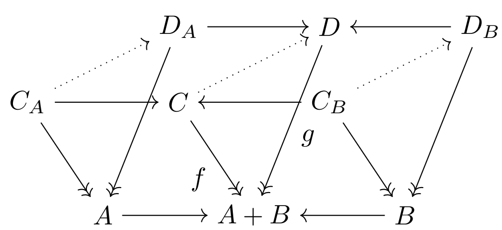
where the vertical arrows are display maps and the bottom row is a coproduct diagram, if both squares are pullbacks then the top row is a coproduct diagram. N.B. This is not an if-and-only-if condition as for an extensive category: we do not require that the if the top row of (1.1) is a coproduct then the two squares are pullbacks.

Lemma 1.7.7. *A well-rooted display map category has strong finite sum types if and only if it is semi-extensive.*

Proof. Suppose that $(\mathbb{B}, \mathcal{F})$ is semi-extensive and let $f : C \twoheadrightarrow A+B$ be a display map into a sum. Then the diagram below, where both squares are pullbacks,

$$\begin{array}{ccccc} C_A & \longrightarrow & C & \longleftarrow & C_B \\ \big\downarrow\!\!\!\downarrow & & \big\downarrow\!\!\!\downarrow f & & \big\downarrow\!\!\!\downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array}$$

is of the form (1.1) and hence the top row is a coproduct diagram. Hence if we have another display map $g : D \twoheadrightarrow A+B$ then similarly $D \cong D_A + D_B$ and it is easy to check from the diagram



that the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ is full and faithful.

Conversely, suppose that $(\mathbb{B}, \mathcal{F})$ has strong finite sum types and consider a diagram of the form (1.1) where the vertical arrows are display maps, the bottom row is a coproduct diagram, and both squares are pullbacks. Let D be an object of $\mathbb{B}$. Then the projection $(A + B) \times D \to A + B$ is an object of $\mathcal{F}/(A+B)$, and maps $C \to D$ correspond naturally to maps $C \to D \times (A+B)$ over $A+B$. Since the pullback of $D \times (A+B) \to A+B$ along $A \to A+B$ is $D \times A \to A$ (and similarly for $B \to A+B$) and using the assumption that the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ is full and faithful, we see that in turn such maps correspond naturally to pairs of maps $X \to D \times A$ over A and $Y \to D \times B$ over B, i.e. to pairs of maps $X \to D$ and $Y \to D$. Hence the top row is indeed a coproduct diagram. $\square$

The next strengthening of the notion of finite sum type allows us to define a dependent type over a sum $A + B$ by specifying a type over each of A and B. Thinking of a display map into C as a family of types indexed by the 'set' C, this corresponds to the fact that a family of sets indexed by a disjoint union $A \sqcup B$ is the same thing as a family of sets indexed by A together with a family of sets indexed by B. The terminology is drawn from [46].

Definition 1.7.8. A display map category $(\mathbb{B}, \mathcal{F})$ has *strong sums for types* if it has weak finite sum types and, for all objects $A, B \in \mathbb{B}$, the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ is an equivalence. Equivalently, if it has strong finite sum types and the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ is essentially surjective.

Lemma 1.7.9. *Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category modelling strong sum types. Then the following are equivalent.*

- *(a)* $(\mathbb{B}, \mathcal{F})$ *models strong sums for types.*
- *(b) Display maps are closed under addition and the functor $+ : \mathcal{F}/A \times \mathcal{F}/B \to \mathcal{F}/(A+B)$ is a quasi-inverse to the functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ induced by reindexing along the coproduct inclusions.*
- *(c) Display maps are closed under addition and for any diagram of the form* (1.1) *where the vertical arrows are display maps and the bottom row is a coproduct diagram, if the top row is also a coproduct diagram then both squares are pullbacks.*

(d) *For any diagram of the form*

$$\begin{array}{ccccc} X & \longrightarrow & C & \longleftarrow & Y \\ \Big\Downarrow & & \Big\downarrow & & \Big\Downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array} \tag{1.2}$$

where the outer two vertical arrows are display maps and the bottom row is a coproduct diagram, if the top row is also a coproduct diagram then the middle vertical arrow is a display map and both squares are pullbacks.

Proof. Trivially (b) implies (a), Let us assume (a) and prove (b). Let $g : X \twoheadrightarrow A$ and $h : Y \twoheadrightarrow B$ be two display maps. Since pulling back along the coproduct inclusions induces an essentially surjective functor $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$, there exists a display map $f : C \twoheadrightarrow A+B$ and a diagram

$$\begin{array}{ccccc} X & \longrightarrow & C & \longleftarrow & Y \\ g\Big\Downarrow & & \Big\Downarrow f & & \Big\Downarrow h \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array}$$

in which both squares are pullbacks. By Lemma 1.7.7, the top row is a coproduct diagram. By inspection, we see that with respect to this coproduct structure $f = g+h$. Hence the sum of two display maps is a display map. It is clear from the above that summing display maps gives the quasi-inverse to $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$.

Let us assume (b) and prove (c). We already have that display maps are closed under addition. Suppose we have a diagram of the from (1.1) where both rows are coproducts.

$$\begin{array}{ccccc} X & \longrightarrow & C & \longleftarrow & Y \\ g\Big\Downarrow & & \Big\Downarrow f & & \Big\Downarrow h \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array}$$

Then with respect to these two coproduct structures, $f = g+h$. Since addition of display maps is quasi-inverse to pulling back along the coproduct inclusions, we see that the two squares are pullbacks, as required.

Let us assume (c) and prove (d). Given a diagram 1.2 where both rows are coproduct diagrams, we see that, with respect to the coproduct structures, the middle vertical arrow is the sum of the two outer ones. Since display maps are closed under addition, the middle vertical arrow is a display map. Hence we have a diagram of the form 1.1, thus by (c) the two squares are pullbacks.

Let us assume (d) and prove (a). By definition of strong finite sum types, we already have that $\mathcal{F}/(A+B) \to \mathcal{F}/A \times \mathcal{F}/B$ is full and faithful. Given display maps $g : X \twoheadrightarrow A$ and $h : Y \twoheadrightarrow B$, it suffices to find a display map $f : C \to A+B$ whose pullbacks along the coproduct inclusions are f and g respectively. We take the sum $f+g : X+Y \to A+B$, which by (d) is a display map with the required property. □

Recall (from A1.4.4 of [25]) that a category $\mathbb{C}$ with finite coproducts *has disjoint coproducts* if binary coproduct inclusions are monic and for any objects $A, B \in \mathbb{C}$, the square

$$\begin{array}{ccc} 0 & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ A & \hookrightarrow & A+B \end{array}$$

is a pullback.

Proposition 1.7.10. *Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category which models strong sums for types. Then $\mathcal{F}$ is closed under addition, contains the coproduct inclusions and coproducts are disjoint.*

Proof. From Lemma 1.7.9, we get that the display maps are closed under addition. It follows that the coproduct inclusions are display maps, since $A \to A+B$ is the sum of the identity $1_A : A \to A$ and $!_B : 0 \to B$. Now for any objects A and B, by (b), the two squares in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & A & \longleftarrow & 0 \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array}$$

are both pullbacks, the left-hand of which implies that $A \to A+B$ is monic. Thus the coproduct $A+B$ is disjoint, as required. □

In order to give the last notion of finite sum, which is the strongest of those we will consider, let us recall the following definition from [8].

Definition 1.7.11. Let $\mathbb{C}$ be a category with finite coproducts. Then $\mathbb{C}$ is *extensive* if for each pair of objects A and B, the functor

$$\mathbb{C}/A \times \mathbb{C}/B \to \mathbb{C}/(A+B)$$

is an equivalence.

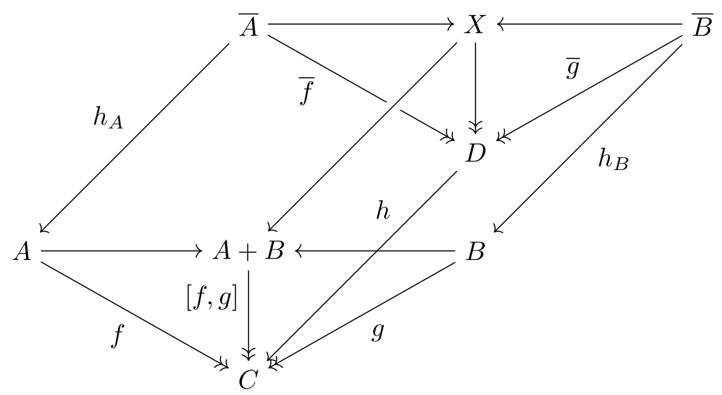
Proposition 2.2 in [8] states that a category $\mathbb{C}$ with finite coproducts is extensive if and only if it has pullbacks of coproduct inclusions and, in every diagram of the form (1.1) with arbitrary vertical arrows and the bottom row a coproduct diagram, the top row is a coproduct diagram if and only if the two squares are pullbacks. It is also shown, in Proposition 2.8, that the initial object in an extensive category is strict. The characterization (b) of strong sums for types from Lemma 1.7.9 shows us that we have been considering a generalized version of extensivity for display map categories.

Definition 1.7.12. Let $(\mathbb{B}, \mathcal{F})$ be a display map category. Then $(\mathbb{B}, \mathcal{F})$ has *extensive finite sums* if it has strong sums for types and each category $\mathcal{F}/A$ is extensive.

Lemma 1.7.13. *Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category. Then $(\mathbb{B}, \mathcal{F})$ has extensive finite sums if and only if $\mathbb{B}$ is extensive, and copairing and addition preserve display maps.*

Proof. Let us suppose that $(\mathbb{B}, \mathcal{F})$ has extensive finite sums. Then in particular it has strong sums for types, so by Lemma 1.7.9, the display maps are closed under addition. By Lemma 1.7.3, every map $!_X : 0 \to X$ is a display map and copairing preserves display maps. Since $\mathbb{B} \simeq \mathcal{F}/\top$, we see that $\mathbb{B}$ is extensive, as required.

Let us prove the converse. To use Lemma 1.7.3 to get weak finite sums, we need to check the pullback-stability of copairing. Let $f : A \twoheadrightarrow C$ and $g : B \twoheadrightarrow C$ be two display maps and $h : D \to C$ an arbitrary morphism. Then pulling back the coproduct diagram for $A + B$ along h gives us



where all five squares are pullbacks. Since the top two squares are pullbacks, extensivity implies that the top row is a coproduct diagram. Inspection shows that $X \to D$ is the copairing map $[\overline{f}, \overline{g}] : \overline{A} + \overline{B}$, as required.

It is clear that extensivity of $\mathbb{B}$ implies semi-extensivity of $(\mathbb{B}, \mathcal{F})$, hence $(\mathbb{B}, \mathcal{F})$ models strong sums. Moreover, it is clear that extensivity plus closure of display maps under addition implies condition (b) in Lemma 1.7.9, so $(\mathbb{B}, \mathcal{F})$ models strong sums for types.

Finally, we must show that each category $\mathcal{F}/I$ is extensive. As in Remark 1.6.3, pullbacks of $\mathcal{F}$-maps in $\mathcal{F}/I$ exist and are preserved and reflected by the forgetful functor $\mathrm{dom} : \mathcal{F}/I \to \mathbb{B}$. Since $\mathrm{dom} : \mathcal{F}/I \to \mathbb{B}$ also preserves and reflects coproducts, extensivity of $\mathcal{F}/I$ reduces to extensivity of $\mathbb{B}$. □

Proposition 1.7.14. *Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category modelling dependent sums. Then $(\mathbb{B}, \mathcal{F})$ has extensive finite sums if and only if $\mathbb{B}$ is extensive, every coproduct inclusion is in $\mathcal{F}$, and copairing preserves $\mathcal{F}$..*

Proof. The forward direction follows from Lemma 1.7.13 by observing that every map of the form $!_X : 0 \to X$ is a display map and hence, since addition preserves display maps, all coproduct inclusions are display maps. The reverse direction also follows from 1.7.13. Addition of maps is built out of copairing and composing with the coproduct inclusions, and by hypothesis both of these operations preserve $\mathcal{F}$. □

Let us give an application of finite sum types, which we will use to construct the polynomial model of [46].

Definition 1.7.15. Let $\mathbb{C}$ be a category with finite coproducts. For any two objects A and B, a *partial map* from A to B consists of a coproduct decomposition $A \cong X + Y$ together with a map $X \to B$.

Proposition 1.7.16. *Let $(\mathbb{B}, \mathcal{F})$ be an extensive display map category and let $f : A \twoheadrightarrow I$ and $g : B \twoheadrightarrow I$ be two display maps. Then partial maps from f to g in $\mathcal{F}/I$ are in bijective correspondence with maps $f \to g +_I 1_I$ in $\mathcal{F}/I$. Moreover, this bijection is stable under pullback in I.*

Proof. We will give the bijection in an extensive category $\mathbb{C}$ with a terminal object — when it is specialized to $\mathcal{F}/I$ it is clearly stable under pullback in I. Given a map $s : A \to B + \top$, we use s to pullback the coproduct diagram for $B + I$ to decompose A as $A : A_B + A_\top$ This decomposition comes with a map $A_B \to B$, so we get partial map from A to B. Conversely, given $A \cong X + Y$ and $t : X \to B$, we take

$$A \cong X + Y \xrightarrow{t+!_Y} B + I.$$

It is easy to see that these operations are mutually inverse. □

1.8 Identity types

Identity types give an internalized notion of equality between terms of a type. This internal notion is much coarser than the equality between arrows in the category of contexts and, indeed, it was shown in [15] that identity types give rise to a weak factorization system on the syntactic category of a type theory. Hence it is thought of as a notion of *intensional equality*, meaning that there may be many different terms of an identity type and these may contain non-trivial information about different identifications between two individuals. In Homotopy Type Theory (see [42]), where a type represents a space or homotopy type, the identity type is interpreted as a space of paths between two points. The idea that a category with a weak factorization system should be a model of intensional type theory appears in [1].

We begin with a formulation of identity types for full split comprehension categories, as it appears in [32], although we assume that we have binary products. This definition matches the syntactic definition quite closely.

Definition 1.8.1. Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a full split comprehension category with strictly stable binary product types. Then when (p, χ) has *strictly stable identity types* it is equipped with, for every $I \in \mathbb{B}$ and $A \in \mathbb{E}(I)$, a type $\mathsf{Id}_{I,A} \in \mathbb{E}(\{A \times_I A\})$ and a map $\mathsf{refl}_{I,A} : \{A\} \to \{\mathsf{Id}_{I,A}\}$ making

$$\begin{array}{ccc} & & \{\mathsf{Id}_{I,A}\} \\ & \overset{\mathsf{refl}_{I,A}}{\nearrow} & \downarrow \chi(\mathsf{Id}_{I,A}) \\ \{A\} & \xrightarrow[\Delta_A]{} & \{A \times_I A\} \end{array}$$

commute, where Δ_A is the diagonal map $\{A\} \to \{A\} \times_I \{A\} \cong \{A \times_I A\}$. There is, for every $C \in \mathbb{E}(\mathsf{Id}_{I,A})$ and map $f : \{A\} \to \{C\}$ making

$$\begin{array}{ccc} & & \{C\} \\ & \overset{f}{\nearrow} & \downarrow \chi(C) \\ \{A\} & \xrightarrow[\mathsf{refl}_{I,A}]{} & \{\mathsf{Id}_{I,A}\} \end{array}$$

commute, a map $J_{I,A,C} : \{\mathsf{Id}_{I,A}\} \to \{C\}$ making the diagram

$$\begin{array}{ccc} \{A\} & \xrightarrow{f} & \{C\} \\ {\scriptstyle \mathsf{refl}_{I,A}} \downarrow & \nearrow {\scriptstyle J_{I,A,C}} & \downarrow {\scriptstyle \chi(C)} \\ \{\mathsf{Id}_{I,A}\} & = & \{\mathsf{Id}_{I,A}\} \end{array}$$

commute. Moreover, all of this data is stable under reindexing along maps $J \to I$ in $\mathbb{B}$.

Remark 1.8.2. This is a standard definition of identity type, see for instance section 2.3 of [32]. However, it is formally weaker than the most natural definition of identity type, see section 3.4.3 of [32]. The two definitions are interderivable, provided we have dependent products (or even weak dependent products, see 2.2.1).

We omit any formulation of identity types for more general comprehension categories. However, we will recall and make use of a convenient way of modelling identity types in a well-rooted display map category. The arguments of [15] may be ported to more general settings to show that a model of type theory with identity types in the sense of Definition 1.8.1 admits a weak factorization system (see, for example, [38] and [14]). We will not consider here the issue of defining identity types in display map categories which are not well-rooted other than to say that it should mean having identity types in each slice which are preserved by change-of-base between slices — for a well-rooted display map category this slice-wise definition follows from the formally weaker definition we give here.

Definition 1.8.3. Let $(\mathbb{B}, \mathcal{F})$ be a display map category. An *acyclic cofibration* or *left map* is a map $m : I \to J$ such that, for any display map $f : A \twoheadrightarrow J$ and morphism $u : I \to A$ making the diagram

$$\begin{array}{ccc} I & \xrightarrow{u} & A \\ & {\scriptstyle m} \searrow \quad \swarrow {\scriptstyle f} & \\ & J & \end{array}$$

commute, there exists a map $s : J \to A$ such that $fs = 1_J$ and $sm = u$. Such a map m is said to have the *left lifting property* with respect to $\mathcal{F}$, and the class of such maps is denoted ${}^{\square}\mathcal{F}$. An acyclic cofibration $u : A \to B$ is *stable* if whenever we have display maps $f : A \twoheadrightarrow I$ and $g : B \twoheadrightarrow I$ such that $fu = g$ and a map $h : J \to I$, the pullback $h^*(u)$ is again an acyclic cofibration. A pair of classes $(\mathcal{L}, \mathcal{R})$ of classes of arrows in $\mathbb{B}$ is *factorizing* if for any $f : A \to B$ in $\mathbb{B}$ there exists a factorization

$$A \xrightarrow{\in \mathcal{L}} X \xrightarrow{\in \mathcal{R}} B$$

of f as the composite of an $\mathcal{L}$-arrow followed by an $\mathcal{R}$-arrow. The display map category $(\mathbb{B}, \mathcal{F})$ has *identity types* if $({}^{\square}\mathcal{F}, \mathcal{F})$ is factorizing and the class of acyclic cofibrations is stable (that is, if every acyclic cofibration is stable).

Example 1.8.4. The example of 1.2.2 is a display map category $(\mathbb{B}, \mathcal{F})$ with identity types. It is easy to check that the class ${}^{\square}\mathcal{F}$ consists of precisely the split monomorphisms. The factorization of a map $f : A \to B$ is given by

$$A \xrightarrow{(1_A, f)} A \times B \xrightarrow{\pi_B} B$$

where $(1_A, f)$ is split with retraction $\pi_A : A \times B \to A$.

Chapter 2

Adding the η-rule

In Martin-Löf's 1984 formulation of dependent type theory in [34], the dependent product of a family of types

$$\frac{x : A \vdash B(x) \ \text{ type}}{\vdash \Pi_{x:A} B(x) \ \text{ type}}(\Pi\mathsf{-Form})$$

with introduction and elimination rules

$$\frac{x : A \vdash t(x) : B(x)}{\lambda x.t(x) : \Pi_{x:A} B(x)}(\Pi\mathsf{-Int}) \qquad \frac{a : A \qquad f : \Pi_{x:A} B(x)}{\mathsf{app}(f, a) : B(a)}(\Pi\mathsf{-Elim})$$

enjoys both the β and η computation rules

$$\frac{\vdash a : A \qquad x : A \vdash t(x) : B(x)}{\mathsf{app}(\lambda x.t(x), a) = t(a) : B(a)}(\Pi-\beta) \qquad \frac{\vdash f : \Pi_{x:A} B(x)}{f = \lambda x.\mathsf{app}(f, x) : \Pi_{x:A} B(x)}(\Pi-\eta),$$

since this is the behaviour we expect from the usual cartesian product of a family of sets. Indeed, these two computation rules are standard in formulations of Homotopy Type Theory (see [42]).

In some contexts it is appropriate to consider the terms of the Π-type as *programs* or *codes* for functions. Then λ-abstraction is an operation which turns a *program specification* — or description of what a program is supposed to do — into a canonical program satisfying it. The β-rule is what tells us that the result of λ does indeed match the given specification. The η-rule would tell us that this program is the only such, which is clearly an unreasonable situation when terms are interpreted as program code, but acceptable if we think of its terms as *functions-in-extension* or *graphs of functions*. Thus it is common to use the qualifier *extensional* for Π-types satisfying the η-rule. (N.B. This is not what 'function extensionality' means in [42]: Homotopy Type Theory relies on its identity types for its intensional content.)

Hence the η-rule is often omitted, as it was in Martin-Löf's original 1972 formulation [35]. This has the effect of making the 'true' type of dependent functions a retract of $\Pi_{x:A}B(x)$, where we think of 'true functions' as being in correspondence with the dependent terms $x : A \vdash t(x) : B(x)$ up to judgemental equality. The inclusion of dependent terms into $\Pi_{x:A}B(x)$ is given by λ-abstraction, and the retraction in the other direction is given by $f : \Pi_{x:A}B(x)$ goes to $x : A \vdash \mathsf{app}(f, x) : B(x)$.

We specified in 1.5.3 that, in a display map category, the full Π-type (with both β- and η-rules) of $B \twoheadrightarrow A$ along a display map $A \twoheadrightarrow \Gamma$ is modelled by applying to $B \twoheadrightarrow A$ the right adjoint to the functor induced by pulling back along $A \twoheadrightarrow \Gamma$. To model the weak Π-type case, we need to consider a weakened notion of adjoint which gives a retraction between two hom-sets rather than an isomorphism.

2.1 Weak adjunctions

Definition 2.1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let B be an object of $\mathcal{D}$. Then a *weak coreflection of B along F* is an object GB of $\mathcal{C}$ together with a family of retractions

$$(r_{A,B}, i_{A,B}) : \mathcal{D}(FA, B) \lhd \mathcal{C}(A, GB)$$

which are natural in A.

Observe that unlike usual right adjoints, a weak coreflection of an object is not necessarily unique. However, we can still phrase the definition in terms of a 'counit'. For instance, here is a characterization corresponding to the universal arrow characterization of the right adjoint.

Proposition 2.1.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories and let B be an object of $\mathcal{D}$. To give a weak coreflection of B along F is equivalently to give:*

- *an object $GB \in \mathcal{C}$,*
- *an arrow $\epsilon_B : FGB \to B$ in $\mathcal{D}$,*
- *and an arrow $\lambda_B : GB \to GB$*

such that

- $\lambda_B \circ \lambda_B = \lambda_B$*,*

- *and for every $h : FA \to B$ in $\mathcal{D}$, there is a unique $\bar{h} : A \to GB$ such that $\lambda_B \circ \bar{h} = \bar{h}$ and $h = \epsilon_B \circ F(\bar{h})$.*

Proof. Suppose we are given a weak coreflection of B along F. Then let $\epsilon_B = r_{GB,B}(1_{GB})$, (cf. the usual definition of the counit) and let $\lambda_B = i_{GB,B}(\epsilon_B)$. Now for any object A of $\mathcal{C}$, the image of each $i_{A,B}$ is given by precomposition with λ_B; since for any $f : A \to GB$, the following diagram commutes by naturality of the retractions

$$\begin{array}{ccccc}
\mathcal{C}(GB, GB) & \xrightarrow{r_{GB,B}} & \mathcal{D}(FGB, B) & \xrightarrow{i_{GB,B}} & \mathcal{C}(GB, GB) \\
{\scriptstyle - \circ f}\downarrow & & \downarrow{\scriptstyle - \circ Ff} & & \downarrow{\scriptstyle - \circ f} \\
\mathcal{C}(A, GB) & \xrightarrow[r_{A,B}]{} & \mathcal{D}(FA, B) & \xrightarrow[i_{A,B}]{} & \mathcal{C}(A, GB)
\end{array}$$

and by following 1_{GB} around the outside of the diagram, we see that $\lambda_B \circ f = i_{A,B}(r_{A,B}(f))$. In particular we see that λ_B is idempotent.

Now, given $h : FA \to B$, let $\bar{h} = i_{A,B}(h)$. Then $\lambda_B \circ \bar{h} = i_{A,B}(r_{A,B}(\bar{h})) = \bar{h}$. Moreover, by following 1_{GB} through the middle path and through the top-right corner in the diagram above with $f = \bar{h}$, we see that $\bar{h} = \lambda_B \circ \bar{h} = i_{A,B}(\epsilon_B \circ F\bar{h})$. Apply $r_{A,B}$ to this equation we see that $\epsilon_B \circ F\bar{h} = r_{A,B}(\bar{h}) = h$. This $\bar{h}$ is suitably unique since if $\epsilon_B \circ Fg_1 = \epsilon_B \circ Fg_2$, then by a similar diagram chase we deduce that $\lambda_B \circ g_1 = \lambda_B \circ g_2$.

Conversely, suppose we are given an object GB, an idempotent λ_B and an 'almost-universal' arrow $\epsilon_B : FGB \to B$. Then for any object A of $\mathcal{C}$, define $i_{A,B}(f : FA \to B)$ to be the unique arrow $\bar{f} : A \to GB$ such that $\lambda_B \circ \bar{f} = \bar{f}$ and $f = \epsilon_B \circ F(\bar{f})$. Define $r_{A,B}(g : A \to GB) = \epsilon_B \circ F(g)$. This clearly gives the desired retraction, natural in A. These two constructions are clearly mutually inverse. □

Remark 2.1.3. Since we have essentially dropped the uniqueness condition, we also lose the result that a complete set of choices of coreflections along F assembles into a functor $G : \mathcal{D} \to \mathcal{C}$. However, we still see that such a set of choices of weak coreflections assembles into a *semifunctor* (a mapping that preserves binary composition but not necessarily identities).

We could equally well express this by saying that to choose a weak coreflection along F for every object in $\mathcal{D}$ is equivalently to choose a functor $G : \mathcal{D} \to \overline{\mathcal{C}}$ such that F is a left J-relative adjoint to G (see [45]). Here $\overline{\mathcal{C}}$ is the Karoubi envelope (see Definition 2.3.1) of $\mathcal{C}$, and $J : \mathcal{C} \to \overline{\mathcal{C}}$ is the inclusion. Note that each coreflection along F is not unique, and this is reflected in the asymmetry of the relative adjunction: G determines F uniquely (up to natural isomorphism) but F does not determine G uniquely (even up to natural isomorphism).

Definition 2.1.4. A *weak right adjoint* for a functor $F : \mathcal{C} \to \mathcal{D}$ is a functor $G : \mathcal{D} \to \overline{\mathcal{C}}$ such that F is a left J-relative adjoint to G.

2.2 Modelling weak dependent products

We give what it means for a model of type theory to model weak dependent products, starting with the simpler case of display map categories.

Definition 2.2.1. Let $\mathcal{C}$ be a category with class of display maps $\mathcal{F}$. Then given $\mathcal{F}$-maps $f : B \twoheadrightarrow A$ and $g : C \twoheadrightarrow B$, a *weak dependent product* of g along f is a weak coreflection $(\Pi_f(g), \lambda_{f,g}, \epsilon_{f,g})$ of the object $g \in \mathcal{F}/B$ along the pullback functor $f^* : \mathcal{F}/A \to \mathcal{F}/B$. Such a dependent product is weakly stable (in the sense of [32]) if it is stable under pullback: for every map $h : C \to A$ and pullback square

$$\begin{array}{ccc} D & \xrightarrow{\bar{h}} & B \\ {\scriptstyle \bar{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ C & \xrightarrow[h]{} & A \end{array}$$

the triple $(h^*(\Pi_f(g)), h^*(\lambda_{f,g}), \bar{h}^*(\epsilon_{f,g}))$ is a weak coreflection of $\bar{h}^*(g)$ along $\bar{f}^* : \mathcal{F}/C \to \mathcal{F}/D$. Then $(\mathcal{C}, \mathcal{F})$ *has weak dependent products* if there is a weakly-stable weak dependent product for every composable pair of display maps. (Equivalently, for every display map $f : B \to A$ there is a weak right adjoint Π_f to the pullback functor $f^* : \mathcal{F}/A \to \mathcal{F}/B$ satisfying the appropriate version of the Beck-Chevalley condition.)

Of course, the usual notion of dependent product can be recovered from the weak version by requiring that $\lambda_{f,g}$ be an identity.

For reasons related to the fact that they do not correspond to the strict syntax of type theory, display map categories are not a good setting to consider the idempotent completion of a model of type theory. We will instead work directly with split comprehension categories and head straight to the definition of the strictly stable version.

Definition 2.2.2. Let $(\mathcal{C}, \mathbb{E}, \chi)$ be a split comprehension category. Then $(\mathcal{C}, \mathbb{E}, \chi)$ is *equipped with weak* Π-*types* if for each $A \in \mathcal{C}$, each $\beta \in \mathbb{E}(A)$ with comprehension $f : B \to A$ and each $\gamma \in \mathbb{E}(B)$ we are given: a type $\Pi_{A,\beta}(\gamma) \in \mathbb{E}(A)$, an idempotent $\lambda_{A,\beta,\gamma}$ on $\Pi_{A,\beta}(\gamma)$ in $\mathbb{E}$, and a map $\epsilon_{A,\beta,\gamma} : f^*(\Pi_{A,\beta}(\gamma)) \to \gamma$ in $\mathbb{E}(B)$ such that $(\Pi_{A,\beta}(\gamma), \lambda_{A,\beta,\gamma}, \epsilon_{A,\beta,\gamma})$ is a weak coreflection of γ along $f^* : \mathbb{E}(A) \to \mathbb{E}(B)$. Moreover, we require for any $h : C \to A$ the equations $\Pi_{C,h^*(\beta)}(\bar{h}^*(\gamma)) = h^*(\Pi_{A,\beta}(\gamma))$, $\lambda_{C,h^*(\beta),\bar{h}^*(\gamma)} = h^*(\lambda_{A,\beta,\gamma})$ and $\epsilon_{C,h^*(\beta),\bar{h}^*(\gamma)} = \bar{h}^*(\epsilon_{A,\beta,\gamma})$, where $\bar{h}$ comes from the pullback square

$$\begin{array}{ccc} D & \xrightarrow{\bar{h}} & B \\ {\scriptstyle \chi(h^*(\beta))}\downarrow & & \downarrow{\scriptstyle f = \chi(\beta)} \\ C & \xrightarrow[h]{} & A \end{array}$$

which is the comprehension of the cartesian arrow with codomain β that lifts h.

We will see in Chapter 7 that there is an 'error Dialectica' model which has weak Π-types. It is clear that the syntactic model of type theory with weak dependent products has weak Π-types.

2.3 The category of retracts

Recall from 1.2.2 that if a category $\mathcal{C}$ has finite products then it admits a class of display maps given by (maps isomorphic to) product projections. In this case, the corresponding dependent type theory is really a non-dependent one: the type family corresponding to the projection $A \times B \to A$ is the constant one which provides the type B independently of the parameter $x : A$. Now for $\mathcal{C}$ to have dependent products is equivalent to its having a cartesian closed structure, and in this case it is a model of the simply typed λ-calculus (with β- and η-rules). The correspondence between simply typed λ-calculi and cartesian closed categories can be found in [29].

The rest of this chapter serves to generalize the following idea from untyped λ-calculus. Informally, given some λ-calculus subject to a β-equality rule, we can construct a cartesian closed category whose objects are the idempotent terms in the theory, i.e. those closed terms A satisfying $A = \lambda x.(A(Ax))$ (see [37]). Morphisms $A \to B$ are (equivalence classes of) those closed terms f such that $f = \lambda x.B(f(Ax))$. Since this category is cartesian closed and the object $U = \lambda x.x$ is a *reflexive object* – i.e. there is a retraction $(U \Rightarrow U) \lhd U$ — the set of maps $U \to U$ is itself a λ-theory with β-equality. This theory, *the theory of* U, is precisely the original untyped calculus with which we began. Hence, this 'category of retracts' construction shows that an untyped $\lambda\beta$-calculus admits a conservative embedding into a *typed* $\lambda\beta\eta$-calculus.

What if we begin with a typed calculus lacking the η-rule? As shown by S. Hayashi [18], if we start with a category $\mathcal{C}$, then its category of retracts (also known as its *Karoubi envelope* after M. Karoubi [26], or its *Cauchy completion* following [30]) is a cartesian closed category if and only if $\mathcal{C}$ is a *semi cartesian closed category.* This condition on $\mathcal{C}$ corresponds in type theory to having weakened finite product types as well as weakened function spaces. For our present purposes we will not need to weaken the finite product types, so below we recall in Proposition 2.3.4 the special case of the result where $\mathcal{C}$ already has true categorical products.

Definition 2.3.1. Let $\mathcal{C}$ be a category. Then the *Karoubi envelope* ([26]) of $\mathcal{C}$ is the category $\widetilde{\mathcal{C}}$ having

- as objects pairs (A, α) where A is an object of $\mathcal{C}$ and α is an arrow $A \to A$ with $\alpha \circ \alpha = \alpha$, and
- as arrows $(A, \alpha) \to (B, \beta)$ those arrows $f : A \to B$ satisfying $\beta f \alpha = f$.

Observe that $\mathcal{C}$ admits a full and faithful embedding into $\widetilde{\mathcal{C}}$, via the functor $A \mapsto (A, 1_A)$.

Remark 2.3.2. It can be helpful to see that the Karoubi envelope of $\mathcal{C}$ is equivalent to the closure under retracts of the image of $\mathcal{C}$ under the Yoneda embedding $\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$.

Definition 2.3.3. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is a *weakly cartesian closed category* if it has finite products and for each object A of $\mathcal{C}$, the functor $A \times (-)$ has a weak right adjoint.

The following well-known result is a special case of a result from [18]. We do not concern ourselves here with the extra generality of starting from 'weak finite products' which can be found in that article.

Proposition 2.3.4. *Let $\mathcal{C}$ be a category with finite products. Then $\mathcal{C}$ is a weakly cartesian closed category if and only if its Karoubi envelope $\widetilde{\mathcal{C}}$ is a cartesian closed category.*

Proof. Firstly, it is easy to check that $\widetilde{\mathcal{C}}$ has finite products, merely on the assumption that $\mathcal{C}$ has them. The terminal is given by $(1, 1_1)$ and the product of (A, α) with (B, β) is given by $(A \times B, \alpha \times \beta)$, and the product projections are, as maps in $\mathcal{C}$, given by $\alpha\pi_A : A \times B \to A$ and $\beta\pi_B : A \times B \to B$, where π_A and π_B are the product projections in $\mathcal{C}$.

Now, suppose $\mathcal{C}$ is a weakly cartesian closed category. Then given objects (B, β) and (C, γ) of $\widetilde{\mathcal{C}}$, suppose that the weak function space of maps from B to

C is given by the object $B \Rightarrow C$ with idempotent λ and evaluation $\epsilon : B \times (B \Rightarrow C) \to C$. Then we define the function space $(B, \beta) \Rightarrow (C, \gamma)$ to be $B \Rightarrow C$ in $\mathcal{C}$ equipped with idempotent given by transposing the composite

$$B \times (B \Rightarrow C) \xrightarrow{\beta \times 1} B \times (B \Rightarrow C) \xrightarrow{\epsilon} C \xrightarrow{\gamma} C$$

and this composite is also the evaluation map.

Conversely, if $(B \Rightarrow C, \lambda)$ is the function space $(B, 1_B) \Rightarrow (C, 1_C)$, with evaluation $\epsilon : B \times (B \Rightarrow C) \to C$, then it is easy to check that $B \Rightarrow C$ is a weak function space for B into C in $\mathcal{C}$, with idempotent λ and evaluation ϵ. □

Note that since $\mathcal{C}$ embeds fully and faithfully inside $\widetilde{\mathcal{C}}$, this gives us a conservativity result: adding extensional Π-types to a type theory with weak Π-types does not add new typing or equality judgements between terms and types that do not feature the new Π-types.

2.4 The split comprehension category of retracts

We have seen that a typed $\lambda\beta$-theory can be conservatively embedded into a typed $\lambda\beta\eta$-theory. We aim to extend this story to dependently typed theories. We will see that every split comprehension category $(\mathcal{C}, \mathbb{E}, \chi)$ over a base category $\mathcal{C}$ embeds fully and faithfully into a split comprehension category category $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$, where $\widetilde{\mathcal{C}}$ is the Karoubi envelope of $\mathcal{C}$; and $(\mathcal{C}, \mathbb{E}, \chi)$ models weak dependent products if and only if $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ models (strict) dependent products.

In the rest of this section, $(\mathcal{C}, \mathbb{E}, \chi)$ denotes a split comprehension category. The Karoubi envelope of $\mathcal{C}$ will be denoted $\widetilde{\mathcal{C}}$. We work towards a definition of $\mathbb{F}$ and $\widetilde{\chi}$.

Notation 2.4.1. For $A, B \in \mathcal{C}$, $f : B \to A$, $\phi \in \mathbb{E}(A)$, denote by $f^{\phi} : f^*(\phi) \to \phi$ the (chosen) cartesian arrow over f with codomain ϕ. We are particularly interested in the special case where we have $A \in \mathcal{C}$, $\alpha : A \to A$ with $\alpha \circ \alpha = \alpha$, $\phi \in \mathbb{E}(A)$, so that $\alpha^{\phi} : \alpha^*(\phi) \to \phi$ is the cartesian lift of α with codomain ϕ. Whenever $\alpha^*(\phi) = \phi$, by functoriality of the chosen cartesian arrows, we see that α^{ϕ} is idempotent.

Definition 2.4.2. For an object (A, α) of $\widetilde{\mathcal{C}}$, the fibre category $\mathbb{F}(A, \alpha)$ is the Karoubi envelope of the (not necessarily full) subcategory of $\mathbb{E}(A)$ which is fixed (on-the-nose) by reindexing under α.

Let us spell this out. A type over (A, α) is a type $\phi \in \mathbb{E}(A)$ over A for which $\alpha^*(\phi) = \phi$, equipped with an vertical idempotent $b : \phi \to \phi$ in $\mathbb{E}(A)$ such that, in $\mathbb{E}$, the equation $\alpha^{\phi} b = b\alpha^{\phi}$ holds (which is saying that $\alpha^*(b) = b$). A morphism of types $(\phi, b) \to (\psi, c)$ over (A, α) is given by a morphism $w : \phi \to \psi$ in $\mathbb{E}(A)$ such that $cwb = w$ (in $\mathbb{E}(A)$) and $\alpha^{\psi} w = w\alpha^{\phi}$ (in $\mathbb{E}$).

Definition 2.4.2 (continued)**.** Given an arrow $h : (C, \gamma) \to (A, \alpha)$ in $\widetilde{\mathcal{C}}$, the reindexing of an object (ϕ, b) of $\mathbb{F}(A, \alpha)$ to $\mathbb{F}(C, \gamma)$ is $(\overline{\phi}, \overline{b})$, where $\overline{\phi} = h^*(\phi)$ and $\overline{b} = h^*(b)$ in $\mathbb{E}$, and this extends to morphisms in the obvious way.

We need to check that $(\overline{\phi}, \overline{b})$ is a valid object in $\mathbb{F}(C, \gamma)$. Let $\xi : \overline{\phi} \to \phi$ be the chosen cartesian arrow in $\mathbb{E}$ lifting h with codomain ϕ. Then $\alpha^{\phi}\xi = \xi = \xi\gamma^{\overline{\phi}}$, since all three arrows lift h. It follows that $\gamma^*(\overline{\phi}) = \overline{\phi}$ and $\gamma^{\overline{\phi}}$ is idempotent. By definition of $\overline{b}$, $b\xi = \xi\overline{b}$. Moreover,

$$\xi\overline{b}\gamma^{\overline{\phi}} = b\xi\gamma^{\overline{\phi}} = b\xi = \xi\overline{b} = \xi\gamma^{\overline{\phi}}\overline{b}$$

and as these arrows lie over s, we deduce that $\gamma^{\overline{\phi}}\overline{b} = \overline{b}\gamma^{\overline{\phi}}$. A similar calculation shows that the reindexing of a vertical morphism is well-defined. This is enough to give $\mathbb{F} \to \widetilde{\mathcal{C}}$ the structure of a split fibration.

Notation 2.4.3. The comprehension functor χ sends arrows in $\mathbb{E}$ to commuting squares in $\mathcal{C}$. Let χ_1 denote the composite of χ with the domain functor $\mathcal{C}^{\to} \to \mathcal{C}$.

Definition 2.4.2 (continued)**.** The comprehension $\widetilde{\chi} : \mathbb{F} \to \widetilde{\mathcal{C}}^{\to}$ is given as follows. On types: if the comprehension of $\phi \in \mathbb{E}(A)$ is $f : B \to A$ in $\mathcal{C}$ then the comprehension of $(\phi, b) \in \mathbb{F}(A, \alpha)$ is $\alpha f : (B, \chi_1(\alpha^{\phi} b)) \to (A, \alpha)$. On vertical arrows: given $(\phi, b), (\psi, c) \in \mathbb{F}(A, \alpha)$, where $\phi, \psi \in \mathbb{E}(A)$ have comprehensions $f : B \to A$ and $g : C \to A$ respectively, an arrow $w : (\phi, b) \to (\psi, c)$ (i.e. an arrow $w : \phi \to \psi$ in $\mathbb{E}(A)$ with $cwb = w$) has comprehension given by $\widetilde{\chi}_1(w) = \chi_1(w\alpha^{\phi})$.

We will return to define the comprehension of cartesian arrows shortly. For now, we easily verify that

$$\alpha \circ (\alpha f) \circ \chi_1(\alpha^{\phi} b) = \alpha \circ (f \circ \chi_1(\alpha^{\phi} b)) = \alpha f$$

since α is idempotent and $\chi_1(\alpha^{\phi} b)$ is a map $f \to f$ in the slice category $\mathcal{C}/A$. Hence the given comprehension arrow is a valid arrow in $\widetilde{\mathcal{C}}$.

We also need to check that the action of the comprehension on arrows is well-defined. On vertical arrows, first note that $\widetilde{\chi}_1(w) = \chi_1(w\alpha^{\phi}) = \chi_1(\alpha^{\psi} w) = \chi_1(\alpha^{\psi} w\alpha^{\phi})$, since w is an arrow in $\mathbb{F}(A, \alpha)$. Now,

$$\chi_1(\alpha^{\psi} c)\chi_1(w\alpha^{\phi})\chi_1(\alpha^{\phi} b) = \chi_1(cwb\alpha^{\phi}) = \chi_1(w\alpha^{\phi}),$$

as required.

Definition 2.4.2 (continued)**.** On cartesian arrows, say on the lift $s^{(\phi,b)}$ of $s : (C, \gamma) \to (A, \alpha)$ with codomain (ϕ, b), the comprehension gives the square

$$\begin{CD} (D, \chi_1(\gamma^{\overline{\phi}}\overline{b})) @>{\chi_1(\alpha^{\phi}b) \circ t \circ \chi_1(\gamma^{\overline{\phi}}\overline{b})}>> (B, \chi_1(\alpha^{\phi}b)) \\ @V{\gamma g}VV @VV{\alpha f}V \\ (C, \gamma) @>>{s}> (A, \alpha) \end{CD} \tag{2.1}$$

where $g : D \to C$ is the comprehension of $\overline{\phi} \in \mathbb{E}(C)$, and

$$\begin{CD} D @>{t}>> B \\ @V{g}VV @VV{f}V \\ C @>>{s}> A \end{CD}$$

is the comprehension of the reindexing of $\phi \in \mathbb{E}(A)$ along $s : C \to A$.

This concludes the definition. We check that the comprehension of cartesian arrows is well-defined. We see that $\chi_1(b)$ and $\chi_1(\overline{b})$ are idempotents fitting into the pullback square

$$\begin{CD} D @>{t}>> B \\ @V{\chi_1(\overline{b})}VV @VV{\chi_1(b)}V \\ D @>>{t}> B \end{CD}$$

in $\mathcal{C}$ since χ sends cartesian arrows to pullback squares. Now $\chi_1(\alpha^{\phi})$ and $\chi_1(\gamma^{\overline{\phi}})$ are also idempotent, and they fit into pullback squares

$$\begin{CD} B @>{\chi_1(\alpha^{\phi})}>> B \\ @V{f}VV @VV{f}V \\ A @>>{\alpha}> A \end{CD} \qquad \begin{CD} D @>{\chi_1(\gamma^{\overline{\phi}})}>> D \\ @V{g}VV @VV{g}V \\ C @>>{\gamma}> C \end{CD}$$

in $\mathcal{C}$, and moreover since $\alpha s = s\gamma$, functoriality of the reindexing tells us that we also get the equation $\chi_1(\alpha^{\phi})t = t\chi_1(\gamma^{\overline{\phi}})$. From the mere commutativity of these squares, it is easy to check that the square (2.1) commutes, and that all four arrows are valid morphisms in $\widetilde{\mathcal{C}}$ (for the top arrow one uses that $\chi_1(b)$ and $\chi_1(\alpha^\phi)$ commute, as do $\chi_1(\overline{b})$ and $\chi_1(\overline{\gamma^\phi})$). Since we may also write the top arrow as either $\chi_1(\alpha^\phi)t$ or $t\chi_1(\overline{\gamma^\phi})$, we see that this comprehension is indeed a strict functor.

Finally, it remains to check that the square (2.1) is indeed a pullback square in $\widetilde{\mathcal{C}}$. It is convenient to check this using generalized element notation. Suppose we have $x \in B$ and $y \in C$ satisfying

$$\chi_1(\alpha^\phi)(x) = x, \chi_1(b)(x) = x, \gamma(y) = y, \alpha f(x) = s(y).$$

Then $f(x) = f(\chi_1(\alpha^\phi)(x)) = \alpha f(x) = s(y)$. Hence there is a unique $z \in D$ with $t(z) = x$ and $g(z) = y$. Now $t(\chi_1(\overline{\gamma^\phi})\chi_1(\overline{b})z) = \chi_1(\alpha^\phi)\chi_1(b)t(z) = x$ and $g(\chi_1(\overline{\gamma^\phi})\chi_1(\overline{b})z) = \gamma g(z) = y$, hence z is indeed fixed by $\chi_1(\overline{\alpha^\phi}\chi_1(\overline{b}))$.

We have in the course of the above already asserted that $\chi_1(\alpha^\phi)\chi_1(b)t(z) = x$ and $\gamma g(z) = y$. Suppose there is some $w \in D$ fixed by $\chi_1(\overline{\gamma^\phi})\chi_1(\overline{b})$ with $\chi_1(\alpha^\phi)\chi_1(b)t(w) = x$ and $\gamma g(w) = y$. Then it is straightforward to check that $t(w) = x$ and $g(w) = y$, so that $w = z$. Hence, (2.1) is a pullback square, as required.

Proposition 2.4.4. *For any split comprehension category* $(\mathcal{C}, \mathbb{E}, \chi)$*, there is a split comprehension category* $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ *into which it embeds via the morphism* $(F_1, F_0) : (\mathcal{C}, \mathbb{E}, \chi) \to (\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$*, where* $F_0 : \mathcal{C} \to \widetilde{\mathcal{C}}$ *and, for each* $A \in \mathcal{C}$*,* $(F_1)_A : \mathcal{C}(A) \to \widetilde{\mathcal{C}}(F_0(A))$ *are each the inclusion of a category into its Karoubi envelope.*

Proof. The construction of $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ is as above, and the statement about F_0 is immediate. To verify that each $(F_1)_A$ is the inclusion into a Karoubi envelope, just observe that for $\phi \in \mathbb{E}(A)$ we always have $1_A^*(\phi) = \phi$ and every idempotent $b : \phi \to \phi$ commutes with $(1_A)^\phi = 1_\phi$. We need to verify that (F_0, F_1) behaves well with respect to reindexing and comprehension, but this is a trivial check. □

Let us refer to this split comprehension category as the *retract completion* of $(\mathcal{C}, \mathbb{E}, \chi)$ or *idempotent splitting model.*

Proposition 2.4.5. *For any split comprehension category* $(\mathcal{C}, \mathbb{E}, \chi)$ *with finite product types, the idempotent splitting model* $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ *also has finite product types. Moreover the inclusion functor* (F_0, F_1) *preserves them.*

Proof. This follows easily from the construction of products in the Karoubi envelope from Proposition 2.3.4. □

Remark 2.4.6. We could perform an analogous construction for display map categories. Seen this way, our construction of the idempotent splitting model is a generalization of the construction in Proposition 2.5.7 in [36]. Given a display map category $(\mathcal{C}, \mathcal{F})$, we could consider the problem of finding a class $\widetilde{\mathcal{F}}$ of display maps in $\widetilde{\mathcal{C}}$. Whenever $f : B \twoheadrightarrow A$ is in $\mathcal{F}$ and $\alpha : A \to A$ is an idempotent for which there exists a pullback square

$$\begin{array}{ccc} B & \xrightarrow{\overline{\alpha}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[\alpha]{} & A \end{array} \tag{2.2}$$

where $\overline{\alpha}$ is an idempotent, then we need

$$\alpha f : (B, \overline{\alpha}) \to (A, \alpha)$$

to be in $\widetilde{\mathcal{F}}$. If, moreover, there is an idempotent $\beta : B \to B$ satisfying $f\beta = f$ and $\overline{\alpha}\beta = \beta\overline{\alpha}$, then we also need

$$\alpha f : (B, \overline{\alpha}\beta) \to (A, \alpha)$$

to be in $\widetilde{\mathcal{F}}$. This might seem to correspond to the construction we have just carried out, however, a class of display maps needs to be closed under composition with isomorphisms, and the class of isomorphisms in $\widetilde{\mathcal{C}}$ might contain some unexpected members: it contains not just the isomorphisms of $\mathcal{C}$ (between objects in the image of the embedding of $\mathcal{C}$) but also the map α as the identity on (A, α). Even more generally, whenever two idempotents (C, γ) and (D, δ) represent the same presheaf (using the viewpoint of Remark 2.3.2) they are isomorphic, even though C and D might seem quite far from isomorphic. Hence we would have to close our tentative class of display maps under composition with isomorphism, and be satisfied with no better description than that.

There is another advantage of using comprehension categories: they allow us to consider 'being a display map' as a structure rather than a property of maps. A full comprehension category can be seen as a category where every morphism comes equipped with a possibly empty set of display map *structures*, where the existence of a structure on a morphism implies the existence of all pullbacks of that morphism and the existence of a structure on those pullbacks, hence the class of morphisms with non-empty sets of structures form a class of display maps. A full split comprehension category includes strictly functorial choice of pullback of structures, which may exist even when the underlying category does not admit a strictly functorial choice of pullbacks for all display maps. The fibre categories of *non-full* comprehension categories may be seen as encoding more information about these structures, if they contain only 'structure-preserving' morphisms or morphisms with structure themselves. However, we will largely concern ourselves here with making use of the existence of structure, and from §2.7 we will restrict to full comprehension categories. In the situation of Definition 2.4.2, a map $h : (C, \gamma) \to (A, \alpha)$ is a display map in $\widetilde{\mathcal{C}}$ because it factorizes as some $g : C \to B$ followed by some display map $f : B \twoheadrightarrow A$ followed by $\alpha : A \to A$, where there exists a pullback square as in (2.2) where $\overline{\alpha}$ is idempotent, and a $\beta : B \to B$ which is idempotent and satisfies $f\beta = f$ and $\overline{\alpha}\beta = \beta\overline{\alpha}$, and g is an isomorphism $(C, \gamma) \cong (B, \overline{\alpha}\beta)$. There is no reason to expect any such choice of $(f, g, \overline{\alpha}, \beta)$ to be unique in any useful sense, yet all constructions on display maps involve making a choice of such 'structure'.

Remark 2.4.7. In principle, one should be able to give analogues to 2.4.5 for the various notions of finite sum types. We omit them here, as to include them would require either formulating the definition of finite sums for comprehension categories or formulating the idempotent splitting model for display map categories.

2.5 Dependent products

We have already restricted to the case of split comprehension categories, but the following argument can be made to work in the case of a well-rooted display map category. In that case, the following may be seen as a generalization of Proposition 2.5.11 in [36]. Seen in the present context, it is clear that the use of identity types in the proof of that proposition is to make a canonical choice of presentation as a retract of a display map for each map in the saturation of the class of display maps.

Proposition 2.5.1. *Suppose that* $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ *has strictly stable dependent products. Then* $(\mathcal{C}, \mathbb{E}, \chi)$ *has strictly stable weak dependent products.*

Proof. Let $A \in \mathcal{C}$ be a base-type, $\phi \in \mathbb{E}(A)$ a type over A with comprehension $f : B \to A$, and $\psi \in \mathbb{E}(B)$ a type over B with comprehension $g : C \to B$. Then this data includes into the idempotent splitting model as $(A, 1_A) \in \widetilde{\mathcal{C}}$, $(\phi, 1_\phi) \in \mathbb{F}(A, 1_A)$ and $(\psi, 1_\psi) \in \mathbb{F}(B, 1_B)$. It is easy to check that the data of a Π-type for this triple of types in the idempotent splitting model is precisely the data of a weak Π-type for the original triple of types. Moreover, strict stability under reindexing of these weak Π-types follows directly from the strict stability of the Π-types in the idempotent splitting model. □

The main result of this chapter is the following converse to Proposition 2.5.1.

Theorem 2.5.2. *Suppose that* $(\mathcal{C}, \mathbb{E}, \chi)$ *has weak* Π*-types. Then* $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ *has* Π*-types.*

Notation 2.5.3. Let (A, α) be an object of $\widetilde{\mathcal{C}}$. Let (ϕ, b) an object of $\mathbb{F}(A, \alpha)$, where the comprehension of ϕ is $f : B \to A$ in $\mathcal{C}$ and $\chi_1(b) = \beta$. Finally, let (ψ, c) be an object of $\mathbb{F}(B, \chi(\alpha^\phi b))$. If we consider the underlying types then we can form the weak Π-type in $(\mathcal{C}, \mathbb{E}, \chi)$, so let $\Pi_\phi \psi$ be $(\pi \in \mathbb{E}(A), \lambda : \pi \to \pi, \epsilon : f^*(\pi) \to \psi)$.

We must now find an idempotent on π making it into $\Pi_{(\phi,b)}(\psi, c)$. We take the idempotent ρ, which corresponds to "postcompose with c". More precisely, ρ is the unique map with $\lambda\rho = \rho$ and $\epsilon \circ f^*(\rho)$ given by the following composite

$$f^*(\pi) = \beta^*(f^*(\pi)) \xrightarrow{\beta^*(\epsilon)} \beta^*(\psi) = \psi \xrightarrow{c} \psi.$$

Note that ϵ need not be fixed by reindexing along β, but $\beta^*(\epsilon)$ is fixed by it.

Proposition 2.5.4. (π, ρ) *gives an object of* $\mathbb{F}(A, \alpha)$*.*

Proof. We use the fact that weak Π-types in $(\mathcal{C}, \mathbb{E}, \chi)$ are stable under reindexing. We are given that reindexing along α preserves ϕ. That ψ is fixed by reindexing along $\chi_1(\alpha^\phi)$ (the pullback of ϕ along f) follows from the hypothesis that ψ is fixed by reindexing along $\chi_1(\alpha^\phi b)$. Hence π is fixed by reindexing along α.

It remains to check that ρ is idempotent and commutes with α^ϕ. We are given that the weak Π-type (π, ρ, ϵ) is stable under reindexing, in particular by α. Since $\chi_1(\alpha^\phi)\beta = \beta\chi_1(\alpha^\phi)$, we see that $\chi_1(\alpha^\phi)^*(\beta^*(\epsilon)) = \beta^*(\epsilon)$. We are given that $\chi_1(\alpha^\phi b)^*(c) = c$ and hence $\chi_1(\alpha^\phi)^*(c) = c$. Hence ρ is indeed fixed by reindexing along α, i.e. it commutes with α^ϕ.

Let us consider the transpose of $\rho\rho$:

$$\begin{aligned}
\epsilon \circ f^*(\rho\rho) &= \epsilon \circ f^*(\rho) \circ f^*(\rho) \\
&= c \circ \beta^*(\epsilon) \circ f^*(\rho) \\
&= c \circ \beta^*(\epsilon \circ f^*(\rho)) \\
&= c \circ \beta^*(c \circ \beta^*(\epsilon)) \\
&= c \circ \beta^*(\epsilon) \\
&= \epsilon \circ f^*(\rho)
\end{aligned}$$

which is the transpose of ρ. □

We need to give an evaluation map.

Proposition 2.5.5. *The map* $\mathsf{ev} = c \circ \beta^*(\epsilon)$ *defines a map* $(\alpha \circ f)^*(\pi, \rho) \to (\psi, c)$.

Proof. Note that $(\alpha f)^*(\pi, \rho) = (f^*(\pi), f^*(\rho))$, since π and ρ are fixed by reindexing along α. So we need to check that $c \circ \mathsf{ev} \circ f^*(\rho) = \mathsf{ev}$, and that $\mathsf{ev} \circ (\chi_1(\alpha^\phi b))^{f^*(\pi)} = (\chi_1(\alpha^\phi b))^\psi \circ \mathsf{ev}$. The first equation is similar to the calculation above in Proposition 2.5.4. The second is just the statement that ev is fixed by reindexing along $\chi_1(\alpha^\phi b)$, which was also checked in Proposition 2.5.4. □

Proposition 2.5.6. $((\pi, \rho) \in \mathbb{F}(A, \alpha), \mathsf{ev} : (\alpha \circ f)^*((\pi, \rho)) \to (\psi, c))$ *is a* Π*-type for* $(\phi, b) \in \mathbb{F}(A, \alpha)$, $(\psi, c) \in \mathbb{F}(B, \chi_1(\alpha^\phi \beta))$.

Proof. Let $(\omega, w) \in \mathbb{F}(A, \alpha)$. We are required to show that the map

$$\mathbb{F}(A, \alpha)((\omega, w), (\pi, \rho)) \to \mathbb{F}(B, \chi_1(\alpha^\phi b))((\alpha \circ f)^*(\omega, w), (\psi, c))$$
$$t \mapsto \mathsf{ev} \circ (\alpha \circ f)^*(t)$$

is a bijection. Let us express the map in a more convenient form. Given $t : (\omega, w) \to (\pi, \rho)$, we have $(\alpha \circ f)^*(t) = f^*(t)$ since t is fixed under reindexing by α. Also since $\epsilon \circ f^*(\rho) = \mathsf{ev}$ and $\rho \circ t = t$, we have $\mathsf{ev} \circ f^*(t) = \epsilon \circ f^*(t)$. Hence the map in question is the restriction of the weak adjunction map $\mathbb{E}(A)(\omega, \pi) \to \mathbb{E}(B)(f^*(\omega), \psi)$.

Given $t_1, t_2 : (\omega, w) \to (\pi, \rho)$, if $\epsilon \circ f^*(t_1) = \epsilon \circ f^*(t_2)$, then $\lambda \circ t_1 = \lambda \circ t_2$. But $t_i = \rho \circ t_i = \lambda \circ \rho \circ t_i = \lambda \circ t_i$, so indeed $t_1 = t_2$. To complete the check that this map is a bijection, it is enough to check that the section of $t \mapsto \epsilon \circ f^*(t)$ given by the weak adjunction restricts to a map $\mathbb{F}(B, \chi_1(\alpha^\phi b))((\alpha \circ f)^*(\omega, w), (\psi, c)) \to \mathbb{F}(A, \alpha)((\omega, w), (\pi, \rho))$. So suppose that $t : \omega \to \pi$ satisfies $\lambda \circ t = t$ and $s = \epsilon \circ f^*(t)$ satisfies $\gamma s f^*(w) = s$, $\beta^*(s) = s$ and $\chi_1(\alpha^\phi)^*(s) = s$. Then

$$\begin{aligned}
\epsilon \circ f^*(\rho \circ t) &= \gamma \circ \beta^*(\epsilon) \circ (f \circ \beta)^*(t) \\
&= \gamma \circ \beta^*(\epsilon f^*(t)) \\
&= \gamma \circ \beta^*(s) \\
&= \gamma \circ s \\
&= s \\
&= \epsilon \circ f^*(t)
\end{aligned}$$

and since $\lambda \circ \rho \circ t = \rho \circ t$, we deduce that $\rho \circ t = t$. We see trivially that $\epsilon \circ f^*(t \circ w) = \epsilon \circ f^*(t)$ and hence that $t \circ w = t$. Finally, we have $\epsilon \circ f^*(\alpha^*(t)) = \epsilon \circ \chi_1(\alpha^\phi)^*(f^*(t))$ and $\chi_1(\alpha^\phi)^*(\epsilon) = \epsilon$ (because the weak Π-type is stable under reindexing along α), so $\alpha^*(t) = t$. Thus t is a valid map $(\omega, w) \to (\pi, \rho)$ in $\mathbb{F}(A, \alpha)$. $\square$

Proof of Theorem 2.5.2. It remains to check that the Π-types we have just constructed are stable under reindexing. This follows immediately from their definition in terms of the reindexing-stable weak Π-types. $\square$

2.6 Fullness and Ehrhard comprehension

Proposition 2.6.1. *If the split comprehension category $(\mathcal{C}, \mathbb{E}, \chi)$ is full, then the comprehension category of retracts $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$ is also full.*

Proof. Let $(A, \alpha) \in \widetilde{\mathcal{C}}$ be a base-type, and suppose that we have dependent types $(\phi, b), (\psi, c) \in \mathbb{F}(A, \alpha)$ where the comprehensions in $(\mathcal{C}, \mathbb{E}, \chi)$ of $\phi, \psi \in \mathbb{E}(A)$ are $f : B \to A$ and $g : C \to A$ respectively. Then the comprehensions of (ϕ, b) and (ψ, c) are $\alpha f : (B, \chi_1(\alpha^\phi b)) \to (A, \alpha)$ and $\alpha g : (C, \chi_1(\alpha^\psi c)) \to (A, \alpha)$ respectively.

Recall that the comprehension of an arrow $w : (\phi, b) \to (\psi, c)$ is $\chi_1(w)\chi_1(\alpha^\phi) = \chi_1(\alpha^\psi)\chi_1(w)$. given that χ_1 is a faithful functor, it follows that this operation is injective. Conversely, given $s : (B, \chi_1(\alpha^\phi b)) \to (C, \chi_1(\alpha^\psi c))$. Then since

$$\begin{aligned} gs &= g\chi_1(\alpha^\psi)s \\ &= \alpha g s \\ &= \alpha f \end{aligned}$$

there is a unique $t : B \to C$ such that $\chi_1(\alpha^\psi)t = s$ and $gt = f$. It is then an easy exercise to check that t satisfies $t \circ t = t$, $\chi_1(c)t\chi_1(b) = t$ and $\chi_1(\alpha^\psi)t = t\chi_1(\alpha^\phi)$. Hence t corresponds to a map $\hat{t} : \phi \to \psi$ in $\mathbb{E}(A)$ giving rise to a map $(\phi, b) \to (\psi, c)$ in $\mathbb{F}(A, \alpha)$ whose comprehension is s, as required. $\square$

Proposition 2.6.2. *If the split comprehension category $(\mathcal{C}, \mathbb{E}, \chi)$ has (strong unit types and) Ehrhard comprehension, then so does $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$.*

Proof. We apply 1.4.5. We have already seen that the idempotent splitting model has unit types in 2.4.5. let $h : (C, \gamma) \to (A, \alpha)$ be a map in $\widetilde{\mathcal{C}}$ and (ϕ, b) a type in $\mathbb{F}(A, \alpha)$ where the comprehension of ϕ is $f : B \to A$. Given a map $t : (\top_C, 1_{\top_C}) \to (\phi, b)$ lying over h, i.e. a map $t : \top_C \to \phi$ satisfying $\alpha^\phi b t \gamma^{\top_C} = t$, the induced map $\check{t} : (C, \gamma) \to (B, \chi_1(\alpha^\phi b))$ is given by $\chi_1(\alpha^\phi b)\widehat{t}\gamma :$ $C \to B$, where $\widehat{t}$ is the map $C \to B$ over A corresponding to t via the operation in $(\mathcal{C}, \mathbb{E}, \chi)$. Now $\check{t}$ does indeed satisfy $\alpha f \check{t} = h$ and $\chi_1(\alpha^\phi b)\check{t}\gamma = \check{t}$. We are required to show that this is a bijection.

It is an injection because if $\check{t}_1 = \check{t}_2$ then

$$\widehat{t_1} = \alpha^{\phi}\widehat{bt_1\gamma^{\top_C}} = \chi_1(\alpha^{\phi}b)\widehat{t_1}\gamma = \chi_1(\alpha^{\phi}b)\widehat{t_2}\gamma = \alpha^{\phi}\widehat{bt_2\gamma^{\top_C}} = \widehat{t_2}$$

and hence $t_1 = t_2$. To see that it is a surjection, consider some $s : C \to B$ which satisfies $\alpha f s = h$ and $\chi_1(\alpha^{\phi}b)s\gamma = s$. It is easy seen that it also satisfies $fs = h$, and so using the fact that $(\mathcal{C}, \mathbb{E}, \chi)$ has Ehrhard comprehension we can write $s = \widehat{t}$ for some unique $t : \top_C \to A$ lying over h. It remains to check that t satisfies $\alpha^{\phi}bt\gamma^{\top_C} = t$ and that $\check{t} = s$. But

$$\alpha^{\phi}\widehat{bt\gamma^{\top_C}} = \chi_1(\alpha^{\phi}b)s\gamma = s = \widehat{t}$$

and hence we get the first equation. Finally,

$$\check{t} = \chi_1(\alpha^{\phi}b)\widehat{t}\gamma = \chi_1(\alpha^{\phi}b)s\gamma = s$$

as required. □

2.7 Dependent sums

Henceforth we will assume that our split comprehension categories are full.

Theorem 2.7.1. *If the original comprehension category $(\mathcal{C}, \mathbb{E}, \chi)$ is equipped with dependent sums then so is the idempotent splitting comprehension category $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$.*

Proof. We use the notation of Definition 1.6.7 for dependent sums. Let $(A, \alpha) \in \widetilde{\mathcal{C}}$ be a base-type, $(\phi, b) \in \mathbb{F}(A, \alpha)$ a dependent type where the comprehension of $\phi \in \mathbb{E}(A)$ is $f : B \to A$, and $(\psi, c) \in \mathbb{F}(B, \beta)$ where the comprehension of $\psi \in \mathbb{E}(B)$ is $g : C \to B$. Then letting

$$\sigma = \theta_{A,\phi,\psi} \circ (\chi_1(\alpha^{\phi}))^{\psi} \circ \chi_1(\chi_1(b)^{\psi}) \circ \chi_1(c) \circ \theta^{-1}_{A,\phi,\psi},$$

there is a unique $s : \Sigma_{A,\phi}\psi \to \Sigma_{A,\phi}\psi$ such that $\chi_1(\alpha^{\Sigma_{A,\phi}\psi}) \circ \chi_1(s) = \sigma$ and this s satisfies $s \circ s = s$ and $\alpha^*(s) = s$ (as in the proof of Proposition 2.6.1). Hence we take $(\Sigma_{A,\phi}\psi, s) \in \mathbb{F}(A, \alpha)$ as the underlying type, and the θ-isomorphism is given by $\sigma \circ \theta_{A,\phi,\psi}$. Stability under reindexing is immediate from the construction. □

2.8 Identity types

We would like the idempotent splitting construction to preserve the existence of identity types. We cannot expect this in general, but it does hold given a mild additional condition on the identity types of the original model. We say that the identity types in these well-behaved models *preserve idempotents*. The construction given here is closely related to the one in Proposition 2.5.10 in [36]. We formulate this result for *full* split comprehension categories.

Notation 2.8.1. For convenience and since we always work with full comprehension categories, in the following we will assume that the hom-sets of the original fibre categories $\mathbb{E}(A)$ are actually identified with the hom-sets of the slice categories $\mathcal{C}/A$ of the original base category. We shall not do this for the idempotent splitting comprehension category, as this would induce two incompatible ways of identifying the arrows of the fibre categories $\mathbb{F}(A, \alpha)$ with arrows in $\mathcal{C}$ (see the definition of $\widetilde{\chi}$). Instead, we shall continue to identify arrows of each $\mathbb{F}(A, \alpha)$ with arrows in $\mathbb{E}(A)$ satisfying some conditions, and hence with certain arrows in $\mathcal{C}/A$.

Definition 2.8.2. Given a full split comprehension category $(\mathcal{C}, \mathbb{E}, \chi)$ with binary product types and identity types, we say that *identity types preserve idempotents* if it is equipped with, for each $A \in \mathcal{C}$, $\phi \in \mathbb{E}(A)$ and idempotent map $b : \phi \to \phi$ in $\mathbb{E}(A)$, an idempotent map $K_{A,\phi}(b) : I_{A,\phi} \to I_{A,\phi}$ which fits into the commutative diagram

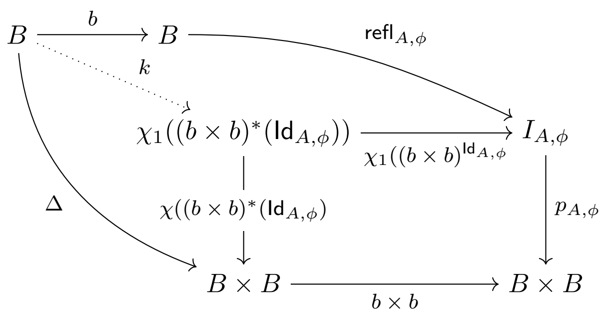
$$\begin{array}{ccc} B & \xrightarrow{\quad b \quad} & B \\ \big\downarrow{\scriptstyle \mathsf{refl}_{A,\phi}} & & \big\downarrow{\scriptstyle \mathsf{refl}_{A,\phi}} \\ I_{A,\phi} & \overset{K_{A,\phi}(b)}{\dashrightarrow} & I_{A,\phi} \\ \big\downarrow{\scriptstyle p_{A,\phi}} & & \big\downarrow{\scriptstyle p_{A,\phi}} \\ B \times B & \xrightarrow[b \times b]{} & B \times B \end{array} \tag{2.3}$$

where we have written $\chi_1(\phi \times \phi)$ as $B \times B$, since it is indeed a product in $\mathcal{C}/A$. Moreover, we require the maps $K_{A,\phi}(b)$ to be stable under reindexing in A.

Remark 2.8.3. There is a natural candidate for the dotted map in (2.3), which is stable under reindexing. It is the map induced by the structure of identity types, or more precisely, $K_{A,\phi}(b)$ is given by

$$K_{A,\phi}(b) = \chi_1((b \times b)^{\mathsf{Id}_{A,\phi}}) \circ J_{A,\phi,\chi_1((b \times b)^{\mathsf{Id}_{A,\phi}})^*(\mathsf{Id}_{A,\phi})}(k)$$

where $k : B \to \chi_1((b \times b)^{\mathsf{Id}_{A,\phi}})$ is given by factorizing through a pullback as in the diagram



in $\mathcal{C}$.

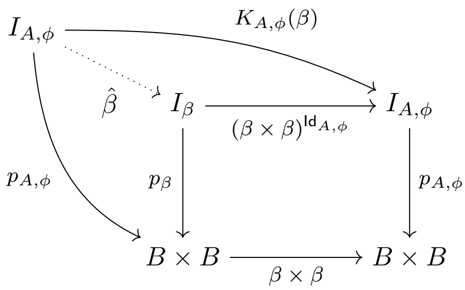
We would not expect this map to be an idempotent in a general intensional type theory: we would only expect the idempotence up to propositional equality. However, we might expect it to be idempotent if the identity types are of the special kind that arise from forming function spaces with an interval object. Whether the $K_{A,\phi}(b)$ are constructed in this canonical way is irrelevant to the argument below, all we use is that the $K_{A,\phi}(b)$ fit into the diagram (2.3), are idempotent, and are stable under reindexing.

Theorem 2.8.4. *Let $(\mathcal{C}, \mathbb{E}, \chi)$ be a full split comprehension category with binary product types and identity types which preserve idempotents. Then the idempotent splitting model $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$. is a full split comprehension category with binary product types and identity types.*

Let us give the construction of the identity types in $(\widetilde{\mathcal{C}}, \mathbb{F}, \widetilde{\chi})$. Let $(A, \alpha) \in \widetilde{\mathcal{C}}$ be a base type and suppose we have a type $(\phi, \beta) \in \mathbb{F}(A, \alpha)$, where the comprehension of $\phi \in \mathbb{E}(A)$ is $f : B \to A$. Let us write $\chi_1(\phi \times \phi)$ as $B \times B$ (it is indeed a product of B with itself in $\mathcal{C}/A$). Now $\mathsf{Id}_{A,\phi}$ is not necessarily stable under reindexing along $\beta \times \beta$. But if we let $\mathsf{Id}_\beta \in \mathbb{E}(B \times B)$ be $(\beta \times \beta)^*(\mathsf{Id}_{A,\phi})$, then Id_β is preserved by reindexing along the idempotent $\beta \times \beta$. Using the interpretation of types as spaces (see [42]), we can give some topological intuition for this construction. For points x, y of ϕ, the type $\mathsf{Id}_{A,\phi}(x, y)$ is the space of paths in ϕ from x to y, and the type $\mathsf{Id}_\beta(x, y)$ is the space of paths in ϕ from $\beta(x)$ to $\beta(y)$.

We will take Id_β to be the underlying object of the identity type. Observe that since identity types are stable under reindexing, we have $(\alpha^{\phi\times\phi})^*(\mathsf{Id}_{A,\phi}) = \mathsf{Id}_{A,\phi}$ and also $(\alpha^{\phi\times\phi})^*(\mathsf{Id}_\beta) = \mathsf{Id}_\beta$ since β and $\beta \times \beta$ are stable under reindexing along α. Denote the comprehensions of $\mathsf{Id}_{A,\phi}$ and Id_β by $I_{A,\phi}$ and I_β respectively. By assumption, we have an idempotent map $K_{A,\phi}(\beta) : I_{A,\phi} \to I_{A,\phi}$, which we think of as applying β pointwise to each path. Define the map $\hat{\beta} : \mathsf{Id}_{A,\phi} \to \mathsf{Id}_\beta$

in $\mathbb{E}(B \times B)$ to be given by the dotted arrow induced by the pullback in the diagram in $\mathcal{C}$ below.



In terms of the intuitive topological picture, the map $\hat{\beta}$ maps the triple $(x, y, p : x \to y)$ to $(x, y, \beta \circ p : \beta(x) \to \beta(y))$. Now define $\beta_{\mathsf{Id}} : \mathsf{Id}_\beta \to \mathsf{Id}_\beta$ as $(\beta \times \beta)^*(\hat{\beta})$. Equivalently, β_{Id} is $(\beta \times \beta)^*(K_{A,\phi}(\beta))$. Hence β_{Id} is idempotent, and preserved by reindexing along $\alpha^{\phi\times\phi}$ and along $\beta \times \beta$. Intuitively, the map β_{Id} corresponds to sending $(x, y, p : \beta(x) \to \beta(y))$ to $(x, y, \beta \circ p : \beta(x) \to \beta(y))$ where $\beta \circ p$ is the result of apply β pointwise to the path p.

Proposition 2.8.5. $(\mathsf{Id}_\beta, \beta_{\mathsf{Id}})$ *is an identity type for* $(\phi, \beta) \in \mathbb{F}(A, \alpha)$.

Proof. The 'reflexivity' map,

$$\mathsf{refl}_\beta : (B, \alpha^\phi \circ \beta) \to (I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}}),$$

is given by $\mathsf{refl}_\beta = \hat{\beta} \circ \mathsf{refl}_{A,\phi} \circ \alpha^\phi \circ \beta$. It is easily checked that the map refl_β really does define a map

$$(B, \alpha^\phi \circ \beta) \to (I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}})$$

if one first checks the equations $\hat{\beta} \circ K_{A,\phi}(\beta) = (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \hat{\beta}$ and $\beta_{\mathsf{Id}} \circ \hat{\beta} = \hat{\beta}$. Note that it follows from the former of these that refl_β may also be written as $(\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \hat{\beta} \circ \mathsf{refl}_{A,\phi}$.

Now suppose that we have a type $(\psi, \gamma) \in \mathbb{F}(I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}})$, where the comprehension of $\psi \in \mathbb{E}(\mathsf{Id}_\beta)$ is $g : C \to I_\beta$, and a map

$$h : (B, \alpha^\phi \beta) \to (C, ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ ((\beta \times \beta)^{\mathsf{Id}_\beta})^\psi \circ (\beta_{\mathsf{Id}})^\psi \circ \gamma)$$

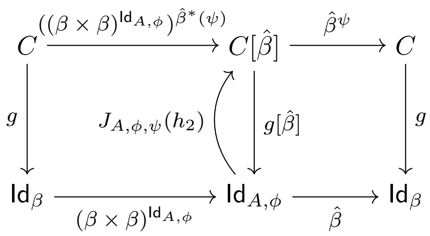
with $((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}} \circ g) \circ h = \mathsf{refl}_\beta$. We may rewrite the left-hand side of this equation simply as $g \circ h$. Now ψ is preserved by reindexing along $(\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta}$ and we have $((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ ((\beta \times \beta)^{\mathsf{Id}_\beta})^\psi \circ h = h$, so let $h_1 : B \to C$ be the unique factorization of $h : B \to C$ through $((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ ((\beta \times \beta)^{\mathsf{Id}_\beta})^\psi$ satisfying $gh_1 = \hat{\beta} \circ \mathsf{refl}_{A,\phi}$. Let $h_2 : B \to C[\hat{\beta}]$ be the unique factorization of h_1 through $\hat{\beta}^{\psi} : C[\hat{\beta}] \to C$ satisfying $g[\hat{\beta}] \circ h_2 = \mathsf{refl}_{A,\phi}$, where $g[\hat{\beta}] : C[\hat{\beta}] \to I_{A,\phi}$ is the comprehension of $\hat{\beta}^*(\psi) \in \mathbb{E}(I_{A,\phi})$.

We are now in a situation where we may use the eliminator for the original identity types. The map $J_{A,\phi,\psi}(h_2) : \mathsf{Id}_{A,\phi} \to C$ satisfies $J_{A,\phi,\psi}(h_2) \circ \mathsf{refl}_{A,\phi} = h_2$ and $g[\hat{\beta}] \circ J_{A,\phi,\psi}(h_2) = 1_{\mathsf{Id}_{A,\phi}}$.

Now, the required extension of h is given by

$$j = ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^{\psi} \circ \gamma \circ \left(\hat{\beta}^{\psi} \circ J_{A,\phi,\psi}(h_2) \circ (\beta \times \beta)^{\mathsf{Id}_{A,\phi}}\right) \circ (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}}$$

Note that the inner bracketed composite is given by traversing from bottom-left to top-right in the diagram



containing two pullback squares, since ψ is fixed by reindexing along $(\beta \times \beta)^{\mathsf{Id}_\beta}$ and β_{Id}, and $\hat{\beta} \circ (\beta \times \beta)^{\mathsf{Id}_{A,\phi}} = (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}}$. Hence j factorizes through the map

$$(\hat{\beta} \circ (\beta \times \beta)^{\mathsf{Id}_{A,\phi}})^{\psi} = ((\beta \times \beta)^{\mathsf{Id}_\beta})^{\psi} \circ (\beta_{\mathsf{Id}})^{\psi}$$

so that $((\beta \times \beta)^{\mathsf{Id}_\beta})^{\psi} \circ (\beta_{\mathsf{Id}})^{\psi} \circ j = j$. Hence, j is a valid map

$$(I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}}) \to (C, ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^{\psi} \circ ((\beta \times \beta)^{\mathsf{Id}_\beta})^{\psi} \circ (\beta_{\mathsf{Id}})^{\psi} \circ \gamma)$$

in the Karoubi envelope. It is trivial to check that

$$\begin{aligned}((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}} \circ g) \circ j &= 1_{(I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta\times\beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}})} \\ &= (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}}\end{aligned}$$

i.e., that j is a section of the comprehension of $(\psi, \gamma) \in \mathbb{F}(I_\beta, (\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta} \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ \beta_{\mathsf{Id}})$.

We have only to show that $j \circ \mathsf{refl}_\beta = h$. So we calculate

$$\begin{aligned}
j \circ \mathsf{refl}_\beta &= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ \hat{\beta}^\psi \circ J_{A,\phi,\psi}(h_2) \circ (\beta \times \beta)^{\mathsf{Id}_{A,\phi}} \circ \beta_{\mathsf{Id}} \circ \hat{\beta} \circ \mathsf{refl}_{A,\phi} \circ \alpha^\phi \circ \beta \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ \hat{\beta}^\psi \circ J_{A,\phi,\psi}(h_2) \circ (\beta \times \beta)^{\mathsf{Id}_{A,\phi}} \circ \hat{\beta} \circ \mathsf{refl}_{A,\phi} \circ \alpha^\phi \circ \beta \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ \hat{\beta}^\psi \circ J_{A,\phi,\psi}(h_2) \circ K_{A,\phi}(\beta) \circ \mathsf{refl}_{A,\phi} \circ \alpha^\phi \circ \beta \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ \hat{\beta}^\psi \circ J_{A,\phi,\psi}(h_2) \circ \mathsf{refl}_{A,\phi} \circ \beta \circ \alpha^\phi \circ \beta \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ \hat{\beta}^\psi \circ h_2 \circ \beta \circ \alpha^\phi \circ \beta \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ h_1 \circ \beta \circ \alpha^\phi \\
&= \gamma \circ ((\alpha^{\phi\times\phi})^{\mathsf{Id}_\beta})^\psi \circ (\beta \times \beta)^{\mathsf{Id}_\beta} \circ h_1 \circ \alpha^\phi \\
&= \gamma \circ h \circ \alpha^\phi \\
&= h
\end{aligned}$$

as required. □

Proof of Theorem 2.8.4. We have just constructed the identity types in Proposition 2.8.5 and it remains to complete the check that they are stable under reindexing. This is clear from the construction given. □

To conclude this chapter, we have shown how to take a model of the basic Martin-Löf type theory minus the η-rule for dependent products and build another model which includes the η-rule. Here, one may take Martin-Löf type theory to be dependent type theory with Π-types, Σ-types, intensional identity types, and finite product types. The only extra condition required on the original model is the condition that *identity types preserve idempotents* from Definition 2.8.2.

Chapter 3

Biproducts of algebras

A biproduct of two objects A and B is a product which is also a sum in such a way that the product and sum structures are compatible. It makes sense to postpone investigation of the Diller-Nahm category until after we have made a careful investigation of biproducts — and in particular biproducts in Kleisli categories — in the present chapter. Much of this chapter can be found in less detail in, for example, section 4 of [10].

3.1 Biproducts in a category with a zero object

Let us recall some definitions from [33].

Definition 3.1.1. A category $\mathbb{A}$ has a *zero object* if it has an object $0 \in \mathbb{C}$ which is both an initial and a terminal object.

Clearly any zero object is unique up to unique isomorphism.

Definition 3.1.2. Let $\mathbb{A}$ be a category with a zero object. Then for any two objects $X, Y \in \mathbb{C}$, the *zero morphism*, denoted $0 : X \to Y$, from X to Y is the unique map $X \to Y$ which factorizes through 0. That is to say, it is the composite

$$X \xrightarrow{!_X} 0 \xrightarrow{!_Y} Y$$

where the first arrow is induced by the fact that 0 is a terminal object and the second is induced by the fact that 0 is an initial object.

Clearly, zero morphisms form a two-sided ideal in the morphisms of $\mathbb{A}$ in the sense that the composite of any morphism with a zero morphism is again a zero morphism. Henceforth, let $\mathbb{A}$ be a category with a zero object 0.

Definition 3.1.3. Suppose we are given objects $X, Y \in \mathbb{A}$. Then a *biproduct* for X and Y consists of an object $X \oplus Y$, together with maps

$$\iota_1 : X \to X \oplus Y \qquad \iota_2 : Y \to X \oplus Y$$
$$\pi_1 : X \oplus Y \to X \qquad \pi_2 : X \oplus Y \to Y$$

satisfying the equations

$$\pi_1 \iota_1 = 1_X \qquad \pi_1 \iota_2 = 0$$
$$\pi_2 \iota_1 = 0 \qquad \pi_2 \iota_2 = 1_Y,$$

making

$$X \xrightarrow{\iota_1} X \oplus Y \xleftarrow{\iota_2} Y$$

a coproduct diagram, and making

$$X \xleftarrow{\pi_1} X \oplus Y \xrightarrow{\pi_2} Y$$

a product diagram.

Remark 3.1.4. A biproduct is not simply a coproduct which coincides with a product. There needs to be a compatibility between the product and coproduct structures. Using exercise 4.(a) of VIII.2 in [33], we can express this compatibility condition as follows. We use the matrix notation from [33], described for maps from a coproduct to a product in III.5 and for the special case of maps between biproducts in VIII.2.

Proposition 3.1.5. *Objects X and Y in $\mathbb{A}$ admitting both a product and a coproduct admit a biproduct if and only if the canonical map*

$$c : X + Y \to X \times Y$$

given by

$$\begin{pmatrix} \pi_1 c \iota_1 & \pi_1 c \iota_2 \\ \pi_2 c \iota_1 & \pi_2 c \iota_2 \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix}$$

is an isomorphism.

Proof. In the forward direction, it is easy to check that if we take the biproduct $X \oplus Y$ for both the coproduct $X + Y$ and the product $X \times Y$, then the map c is actually an identity. Moreover, for any choice of coproduct and product of X and Y, the map c factorizes as

$$X + Y \xrightarrow{[\iota_X, \iota_Y]} X \oplus Y \xrightarrow{(\pi_X, \pi_Y)} X \times Y$$

both of which are isomorphisms.

For the converse, it is easy to check that if we equip with $X \times Y$ with the coprojections

$$X \xrightarrow{\iota_X} X + Y \xrightarrow{c} X \times Y$$
$$Y \xrightarrow{\iota_Y} X + Y \xrightarrow{c} X \times Y$$

then it is a coproduct and moreover the equations of 3.1.3 are satisfied. □

Remark 3.1.6. It is not sufficient that *some* isomorphism $X + Y \cong X \times Y$ exists. For example, consider the category of pointed sets, which has zero object $(\{*\}, *)$. The product of (X, x_0) and (Y, y_0) is simply $(X \times Y, (x_0, y_0))$ and the coproduct is $X \amalg Y/\sim$, where $\sim$ identifies x_0 and y_0, where the chosen element is the equivalence class $[x_0] = [y_0]$. Then for any infinite set X and $x_0 \in X$, there is an isomorphism

$$(X, x_0) + (X, x_0) \cong (X, x_0) \times (X, x_0).$$

However, the canonical map

$$(X, x_0) + (X, x_0) \to (X, x_0) \times (X, x_0).$$

is never an isomorphism for $|X| \geq 2$.

3.2 Naturality of the product-coproduct isomorphism

We will check one of the basic properties of biproducts, namely, that when every pair of objects in $\mathbb{A}$ has a biproduct, then the functorial operations on arrows between pairs of objects of induced by the coproduct agrees with the one induced by the product. This will be used in the construction of function and dependent product types in Proposition 4.5.7 and Lemma 7.3.5.

Proposition 3.2.1. *Let A, B, X, Y be objects of $\mathbb{A}$ and let $f : A \to X$ and $g : B \to Y$ be arrows. Then $f + g = f \times g$ are equal arrows $A \oplus B \to X \oplus Y$.*

Proof. It is easy to calculate that

$$\begin{aligned}
&\pi_{1,X,Y} \circ ((f + g) \circ \iota_{1,A,B}) \\
&\qquad = \pi_{1,X,Y} \circ (\iota_{1,X,Y} \circ f) = f = (f \circ \pi_{1,A,B}) \circ \iota_{1,A,B} \\
&\qquad\qquad = (\pi_{1,X,Y} \circ ((f \times g) \circ \iota_{1,A,B})
\end{aligned}$$

and so on. $\square$

We remarked in 3.1.6 that it does not suffice to exhibit an arbitrary isomorphism $X + Y \cong X \times Y$ to obtain a biproduct for X and Y. However, if we assume that every pair of objects X and Y is equipped with a chosen binary coproduct and product (including chosen coprojections and projections), then we can deduce that every pair of objects admits a biproduct provided we can find a family of isomorphisms

$$\alpha_{X,Y} : X + Y \to X \times Y$$

natural in X and Y (not necessarily the canonical map).

Let us record this result precisely. It will allow us to avoid some spurious generality when we come to the Diller-Nahm category.

Proposition 3.2.2. *Suppose that every pair of objects X and Y in $\mathbb{A}$ are equipped with objects $X + Y$ and $X \times Y$ together with maps*

$$\begin{array}{ll} \iota_{1,X,Y} : X \to X + Y & \qquad \iota_{2,X,Y} : Y \to X + Y \\ \pi_{1,X,Y} : X \times Y \to X & \qquad \pi_{2,X,Y} : X \times Y \to Y \end{array}$$

making $X + Y$ and $X \times Y$ into a coproduct and product respectively, and an isomorphism

$$\alpha_{X,Y} : X + Y \to X \times Y$$

which is natural in X and Y where the coproduct and product functors are determined by our choice of coprojections and projections. Then every pair of objects in $\mathbb{A}$ admits a biproduct.

Proof. The underlying object of the biproduct will be $X \oplus Y = X + Y$, and the coprojections will be given by $\iota_{1,X,Y} : X \to X + Y$ and $\iota_{2,X,Y} : Y \to X + Y$.

Observe that if we were to take $\pi_{1,X,Y} \circ \alpha_{X,Y} : X + Y \to X$ and $\pi_{2,X,Y} \circ \alpha_{X,Y} : X + Y \to Y$ as the projections, this would almost give us a biproduct in the sense that the two composites from 3.1.3 that are required to be zero morphisms are indeed so. For instance, the following diagram commutes

$$\begin{array}{ccccccc} X & \xrightarrow{\iota_{1,X,Y}} & X + Y & \xrightarrow{\alpha_{X,Y}} & X \times Y & \xrightarrow{\pi_{2,X,Y}} & Y \\ \| & & \downarrow 1_X + 0 & & \downarrow 1_X \times 0 & & \downarrow 0 \\ X & \xrightarrow[\iota_{1,X,Y}]{} & X + Y & \xrightarrow[\alpha_{X,Y}]{} & X \times Y & \xrightarrow[\pi_{2,X,Y}]{} & Y \end{array}$$

by naturality of $\alpha_{X,Y}$, coprojections and projections, which shows that $\pi_{2,X,Y} \circ \alpha_{X,Y} \circ \iota_{1,X,Y}$ is a zero morphism, and similarly for $\pi_{1,X,Y} \circ \alpha_{X,Y} \circ \iota_{2,X,Y}$.

We do not necessarily have the composite $\pi_{1,X,Y} \circ \alpha_{X,Y} \circ \iota_{1,X,Y}$ being the identity on X. However, it is an isomorphism: the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\iota_{1,X,0}} & X+0 & \xrightarrow{\alpha_{X,0}} & X \times 0 & \xrightarrow{\pi_{1,X,0}} & X \\ \| & & \downarrow{\scriptstyle 1_X+0} & & \downarrow{\scriptstyle 1_X \times 0} & & \| \\ X & \xrightarrow[\iota_{1,X,Y}]{} & X+Y & \xrightarrow[\alpha_{X,Y}]{} & X \times Y & \xrightarrow[\pi_{1,X,Y}]{} & X \end{array}$$

commutes by naturality of $\alpha_{X,Y}$, coprojections and projections, but the three arrows along the top are all isomorphisms, hence so is the composite of the three arrows along the bottom. We denote this automorphism $\pi_{1,X,Y} \circ \alpha_{X,Y} \circ \iota_{1,X,Y}$ of X by $\lambda_{X,Y} : X \to X$, and denote the analogous automorphism $\pi_{2,X,Y} \circ \alpha_{X,Y} \circ \iota_{2,X,Y}$ of Y by $\rho_{X,Y} : Y \to Y$. (Aside: we have also shown that $\lambda_{X,Y}$ is independent of Y and that $\rho_{X,Y}$ is independent of X). Now it is easy to check that taking for the projections

$$\lambda_{X,Y}^{-1} \circ \pi_{1,X,Y} \circ \alpha_{X,Y} : X+Y \to X \qquad \rho_{X,Y}^{-1} \circ \pi_{2,X,Y} \circ \alpha_{X,Y} : X+Y \to Y$$

makes $X+Y$ into a biproduct of X and Y. □

Proposition 3.2.3. *Suppose the category $\mathbb{A}$ has zero objects and binary coproducts. Then for any $X, Y \in \mathbb{A}$, X and Y admit a biproduct in $\mathbb{A}$ if and only if the maps*

$$1_X + 0 : X+Y \to X+0 \cong X \qquad 0 + 1_Y : X+Y \to 0+Y \cong Y$$

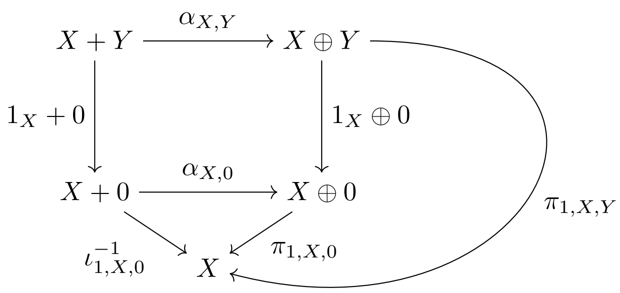
make $X+Y$ into a product of X and Y.

Proof. The given maps clearly validate the equations in 3.1.3. So if they make $X+Y$ into a product, then we clearly have a biproduct.

Conversely, if $\mathbb{A}$ has biproducts, then there is a canonical natural isomorphism

$$\alpha_{X,Y} : X+Y \to X \oplus Y$$

which intertwines the coprojections. Now, it is easy to see that the diagram



commutes and similarly for a diagram with sink Y. Hence the maps given in the proposition are isomorphic to a pair of product projections, so do indeed give the structure of a product to $X+Y$. □

3.3 Biproducts in Kleisli categories

Henceforth, instead of an arbitrary category $\mathbb{A}$ with zero object, we will suppose we have a category $\mathbb{C}$ with finite products and coproducts and a monad M over it. We will look at categories of algebras for this monad, starting with the Kleisli category. The reader may refer to [33] for background. Our task is to derive conditions on M that ensure that $\mathbb{C}$ does indeed have a zero object, and then to ensure that it has biproducts.

Lemma 3.3.1. *The Kleisli category* $\mathbb{C}_{\mathrm{M}}$ *has finite coproducts.*

Proof. The inclusion $\mathbb{C} \to \mathbb{C}_{\mathrm{M}}$ preserves coproducts since it is a left adjoint. Moreover it is surjective on objects. □

In light of Lemma 3.3.1, if $\mathbb{C}_{\mathrm{M}}$ has a zero object, then the underlying object of that zero object must be 0, the initial object of $\mathbb{C}$. Hence we need, for each $X \in \mathbb{C}$, a bijection

$$\mathbb{C}_{\mathrm{M}}(X, 0) = \mathbb{C}(X, \mathrm{M}\,0) \cong \{*\}.$$

In other words, we need $\mathrm{M}\,0$ to be a terminal object of $\mathbb{C}$.

Axiom (M-0). There is an isomorphism $\mathrm{M}\,0 \cong \top$.

Here $\top$ is the terminal object of $\mathbb{C}$. We have shown:

Proposition 3.3.2. *The Kleisli category* $\mathbb{C}_{\mathrm{M}}$ *has a zero object if and only if (M-0) holds.*

For the rest of this section we assume that (M-0) holds. In light of Proposition 3.2.3, in our situation the Kleisli category $\mathbb{C}_{\mathrm{M}}$ having biproducts is merely a property. We know that the biproduct projections must be given as in the following.

Axiom (M-biprod). The Kleisli arrows $X + Y \twoheadrightarrow X$ and $X + Y \twoheadrightarrow Y$

$$\widetilde{\pi}_{1,X,Y} : X + Y \xrightarrow{\eta_X + !_Y} \mathrm{M}\,X + \mathrm{M}\,0 \to \mathrm{M}(X + 0) \cong \mathrm{M}\,X$$
$$\widetilde{\pi}_{2,X,Y} : X + Y \xrightarrow{!_X + \eta_X} \mathrm{M}\,0 + \mathrm{M}\,Y \to \mathrm{M}(0 + Y) \cong \mathrm{M}\,Y$$

give the structure of a product of X and Y to $X + Y$ in $\mathbb{C}_{\mathrm{M}}$.

Let us reformulate this axiom in terms of $\mathbb{C}$ rather than $\mathbb{C}_{\mathrm{M}}$. Suppose that $\widetilde{\pi}_{1,X,Y} : X+Y \to \mathrm{M}\,X$ and $\widetilde{\pi}_{2,X,Y} : X+Y \to \mathrm{M}\,Y$ make $X+Y$ into a product of X and Y in $\mathbb{C}_{\mathrm{M}}$. Then Kleisli-composing a map $F : A \to \mathrm{M}(X+Y)$ with the first projection is the same as composing (in $\mathbb{C}$) F with $\tau_{1,X,Y} : \mathrm{M}(X + Y) \to \mathrm{M}\,X$ where

$$\tau_{1,X,Y} = \mu_X \circ \mathrm{M}\,\widetilde{\pi}_{1,X,Y}$$

and similarly for $\widetilde{\pi}_{2,X,Y}$ where

$$\tau_{2,X,Y} = \mu_Y \circ \mathrm{M}\,\widetilde{\pi}_{2,X,Y}.$$

The product property of $\widetilde{\pi}_{1,X,Y}$ and $\widetilde{\pi}_{2,X,Y}$ then amounts to saying that the map

$$\tau_{X,Y} : \mathrm{M}(X + Y) \xrightarrow{(\tau_{1,X,Y},\tau_{2,X,Y})} \mathrm{M}\,X \times \mathrm{M}\,Y$$

is an isomorphism.

Axiom (M-+-×). The maps

$$\mathrm{M}(X + Y) \xrightarrow{\mathrm{M}(\eta_X + !_Y)} \mathrm{M}(\mathrm{M}\,X + \mathrm{M}\,0) \to \mathrm{M}\,\mathrm{M}(X + 0) \cong \mathrm{M}\,\mathrm{M}\,X \xrightarrow{\mu_X} \mathrm{M}\,X$$
$$\mathrm{M}(X + Y) \xrightarrow{\mathrm{M}(!_X + \eta_Y)} \mathrm{M}(\mathrm{M}\,0 + \mathrm{M}\,Y) \to \mathrm{M}\,\mathrm{M}(0 + Y) \cong \mathrm{M}\,\mathrm{M}\,Y \xrightarrow{\mu_Y} \mathrm{M}\,Y$$

induce an isomorphism $\mathrm{M}(X + Y) \cong \mathrm{M}\,X \times \mathrm{M}\,Y$.

The following is immediate from the foregoing discussion.

Proposition 3.3.3. *Assume (M-0). Then the following are equivalent:*

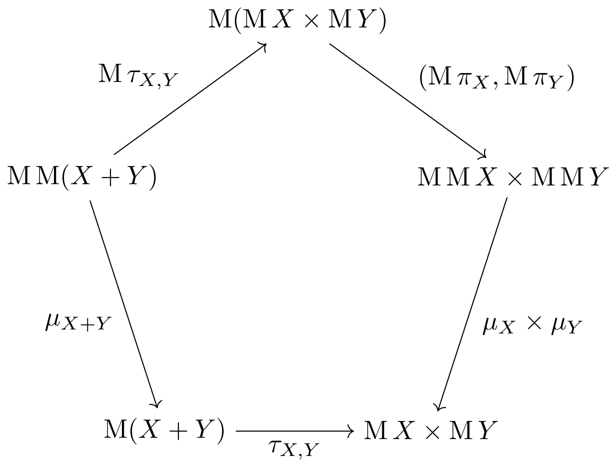
- *(M-biprod) holds,*
- *(M-+-×) holds,*
- $\mathbb{C}_{\mathrm{M}}$ *has biproducts.*

A monad satisfying (M-0) and (M-+-$\times$) is called an *additive monad* in [10]. Proposition 3.3.3 follows from Theorem 19 therein.

3.4 Coherence

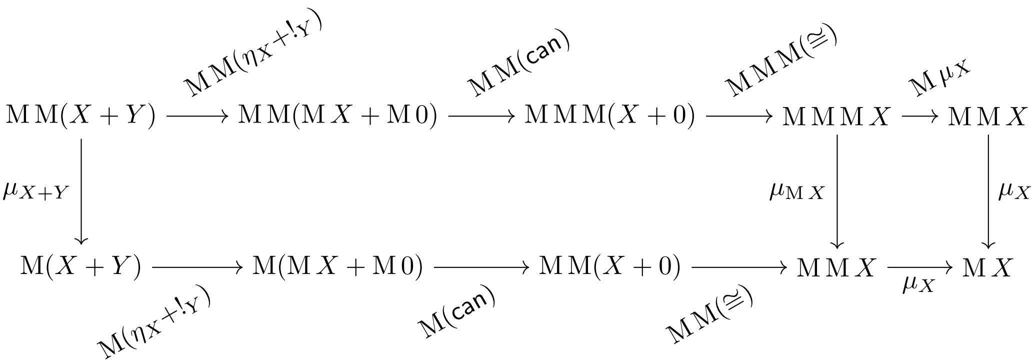
It is worth collecting here some equations satisfied by the maps $\tau_{1,X,Y}$, $\tau_{2,X,Y}$ and $\tau_{X,Y}$.

Proposition 3.4.1. *The diagram*



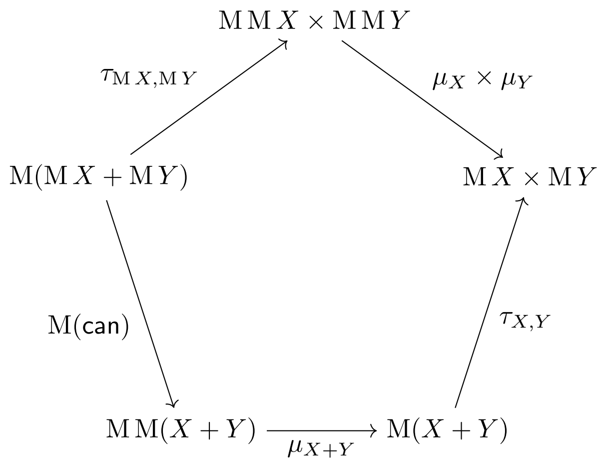
commutes.

Proof. Without loss of generality, let us just consider the projection onto M X. Then we are required to check that the outer rectangle of



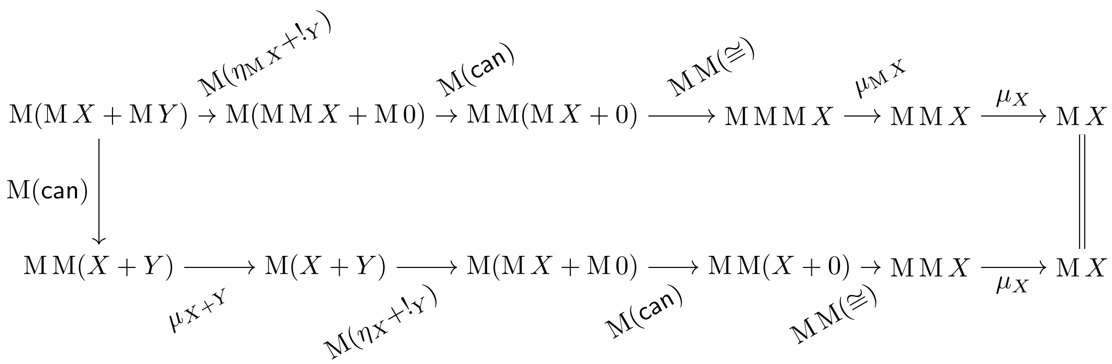
commutes. But the left-hand rectangle is a naturality square for the multiplication and the right-hand square is the associativity axiom for monads. $\square$

Proposition 3.4.2. *The diagram*



commutes.

Proof. Without loss of generality we need only prove this for its projection to $\mathrm{M}\,X$. This amounts to checking that the diagram



commutes. Using the naturality of the monad multiplication, and the monad associativity law, we can rewrite this equation as

$$\begin{array}{ccccccccccc}
\mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,Y) & \xrightarrow{\mathrm{M}(\eta_{\mathrm{M}\,X}+!_Y)} & \mathrm{M}(\mathrm{M}\,\mathrm{M}\,X+\mathrm{M}\,0) & \xrightarrow{\mathrm{M}(\mathsf{can})} & \mathrm{M}\,\mathrm{M}(\mathrm{M}\,X+0) & \xrightarrow{\mathrm{M}\,\mathrm{M}(\cong)} & \mathrm{M}\,\mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mathrm{M}\,\mu_X} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X \\
\downarrow{\scriptstyle \mathrm{M}(\mathsf{can})} & & & & & & & & & & \| \\
\mathrm{M}\,\mathrm{M}(X+Y) & \xrightarrow{\mathrm{M}\,\mathrm{M}(\eta_X+!_Y)} & \mathrm{M}\,\mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,0) & \xrightarrow{\mathrm{M}\,\mathrm{M}(\mathsf{can})} & \mathrm{M}\,\mathrm{M}\,\mathrm{M}(X+0) & \xrightarrow{\mathrm{M}\,\mathrm{M}\,\mathrm{M}(\cong)} & \mathrm{M}\,\mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mathrm{M}\,\mu_X} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X
\end{array}$$

To prove this equation, it is sufficient to prove that the diagram

$$\begin{array}{ccccccccc}
\mathrm{M}\,X+\mathrm{M}\,Y & \xrightarrow{\eta_{\mathrm{M}\,X}+!_Y} & \mathrm{M}\,\mathrm{M}\,X+\mathrm{M}\,0 & \xrightarrow{\mathsf{can}} & \mathrm{M}(\mathrm{M}\,X+0) & \xrightarrow{\mathrm{M}(\cong)} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X \\
\downarrow{\scriptstyle \mathsf{can}} & & & & & & & & \| \\
\mathrm{M}(X+Y) & \xrightarrow[\mathrm{M}(\eta_X+!_Y)]{} & \mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,0) & \xrightarrow[\mathrm{M}(\mathsf{can})]{} & \mathrm{M}\,\mathrm{M}(X+0) & \xrightarrow[\mathrm{M}\,\mathrm{M}(\cong)]{} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow[\mu_X]{} & \mathrm{M}\,X
\end{array}$$

commutes, since we get back the equation above by applying M and postcomposing with μ_X. Now to prove this equation, we need only check it upon composing with each of the coproduct inclusions. Firstly, on composing with $\iota_{\mathrm{M}\,X} : \mathrm{M}\,X \to \mathrm{M}\,X + \mathrm{M}\,Y$ we get the identity both ways round, since both of the diagrams

$$\begin{array}{ccccccccc}
\mathrm{M}\,X+\mathrm{M}\,Y & \xrightarrow{\eta_{\mathrm{M}\,X}+!_Y} & \mathrm{M}\,\mathrm{M}\,X+\mathrm{M}\,0 & \xrightarrow{\mathsf{can}} & \mathrm{M}(\mathrm{M}\,X+0) & \xrightarrow{\mathrm{M}(\cong)} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X \\
\uparrow{\scriptstyle \iota_{\mathrm{M}\,X}} & & \uparrow{\scriptstyle \iota_{\mathrm{M}\,\mathrm{M}\,X}} & & \uparrow{\scriptstyle \mathrm{M}\,\iota_{\mathrm{M}\,X}} & & \| & & \| \\
\mathrm{M}\,X & \xrightarrow[\eta_{\mathrm{M}\,X}]{} & \mathrm{M}\,\mathrm{M}\,X & = & \mathrm{M}\,\mathrm{M}\,X & = & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow[\mu_X]{} & \mathrm{M}\,X
\end{array}$$

and

$$
\begin{array}{ccccccccc}
\mathrm{M}\,X & \xrightarrow{\mathrm{M}\,\eta_X} & \mathrm{M}\,\mathrm{M}\,X & = & \mathrm{M}\,\mathrm{M}\,X & = & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X \\
{\scriptstyle \mathrm{M}\,\iota_X}\downarrow & & {\scriptstyle \mathrm{M}\,\iota_{\mathrm{M}\,X}}\downarrow & & {\scriptstyle \mathrm{M}\,\mathrm{M}\,\iota_X}\downarrow & & \Vert & & \Vert \\
\mathrm{M}(X+Y) & \xrightarrow[\mathrm{M}(\eta_X+!_Y)]{} & \mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,0) & \xrightarrow[\mathrm{M}(\mathsf{can})]{} & \mathrm{M}\,\mathrm{M}(X+0) & \xrightarrow[\mathrm{M}\,\mathrm{M}(\cong)]{} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow[\mu_X]{} & \mathrm{M}\,X
\end{array}
$$

commute, and $\mu_X \circ \eta_{\mathrm{M}\,X} = 1_{\mathrm{M}\,X}$. Secondly, on composing with $\iota_{\mathrm{M}\,Y} : \mathrm{M}\,Y \to \mathrm{M}\,X + \mathrm{M}\,Y$, we must check the commutativity of the diagram

$$
\begin{array}{ccccccccc}
\mathrm{M}\,Y & \xrightarrow{!_Y} & \mathrm{M}\,0 & \xrightarrow{\mathrm{M}\,\iota_0} & \mathrm{M}(\mathrm{M}\,X+0) & \xrightarrow{\mathrm{M}(\cong)} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow{\mu_X} & \mathrm{M}\,X \\
{\scriptstyle \mathrm{M}\,\iota_Y}\downarrow & & & & & & & & \Vert \\
\mathrm{M}(X+Y) & \xrightarrow[\mathrm{M}(\eta_X+!_Y)]{} & \mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,0) & \xrightarrow[\mathrm{M}(\mathsf{can})]{} & \mathrm{M}\,\mathrm{M}(X+0) & \xrightarrow[\mathrm{M}\,\mathrm{M}(\cong)]{} & \mathrm{M}\,\mathrm{M}\,X & \xrightarrow[\mu_X]{} & \mathrm{M}\,X
\end{array}
$$

of which the top composite simplifies to $\mu_X \circ \mathrm{M}(!_{\mathrm{M}\,X}) \circ !_Y$ and the bottom composite simplifies to $\mu_X \circ \mathrm{M}\,\mathrm{M}(!_X) \circ \mathrm{M}(!_Y) = \mathrm{M}(!_X) \circ \mu_0 \circ \mathrm{M}(!_Y) = \mathrm{M}(!_X) \circ !_Y$. It remains to check that $\mu_X \circ \mathrm{M}(!_{\mathrm{M}\,X}) = \mathrm{M}(!_X)$. But we obtain this from the equation $\eta_X \circ !_X = !_{\mathrm{M}\,X}$ by applying M and postcomposing with μ_X. □

Proposition 3.4.3. *The family of maps* $\tau_{X,Y} : \mathrm{M}(X+Y) \to \mathrm{M}(X) \times \mathrm{M}(Y)$ *is natural in* X *and* Y*.*

Proof. This is obvious from the definition of $\tau_{1,X,Y} : \mathrm{M}(X+Y) \to \mathrm{M}\,X$ as

$$\mathrm{M}(X+Y) \xrightarrow{\mathrm{M}(\eta_X+!_Y)} \mathrm{M}(\mathrm{M}\,X+\mathrm{M}\,0) \xrightarrow{\mathrm{M}(\mathsf{can})} \mathrm{M}\,\mathrm{M}(X+0) \xrightarrow{\mathrm{M}\,\mathrm{M}(\cong)} \mathrm{M}\,\mathrm{M}\,X \xrightarrow{\mu_X} \mathrm{M}\,X$$

and similarly for $\tau_{2,X,Y} : \mathrm{M}(X+Y) \to \mathrm{M}\,Y$. □

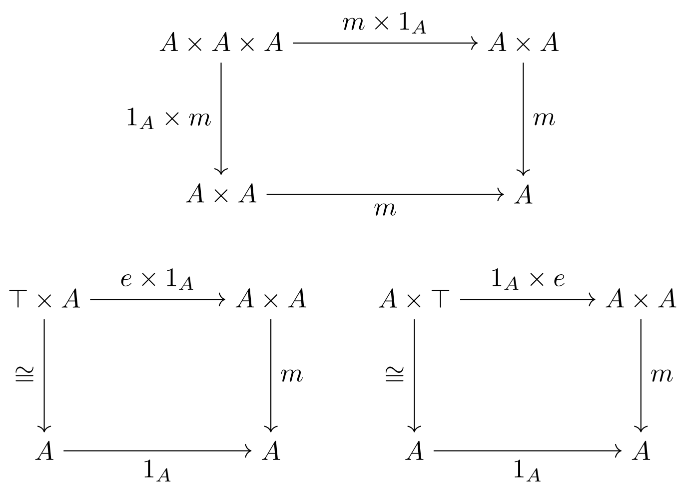
3.5 Commutative monoids

Let us recall the following definition.

Definition 3.5.1. In a category $\mathbb{C}$ with finite products, a *commutative monoid* is an object $A \in \mathbb{C}$ equipped with morphisms

$$m : A \times A \to A$$
$$e : \top \to A$$

such that the diagrams



commute.

Let us assume (M-0) and (M-+-×). Then any free M-algebra

$$(\mathrm{M}\,A, \mu_A : \mathrm{M}\,\mathrm{M}\,A \to \mathrm{M}\,A)$$

is a commutative monoid with addition $\oplus_A : \mathrm{M}\,A \times \mathrm{M}\,A \to \mathrm{M}\,A$ given by

$$\mathrm{M}\,A \times \mathrm{M}\,A \cong \mathrm{M}(A + A) \xrightarrow{\mathrm{M}([1_A, 1_A])} \mathrm{M}\,A \tag{3.1}$$

and unit $e_A : \top \to \mathrm{M}\,A$ given by

$$\top \cong \mathrm{M}\,0 \xrightarrow{\mathrm{M}\,!_A} \mathrm{M}\,A. \tag{3.2}$$

Proposition 3.5.2. *The morphisms above equip* $\mathrm{M}\,A$ *with the structure of a commutative monoid.*

Proof. Associativity is clear once one checks that the two maps

$$\mathrm{M}(A + A + A) \to \mathrm{M}\,A \times \mathrm{M}\,A \times \mathrm{M}\,A$$

given by the two different ways of associating the binary operations are equal. Unitality follows easily using the definition of $\tau_{X,Y}$, commutativity is trivial. □

Proposition 3.5.3. *For any map* $f : A \to B$*, the induced morphism of free* M*-algebras* $\mathrm{M}\,f : \mathrm{M}\,A \to \mathrm{M}\,B$ *is also homomorphism of monoids.*

Proof. We must check that the diagrams

$$\begin{array}{ccc} & & \mathrm{M}\,A \\ & \overset{\mathrm{M}\,!_A}{\nearrow} & \\ \top \cong \mathrm{M}\,0 & & \Big\downarrow \mathrm{M}\,f \\ & \underset{\mathrm{M}\,!_B}{\searrow} & \\ & & \mathrm{M}\,B \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{M}\,A \times \mathrm{M}\,A & \xrightarrow{\cong} & \mathrm{M}(A+A) & \xrightarrow{\mathrm{M}([1_A, 1_A])} & \mathrm{M}\,A \\ \Big\downarrow \mathrm{M}\,f \times \mathrm{M}\,f & & & & \Big\downarrow \mathrm{M}\,f \\ \mathrm{M}\,B \times \mathrm{M}\,B & \xrightarrow[\cong]{} & \mathrm{M}(B+B) & \xrightarrow[\mathrm{M}([1_B, 1_B])]{} & \mathrm{M}\,B \end{array}$$

commute. The first is clear since $f \circ !_A = !_B$. The second is also easy, using the naturality of $\mathrm{M}\,A \times \mathrm{M}\,A \cong \mathrm{M}(A+A)$ and the naturality of the copairing. □

What we have just shown is that the functor $\mathrm{M} : \mathbb{C} \to \mathbb{C}$ lifts through the forgetful functor $V : \mathsf{CMon}(\mathbb{C}) \to \mathbb{C}$, i.e. there is a functor $\widehat{\mathrm{M}} : \mathbb{C} \to \mathsf{CMon}(\mathbb{C})$ making the diagram

$$\begin{array}{ccc} & & \mathsf{CMon}(\mathbb{C}) \\ & \overset{\widehat{\mathrm{M}}}{\nearrow} & \Big\downarrow V \\ \mathbb{C} & \xrightarrow[\mathrm{M}]{} & \mathbb{C} \end{array}$$

commute. Our aim now is to show that this lift admits an extension to the Eilenberg-Moore category $\mathbb{C}^{\mathrm{M}}$.

Theorem 3.5.4. *There exists a functor $\mathbb{C}^{\mathrm{M}} \to \mathsf{CMon}(\mathbb{C})$ making the following diagram commute*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widehat{\mathrm{M}}} & \mathsf{CMon}(\mathbb{C}) \\ F \Big\downarrow & \nearrow & \Big\downarrow V \\ \mathbb{C}^{\mathrm{M}} & \xrightarrow[U]{} & \mathbb{C} \end{array}$$

with $\widehat{\mathrm{M}}$ the functor described above.

Rather than simply producing a formula for the commutative monoid structure induced by an algebra, we will take a slightly abstract approach. We will give a variant of the argument of Beck's monadicity theorem (see Volume 2 §4 of [6], §I.V of [33], or Chapter 3 of [2]).

Let us recall some definitions.

Definition 3.5.5. Let $V : \mathbb{B} \to \mathbb{A}$ be a functor. Then a parallel pair $f, g : X \rightrightarrows Y$ in $\mathbb{B}$ is called *V-split* if the pair $Vf, Vg : VX \rightrightarrows VY$ has a split coequalizer in $\mathbb{A}$, or more precisely, if there is an object $Z \in \mathbb{A}$ and arrows

$$VY \xrightarrow{h} Z$$
$$Z \xrightarrow{s} VY$$
$$VY \xrightarrow{t} VX$$

such that $hs = 1_Z$, $Vf \circ t = 1_{VY}$, and $Vg \circ t = sh$. The functor V *creates coequalizers of V-split pairs* if every V-split pair admits a coequalizer in $\mathbb{B}$ which is preserved by V and moreover every diagram of shape

$$\bullet \rightrightarrows \bullet \to \bullet$$

in $\mathbb{B}$ which is mapped by V to a split coequalizer in $\mathbb{A}$ is already a coequalizer in $\mathbb{B}$.

Now let us recall the monadicity theorem.

Theorem 3.5.6 (Beck's monadicity theorem)**.** *A functor $V : \mathbb{B} \to \mathbb{A}$ is monadic if and only if V has a left adjoint and V creates coequalizers of V-split pairs.*

A key step in the proof of Theorem 3.5.6 is to construct a functor from an Eilenberg-Moore category to the domain of V, using the fact that every Eilenberg-Moore algebra is a reflexive coequalizer of free algebras. We will deduce 3.5.4 from the following.

Proposition 3.5.7. *Let $W : \mathbb{D} \to \mathbb{C}$ be a faithful functor which creates coequalizers of W-split pairs and suppose we have a commutative diagram*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\widehat{\mathrm{M}}} & \mathbb{D} \\ {\scriptstyle F}\downarrow & \nearrow & \downarrow{\scriptstyle W} \\ \mathbb{C}^{\mathrm{M}} & \xrightarrow[U]{} & \mathbb{C}\ . \end{array}$$

If there exists a lift $\widehat{\mu} : \widehat{\mathrm{M}}\,\mathrm{M} \Rightarrow \widehat{\mathrm{M}}$ of $\mu : \mathrm{M}\,\mathrm{M} \Rightarrow \mathrm{M}$ then there exists a dotted functor filling in the diagram.

Proof. We define a functor $S : \mathbb{C}^{\mathrm{M}} \to \mathbb{D}$ on objects by sending an algebra (A, α) to the object $S(A, \alpha)$ which is given by the coequalizer diagram

$$\widehat{\mathrm{M}}\,\mathrm{M}\,A \underset{\widehat{\mu}_A}{\overset{\widehat{\mathrm{M}}\alpha}{\rightrightarrows}} \widehat{\mathrm{M}}A \xrightarrow{\widehat{\alpha}} S(A, \alpha)$$

which exists since

$$\mathrm{M}\,\mathrm{M}\,A \underset{\widehat{\mu}_A}{\overset{\mathrm{M}\,\alpha}{\rightrightarrows}} \mathrm{M}\,A \xrightarrow{\alpha} A$$

is a split coequalizer, so indeed $VS(A, \alpha) = A$ (and $\widehat{\alpha}$ lies over α). It is now easy to check, using the fact that $\widehat{\mu}$ is a natural transformation, that this assignment on objects extends to a functor on algebra morphisms.

It remains to check that we can choose our coequalizer diagrams in $\mathbb{D}$ in such a way that $SF = \widehat{M}$. For this it suffices to observe that, for an object $A \in \mathbb{C}$, the fork

$$\widehat{\mathrm{M}}\,\mathrm{M}\,\mathrm{M}\,A \underset{\widehat{\mu}_{\mathrm{M}\,A}}{\overset{\widehat{\mathrm{M}}\mu_A}{\rightrightarrows}} \widehat{\mathrm{M}}\,\mathrm{M}\,A \xrightarrow{\widehat{\mu}_A} \widehat{\mathrm{M}}A$$

which lies over a split coequalizer diagram in $\mathbb{C}$ (split by $\eta_{\mathrm{M}\,A}$ and $\eta_{\mathrm{M}\,\mathrm{M}\,A}$) is a possible choice of coequalizer for the W-split pair $\widehat{\mathrm{M}}\mu_A, \widehat{\mu}_{\mathrm{M}\,A} : \widehat{\mathrm{M}}\,\mathrm{M}\,\mathrm{M}\,A \rightrightarrows \widehat{\mathrm{M}}\,\mathrm{M}\,A$, since W creates coequalizers of W-split pairs. Actually, we must consider the case when $A \neq B$ but $(\mathrm{M}\,A, \mu_A) = (\mathrm{M}\,B, \mu_B)$. In this case, since W is faithful we have $\widehat{\mu}_{\mathrm{M}\,A} = \widehat{\mu}_{\mathrm{M}\,B}$ so the two choices of coequalizer diagram are actually the same. ☐

To deduce Theorem 3.5.4, we must check that $V : \mathsf{CMon}(\mathbb{C}) \to \mathbb{C}$ satisfies the conditions of 3.5.7. We can check that V creates coequalizers of V-split pairs by inspection, and the same argument will work for the category of models in $\mathbb{C}$ of any algebraic theory in the sense of [31], say. Alternatively, we can use Theorem 3.5.6 to deduce this fact, by embedding $\mathbb{C}$ in its free cocompletion $\widehat{\mathbb{C}} = [\mathbb{C}^{\mathrm{op}}, \mathsf{Set}]$. The functors $\mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ and $\mathsf{CMon}(\mathbb{C}) \hookrightarrow \mathsf{CMon}(\widehat{\mathbb{C}})$ are inclusions of full subcategories making the square

$$\begin{array}{ccc} \mathsf{CMon}(\mathbb{C}) & \hookrightarrow & \mathsf{CMon}(\widehat{\mathbb{C}}) \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle \widehat{V}} \\ \mathbb{C} & \hookrightarrow & \widehat{\mathbb{C}} \end{array}$$

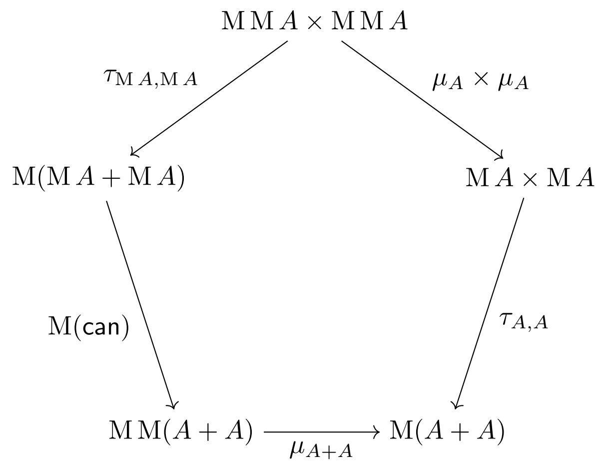
commutative and moreover the square is a pullback. Since $\widehat{\mathbb{C}}$ is cocomplete, one can see that $\widehat{\mathbb{C}}$ admits free commutative monoids. Hence $\widehat{V} : \mathsf{CMon}(\widehat{\mathbb{C}}) \to \widehat{\mathbb{C}}$ is monadic, and so by 3.5.6 it has a left adjoint and creates coequalizers of $\widehat{V}$-split pairs. Since split coequalizers are preserved by any functor, and in particular the inclusion $\mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$, any V-split pair in $\mathsf{CMon}(\mathbb{C})$ maps to a $\widehat{V}$-split pair in $\mathsf{CMon}(\widehat{\mathbb{C}})$, hence that pair has a coequalizer in $\mathsf{CMon}(\widehat{\mathbb{C}})$ lying over an object of $\mathbb{C}$, hence lying in $\mathsf{CMon}(\mathbb{C})$. Moreover, any fork in $\mathsf{CMon}(\mathbb{C})$ lying over a split coequalizer gives a fork in $\mathsf{CMon}(\widehat{\mathbb{C}})$ also lying over a split coequalizer, which is therefore a coequalizer, and since $\mathsf{CMon}(\mathbb{C})$ is a full subcategory of $\mathsf{CMon}(\widehat{\mathbb{C}})$, it is also a coequalizer in $\mathsf{CMon}(\mathbb{C})$.

To complete the proof of Theorem 3.5.4, we need to verify that $\mu : \mathrm{M}\,\mathrm{M} \Rightarrow \mathrm{M}$ lifts to a natural transformation $\widehat{\mu} : \widehat{\mathrm{M}}\,\mathrm{M} \Rightarrow \widehat{\mathrm{M}}$. This amounts to checking the following.

Lemma 3.5.8. *For every $A \in \mathbb{C}$, the map $\mu_A : \mathrm{M}\,\mathrm{M}\,A \to \mathrm{M}\,A$ is a monoid homomorphism for the commutative monoid structures given in* (3.1) *and* (3.2).

Proof. For the unit, we need to check that $\mu_A \circ \mathrm{M}(!_{\mathrm{M}\,A}) = \mathrm{M}(!_A)$. But $\eta_A \circ !_A = !_{\mathrm{M}\,A}$, since both sides are maps $0 \to \mathrm{M}\,A$. By apply M and postcomposing with μ_A we get the desired equation.

For the addition, we start with



which we obtain from 3.4.2 by reversing the direction of the isomorphisms and substituting A for X and Y. Postcomposing with $\mathrm{M}([1_A, 1_A])$ and a little rewriting gives us

$$\begin{array}{ccccc} \mathrm{M}\,\mathrm{M}\,A \times \mathrm{M}\,\mathrm{M}\,A & \xrightarrow{\tau^{-1}_{\mathrm{M}\,A,\mathrm{M}\,A}} & \mathrm{M}(\mathrm{M}\,A + \mathrm{M}\,A) & \xrightarrow{\mathrm{M}([1_{\mathrm{M}\,A}, 1_{\mathrm{M}\,A}])} & \mathrm{M}\,\mathrm{M}\,A \\ \downarrow{\scriptstyle \mu_A \times \mu_A} & & & & \downarrow{\scriptstyle \mu_A} \\ \mathrm{M}\,A \times \mathrm{M}\,A & \xrightarrow[\tau^{-1}_{A,A}]{} & \mathrm{M}(A + A) & \xrightarrow[\mathrm{M}[1_A, 1_A]]{} & \mathrm{M}\,A \end{array} \tag{3.3}$$

as required. ☐

This completes the proof of 3.5.4. If one calculates the coequalizers of V-split pairs, then one can give the commutative monoid structure on an M-algebra $(A, \alpha : \mathrm{M}\,A \to A)$ explicitly. The unit e_α is given by

$$\top \cong \mathrm{M}\,0 \xrightarrow{\mathrm{M}\,!_A} \mathrm{M}\,A \xrightarrow{\alpha} A$$

and the addition $\oplus_\alpha$ is given by

$$A \times A \xrightarrow{\eta_A \times \eta_A} \mathrm{M}\,A \times \mathrm{M}\,A \cong \mathrm{M}(A + A) \xrightarrow{\mathrm{M}([1_A, 1_A])} \mathrm{M}\,A \xrightarrow{\alpha} A.$$

Corollary 3.5.9. *Let $f : (A, \alpha) \to (B, \beta)$ be a homomorphism of* M*-algebras. Then f is also a homomorphism of monoids $(A, e_\alpha, \oplus_\alpha) \to (B, e_\beta, \oplus_\beta)$.*

Corollary 3.5.10. *For free algebras, we have two equal formulations of the commutative monoid structure, More precisely, for any object A, we have $\oplus_{\mu_A} = \oplus_{\mathrm{M}\,A} : \mathrm{M}\,A \times \mathrm{M}\,A \to \mathrm{M}\,A$, i.e.*

$$\mathrm{M}\,A \times \mathrm{M}\,A \cong \mathrm{M}(A + A) \xrightarrow{\mathrm{M}([1_A, 1_A])} \mathrm{M}\,A$$

is equal to

$$\mathrm{M}\,A \times \mathrm{M}\,A \xrightarrow{\eta_{\mathrm{M}\,A} \times \eta_{\mathrm{M}\,A}} \mathrm{M}\,\mathrm{M}\,A \times \mathrm{M}\,\mathrm{M}\,A \cong \mathrm{M}(\mathrm{M}\,A + \mathrm{M}\,A) \xrightarrow{\mathrm{M}([1_{\mathrm{M}\,A}, 1_{\mathrm{M}\,A}])} \mathrm{M}\,\mathrm{M}\,A \xrightarrow{\mu_A} \mathrm{M}\,A$$

and also $e_{\mu_A} = e_{\mathrm{M}\,A} : \top \to \mathrm{M}\,A$, i.e.

$$\top \cong \mathrm{M}\,0 \xrightarrow{\mathrm{M}\,!_A} \mathrm{M}\,A$$

is equal to

$$\top \cong \mathrm{M}\,0 \xrightarrow{\mathrm{M}\,!_{\mathrm{M}\,A}} \mathrm{M}\,\mathrm{M}\,A \xrightarrow{\mu_A} \mathrm{M}\,A.$$

Proof. For the unit, observe that $\eta_A \circ !_A = !_{MA}$ since both sides are maps $0 \to \mathrm{M}\,A$. By applying M and postcomposing with α we get the desired equation. ☐

We end this section with the following useful result.

Proposition 3.5.11. *For any* M*-algebra* (A, α)*, the unit* e_α *and addition* $\oplus_\alpha$ *are both algebra homomorphisms (from the terminal algebra and product algebra* $(A, \alpha) \times (A, \alpha)$*, respectively).*

Proof. For the unit, we are required to show that

$$\begin{array}{ccccc}
\mathrm{M}\top \cong \mathrm{M}\,\mathrm{M}\,0 & \xrightarrow{\mathrm{M}\,\mathrm{M}\,!_A} & \mathrm{M}\,\mathrm{M}\,A & \xrightarrow{\mathrm{M}\,\alpha} & \mathrm{M}\,A \\
\downarrow{\scriptstyle \mu_0} & & & & \downarrow{\scriptstyle \alpha} \\
\top \cong \mathrm{M}\,0 & \xrightarrow[\mathrm{M}\,!_A]{} & \mathrm{M}\,A & \xrightarrow[\alpha]{} & A
\end{array}$$

commutes, but this is equivalent to the commutativity of the outer rectangle in

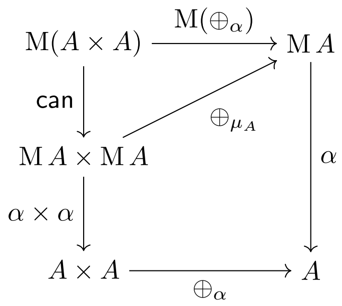
$$\begin{array}{ccccc}
\mathrm{M}\top \cong \mathrm{M}\,\mathrm{M}\,0 & \xrightarrow{\mathrm{M}\,\mathrm{M}\,!_A} & \mathrm{M}\,\mathrm{M}\,A & \xrightarrow{\mu_A} & \mathrm{M}\,A \\
\downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \alpha} \\
\top \cong \mathrm{M}\,0 & \xrightarrow[\mathrm{M}\,!_A]{} & \mathrm{M}\,A & \xrightarrow[\alpha]{} & A
\end{array}$$

by the algebra axioms, and this square commutes by the naturality of the monad multiplication.

For the multiplication, the result follows if we can prove the commutativity of the diagram

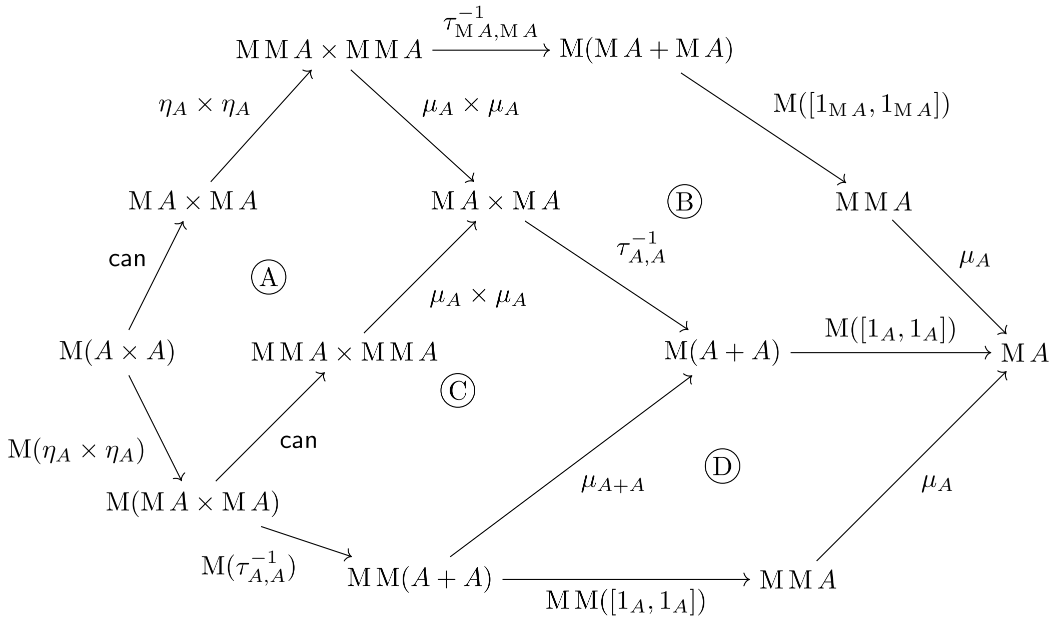
$$\begin{array}{ccc}
\mathrm{M}(A \times A) & \xrightarrow{\mathsf{can}} & \mathrm{M}\,A \times \mathrm{M}\,A \\
\downarrow{\scriptstyle \mathrm{M}(\oplus_\alpha)} & & \downarrow{\scriptstyle \oplus_{\mu_A}} \\
\mathrm{M}\,A & & \mathrm{M}\,A \\
& {\scriptstyle \alpha}\searrow \quad \swarrow{\scriptstyle \alpha} & \\
& A &
\end{array} \tag{3.4}$$

for then we can break down the homomorphism equation as



where the upper triangle commutes after composition with α by (3.4) and the trapezium commutes by 3.5.9 since α is an algebra homomorphism $(\mathrm{M}\,A, \mu_A) \to (A, \alpha)$.

Now, equation (3.4) follows from the equality of the two outer composites of



by postcomposing with α and using the associativity axiom for algebras once. But in fact the whole diagram is commutative. The subdiagram Ⓐ commutes by a trivial calculation, and Ⓑ commutes by (3.3) from 3.5.8. The pentagon Ⓒ is 3.4.1 and Ⓓ is just naturality of the monad multiplication. □

3.6 Biproducts in Eilenberg-Moore categories

To complete our survey of biproducts of algebras, let us consider when biproducts exists in Eilenberg-Moore categories. Corollary 3.6.2 below follows from Theorem 19 in [10], but here we provide the detail that had been left to the reader. Given M-algebras (A, α) and (B, β), their product is given by $A \times B$ together with the structure map

$$\mathrm{M}(A \times B) \to \mathrm{M}\,A \times \mathrm{M}\,B \xrightarrow{\alpha \times \beta} A \times B.$$

If we assume (M-0) then the category of M-algebras has a zero object: $(1 \cong \mathrm{M}\,0, !_{\mathrm{M}\,1} = \mu_0)$. Hence, by the dual of proposition 3.2.3, the category of algebras has biproducts if and only if $(A, \alpha) \times (B, \beta)$ is given the structure of a coproduct by the maps

$$A \cong A \times \mathrm{M}\,0 \xrightarrow{1_A \times \mathrm{M}\,!_B} A \times \mathrm{M}\,B \xrightarrow{1_A \times \beta} A \times B$$
$$B \cong \mathrm{M}\,0 \times B \xrightarrow{\mathrm{M}\,!_A \times 1_B} \mathrm{M}\,A \times B \xrightarrow{\alpha \times 1_B} A \times B$$

or alternatively we can write these as

$$A \cong A \times \top \xrightarrow{1_A \times e_\beta} A \times B$$
$$B \cong \top \times B \xrightarrow{e_\alpha \times 1_B} A \times B.$$

which are clearly algebra homomorphisms.

Suppose we are given another algebra (C, γ) and two algebra homomorphisms $f : A \to C$ and $g : B \to C$. Then there is a map

$$A \times B \xrightarrow{f \times g} C \times C \xrightarrow{\oplus_\gamma} C$$

or, equivalently,

$$A \times B \xrightarrow{\eta_A \times \eta_B} \mathrm{M}\,A \times \mathrm{M}\,B \cong \mathrm{M}(A + B) \xrightarrow{\mathrm{M}[f,g]} \mathrm{M}\,C \xrightarrow{\gamma} C.$$

Proposition 3.6.1. *Assuming (M-+-$\times$), the structure on $(A, \alpha) \times (B, \beta)$ given above does indeed give it the structure of a coproduct of (A, α) and (B, β).*

Proof. By 3.5.9, g is a monoid homomorphism $(B, e_\beta, \oplus_\beta) \to (C, e_\gamma, \oplus_\gamma)$ and hence $g \circ e_\beta = e_\gamma$. Thus,

$$\oplus_\gamma \circ (f \times g) \circ (1_A \times e_\beta) = \oplus_\gamma \circ (f \times e_\gamma) = f$$

and similarly $\oplus_\gamma \circ (f \times g) \circ (e_\alpha \times 1_B) = g$. It remains to check that the coprojections and the copairing are indeed algebra homomorphisms. However, by 3.5.11 the units e_α and e_β are algebra homomorphisms $(\top, !_\top) \to (A, \alpha)$ and $(\top, !_\top) \to (B, \beta)$ and also the addition $\oplus_\gamma$ is an algebra homomorphism $(C, \gamma) \times (C, \gamma) \to (C, \gamma)$. We finish by observing that the product and composite of algebra homomorphisms are again algebra homomorphisms. □

Corollary 3.6.2. *Assuming (M-0) and (M-+-×), the Eilenberg-Moore category for* M *admits finite biproducts.*

Chapter 4

The Diller-Nahm category

Our goal here is to give a detailed presentation of the construction of the category Dill proposed in [21], which is the motivating idea in this thesis. Thus we should take some time to elucidate the construction and to list the subtle hypotheses required to show that we do indeed obtain a cartesian closed category. We aim to analyse the construction in terms of fibrations, with a view to understanding it as a construction which glues together categorical structure in the fibre categories into structure on the total category. We shall take an abstract view on the finite multisets monad as one whose Kleisli category admits biproducts, as in Chapter 3, rather than assuming our base category admits free commutative monoids. In contrast to [11] and [21], but in line with [4], we shall not assume that the fibration of predicates is preordered.

4.1 Three settings

In [21] the basic setting is that we have a system of types, represented by a category $\mathbb{T}$ with finite products. There is also a strong monad $(-)^\bullet$ on $\mathbb{T}$. The basic example is that of the category of sets equipped with the free commutative monoid monad, i.e. $X^\bullet$ is the set of finite multisets with elements from X. We also assume there is a preordered fibration $p : \mathbb{P} \to \mathbb{T}$, representing a system of predicates on the types of $\mathbb{T}$. Moreover, the monad $(-)^\bullet$ admits some kind of extension to $\mathbb{P}$, also denoted $(-)^\bullet$. This is enough data to define the category Dill as a category, provided we are more precise about what we mean by the extension of $(-)^\bullet$ to $\mathbb{P}$. In fact, we do not merely define a category, because Dill will naturally come equipped with a fibration $s : \mathsf{Dill} \to \mathbb{T}$, which we will refer to as *the Diller-Nahm fibration.*

As one of the main goals of this thesis is to define a Diller-Nahm *model of type theory*, it is worth considering how the construction in the simply-typed case already uses dependently-typed concepts. In order to better clarify how the simply-typed construction is a special case of what we shall do later, and in order to keep the calculations manageable, we will introduce three *settings* (A), (B) and (C) in increasing order of abstraction, each of whose definitions will be spread across this chapter. We will define and investigate $s : \mathsf{Dill} \to \mathbb{T}$ in setting (C). Each setting is just a list of hypothesized objects and properties thereof. The settings may be briefly described as follows:

(A) The simply-typed case. We assume, as in [21], that we have a 'system of types' (category) $\mathbb{T}$, a 'system of predicates' (fibration) $p : \mathbb{P} \to \mathbb{T}$, and a 'finite multisets monad' $(-)^\bullet$.

(B) A more dependently-typed case. We have an indexing category $\mathbb{C}$, a fibration $q : \mathbb{E} \to \mathbb{C}$ each of whose fibres is a 'system of types', a fibration $r : \mathbb{Q} \to \mathbb{E}$ thought of as a 'fibred system of predicates', and indexed 'finite multisets' monads on $\mathbb{E}$ and $\mathbb{Q}$.

(C) We abstract away the finite multisets monad, replacing it with its Kleisli category. Hence we assume a fibration $q : \mathcal{E} \to \mathcal{C}$ with fibred biproducts and a fibration $r : \mathcal{P} \to \mathcal{E}$.

We also give two *translations* (A) $\longrightarrow$ (B) and (B) $\longrightarrow$ (C). That is, we will show how the objects and properties assumed in setting (A) give rise to an example of the objects postulated in (B) satisfying the relevant hypotheses, and similarly for (B) and (C).

Furthermore, in order to describe the most basic features of the Diller-Nahm category independently of the assumptions needed for the more complicated ones, we will define three settings by cumulatively through this chapter. Each time new hypotheses are added we will also check that the translations can be expanded to validate them.

As well as describing the precise assumptions in detail as we go, we will also spell out some of the constructions in the more concrete settings (A) and (B) after giving them in (C). Finally, we the reader may also have in mind one more setting, (0), which is the main example of (A). In (0) we consider the category of sets equipped with the fibration of subsets and the finite multisets monad.

4.2 The Diller-Nahm fibration

Let us first consider minimal assumptions needed in each of our three settings in order to merely define the Diller-Nahm category and fibration.

Since these are so modest for setting (C), we will simply give them here: we require two fibrations $q : \mathcal{E} \to \mathcal{C}$ and $r : \mathcal{P} \to \mathcal{E}$.

Definition 4.2.1. In setting (C), the *Diller-Nahm fibration* $s : \mathsf{Dill} \to \mathcal{C}$ is $(q \circ r^{\mathrm{op}})^{\mathrm{op}}$. The *Diller-Nahm category* is the total category of this fibration.

The definition of the Diller-Nahm category in settings (A) and (B) will follow as soon as we have described these settings precisely and the translations (A) $\to$ (B) and (B) $\to$ (C).

4.2.1 Assumptions for setting (A)

Let us begin by recalling a definition (see, for instance, [27]).

Definition 4.2.2. A *strong monad* (T, η, μ, σ) on a category $\mathbb{C}$ with finite products is a monad (T, η, μ) together with a natural transformation $\sigma_{A,B} : A \times TB \to T(A \times B)$ making the following diagrams commute,

$$
\begin{array}{ccc}
\top \times TA & \xrightarrow{\sigma_{\top,A}} & T(\top \times A) \\
 & {\scriptstyle \cong}\searrow \quad \swarrow{\scriptstyle T(\cong)} & \\
 & TA &
\end{array}
\qquad
\begin{array}{ccc}
A \times B & \xrightarrow{1_A \times \eta_B} & A \times TB \\
 & {\scriptstyle \eta_{A\times B}}\searrow \quad \swarrow{\scriptstyle \sigma_{A,B}} & \\
 & T(A \times B) &
\end{array}
$$

$$
\begin{array}{ccc}
A \times B \times TC & \xrightarrow{\sigma_{A\times B,C}} & T(A \times B \times C) \\
 & {\scriptstyle 1_A \times \sigma_{B,C}}\searrow \quad \nearrow{\scriptstyle \sigma_{A,B\times C}} & \\
 & A \times T(B \times C) &
\end{array}
$$

$$
\begin{array}{ccccc}
A \times TTB & \xrightarrow{\sigma_{A,TB}} & T(A \times TB) & \xrightarrow{T(\sigma_{A,B})} & TT(A \times B) \\
{\scriptstyle 1_A \times \mu_B}\downarrow & & & & \downarrow{\scriptstyle \mu_{A\times B}} \\
A \times TB & & \xrightarrow[\sigma_{A,B}]{} & & T(A \times B)
\end{array}
$$

where we have suppressed the associativity of the product.

The category of types $\mathbb{T}$ must have finite products and a strong monad $(-)^\bullet$, which admits a fibred extension to $\mathbb{P}$. By such a fibred extension, we mean a monad $(-)^\bullet$ on $\mathbb{P}$ such that the functor part satisfies $p \circ (-)^\bullet = (-)^\bullet \circ p$ and preserves cartesian arrows and the unit and multiplication in $\mathbb{P}$ are p-cartesian arrows lying over the unit and multiplication in $\mathbb{T}$.

4.2.2 Assumptions for setting (B)

Let us spell out (B) more precisely. In saying that it resembles the dependently-typed case, for now we mean that rather than considering merely a system of types and a system of predicates on those types, we consider an abstract 'indexing category' $\mathbb{C}$ and for each indexing object $I \in \mathbb{C}$ we have a system of 'I-indexed type-families', which is to say we have a fibration $q : \mathbb{E} \to \mathbb{C}$. Moreover, we have a system of 'predicates' on each I-indexed type-family, which is to say that we have a fibration $r : \mathbb{Q} \to \mathbb{E}$.

We deal with the finite multisets monad as follows. Let us suppose we have a functor $M_{\mathbb{E}} : \mathbb{E} \to \mathbb{E}$ making a morphism of fibrations

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{M_{\mathbb{E}}} & \mathbb{E} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{C} & = & \mathbb{C} \end{array}$$

and a functor $M_{\mathbb{Q}} : \mathbb{Q} \to \mathbb{Q}$ making a morphism of fibrations

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{M_{\mathbb{Q}}} & \mathbb{Q} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r} \\ \mathbb{E} & \xrightarrow[M_{\mathbb{E}}]{} & \mathbb{E} \end{array} .$$

We suppose that $M_{\mathbb{E}}$ is the functor part of a fibrewise monad on q in the sense that there are natural transformations $\eta_{\mathbb{E}} : \mathrm{id}_{\mathbb{E}} \Rightarrow M_{\mathbb{E}}$ and $\mu_{\mathbb{E}} : M_{\mathbb{E}} M_{\mathbb{E}} \Rightarrow M_{\mathbb{E}}$ with q-vertical components making $(M_{\mathbb{E}}, \eta_{\mathbb{E}}, \mu_{\mathbb{E}})$ into a monad on $\mathbb{E}$. Equivalently, we have a monad in each fibre of q which is preserved by reindexing. We also suppose that $M_{\mathbb{Q}}$ is the functor part of a monad on $\mathbb{Q}$ which is a 'cartesian lift' of $(M_{\mathbb{E}}, \eta_{\mathbb{E}}, \mu_{\mathbb{E}})$. This means that there are natural transformations $\eta_{\mathbb{Q}} : \mathrm{id}_{\mathbb{Q}} \Rightarrow M_{\mathbb{Q}}$ and $\mu_{\mathbb{Q}} : M_{\mathbb{Q}} M_{\mathbb{Q}} \Rightarrow M_{\mathbb{Q}}$ where for each $I \in \mathbb{C}$, $X \in \mathbb{E}(I)$, $\alpha \in \mathbb{Q}(I, X)$, the components $\eta_{\mathbb{Q},\alpha}$ and $\mu_{\mathbb{Q},\alpha}$ are r-cartesian over $\eta_{\mathbb{E},X}$ and $\mu_{\mathbb{E},X}$ respectively.

4.2.3 From (A) to (B): the simple slice fibration

Let us spell out how (B) generalizes (A). The idea here is that from a system of simple types $\mathbb{T}$ we can produce an indexed system of types $q : \mathbb{E} \to \mathbb{C}$ where for $I \in \mathbb{C}$, each object of $\mathbb{E}(I)$ is a 'constant' type-family. However, the maps between I-indexed families do not need to be constant families of maps. This situation is described by the simple slice fibration of Example 1.1.6. We take the fibration of type-families $q : \mathbb{E} \to \mathbb{C}$ to be the simple slice fibration $P_{\mathbb{T}} : \mathbb{T}_{(-)} \to \mathbb{T}$. The fibration of predicates on type-families $r : \mathbb{Q} \to \mathbb{E}$ is taken to be the fibration $p_{(-)} : \mathbb{P}_{(-)} \to \mathbb{T}_{(-)}$ given by change of base of $p : \mathbb{P} \to \mathbb{T}$ along the functor $\mathbb{T}_{(-)} \to \mathbb{T}$ given by $(I, X) \mapsto I \times X$. Explicitly, an object of $\mathbb{P}_{(-)}$ is a triple (I, X, α) where $I, X \in \mathbb{T}$ and $\alpha \in \mathbb{P}(I \times X)$ and an arrow $(I, X, \alpha) \to (J, Y, \beta)$ lying over $(f, F) : (I, X) \to (J, Y)$ in $\mathbb{T}_{(-)}$ is given by an arrow $\phi : \alpha \to \beta$ in $\mathbb{P}$ lying over $(f \circ \pi_I, F) : I \times X \to J \times Y$ in $\mathbb{T}$.

It remains to show how to extend the strong monad $(-)^\bullet$ on $\mathbb{T}$ with its extension to $\mathbb{P}$ to the relevant sort of indexed monad on $p_{(-)} : \mathbb{P}_{(-)} \to \mathbb{T}_{(-)}$. The monadic strength σ induces a monad on each simple slice category $\mathbb{T}_I$, which in fact assemble into a fibrewise monad. Given $I \in \mathbb{T}$ and $(I, X) \in \mathbb{T}_{(-)}$, we define $M_{\mathbb{E}}(I, X)$ to be $(I, X^\bullet)$ and

$$M_{\mathbb{E}}((f, F) : (I, X) \to (J, Y))$$

to be the pair

$$f : I \to J, \quad F^\bullet \circ \sigma_{I,X} : I \times X^\bullet \to Y^\bullet.$$

This is easily seen to be a cartesian functor $\mathbb{T}_{(-)} \to \mathbb{T}_{(-)}$. We define $\eta_{\mathbb{E},I,X} : (I, X) \to (I, X^\bullet)$ to be the pair

$$1_I : I \to I, \quad \eta_X \circ \pi_X : I \times X \to X^\bullet$$

and $\mu_{I,X} : (I, X^{\bullet\bullet}) \to (I, X^\bullet)$ to be the pair

$$1_I : I \to I, \quad \mu_X \circ \pi_{X^{\bullet\bullet}} : I \times X^{\bullet\bullet} \to X^\bullet.$$

It is trivial to check that these are natural transformations with $\mathbb{T}_{(-)}$-vertical components.

Completing the translation will involve making some choices of cartesian lifts in the fibration $p : \mathbb{P} \to \mathbb{T}$. As an alternative to assuming that p is a cloven fibration, we replace it by an equivalent cloven fibration, which will yield an equivalent category at the end. For $I, X \in \mathbb{T}$ and $\alpha \in \mathbb{P}(I \times X)$, let $\hat{\sigma}_{I,X,\alpha}$ be the chosen cartesian lift of $\sigma_{I,X} : I \times X^\bullet \to (I \times X)^\bullet$ with codomain $\alpha^\bullet$. We define $M_{\mathbb{Q}}(I, X, \alpha)$ to be the domain of $\hat{\sigma}_{I,X,\alpha}$ and for an arrow $(f, F, \phi) : (I, X, \alpha) \to (J, Y, \beta)$ in $\mathbb{P}_{(-)}$ we define $M_{\mathbb{Q}}(f, F, \phi)$ to be $(f, F^\bullet \circ \sigma_{I,X}, \hat{\phi})$ where $\hat{\phi}$ is the unique arrow $\operatorname{dom} \hat{\sigma}_{I,X,\alpha} \to \operatorname{dom} \hat{\sigma}_{J,Y,\beta}$ lying over

$$M_{\mathbb{E}}(f, F) = (f, F^\bullet \circ \sigma_{I,X}) : (I, X^\bullet) \to (J, Y^\bullet)$$

making the diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{\hat{\phi}} & \bullet \\ {\scriptstyle \hat{\sigma}_{J,Y,\beta}}\Big\downarrow & & \Big\downarrow{\scriptstyle \hat{\sigma}_{I,X,\alpha}} \\ \alpha^{\bullet} & \xrightarrow[\phi^{\bullet}]{} & \beta^{\bullet} \end{array}$$

commute.

We define the unit $\eta_{\mathbb{Q}}$ first as a family of arrows in $\mathbb{P}_{(-)}$. For $\alpha \in \mathbb{P}(I \times X)$ we define $\eta_{\mathbb{Q},I,X,\alpha} : \alpha \to M_{\mathbb{Q}}(I, X, \alpha)$ to be the unique factorization of $\hat{\eta}_{I\times X,\alpha} : \alpha \to \alpha^{\bullet}$ through $\hat{\sigma}_{I,X,\alpha}$ lying over $1_I \times \eta_X : I \times X \to I \times X^{\bullet}$, and this factorization exists since $\sigma_{I,X} \circ (1_I \times \eta_X) = \eta_{I\times X}$. Since the unit $\hat{\eta}_{I\times X,\alpha}$ in $\mathbb{P}$ is p-cartesian over $\eta_{I\times X} : I \times X \to (I \times X)^{\bullet}$, and the strength $\hat{\sigma}_{I,X,\alpha}$ is p-cartesian over $\sigma_{I,X}$, it follows that $\eta_{\mathbb{Q},I,X,\alpha}$ is p-cartesian over $1_I \times \eta_X$. Hence it is easy to see that the arrows $\eta_{\mathbb{Q},I,X,\alpha}$ give a natural transformation $\mathrm{id}_{\mathbb{P}} \Rightarrow M_{\mathbb{P}}$ with $p_{(-)}$-cartesian components lying over $\eta_{\mathbb{E},I,X}$.

We define the multiplication $\mu_{\mathbb{Q},I,X,\alpha}$ in a similar manner, by considering the commuting square

$$\begin{array}{ccccc} I \times X^{\bullet\bullet} & \xrightarrow{\sigma_{I,X^{\bullet}}} & (I \times X^{\bullet})^{\bullet} & \xrightarrow{\sigma^{\bullet}_{I,X}} & (I \times X)^{\bullet\bullet} \\ {\scriptstyle 1_X \times \mu_X}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \mu_{I\times X}} \\ I \times X^{\bullet} & & \xrightarrow[\sigma_{I,X}]{} & & (I \times X)^{\bullet} \end{array}$$

in $\mathbb{T}$ and letting $\mu_{\mathbb{Q},I,X,\alpha} : M_{\mathbb{Q}}M_{\mathbb{Q}}(I, X, \alpha) \to M_{\mathbb{Q}}(I, X, \alpha)$ be the factorization of

$$\hat{\mu}_{I\times X,\alpha} \circ \hat{\sigma}^{\bullet}_{I,X,\alpha} \circ \hat{\sigma}_{I,X^{\bullet},(M_{\mathbb{Q}}(I,X,\alpha))^{\bullet}} : M_{\mathbb{Q}}M_{\mathbb{Q}}(I, X, \alpha) \to \alpha^{\bullet}$$

through

$$\hat{\sigma}_{I,X,M_{\mathbb{Q}}(I,X,\alpha)} : M_{\mathbb{Q}}(I, X, \alpha) \to \alpha^{\bullet}.$$

As before, $\mu_{\mathbb{Q},I,X,\alpha}$ is p-cartesian over $1_I \times \mu_X$. Hence it is easy to check that the $\mu_{\mathbb{Q},I,X,\alpha}$ form a natural transformation $M_{\mathbb{Q}}M_{\mathbb{Q}} \Rightarrow M_{\mathbb{Q}}$ with $p_{(-)}$-cartesian components over $\mu_{\mathbb{E},I,X}$.

Remark 4.2.3. In [21], the indexed extension of $(-)^{\bullet}$ to $\mathbb{P}_{(-)}$ is assumed to exist, rather than derived from a monad on $\mathbb{P}$. This gives some additional generality, but at the cost of a rather unnatural way of describing the basic inputs to the construction. Moreover, in setting (0), the indexed monad sends a subset $\alpha \subseteq I \times X$ to

$$M_{\mathbb{Q}}(I, X, \alpha) = \{(i, \chi) \in I \times X^{\bullet} \mid \forall x \in \chi.(i, x) \in \alpha\}$$

which is indeed of our slightly restricted form.

4.2.4 From (B) to (C): the Kleisli fibrations

We obtain the two fibrations in (C) by applying a version of the Kleisli category construction to the fibrations from (B). However, we will not actually be able to prove that the fibrations support fibred biproducts or have cartesian closed fibres at this stage — we will need to add more assumptions to setting (B), which we shall do when we come to proving that Dill admits products and exponentials. When dealing with Kleisli categories, we use the term *pure morphism* to mean a Kleisli arrow $A \twoheadrightarrow B$ which can be expressed, as an arrow $A \to TB$ in the underlying category, in the form

$$A \xrightarrow{f} B \xrightarrow{\eta_B} TB$$

for some morphism $f : A \to B$ in the underlying category.

We now define the Kleisli fibrations, which we give first as mere functors and prove to be fibrations in Propositions 4.2.5 and 4.2.6. Note that the 'M' in 'q^M' and 'r^M' is just notation, and reflects the dependence on the pair $M = (M_{\mathbb{E}}, M_{\mathbb{Q}})$.

Definition 4.2.4. The *first Kleisli fibration* is $q^M : \mathbb{E}_{M_{\mathbb{E}}} \to \mathbb{C}$ where $\mathbb{E}_{M_{\mathbb{E}}}$ is the Kleisli category of $\mathbb{E}$ with respect to $M_{\mathbb{E}}$ and q^M is the functor sending $X \in \mathbb{E}_{M_{\mathbb{E}}}$ to $q(X) \in \mathbb{C}$ and $F : X \to M_{\mathbb{E}}Y$ to

$$q(X) \xrightarrow{q(F)} q(M_{\mathbb{E}}(Y)) = q(Y).$$

The *second Kleisli fibration* $r^M : \mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{E}_{M_{\mathbb{E}}}$ has base $\mathbb{E}_{M_{\mathbb{E}}}$ and total category $\mathbb{Q}_{M_{\mathbb{Q}}}$ the Kleisli category of $\mathbb{Q}$ with respect to $M_{\mathbb{Q}}$ and r^M is the evident functor $\mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{E}_{M_{\mathbb{E}}}$ which sends $\alpha \in \mathbb{Q}_{M_{\mathbb{Q}}}$ to $r(\alpha) \in \mathbb{E}_{M_{\mathbb{E}}}$ and $\phi : \alpha \to M_{\mathbb{Q}}\beta$ to $r(\phi) : r(\alpha) \to r(M_{\mathbb{Q}}(\beta)) = M_{\mathbb{E}}(r(\beta))$.

Let us investigate the extent to which the Kleisli fibrations are fibrewise constructions.

Proposition 4.2.5. *The functor $q^M : \mathbb{E}_{M_{\mathbb{E}}} \to \mathbb{C}$ is a fibration, and the fibre of q^M over $I \in \mathbb{C}$ is the Kleisli category of the fibre over I of $q : \mathbb{E} \to \mathbb{C}$ with respect to the restriction of the monad $M_{\mathbb{E}}$. Moreover the action of reindexing along $f : J \to I$ in $\mathbb{C}$ on an arrow $F : X \twoheadrightarrow Y$ in $\mathbb{E}_{M_{\mathbb{E}}}(I)$ is to send it to the* *arrow* $f^*(X) \to\!\!\!\!\to f^*(Y)$ *represented by*

$$f^*(X) \xrightarrow{f^*(F)} f^*(M_{\mathbb{E}}(Y)) \cong M_{\mathbb{E}}(f^*(Y)).$$

Proof. Given a map $f : J \to I$ in $\mathbb{C}$ and $X \in \mathbb{E}_{M_{\mathbb{E}}}(I)$ (i.e. $X \in \mathbb{E}(I)$) a cartesian lift of f with codomain X is given by the composite

$$f^*(X) \xrightarrow{f^X} X \xrightarrow{\eta_{\mathbb{E},X}} M_{\mathbb{E}}X$$

where f^X is a q-cartesian lift of f with codomain X. To see that this arrow is q^M-cartesian, let $G : Z \to MX$ be an arrow in $\mathbb{E}_{M_{\mathbb{E}}}$ lying over $q^M(G) : K \to I$ and let $h : K \to J$ be an arrow satisfying $f \circ h = q^M(G)$. Then, since $M_{\mathbb{E}}$ preserves cartesian arrows, there is a unique factorization of G through $M_{\mathbb{E}}(f^X)$ lying over h, say $H : Z \to M_{\mathbb{E}}(f^*(X))$. Then this H is the required factorization with respect to Kleisli composition, since

$$\begin{aligned} & \mu_{\mathbb{E},X} \circ M_{\mathbb{E}}(\eta_{\mathbb{E},X}) \circ M_{\mathbb{E}}(f^X) \circ H \\ = \; & M_{\mathbb{E}}(f^X) \circ H \\ = \; & G. \end{aligned}$$

Uniqueness of H is easy to see from this calculation.

The assertion that the fibres are Kleisli categories is trivial. The description of reindexing along $f : J \to I$ of an arrow $X \to\!\!\!\!\to Y$ represented by $F : X \to M_{\mathbb{E}}(Y)$ is represented by the unique arrow $\overline{F} : f^*(X) \to M_{\mathbb{E}}(f^*(Y))$ satisfying

$$\mu_Y \circ M_{\mathbb{E}}(F) \circ (\eta_X \circ f^X) = \mu_Y \circ M_{\mathbb{E}}(\eta_Y \circ f^Y) \circ \overline{F}$$

where $f^Y : f^*(Y) \to Y$ is a cartesian lift of f with codomain Y. The left-hand side simplifies to $F \circ f^X$ and the right-hand side simplifies to

$$f^*(X) \xrightarrow{\overline{F}} M_{\mathbb{E}}(f^*(Y)) \cong f^*(M_{\mathbb{E}}(Y)) \xrightarrow{f^{M_{\mathbb{E}}(Y)}} M_{\mathbb{E}}(Y)$$

so the unique possibility for $\overline{F}$ is the one claimed in the proposition. □

Proposition 4.2.6. *The functor* $r^M : \mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{E}_{M_{\mathbb{E}}}$ *is a fibration, and for* $I \in \mathbb{C}$, $X \in \mathbb{E}(I)$ *(so* $X \in \mathbb{E}_{M_{\mathbb{E}}}(I)$*), the fibre* $\mathbb{Q}_{M_{\mathbb{Q}}}(X)$ *is isomorphic to the fibre* $\mathbb{Q}(X)$*. Moreover, these isomorphisms respect reindexing along* q^M*-cartesian arrows and along arrows* $X \to\!\!\!\!\to Y$ *in* $\mathbb{E}_{M_{\mathbb{E}}}(I)$ *of the form*

$$X \xrightarrow{F} Y \xrightarrow{\eta_{\mathbb{E},Y}} M_{\mathbb{E}}(Y)$$

where F *is an arrow in* $\mathbb{E}_{M_{\mathbb{E}}}(I)$*.*

Proof. Suppose that $X \in \mathbb{E}$, $\alpha \in \mathbb{Q}(X)$, and $F : Y \to M_{\mathbb{E}}X$ represents an arrow $Y \twoheadrightarrow X$ in $\mathbb{E}_{M_{\mathbb{E}}}$. Then a r^M-cartesian lift of F with codomain α is given by a map

$$F^*(M_{\mathbb{Q}}(\alpha)) \xrightarrow{F^{M_{\mathbb{Q}}(\alpha)}} M_{\mathbb{Q}}(\alpha)$$

which is an r-cartesian lift of F with codomain $M_{\mathbb{Q}}(\alpha)$. To see that this is r^M-cartesian, let $Z \in \mathbb{E}$ with $\gamma \in \mathbb{Q}(Z)$, let $\phi : \gamma \to M_{\mathbb{Q}}(\alpha)$ be a Kleisli arrow $\gamma \twoheadrightarrow \alpha$, and let $G : Z \to M_{\mathbb{E}}(Y)$ be a a Kleisli arrow $Z \twoheadrightarrow Y$ such that $F \circ G = M_{\mathbb{Q}}(\alpha)$ using Kleisli composition, i.e. the composite in $\mathbb{E}$

$$Z \xrightarrow{F} M_{\mathbb{E}}(Y) \xrightarrow{M_{\mathbb{E}}(F)} M_{\mathbb{E}}M_{\mathbb{E}}(X) \xrightarrow{\mu_{\mathbb{E},X}} M_{\mathbb{E}}(X)$$

is $r(\phi) : Z \to M_{\mathbb{E}}(X)$. Now Kleisli maps $\psi : \gamma \twoheadrightarrow F^*(M_{\mathbb{Q}}(\alpha))$ lying over $G : Z \twoheadrightarrow Y$ that Kleisli-compose with $F^{M_{\mathbb{Q}}(\alpha)}$ to give $\phi : \gamma \twoheadrightarrow \alpha$ are in bijective correspondence with maps $\chi : \gamma \to M_{\mathbb{Q}}(F^*M_{\mathbb{Q}}(\alpha))$ in $\mathbb{Q}$ lying over $G : Z \to M_{\mathbb{E}}Y$ making the composite

$$\gamma \xrightarrow{\chi} M_{\mathbb{Q}}(F^*(M_{\mathbb{Q}}(\alpha))) \xrightarrow{M_{\mathbb{Q}}(F^{M_{\mathbb{Q}}(\alpha)})} M_{\mathbb{Q}}M_{\mathbb{Q}}(\alpha) \xrightarrow{\mu_{\mathbb{Q},\alpha}} M_{\mathbb{Q}}(\alpha)$$

in $\mathbb{Q}$, but as $M_{\mathbb{Q}}$ preserves cartesian arrows and the components of the multiplication are cartesian, there is a unique such arrow χ. This completes the proof that r^M is a fibration.

If $\alpha, \beta \in \mathbb{Q}(X)$ where $X \in \mathbb{E}(I)$, then arrows $\alpha \twoheadrightarrow \beta$ in $\mathbb{Q}_{M_{\mathbb{Q}}}(X)$ correspond to arrows $\phi : \alpha \to M_{\mathbb{Q}}(\beta)$ in $\mathbb{Q}$ lying over $\eta_{\mathbb{E},X} : X \to M_{\mathbb{E}}(X)$ in $\mathbb{E}(I)$. But $\eta_{\mathbb{Q},\alpha} : \alpha \to M_{\mathbb{Q}}\alpha$ is r-cartesian over $\eta_{\mathbb{E},X}$, hence these arrows correspond to arrows $\alpha \to \beta$ in $\mathbb{Q}(X)$. A short calculation shows that this correspondence respects Kleisli composition and identities so is in fact an isomorphism.

We need only check the statement that reindexing along pure morphisms commutes with this isomorphism, since in the proof of proposition 4.2.5 we saw that every q^M-cartesian arrow is pure. Given $F : Y \to X$ in $\mathbb{E}$, we have seen that reindexing in r^M along $\eta_{\mathbb{E},X} \circ F$ amounts to applying $M_{\mathbb{Q}}$ and then reindexing along $\eta_{\mathbb{E},X} \circ F$ in r. However, since for any $\alpha \in \mathbb{Q}(X)$ the unit $\eta_{\mathbb{Q},\alpha}$ is r-cartesian over $\eta_{\mathbb{E},X}$, this whole process is equivalent just to reindexing in r along F, as required. □

This completes the translation of setting (B) into setting (C).

4.2.5 The Diller-Nahm category in (B)

We can describe the construction of $s : \mathsf{Dill} \to \mathbb{C}$ in setting (B) directly. One takes the opposite fibration of r, composes with q, takes the fibred Kleisli category of the induced fibred monad and finally one takes the opposite fibration.

Let us describe the various steps in more detail. The first point to check is that the monad $(M_{\mathbb{Q}}, \eta_{\mathbb{Q}}, \mu_{\mathbb{Q}})$ over $(M_{\mathbb{E}}, \eta_{\mathbb{E}}, \mu_{\mathbb{E}})$ gives rise to a monad on q^{op} also lying over $(M_{\mathbb{E}}, \eta_{\mathbb{E}}, \mu_{\mathbb{E}})$. This is the case since taking opposites is functorial, so we do indeed get the functor part. The key point for the monad is that the components of the unit and multiplication are r-cartesian arrows, hence they also represent (cartesian) arrows in r^{op}. Thus we get a monad $(M'_{\mathbb{Q}}, \eta'_{\mathbb{Q}}, \mu'_{\mathbb{Q}})$ on r^{op} lying over $(M_{\mathbb{E}}, \eta_{\mathbb{E}}, \mu_{\mathbb{E}})$.

The next step is to check that we get a fibrewise monad on $q \circ r^{\mathrm{op}}$. We do this by forgetting the $\mathbb{E}$ part of the monad so that the monad is just $(M'_{\mathbb{Q}}, \eta'_{\mathbb{Q}}, \mu'_{\mathbb{Q}})$. Subsequently, we take the fibred Kleisli category, $(q \circ r^{\mathrm{op}})^{M'_{\mathbb{Q}}}$. This may be seen as either applying the usual Kleisli category construction on each fibre, with the evident reindexing, or taking the Kleisli category of the total category together with the obvious projection functor to the base category.

The final steps are simply to take the opposite fibration again and then the usual total category.

A concrete description

Let us describe the Diller-Nahm category in more concrete terms for setting (B). An object of Dill is just an object of $\mathbb{Q}$, but we write it as a triple (I, X, α) with $I \in \mathbb{C}$, $X \in \mathbb{E}(I)$ and $\alpha \in \mathbb{Q}(I, X)$. A morphism $(f, F, \phi) : (I, X, \alpha) \to (J, Y, \beta)$ is given by a map

$$f : I \to J,$$

in $\mathbb{C}$, a map

$$F : f^*Y \to M_{\mathbb{E}}(X)$$

in $\mathbb{E}(I)$, together with a map

$$\phi : F^*(M_{\mathbb{Q}}(\alpha)) \to f^*\beta$$

in $\mathbb{Q}(f^*Y)$, where $f^*\beta$ is the reindexing of $\beta \in \mathbb{Q}(Y)$ along $f : I \to J$ with respect to the fibration $q \circ r : \mathbb{Q} \to \mathbb{C}$.

Let us describe the composition. Given objects (I, X, α), (J, Y, β) and (K, Z, γ), and maps

$$(f, F, \phi) : (I, X, \alpha) \to (J, Y, \beta)$$

and

$$(g, G, \psi) : (J, Y, \beta) \to (K, Z, \gamma)$$

then the composite map is constituted by

$$g \circ f : I \to K$$

in $\mathbb{C}$ together with

$$(gf)^*Z \cong f^*g^*Z \xrightarrow{f^*G} f^*M_{\mathbb{E}}(Y) \cong M_{\mathbb{E}}(f^*Y) \xrightarrow{M_{\mathbb{E}}(F)} M_{\mathbb{E}}M_{\mathbb{E}}(X) \xrightarrow{\mu_{\mathbb{E},X}} M_{\mathbb{E}}(X)$$

in $\mathbb{E}(I)$ and

$$\begin{aligned}(\mu_{\mathbb{E},X} \circ M_{\mathbb{E}}(F) \circ (f^*G))^*(M_{\mathbb{Q}}(\alpha)) &\cong (f^*G)^*M_{\mathbb{E}}(F)^*\mu^*_{\mathbb{E},X}M_{\mathbb{Q}}(\alpha)\\ &\cong (f^*G)^*M_{\mathbb{E}}(F)^*M_{\mathbb{Q}}M_{\mathbb{Q}}(\alpha)\\ &\cong (f^*G)^*M_{\mathbb{Q}}(F^*M_{\mathbb{Q}}(\alpha))\\ &\xrightarrow{(f^*G)^*M_{\mathbb{Q}}(\phi)} (f^*G)^*M_{\mathbb{Q}}(f^*\beta)\\ &\cong f^*(G^*M_{\mathbb{Q}}(\beta))\\ &\xrightarrow{f^*\psi} f^*g^*\gamma\\ &\cong (gf)^*\gamma\end{aligned}$$

in $\mathbb{Q}((gf)^*Z)$. The identity on (I, X, α) is given by

$$1_I : I \to I$$

in $\mathbb{C}$,

$$\eta_{\mathbb{E},X} : X \to M_{\mathbb{E}}(X)$$

in $\mathbb{E}(I)$ and

$$\eta_{\mathbb{E},X}{}^*(M_{\mathbb{Q}}(\alpha)) \cong \alpha$$

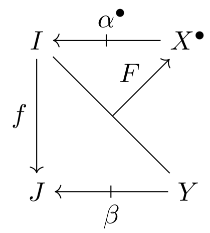
in $\mathbb{Q}(X)$ is the canonical isomorphism induced by the fact that $\eta_{\mathbb{Q},\alpha}$ is cartesian.

We omit a concrete description of the Diller-Nahm category in settings (A) or (0) as it is straightforward to specialize the situation of setting (B). Alternatively, one may consult the treatments given in [11] and [21].

Notation 4.2.7. It may be convenient at some points to represent the objects and maps of Dill pictorially, following the practice in [11] and [21]. An object (I, X, α) is drawn as

$$I \overset{\alpha}{\longleftarrow\!\!\!+\!\!\!\longrightarrow} X\ .$$

A map $(f, F, \phi) : (I, X, \alpha) \to (J, Y, \beta)$ is then represented by a diagram



which unfortunately suppresses the role of ϕ. However, in setting (A), the fibration $p : \mathbb{P} \to \mathbb{T}$ is assumed to be preordered, so it is not necessary to record the name of the map ϕ.

4.3 Finite Products

In this section we will show that Dill has finite products. As we start to explore the logical structure of Dill, we will see that it is useful to keep hold of the Diller-Nahm fibration. In fact, we shall immediately see that it is convenient to define finite products in Dill by constructing fibred finite products in s. Let us give the additional conditions that we need for finite products to exist in each setting in summary form here and then spell them out more precisely afterwards.

(A) We need the base category of types $\mathbb{T}$ to be a distributive category, for each X and Y the evident functor $\mathbb{P}(X + Y) \to \mathbb{P}(X) \times \mathbb{P}(Y)$ (given by reindexing along the coproduct inclusions) to be an isomorphism, and $\mathbb{P}(0) \cong \mathbb{1}$.

(B) The base category $\mathbb{C}$ has finite products, the fibration $q : \mathbb{E} \to \mathbb{C}$ has fibred finite coproducts, for each I, X and Y the evident functor $\mathbb{Q}(X +_I Y) \to \mathbb{Q}(X) \times \mathbb{Q}(Y)$ (given by reindexing along the coproduct inclusions) is an isomorphism and $\mathbb{Q}(0_I) \cong \mathbb{1}$.

(C) The base category $\mathcal{C}$ has finite products, the fibration $q : \mathcal{E} \to \mathcal{C}$ has fibred finite coproducts, for each I, X and Y the evident functor $\mathcal{P}(X +_I Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$ (given by reindexing along the coproduct inclusions) is an isomorphism, and $\mathcal{P}(0_I) \cong \mathbb{1}$.

4.3.1 Finite products and fibred finite products in (C)

Let us spell out the additional assumptions for setting (C) given above. The coproduct inclusions $\iota_X : X \to X +_I Y$ and $\iota_Y : Y \to X +_I Y$ induce functors

$$\iota_X^* : \mathcal{P}(X +_I Y) \to \mathcal{P}(X)$$
$$\iota_Y^* : \mathcal{P}(X +_I Y) \to \mathcal{P}(Y)$$

and hence a functor

$$(\iota_X^*, \iota_Y^*) : \mathcal{P}(X +_I Y) \to \mathcal{P}(X) \times \mathcal{P}(Y).$$

We ask that this functor be an isomorphism. We write the inverse to this isomorphism as

$$\mathsf{ext}_{I,X,Y} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X +_I Y).$$

We also ask that $\mathcal{P}(0_I) \cong \mathbb{1}$, the category with one object and one (identity) morphism.

Proposition 4.3.1. *In* (C)*, the fibration* $s : \mathsf{Dill} \to \mathcal{C}$ *has fibred finite products.*

Proof. Given $I \in \mathcal{C}$, the terminal object in $\mathsf{Dill}(I)$ is given by $(0_I, *)$, where $*$ is the unique object of $\mathbb{1} \cong \mathcal{P}(0_I)$. It is easy to see that this is indeed terminal and stable under reindexing.

Given $I \in \mathcal{C}$, $X, Y \in \mathcal{E}(I)$, $\alpha \in \mathcal{P}(X)$ and $\beta \in \mathcal{P}(Y)$, the fibred product of (X, α) and (Y, β) is

$$(X +_I Y, \mathsf{ext}_{I,X,Y}(\alpha, \beta)).$$

This really is a coproduct in the fibre over I, since given $Z \in \mathcal{E}(I)$, $\gamma \in \mathcal{P}(Z)$, a map from this object to (Z, γ) in $\mathsf{Dill}(I)$ corresponds to a map $[f, g] : X +_I Y \to Z$ in $\mathcal{E}(I)$ together with a map

$$[f, g]^*(\gamma) \to \mathsf{ext}_{I,X,Y}(\alpha, \beta)$$

in $\mathcal{P}(X +_I Y)$, naturally in (Z, γ) as an object of r^{op}. Since the isomorphism $\mathcal{P}(X +_I Y) \cong \mathcal{P}(X) \times \mathcal{P}(Y)$ is induced by coproduct inclusions means that the latter is equivalent to a map

$$(f^*(\gamma), g^*(\gamma)) \to (\alpha, \beta)$$

in $\mathcal{P}(X) \times \mathcal{P}(Y)$, naturally in γ, as required. $\square$

Corollary 4.3.2. *The category* Dill *has finite products.*

Proof. By 1.1.15, this follows immediately from 4.3.1. □

4.3.2 Translation from (B) into (C)

Let us spell out the conditions required for (B). We will suppose that $\mathbb{C}$ has finite products and that q has fibred finite products. This is equivalent to requiring that the categories $\mathbb{C}$ and $\mathbb{E}$ have finite products and that q preserves them. We will also ask that q has fibred finite sums.

Notation 4.3.3. Write $\top$ for the terminal object in $\mathbb{C}$, and $\top_I$ and 0_I respectively for the fibred terminal and initial objects in $\mathbb{E}(I)$. We use the usual $\times$ for products in $\mathbb{C}$, and $\times_I$ and $+_I$ respectively for the fibred product and fibred sums in $\mathbb{E}(I)$.

We assume that the fibration r satisfies $\mathbb{Q}(0_I) \cong \mathbb{1}$ for each $I \in \mathbb{C}$, where $\mathbb{1}$ is the terminal category, and also that for each $X, Y \in \mathbb{E}(I)$, $\mathbb{Q}(X +_I Y) \cong \mathbb{Q}(X) \times \mathbb{Q}(Y)$. To state the latter more precisely, reindexing along the fibred coproduct inclusions $\iota_X : X \to X +_I Y$ and $\iota_Y : Y \to X +_I Y$ gives rise to a functor

$$(\iota_X^*, \iota_Y^*) : \mathbb{Q}(X +_I Y) \to \mathbb{Q}(X) \times \mathbb{Q}(Y)$$

which we require to be an isomorphism with inverse denoted by

$$\mathsf{ext}_{I,X,Y} : \mathbb{Q}(X) \times \mathbb{Q}(Y) \to \mathbb{Q}(X +_I Y).$$

For the actual translation, we must first show the following.

Proposition 4.3.4. *The first Kleisli fibration* $q^M : \mathbb{E}_{M_\mathbb{E}} \to \mathbb{C}$ *admits fibred finite coproducts.*

Proof. This is clear since the construction of a Kleisli category preserves the existence of finite coproducts and the coproduct inclusions are given by pure arrows. Moreover, by 4.2.5, reindexing respects the Kleisli construction and acts in the obvious way on pure arrows, hence reindexing preserves the finite coproducts. □

The other thing we have to show is the 'extensivity' condition for the second Kleisli fibration. Here we use 4.2.6, which tells us that reindexing along the coproduct inclusions (pure arrows) in r^M is the same as reindexing in r. This completes the translation from (B) into (C).

4.3.3 Translation from (A) into (B)

Let us spell out the additional assumptions we use in (A). Let us assume that the base category of types $\mathbb{T}$ has finite products and finite coproducts and moreover that it is a *distributive category* (see [8]). This means that the canonical map

$$X \times Z + Y \times Z \to (X + Y) \times Z$$

is an isomorphism. It follows from this that $0 \times Z \cong 0$ for every object Z.

Notation 4.3.5. Write 0 and $\top$ respectively for the initial and terminal objects in $\mathbb{T}$.

We do not make assumptions of fibred products or coproducts on $p : \mathbb{P} \to \mathbb{T}$, but we do assume a sort of 'extensivity' property. Precisely, for any objects $X, Y \in \mathbb{T}$, reindexing along the coproduct inclusions $\iota_X : X \to X + Y$ and $\iota_Y : Y \to X + Y$ induces a functor

$$(\iota_X^*, \iota_Y^*) : \mathbb{P}(X + Y) \to \mathbb{P}(X) \times \mathbb{P}(Y)$$

and we ask that this functor is an isomorphism. We denote its inverse by

$$\mathsf{ext}_{X,Y} : \mathbb{P}(X) \times \mathbb{P}(Y) \to \mathbb{P}(X + Y).$$

We also insist that $\mathbb{P}(0) \cong \mathbb{1}$, where $\mathbb{1}$ is the terminal category.

Let us check that the new assumptions for setting (A) validate our new assumptions for (B) after the translation. It is easy to see that when $\mathbb{T}$ has products, $P_{\mathbb{T}} : \mathbb{T}_{(-)} \to \mathbb{T}$ has fibred finite products. For an object $I \in \mathbb{T}$, the terminal object in $\mathbb{T}_I$ is $\top$ and the product of X and Y is just the same as the product $X \times Y$ in $\mathbb{T}$.

When $\mathbb{T}$ is a distributive category, $P_{\mathbb{T}} : \mathbb{T}_{(-)} \to \mathbb{T}$ has fibred finite coproducts. For an object $I \in \mathbb{T}$, the initial object in $\mathbb{T}_I$ is 0, (since $I \times 0 \cong 0$ in distributive categories). The coproduct of $X, Y \in \mathbb{T}_I$ is the same as the coproduct $X + Y$ in $\mathbb{T}$, since maps $X + Y \to Z$ in $\mathbb{T}_I$ are given by maps

$$I \times (X + Y) \cong I \times X + I \times Y \to Z$$

in $\mathbb{T}$.

Given $I, X, Y \in \mathbb{T}$, we define

$$\mathsf{ext}_{I,X,Y} : \mathbb{P}_I(X) \times \mathbb{P}_I(Y) \to \mathbb{P}_I(X + Y))$$

to be

$$\mathbb{P}(I \times X) \times \mathbb{P}(I \times Y) \xrightarrow{\mathsf{ext}_{I \times X, I \times Y}} \mathbb{P}(I \times X + I \times Y) \cong \mathbb{P}(I \times (X + Y)).$$

Finally $\mathbb{P}_I(0) \cong \mathbb{P}(I \times 0) \cong \mathbb{P}(0) \cong \mathbb{1}$, since $I \times 0 \cong 0$ in a distributive category.

4.3.4 Concrete description of finite products in (B)

Let us given a concrete description of the finite products in Dill in setting (B). The product of (I, X, α) with (J, Y, β) is

$$(I \times J, \pi_I^* X +_I \pi_J^* Y, \mathsf{ext}_{I \times J, \pi_I^* X, \pi_J^* Y}(\pi_I^* \alpha, \pi_J^* \beta)).$$

The terminal object is $(\top, 0_\top, *)$, where $*$ is the unique object of $\mathbb{Q}(0_\top) \cong \mathbb{1}$.

In the simply-typed case we may write the product as

$$(I \times J, X + Y, \mathsf{ext}_{I \times J \times X, I \times J \times Y}(\pi_{I \times X}^* \alpha, \pi_{J \times Y} * \beta)),$$

where $\pi_{I \times X}$ is the projection $I \times J \times X \to I \times X$ and $\pi_{J \times Y}$ is the projection $I \times J \times Y \to J \times Y$, and we may write the terminal object as

$$(\top, 0, *)$$

where $*$ is the unique object of $\mathbb{P}(\top \times 0) \cong \mathbb{P}(0)$.

4.4 Simple products

Let us summarize the additional assumptions we need in order to define simple products, which we will make more precise below.

(A) No additional assumptions required.

(B) The fibration $q : \mathbb{E} \to \mathbb{C}$ has simple sums which preserve predicates in the sense that the functor $\mathbb{Q}(\Sigma_J X) \to \mathbb{Q}(X)$ given by reindexing along the canonical cocartesian map $X \to \Sigma_J X$ is an isomorphism for each $X \in \mathbb{E}(I \times J)$.

(C) The fibration $q : \mathcal{E} \to \mathcal{C}$ has simple sums which preserve predicates in the sense that the functor $\mathcal{P}(\Sigma_J X) \to \mathcal{P}(X)$ given by reindexing along the canonical cocartesian map $X \to \Sigma_J X$ is an isomorphism for each $X \in \mathcal{E}(I \times J)$.

4.4.1 Simple products in setting (C)

Let us be more precise about the new assumption for (C). The isomorphism $\mathcal{P}(\Sigma_J X) \cong \mathcal{P}(X)$ should not be arbitrary. Instead, we ask that reindexing along the canonical map $\epsilon_{I,J,X} : X \to \Sigma_J X$, which is a functor $\mathcal{P}(\Sigma_J X) \to \mathcal{P}(X)$, is an isomorphism. We write the inverse as

$$\mathsf{sum}_{I,J,X} : \mathcal{P}(X) \to \mathcal{P}(\Sigma_J X).$$

Proposition 4.4.1. *In setting* (C)*,* $s : \mathsf{Dill} \to \mathcal{C}$ *has simple products.*

Proof. It suffices to show that $q \circ r^{\mathrm{op}}$ has simple sums, since then upon taking opposites we get simple products. Given $I, J \in \mathcal{C}$, $X \in \mathcal{E}(I \times J)$, $\alpha \in \mathcal{P}(X)$, the sum $\Sigma_J(X, \alpha)$ is given by

$$(\Sigma_J X, \mathsf{sum}_{I,J,X}(\alpha)).$$

This is indeed a sum, since for any $Y \in \mathcal{E}(I)$, $\beta \in \mathcal{P}(Y)$, maps

$$F : \Sigma_J X \to Y$$

in $\mathcal{E}(I)$ together with

$$\phi : F^*(\beta) \to \mathsf{sum}_{I,J,X}(\alpha)$$

in $\mathcal{P}(\Sigma_J X)$ correspond to maps

$$G : X \to \pi_I^*(Y)$$

in $\mathcal{E}(I \times J)$ together with

$$\psi : G^*(\pi_I^*(\beta)) \to \alpha$$

where G and F are adjuncts via the simple sum adjunction in q, and ψ is given by $\epsilon_{I,J,X}^*(\phi)$, where

$$\epsilon_{I,J,X}^*(F^*(\beta)) \cong G^*(\pi_I^*(\beta))$$

because $F \circ \epsilon_{I,J,K} = \pi_I^Y \circ G$ where π_I^Y is a cartesian lift of $\pi_I : I \times J \to I$ with codomain Y. The bijection between F and G is natural in Y since it comes from a simple sum. The other bijection is also clearly natural since it is given by reindexing. To verify the Beck-Chevalley condition it suffices to check that for any map $g : K \to I$ in $\mathcal{C}$, the induced map

$$(f \times 1_J)^*(X, \alpha) \to f^*(\Sigma_J X, \mathsf{sum}_{I,J,X}\, \alpha)$$

exhibits its codomain as a simple sum along $\pi_K : K \times J \to K$ of its domain. This is easy to verify, using the fact that $\mathsf{sum}_{I,J,X}$ is inverse to reindexing. □

4.4.2 Translation from (B) into (C)

Let us clarify the additional assumption on (B). We assume that $q : \mathbb{E} \to \mathbb{C}$ has simple sums and moreover that for any $I \in \mathbb{C}$ and $X \in \mathbb{E}(I)$, reindexing along the canonical map $\epsilon_{I,J,X} : X \to \Sigma_J X$ gives rise to an isomorphism $\mathbb{Q}(\Sigma_J X) \cong$

$\mathbb{Q}(X)$. We write the inverse as

$$\mathsf{sum}_{I,J,X} : \mathbb{Q}(X) \to \mathbb{Q}(\Sigma_J X).$$

For the translation, we must first check that the first Kleisli fibration has simple sums. It is easy to see that it does, using the same formula for the simple sum, using the description of reindexing in 4.2.5. We must also verify that these sums preserve the fibres of the second Kleisli fibration. This is trivial using the fact that the canonical arrows $\epsilon_{I,X,Y} : X \twoheadrightarrow \Sigma_J X$ are pure morphisms and the description of the fibres and reindexing given in 4.2.6.

4.4.3 Translation from (A) into (B)

We did not add any new assumptions in (A) for this part, so it remains to verify that the translation validates our new assumptions for (B).

Firstly, we must check that the simple slice fibration has simple sums. Let $I, J, X \in \mathbb{T}$. Then considering $(I \times J, X)$ as an object of $\mathbb{T}_{I\times J}$, its simple sum with respect to J is $(I, J \times X)$. It is trivial to verify that this has the correct universal property, together with the map

$$(I \times J, X) \to (I, J \times X)$$

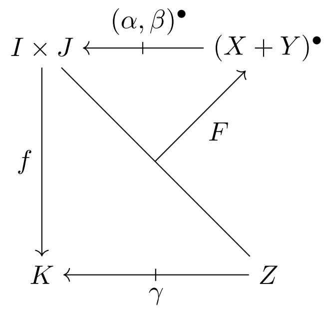
given by the projection $I \times J \times X \to J \times X$. The Beck-Chevalley condition is trivial. For predicates, reindexing along this map is an isomorphism by the associativity of products:

$$\mathbb{P}_I(J \times X) = \mathbb{P}(I \times (J \times X)) \cong \mathbb{P}((I \times J) \times X) = \mathbb{P}_{I\times J}(X),$$

which isomorphism we usually suppress.

4.5 Function spaces

We now turn to the remarkable fact that Dill is a cartesian closed category in our basic example with sets, subsets and the finite multisets monad (setting (0)). This is true in spite of the fact that $s : \mathsf{Dill} \to \mathbb{T}$ does not have fibrewise dependent products even along product projections. Sticking with this concrete example, consider a map $(f, F, \phi) : (I, X, \alpha) \times (J, Y, \beta) \to (K, Z, \gamma)$, or pictorially:



where, (for all $i \in I$, $j \in J$, $z \in Z$),

$$\forall w \in F(i,j,z).(w \in X \Rightarrow \alpha(i,w)) \wedge (w \in Y \Rightarrow \beta(j,w)) \vdash \gamma(f(i,j),z).$$

Now maps $f : I \times J \to K$ correspond to maps $\overline{f} : I \to K^J$. Maps $F : I \times J \times Z \to (X+Y)^\bullet$ correspond to pairs of maps

$$F_1 : I \times J \times Z \to X^\bullet$$
$$F_2 : I \times J \times Z \to Y^\bullet$$

since $(X+Y)^\bullet \cong X^\bullet \times Y^\bullet$. The condition on predicates then just says that we have the sequent

$$\forall x \in F_1(i,j,z).\alpha(i,x) \vdash \big(\forall y \in F_2(i,j,z).\beta(j,y)\big) \Rightarrow \gamma(f(i,j),z).$$

Now we can take the exponential transpose of the map F_2 to get

$$\overline{F_2} : I \to (Y^\bullet)^{J \times Z}.$$

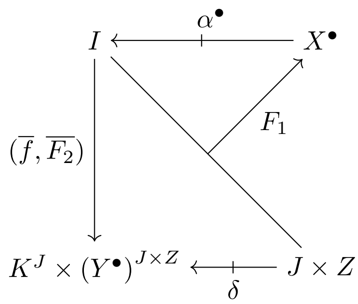
Hence we can give the exponential object $[(J,Y,\beta),(K,Z,\gamma)]$ to be the object

$$K^J \times (Y^\bullet)^{J \times Z} \xleftarrow{\ \delta\ } J \times Z$$

where δ is the subset/predicate on $J^K \times (Y^\bullet)^{J \times Z} \times J \times Z$ given by

$$\delta(g,G,j,z) = \big(\forall y \in G(j,z).\beta(j,y)\big) \Rightarrow \gamma(g(j),z).$$

It is now easy to check that the mapping sending an arrow (f, F, ϕ) as above to



is a bijection.

What is not so clear is that this bijection is natural. We will show that it is axiomatically, but this will require extending all three of our settings (A), (B) and (C) with more assumptions. Let us summarize the required additional assumptions:

(A) The category $\mathbb{T}$ is cartesian closed, the fibration $p : \mathbb{P} \to \mathbb{T}$ is cartesian closed, in the fibres of $\mathbb{P}$ the monad $(-)^\bullet$ preserves fibred finite products, and on $\mathbb{T}$ the monad $(-)^\bullet$ satisfies (M-0) and (M-+-$\times$).

(B) The category $\mathbb{C}$ is cartesian closed, the fibration $q : \mathbb{E} \to \mathbb{C}$ is locally small with fibred finite products, the fibration $r : \mathbb{Q} \to \mathbb{E}$ is cartesian closed, the monad $M_{\mathbb{Q}}$ preserves fibred finite products, and the monad $M_{\mathbb{E}}$ satisfies (M-0) and (M-+-$\times$) when restricted to each fibre of q.

(C) The category $\mathcal{C}$ is cartesian closed, the fibration $q : \mathcal{E} \to \mathcal{C}$ is locally small and has fibred finite biproducts, the fibration $r : \mathcal{P} \to \mathcal{E}$ has fibred finite products, and the fibres of r are cartesian closed with the exponentials being preserved by reindexing along q-cartesian arrows.

4.5.1 Locally small fibrations

We should first explain what is meant by the condition of locally small fibration used in the hypotheses for (B) and (C).

Definition 4.5.1. Let $p : \mathbb{C} \to \mathbb{B}$ be a fibration. Then p is *locally small* if for any $I \in \mathbb{B}$, $X, Y \in \mathbb{C}(I)$ there exists an object $[X, Y]_I \in \mathbb{B}$ (a *function comprehension*) which induces, for any $J \in \mathbb{B}$ and $g : J \to I$, a natural bijection of homsets $\mathbb{B}(J, [X, Y]_I) \cong \mathbb{C}(J)(g^*(X), g^*(Y))$. The bijection is natural in g in the sense that, for an object $K \in \mathbb{B}$ and any map $h : K \to J$, precomposing a map $J \to [X, Y]_I$ with h corresponds under this bijection to reindexing a map $g^*(X) \to g^*(Y)$ along h.

From the naturality of the bijection we can deduce that there is a *generic arrow*

$$\mathsf{gen}_{I,X,Y} : \pi_I^*(X) \to \pi_I^*(Y) \in \mathbb{C}(I \times [X,Y]_I),$$

namely the one corresponding to the product projection $I \times [X,Y]_I \to [X,Y]_I$, such that for $g : J \to I$ the bijection is given by the operation taking a map $f : J \to [X,Y]_I$ in $\mathbb{B}$ to

$$(g,f)^*(\mathsf{gen}_{I,X,Y}) : g^*(X) \to g^*(Y)$$

in $\mathbb{C}(J)$.

Remark 4.5.2. Built into this notion of locally small is the idea that the objects of the fibre categories $\mathbb{C}(I)$ are constant I-indexed (type-)families. A map between constant families $(X)_{i\in I} \to (Y)_{i\in I}$ is just an I-indexed family of maps $X \to Y$. In Chapter 6, we will deal with non-constant type-families. The treatment we give there in terms of quasifibred dependent products, which generalize the quasifibred exponentials below, as applied to the examples of Chapter 7 can be analysed in terms of a generalization of this notion of local smallness.

4.5.2 Function spaces from quasifibred exponentials

The construction of function spaces in setting (C) will use the follow general construction, reducing the problem to finding *quasifibred exponentials* in $s : \mathsf{Dill} \to \mathcal{C}$.

Let $\mathbb{B}$ be a cartesian closed category and let $p : \mathbb{C} \to \mathbb{B}$ be a fibration over it. We will suppose that p has fibred finite products. It is well-known that the following are equivalent:

- Each fibre of p is cartesian closed and reindexing preserves the cartesian closed structure.
- $\mathbb{C}$ is cartesian closed and p preserves the cartesian closed structure.

However, in the situation of interest to us here we wish to show that $\mathbb{C}$ is cartesian closed but we do not expect p to preserve the cartesian closed structure. We will describe a more general situation in which $\mathbb{C}$ is cartesian closed. We will assume that p has simple products.

The key condition that we will use now is the following, which appears to be novel.

Definition 4.5.3. A fibration $p : \mathbb{C} \to \mathbb{B}$ has *quasifibred exponentials* if for each $I \in \mathbb{B}$, and $Y, Z \in \mathbb{C}(I)$, there exists an object

$$[\![Y, Z]\!]_I \in \mathbb{B}$$

and an object

$$(\!|Y, Z|\!)_I \in \mathbb{C}(I \times [\![Y, Z]\!]_I)$$

such that for any object $J \in \mathbb{B}$, any arrow $f : J \to I$ in $\mathbb{B}$ and any object $X \in \mathbb{C}(J)$, there is a natural bijection between the set of arrows

$$F : X \times_J f^*(Y) \to f^*(Z)$$

in $\mathbb{C}(J)$ and the set of pairs (g, ϕ) where

$$g : J \to [\![Y, Z]\!]_I$$

is an arrow in $\mathbb{B}$ and

$$\phi : X \to (f, g)^*((\!|Y, Z|\!)_I)$$

is an arrow in $\mathbb{C}(J)$. The naturality has two components. The first, naturality in X, is the requirement that for any $\psi : W \to X$, the composite arrow

$$W \times_J f^*(Y) \xrightarrow{\psi \times_J 1_{f^*(Y)}} X \times_J f^*(Y) \xrightarrow{F} f^*(Z)$$

corresponds to the pair g, $\phi \circ \psi$, where F corresponds to g, ϕ as above. The second, naturality in f is the requirement that for any $h : K \to J$ in $\mathbb{B}$ the map

$$h^*(X) \times_J (fh)^*(Y) \xrightarrow{h^*(F)} (fh)^*(Z)$$

in $\mathbb{C}(K)$ corresponds to the pair $g \circ h$, $h^*(\phi) : W \to (fh, gh)^*((\!|Y, Z|\!)_I)$ where F corresponds to g, ϕ as above. Moreover, this bijection is natural in J and X in the sense that if $K \in \mathbb{B}$, $W \in \mathbb{C}(K)$ and we are given $h : K \to J$ in $\mathbb{B}$ and $\psi : W \to h^*(X)$ in $\mathbb{C}(K)$, then under the bijection sending a map F as above to

$$h^*(F) \circ (\psi \times 1_{(fh)^*(Y)}) : W \times_K (fh)^*(Y) \to (fh)^*(Z)$$

corresponds to sending a pair (g, ϕ) as above to

$$g \circ h : K \to [\![Y, Z]\!]_I$$

together with

$$h^*(\phi) \circ \psi : W \to (fh, gh)^*((\!|Y, Z|\!)_I).$$

Remark 4.5.4. There is an equivalent definition of quasifibred exponential which omits mention of the arrow $f : J \to I$ and the naturality in J — i.e. we only have the bijection for $f = 1_I$. The correct definition then includes a condition of stability under reindexing in I. One might compare this with the alternative form of the Beck-Chevalley condition for display map categories from 1.5.5.

Remark 4.5.5. Having fibred exponentials is the special case of Definition 4.5.3 where $[\![Y, Z]\!]_I$ is always a terminal object.

Proposition 4.5.6. *If $\mathbb{B}$ is cartesian closed and $p : \mathbb{C} \to \mathbb{B}$ has quasifibred exponentials, fibred finite products, and simple products, then the total category $\mathbb{C}$ is cartesian closed.*

Proof. Let $J, K \in \mathbb{B}$, $Y \in \mathbb{C}(J)$ and $Z \in \mathbb{C}(K)$. We will construct the exponential $(Y \Rightarrow Z)$. Writing $\mathsf{ev}_{J,K} : J \times (J \Rightarrow K) \to K$ for the counit of exponentials in $\mathbb{B}$, we consider the objects

$$\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z) \in \mathbb{C}(J \times (J \Rightarrow K))$$

and take the quasifibred exponential

$$(\!|\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)|\!)_{J\times(J\Rightarrow K)} \in \mathbb{C}\Big(J \times (J \Rightarrow K) \times [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}\Big).$$

Our candidate for the exponential object is given by reindexing this object along the map

$$\begin{aligned}(\pi_J, \pi_{(J\Rightarrow K)}, \mathsf{ev}_{J,[\![\pi_J^*(Y),\mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}} \circ \pi_{J\times[\![\pi_J^*(Y),\mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}}) :&\\ J \times (J \Rightarrow K) \times [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)} \to&\\ J \times (J \Rightarrow K) \times (J \Rightarrow [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}),&\end{aligned}$$

which we denote by $E_{J,K,Y,Z}$, and then taking the simple product with respect to J. Our claim is that the exponential object $(Y \Rightarrow Z)$ is

$$\Pi_J\left(E_{J,K,Y,Z}^*(\!|\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)|\!)_{J\times(J\Rightarrow K)}\right)$$

lying over

$$\begin{aligned}(J \Rightarrow K) \times (J \Rightarrow [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)})&\\ \cong (J \Rightarrow K \times [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}).&\end{aligned}$$

To see this, let $I \in \mathbb{B}$ and $X \in \mathbb{C}(I)$ and let us suppose we have a map from

X into the candidate exponential object, so we have

$$\begin{aligned}
&f : I \to (J \Rightarrow K) && \in \mathbb{B} \\
&g : I \to (J \Rightarrow [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}) && \in \mathbb{B} \\
&\phi : X \to (f,g)^*\ \Pi_J\ E_{J,K,Y,Z}^*\Big((\!|\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)|\!)_{J\times(J\Rightarrow K)}\Big) && \in \mathbb{C}(I).
\end{aligned}$$

By the Beck-Chevalley condition, we may replace $(f,g)^*\ \Pi_J$ with $\Pi_J\,(1_J \times (f,g))^*$. But we can easily calculate

$$\begin{aligned}
E_{J,K,Y,Z} \circ (1_J \times (f,g)) = (1_J \times f, g)& \\
: J \times I \to J \times (J \Rightarrow K) \times [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)}&
\end{aligned}$$

Hence our data naturally corresponds to

$$\begin{aligned}
&\overline{f} : J \times I \to K && \in \mathbb{B} \\
&\overline{g} : J \times I \to [\![\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)]\!]_{J\times(J\Rightarrow K)} && \in \mathbb{B} \\
&\overline{\phi} : \pi_I^*(X) \to (1_J \times f, g)^*\Big((\!|\pi_J^*(Y), \mathsf{ev}_{J,K}^*(Z)|\!)_{J\times(J\Rightarrow K)}\Big) && \in \mathbb{C}(J \times I).
\end{aligned}$$

By the definition of quasifibred exponentials, this naturally corresponds to

$$\begin{aligned}
&\overline{f} : J \times I \to K && \in \mathbb{B} \\
&F : \pi_I^*(X) \times_{J\times I} (1_J \times f)^*(\pi_J^*(Y)) \to (1_J \times f)^*(\mathsf{ev}_{J,K}^*(Z)) && \in \mathbb{C}(J \times I)
\end{aligned}$$

as required, since we may simplify the expressions

$$(1_J \times f)^*(\pi_J^*(Y)) \cong \pi_J^*(Y)$$

(where the π_J maps are product projections with codomain J but different domains) and

$$(1_J \times f)^*(\mathsf{ev}_{J,K}^*(Z)) \cong \overline{f}^*(Z)$$

and also the object $\pi_I^*(X) \times_{J\times I} \pi_J^*(Y)$ is indeed the product of X and Y in $\mathbb{C}$. $\square$

4.5.3 Quasifibred exponentials in (C)

To obtain function spaces in $s : \mathsf{Dill} \to \mathcal{C}$ in setting (C), we only need to construct quasifibred exponentials.

Proposition 4.5.7. *In setting* (C) *the fibration* $s : \mathsf{Dill} \to \mathcal{C}$ *admits quasifibred exponentials.*

Proof. Let $I \in \mathcal{C}$, $Y, Z \in \mathcal{E}(I)$, $\beta \in \mathcal{E}(Y)$ and $\gamma \in \mathcal{E}(Z)$. We use the local smallness of $q : \mathcal{E} \to \mathcal{C}$ to define $[\![(Y,\beta),(Z,\gamma)]\!]_I$ to be

$$[Z, Y]_I$$

and $(\!|(Y,\beta),(Z,\gamma)|\!)_I \in \mathsf{Dill}(I \times [\![(Y,\beta),(Z,\gamma)]\!]_I)$ to be

$$(\pi_I^*(Z), \mathsf{gen}^*_{I,Z,Y}(\beta) \Rightarrow \gamma)$$

(where π_I is the projection $I \times [Z,Y]_I \to I$).

To see that this is indeed a quasifibred exponential, let us consider another object $J \in \mathcal{C}$, map $f : J \to I$, and also objects $X \in \mathcal{E}(J)$ and $\alpha \in \mathcal{P}(X)$, and give the required bijection between the two kinds of data. Suppose we have an arrow

$$(X,\alpha) \times_J f^*(Y,\beta) \to f^*(Z,\gamma)$$

in $\mathsf{Dill}(I)$, which is to say we are given the data

$$\begin{aligned} &F : f^*(Z) \to X \oplus_J f^*(Y) && \in \mathcal{E}(J) \\ &\phi : F^*(\pi_X^*(\alpha) \times_{f^*(Z)} \pi^*_{f^*(Y)}(f^*(\beta))) \to f^*(\gamma) && \in \mathcal{P}(f^*(Z)) \end{aligned}$$

where $\oplus_J$ is the biproduct in the fibre of q over J. This data corresponds to

$$\begin{aligned} &F_1 : f^*(Z) \to X && \in \mathcal{E}(J) \\ &F_2 : f^*(Z) \to f^*(Y) && \in \mathcal{E}(J) \\ &\phi : F_1^*(\alpha) \times F_2^*(f^*(\beta)) \to f^*(\gamma) && \in \mathcal{P}(f^*(Z)) \end{aligned}$$

just by using properties of the fibred products. On the other hand, suppose we also have the data classified by our proposed quasifibred exponential:

$$\begin{aligned} &g : J \to [Z,Y]_I && \in \mathcal{C} \\ &G : (f,g)^*\pi_I^*(Z) \to X && \in \mathcal{E}(J) \\ &\chi : G^*(\alpha) \to (f,g)^*(\mathsf{gen}^*_{I,Z,Y}(\beta) \Rightarrow \gamma) && \in \mathcal{P}((f,g)^*\pi_I^*(Z)). \end{aligned}$$

Now maps with the type of F_2 correspond to maps with the type of g, so let us continue to specify the bijection in the situation where F_2 and g are paired under this correspondence. With the silent use of some coherence isomorphisms from the reindexing and the fact that the exponentials in the fibres of r are stable under reindexing along maps of $\mathcal{C}$ we can write this last triple as

$$\begin{aligned} g &: J \to [Z,Y]_I && \in \mathcal{C} \\ G &: f^*(Z) \to X && \in \mathcal{E}(J) \\ \psi &: G^*(\alpha) \to F_2^*(f^*(\beta)) \Rightarrow f^*(\gamma) && \in \mathcal{P}(f^*(Z)). \end{aligned}$$

Now F_1 and G actually have the same type, so we will finish off specifying the bijection in the situation where F_1 and G are actually equal. We need a bijection between maps

$$\phi : F_1^*(\alpha) \times F_2(f^*(\beta)) \to f^*(\gamma) \qquad \in \mathcal{P}(f^*(Z))$$

and maps

$$\psi : F_1^*(\alpha) \to F_2^*(f^*(\beta)) \Rightarrow f^*(\gamma) \qquad \in \mathcal{P}(f^*(Z))$$

But these classes of maps are in bijection because the fibres of r are cartesian closed.

It remains to check that this bijection is natural in J and (X, α). Suppose we are given the map $K \to J$ and a map $(W, \delta) \to (X, \alpha)$ over it, i.e. the data

$$\begin{aligned} h &: K \to J && \in \mathcal{C} \\ H &: h^*(X) \to W && \in \mathcal{E}(K) \\ \omega &: H^*(\delta) \to h^*(\alpha) && \in \mathcal{P}(h^*(X)). \end{aligned}$$

Then the action of this data sends F to

$$(H \oplus_K 1_{(fh)^*(Y)}) \circ h^*(F) : (fh)^*(Z) \to W \oplus_K (fh)^*(Y) \qquad \in \mathcal{E}(K)$$

and sends F_1 and F_2 to

$$\begin{aligned} H \circ h^*(F_1) &: (fh)^*(Z) \to W && \in \mathcal{E}(K) \\ h^*(F_2) &: (fh)^*(Z) \to (fh)^*(Y) && \in \mathcal{E}(K) \end{aligned}$$

which do indeed correspond because with biproducts, crucially, by Proposition 3.2.1, the sum of the arrows H and $1_{(fh)^*(Y)}$ is equal to their product. The maps g and G are sent to

$$\begin{aligned} g \circ h &: K \to [Z,Y]_I && \in \mathcal{C} \\ H \circ h^*(G) &: (fh, gh)^* \pi_I^*(Z) \to X && \in \mathcal{E}(K) \end{aligned}$$

which do indeed correspond to F_2 and F_1 respectively. The action of the data on ϕ is to send it to the composite of

$$
\begin{aligned}
&(h^*(F_1))^*(\omega) \times_{(fh)^*(Z)} 1 : \\
&\qquad (H \circ h^*(F_1))^*(\delta) \times_{(fh)^*(Z)} (h^*(F_2))^*((fh)^*(\beta)) \to \\
&\qquad\qquad (h^*(F_1))^*(h^*(\alpha)) \times_{(fh)^*(Z)} (h^*(F_2))^*((fh)^*(\beta))
\end{aligned}
$$

with

$$
\begin{aligned}
&(h^*(F_1))^*(h^*(\alpha)) \times_{(fh)^*(Z)} (h^*(F_2))^*((fh)^*(\beta)) \\
&\qquad \cong h^*\big(F_1^*(\alpha) \times_{f^*(Z)} F_2^*(f^*(\beta))\big) \xrightarrow{h^*(\phi)} (fh)^*(\gamma).
\end{aligned}
$$

The action of the data on ψ is to send it to the composite of

$$(h^*(F_1))^*(\omega) : (H \circ h^*(F_1))^*(\delta) \to (h^*(F_1))^*(h^*(\alpha))$$

with

$$
\begin{aligned}
&(h^*(F_1))^*(h^*(\alpha)) \cong \\
&\qquad h^*(F_1^*(\alpha)) \xrightarrow{h^*(\psi)} h^*(F_2^*(f^*(\beta)) \Rightarrow f^*(\gamma)) \\
&\qquad\qquad \cong (h^*F_2)^*((fh)^*(\beta)) \Rightarrow (fh)^*(\gamma)
\end{aligned}
$$

which does indeed correspond to the action of the data on ϕ, by the preservation of exponentials under reindexing along h and the naturality of the bijection on homsets induced by exponentials. Finally, it is clear that the action of the data on χ preserves the correspondence of χ with ψ from the naturality of the local-smallness bijection. □

4.5.4 Translation from (B) into (C)

Since the base category $\mathbb{C}$ of (B) is mapped directly to the base category $\mathcal{C}$ of (C) under the translation, we automatically get that $\mathcal{C}$ is cartesian closed.

Proposition 4.5.8. *The first Kleisli fibration $q^M : \mathbb{E}_{M_{\mathbb{E}}} \to \mathbb{C}$ is locally small.*

Proof. Given $I \in \mathbb{C}$, $X, Y \in \mathbb{E}$, recall from 4.2.5 that the fibres of q^M are just the Kleisli categories of the fibres of q. Hence, we define the Kleisli function comprehension $[X, Y]_{M_{\mathbb{E}}, I}$ in terms of the function comprehension for q, namely as the object

$$[X, M_{\mathbb{E}}Y]_I$$

with generic arrow $\mathsf{gen}_{M_{\mathbb{E}},I,X,Y} : \pi_I^*(X) \twoheadrightarrow \pi_I^*(Y)$ being given by

$$\pi_I * (X) \xrightarrow{\mathsf{gen}_{I,X,M_{\mathbb{E}}Y}} \pi_I^*(M_{\mathbb{E}}Y) \cong M_{\mathbb{E}}(\pi_I^*(Y)).$$

It is straightforward to check from the description of the fibres of and reindexing in q^M given in Proposition 4.2.5 that this does satisfying Definition 4.5.1. □

Let us show that the second Kleisli fibration has fibred finite products. This requires the assumption that $M_{\mathbb{Q}}$ preserves fibred finite products in $\mathbb{Q}$. This is a natural one to make since, in the set-theoretic case, if I and X are sets and $\alpha, \beta \subseteq I \times X$ then

$$\begin{aligned}(\alpha \wedge \beta)^\bullet &= \{(i,\xi) \in I \times X^\bullet \mid \forall x \in \xi.\, \alpha(i,x) \wedge \beta(i,x)\} \\ &= \{(i,\xi) \in I \times X^\bullet \mid \forall x \in \xi.\, \alpha(i,x)\} \cap \{(i,\xi) \in I \times X^\bullet \mid \forall x \in \xi.\, \beta(i,x)\} \\ &= \alpha^\bullet \cap \beta^\bullet.\end{aligned}$$

Proposition 4.5.9. *In setting* (B)*, the fibration* $r^M : \mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{E}_{M_{\mathbb{E}}}$ *has fibred finite products.*

Proof. By 4.2.6 each fibre of r^M has finite products given by taking finite products in the corresponding fibre of $\mathbb{Q}$. It remains to check that reindexing preserves these finite products, but this follows from the description of the reindexing in 4.2.6 and the assumption that $M_{\mathbb{Q}}$ preserves fibred finite products. □

It is worth considering the fibred finite products in the composite $q^M \circ r^M : \mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{C}$, which we can calculate using the foregoing results. Given $I \in \mathbb{C}$, the terminal object in the fibre over I is given by $(0_I, *)$, where 0_I is the initial object in $\mathbb{E}(I)$ and $*$ is the unique object in $\mathbb{Q}(0_I) \cong \mathbb{1}$ Given $I \in \mathbb{C}$, $X, Y \in \mathbb{E}(I)$, $\alpha \in \mathbb{Q}(X)$ and $\beta \in \mathbb{Q}(Y)$, the fibred product $(X, \alpha) \times_I (Y, \beta)$ is $(X +_I Y, \mathsf{ext}_{I,X,Y}(\alpha, \beta))$. Note that $\mathsf{ext}_{I,X,Y}(\alpha, \beta)$ is the product of $\mathsf{ext}_{I,X,Y}(\alpha, \top_Y)$ with $\mathsf{ext}_{I,X,Y}(\top_X, \beta)$.

Proposition 4.5.10. *The second Kleisli fibration* $r^M : \mathbb{Q}_{M_{\mathbb{Q}}} \to \mathbb{E}_{M_{\mathbb{E}}}$ *has cartesian closed fibres, and the cartesian closed structure is stable under reindexing along* q^M*-cartesian arrows.*

Proof. Since $r : \mathbb{Q} \to \mathbb{E}$ is a cartesian closed fibration, by 4.2.6 each fibre of r^M is cartesian closed. Moreover, the description of reindexing in 4.2.6 shows that the exponentials are stable under reindexing along pure morphisms in $\mathbb{E}_{M_{\mathbb{E}}}$, so in particular they are stable under reindexing along q^M-cartesian arrows. □

To complete the check that the translation from (B) to (C) validates the assumptions of (C), it remains to see that the first Kleisli fibration admits fibred finite biproducts.

Proposition 4.5.11. *The first Kleisli fibration $q^M : \mathbb{E}_{M_\mathbb{E}} \to \mathbb{C}$ admits fibred finite biproducts.*

Proof. As we saw in 4.3.4, q^M admits fibred finite coproducts. By 4.2.5 the fibres of q^M are just Kleisli categories of the fibres of q, hence we can use 3.3.3 to deduce from the assumption that $M_\mathbb{E}$ satisfies (M-0) and (M-+-×) that each fibre of q^M has finite biproducts. The fact that they are indexing stable is obvious since all ingredients in the construction are preserved by reindexing, in particular the natural map $\tau_{X,Y} : M_\mathbb{E}(X +_I Y) \to M_\mathbb{E}X \times_I M_\mathbb{E}Y$ is stable under reindexing, being defined out of the reindexing-stable fibred coproduct and terminal object structure. □

This completes the translation from settings (B) into setting (C).

4.5.5 Translation from (A) into (B)

Since $\mathbb{T}$ plays the role of the base $\mathbb{C}$ under the translation, we have that $\mathbb{C}$ is cartesian closed.

Proposition 4.5.12. *The simple slice fibration $P_\mathbb{T} : \mathbb{T}_{(-)} \to \mathbb{T}$ is locally small.*

Proof. Given $I, X, Y \in \mathbb{T}$, let the function comprehension be given by

$$[(I,X),(I,Y)]_I = X \Rightarrow Y$$

and let the generic arrow

$$\mathsf{gen}_{I,(I,X),(I,Y)} : (I \times (X \Rightarrow Y), X) \to (I \times (X \Rightarrow Y), Y)$$

be represented by

$$I \times (X \Rightarrow Y) \times X \to (X \Rightarrow Y) \times X \xrightarrow{\mathsf{ev}_{X,Y}} Y.$$

It is straightforward to check that this satisfies Definition 4.5.1. □

Proposition 4.5.13. *The fibration $P_\mathbb{T} : \mathbb{T}_{(-)} \to \mathbb{T}$ has fibred finite products.*

Proof. Suppose we have an object $I \in \mathbb{T}$. It is easy to check that the terminal object in the fibre over I is $(I, \top)$. Given $X, Y \in \mathbb{T}$, it is straightforward to check that their product as objects of $\mathbb{T}_I$ is $X \times Y$. □

Proposition 4.5.14. *The fibration $p_{(-)} : \mathbb{P}_{(-)} \to \mathbb{T}_{(-)}$ is cartesian closed.*

Proof. This is trivial, since $p : \mathbb{P} \to \mathbb{T}$ is cartesian closed and $p_{(-)}$ is given by change of base. □

Recall from §4.2.3 that we construct $M_{\mathbb{Q}}$ and $M_{\mathbb{E}}$ from a strong monad $(-)^\bullet$ which admits a fibred extension to $\mathbb{P}$. Recall that we have added the assumption for this section that $(-)^\bullet$ preserves fibred finite products in the fibres of $\mathbb{P}$.

Proposition 4.5.15. *The monad $M_{\mathbb{Q}}$ on $\mathbb{P}_{(-)}$ preserves fibred finite products.*

Proof. Recall that the monad $M_{\mathbb{Q}}$ on $\mathbb{P}_{(-)}$ is given on (I, X, α) by applying $(-)^\bullet$ to $\alpha \in \mathbb{P}(I \times X)$ to obtain $\alpha^\bullet \in \mathbb{P}((I \times X)^\bullet)$ and then reindexing along the strength $\sigma_{I,X} : I \times X^\bullet \to (I \times X)^\bullet$. By assumption, $(-)^\bullet : \mathbb{P}(I \times X^\bullet) \to \mathbb{P}((I \times X)^\bullet)$ preserves finite products, and reindexing along any map preserves the finite products in each fibre of p. Hence $M_{\mathbb{Q}}$ preserves fibred finite products in $\mathbb{P}_{(-)}$. □

Finally, we must check that the global conditions of (M-0) and (M-+-×) on $(-)^\bullet$ on $\mathbb{T}$ translate to the fibred version for $M_{\mathbb{E}}$ on $\mathbb{T}_{(-)}$.

Proposition 4.5.16. *The monad $M_{\mathbb{E}}$ satisfies (M-0) and (M-+-×) in each fibre of $\mathbb{T}_{(-)}$.*

Proof. Suppose we have $I \in \mathbb{T}$. The initial object in $\mathbb{T}_I$ is $(I, 0)$, and this is sent by $M_{\mathbb{E}}$ to $(I, 0^\bullet) \cong (I, \top)$.

Now suppose we also have $X, Y \in \mathbb{T}$ and recall that their sum as objects of $\mathbb{T}_I$ is $X + Y$. Then we are required to show the composite

$$\begin{aligned} I \times (X+Y)^\bullet &\xrightarrow{\sigma_{I,X+Y}} (I \times (X+Y))^\bullet \cong (I \times X + I \times Y)^\bullet \\ &\xrightarrow{(\pi_X + \pi_Y)^\bullet} (X+Y)^\bullet \xrightarrow{(\eta_X + !_Y)^\bullet} (X^\bullet + 0^\bullet)^\bullet \\ &\xrightarrow{\mathsf{can}} (X+0)^{\bullet\bullet} \cong X^{\bullet\bullet} \xrightarrow{\mu_X} X^\bullet \end{aligned}$$

paired with its dual into $Y^\bullet$ gives a map $I \times (X+Y)^\bullet \to X^\bullet \times Y^\bullet$ which is an isomorphism in $\mathbb{T}_I$. In fact, we can use the axioms for the strength and properties of distributive categories to simplify the composite above to

$$I \times (X+Y)^\bullet \xrightarrow{\pi_{(X+Y)^\bullet}} (X+Y)^\bullet \xrightarrow{\tau_{1,X,Y}} X^\bullet$$

where $\tau_{1,X,Y}$ is the canonical map $(X+Y)^\bullet \to X^\bullet$ in terms of the original monad $(-)^\bullet$ on $\mathbb{T}$. Hence the canonical map

$$M_{\mathbb{E}}((I,X) +_I (I,Y)) \to M_{\mathbb{E}}(I,X) \times_I M_{\mathbb{E}}(I,Y)$$

is given by

$$I \times (X+Y)^\bullet \xrightarrow{\pi_{(X+Y)^\bullet}} (X+Y)^\bullet \xrightarrow{\tau_{X,Y}} X^\bullet \times Y^\bullet$$

which, by assumption, is an isomorphism as a map in $\mathbb{T}_I$. □

4.6 The Diller-Nahm category

Let us collect together the main results of this chapter.

Theorem 4.6.1. *Let $\mathcal{C}$ be a cartesian closed category, let $q : \mathcal{E} \to \mathcal{C}$ be a locally small fibration with fibred finite biproducts, and let $r : \mathcal{P} \to \mathcal{E}$ have fibred finite products. Moreover, suppose that the fibre categories of r are cartesian closed, with exponentials preserved by reindexing along q-cartesian arrows, and that, for all $I \in \mathcal{C}$, r satisfies $\mathcal{P}(0_I) \cong \mathbb{1}$, as well as, for all $I \in \mathcal{C}$, $X, Y \in \mathcal{E}(I)$, the functor $\mathcal{P}(X \oplus_I Y) \to \mathcal{P}(X) \times \mathcal{P}(Y)$, given by reindexing along the coproduct inclusions, is an isomorphism. Suppose also that q has simple sums such that for any $I, J \in \mathcal{C}$, $X \in \mathcal{E}(I \times J)$, the functor $\mathcal{P}(\Sigma_J X) \to \mathcal{P}(X)$, given by reindexing along the canonical cocartesian map $X \to \Sigma_J X$, is an isomorphism, Then the total category of $(q \circ r^{\mathrm{op}})^{\mathrm{op}})$ is cartesian closed.*

Theorem 4.6.2. *Let $\mathbb{C}$ be a cartesian closed category, let $q : \mathbb{E} \to \mathbb{C}$ be a locally small fibration with fibred finite products and fibred finite coproducts and let $q : \mathbb{Q} \to \mathbb{E}$ be a cartesian closed fibration. Suppose that for any $I \in \mathbb{C}$, q satisfies $\mathbb{Q}(0_I) \cong \mathbb{1}$ and also that for any $I \in \mathbb{C}$, $X, Y \in \mathbb{E}(I)$, the functor $\mathbb{Q}(X +_I Y) \to \mathbb{Q}(X) \times \mathbb{Q}(Y)$, given by reindexing along the coproduct inclusions, is an isomorphism. Suppose also that q has simple sums such that for any $I, J \in \mathbb{C}$, $X \in \mathbb{E}(I \times J)$, the functor $\mathbb{Q}(\Sigma_J X) \to \mathbb{Q}(X)$, given by reindexing along the canonical cocartesian map $X \to \Sigma_J X$, is an isomorphism, Moreover over suppose we have monads $M_{\mathbb{E}}$ on $\mathbb{E}$ and $M_{\mathbb{Q}}$ on $\mathbb{Q}$ whose functor parts give rise to morphisms of fibrations $(M_{\mathbb{E}}, 1_{\mathbb{C}}) : q \to q$ and $(M_{\mathbb{Q}}, M_{\mathbb{E}}) : r \to r$, for which the components of the unit and multiplication of $M_{\mathbb{E}}$ are q-vertical and the components of the unit and multiplication of $M_{\mathbb{Q}}$ are r-cartesian morphisms lying over the corresponding components of the unit and multiplication for $M_{\mathbb{E}}$. Suppose also that $M_{\mathbb{Q}}$ preserves fibred finite products, and that $M_{\mathbb{E}}$ is fibrewise additive in the sense that it satisfies (M-0) and (M-+-$\times$) in each fibre of q. Then the Diller-Nahm category* Dill *as described in §4.2 is a cartesian closed category.*

Theorem 4.6.3. *Let $\mathbb{T}$ be a cartesian closed category with finite coproducts and $p : \mathbb{P} \to \mathbb{T}$ a cartesian closed fibration. Suppose that $\mathbb{P}(0) \cong \mathbb{1}$ and that for all $X, Y \in \mathbb{T}$, the functor $\mathbb{P}(X + Y) \to \mathbb{P}(X) \times \mathbb{P}(Y)$, given by reindexing along the coproduct inclusions, is an isomorphism. Moreover, suppose we have a strong monad $(-)^\bullet$ on $\mathbb{T}$ and a monad on $\mathbb{P}$ also denoted $(-)^\bullet$, such that the functor parts satisfy $p \circ (-)^\bullet = (-)^\bullet \circ p$, $(-)^\bullet$ on $\mathbb{P}$ preserves p-cartesian arrows, and the unit and multiplication in $\mathbb{P}$ are p-cartesian arrows lying over the unit and multiplication in $\mathbb{T}$. Finally, suppose that $(-)^\bullet$ preserves fibred finite products in $\mathbb{P}$ and that, in $\mathbb{T}$, the monad $(-)^\bullet$ is additive in that it satisfies (M-0) and*

(M-+-×). Then the Diller-Nahm category Dill *as described in §4.2 is a cartesian closed category.*

Chapter 5

Fibred models of type theory

In this chapter we shall set up the context for the main construction used in this thesis: the gluing construction. Descriptions of versions of this construction for dependent type theory can be found in [38], [43] and [46]. The general gluing construction can be seen as a sort of 'comprehension completion'. We take a 'partial model' of type theory, for which there are some (new) dependent types which do not yet have comprehensions (meaning the context cannot be extended to include variables of those types) then we freely add in such comprehensions, expanding the collection of contexts. We need to know in advance something about what the types over these new contexts should be. This corresponds to knowing about a 'fibred model of type theory': we have some base contexts, and over each of these there is a type theory of new types and types dependent on those new types. We will give the notion of fibred model of type theory, and justify the viewpoint that a fibred model of type theory is a model of type theory which just lacks a comprehension. In the next chapter we will describe the gluing construction for turning a fibred model of type theory into an ordinary model.

5.1 The Fibred fundamental fibration

Definition 5.1.1. The *fibred arrow category* $(p)^{\rightarrow}$ of a fibration $p : \mathbb{E} \rightarrow \mathbb{B}$ is the full subcategory of the arrow category $\mathbb{E}^{\rightarrow}$ of $\mathbb{E}$ whose objects are p-vertical as arrows in $\mathbb{E}$. The *fibred codomain functor* $\mathsf{cod}_p : (p)^{\rightarrow} \rightarrow \mathbb{E}$ is the restriction of the codomain functor $\mathrm{cod} : \mathbb{E}^{\rightarrow} \rightarrow \mathbb{E}$.

Note that for any object $I \in \mathbb{B}$, the restricted functor

$$(\mathsf{cod}_p)_I : (p)^{\to}(I) \to \mathbb{E}(I)$$

is just the usual codomain functor cod from the arrow category $\mathbb{E}(I)^{\to}$ of the fibre category $\mathbb{E}(I)$ to the category $\mathbb{E}(I)$ itself.

Proposition 5.1.2. *The functor* $\mathsf{cod}_p : (p)^{\to} \to \mathbb{E}$ *is a fibration if and only if* p *has fibred pullbacks.*

Proof. It is straightforward to check that the composite

$$(p)^{\to} \xrightarrow{\mathsf{cod}_p} \mathbb{E} \xrightarrow{p} \mathbb{B}$$

is a fibration, where given $f : J \to I$ in $\mathbb{B}$ and an I-vertical arrow $\phi : B \to A$, there is a cartesian lift of f with codomain ϕ given by the square

$$\begin{array}{ccc} f^*(B) & \longrightarrow & B \\ {\scriptstyle f^*(\phi)} \downarrow & & \downarrow {\scriptstyle \phi} \\ f^*(A) & \longrightarrow & A \end{array}$$

where the top and bottom arrows are p-cartesian over f. Moreover, we can see from this that cod_p sends $p \circ \mathsf{cod}_p$-cartesian arrows to p-cartesian arrows. Hence it suffices to show that the remaining two conditions of Proposition 1.1.12 are equivalent to p having fibred pullbacks.

The restricted functors $(\mathsf{cod}_p)_I : (p)^{\to}(I) \to \mathbb{E}(I)$ are by definition just the codomain functors cod : $\mathbb{E}(I)^{\to} \to \mathbb{E}(I)$. These are fibrations if and only each fibre category $\mathbb{E}(I)$ has pullbacks. In fact, cartesian arrows for these functors are precisely pullback squares. Hence, to say that $(\mathsf{cod}_p)_I$-cartesian arrows are preserved by $p \circ (\mathsf{cod}_p)$-reindexing is just to say that pullbacks in $\mathbb{E}(I)$ are preserved by p-reindexing. □

Remark 5.1.3. When a category $\mathbb{C}$ has pullbacks, the functor cod : $\mathbb{C}^{\to} \to \mathbb{C}$ is often referred to as the *fundamental fibration* or *self-indexing* of $\mathbb{C}$. Hence Definition 5.1.1 may be seen as giving a "fibred fundamental fibration" in the situation where p has fibred pullbacks.

5.2 Fibred comprehension categories

Given a fibration $p_0 : \mathbb{E}_1 \to \mathbb{E}_0$, we can use the fibred codomain functor $\mathsf{cod}_{p_0} : (p_0)^{\to} \to \mathbb{E}_1$ to define the notion of fibred comprehension category.

Definition 5.2.1. A *fibred comprehension category* consists of

- a base fibration $p_0 : \mathbb{E}_1 \to \mathbb{E}_0$,
- a fibration of types $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$, and
- a functor $\chi_1 : \mathbb{E}_2 \to (p_0)^{\to}$

such that

- $\mathsf{cod}_{p_0} \circ \chi_1 = p_1$, and
- χ_1 sends p_1-cartesian arrows to cod_{p_0}-cartesian arrows.

These are conditions on χ_1 with respect to the total category. We can replace them with 'fibrewise' conditions, which we give assuming that p_0 and p_1 are cloven fibrations — it is easy to reformulate them in elementary terms.

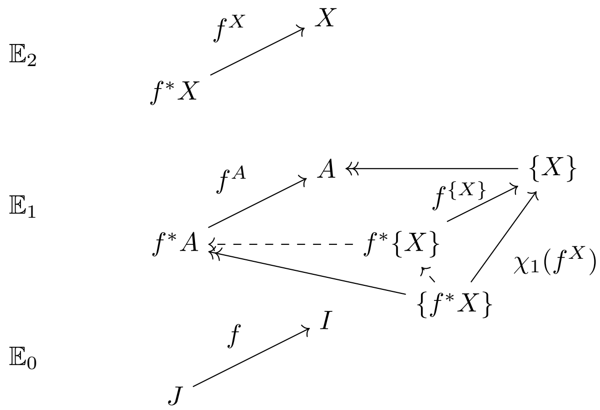
Proposition 5.2.2. *Given a cloven fibration $p_0 : \mathbb{E}_1 \to \mathbb{E}_0$, a cloven fibration $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$ and a functor $\chi_1 : \mathbb{E}_2 \to (p_0)^{\to}$ satisfying $\mathsf{cod}_{p_0} \circ \chi_1 = p_1$, the triple (p_0, p_1, χ_1) is a fibred comprehension category if and only if*

- *for each $I \in \mathbb{E}_0$ the restricted functors $(\chi_1)_I : \mathbb{E}_2(I) \to (p_0)^{\to}(I) = \mathbb{E}_1(I)^{\to}$ make $(p_1)_I : \mathbb{E}_2(I) \to \mathbb{E}_1(I)$ into a comprehension category; and*
- *for each $f : J \to I$ in $\mathbb{E}_0$, the canonical colax morphism of comprehension categories*

  $$((p_1)_I : \mathbb{E}_2(I) \to \mathbb{E}_1(I), (\chi_1)_I) \to ((p_1)_J : \mathbb{E}_2(J) \to \mathbb{E}_1(J), (\chi_1)_J)$$

  given by the action of f by reindexing is moreover a strong morphism.

Let us describe what this canonical colax morphism of comprehension categories is. On base categories we simply have the reindexing functor $f^* : \mathbb{E}_1(I) \to \mathbb{E}_1(J)$, and similarly on total categories we take $f^* : \mathbb{E}_2(I) \to \mathbb{E}_2(I)$. For the 2-cell $(\chi_1)_J \circ f^* \Rightarrow f^* \circ (\chi_1)_I$, we give its component on some $X \in \mathbb{E}_2(A)$ where $A \in \mathbb{E}_1(I)$: it is the canonical vertical map by which $\chi_1(f^X)$ factorizes through the p_0-cartesian arrow $f^{\operatorname{cod} \chi_1(X)}$ (lying over f with codomain $\operatorname{cod} \chi_1(X)$), where f^X is the $p_0 \circ p_1$-cartesian arrow lying over f with codomain X, as in the diagram below.



Proof. We divide the condition that χ_1 preserves p_1-cartesian arrows into two conditions: preserving p_1-cartesian arrows that lie over p_0-vertical arrows, and preserving p_1-cartesian arrows that lie over p_0-cartesian arrows.

Using the description of cartesian-over-vertical arrows in $(p_0)^{\to}$ given in 5.1.2 we see that for χ_1 to preserve cartesian-over-vertical arrows is exactly the condition that each $(\chi_1)_I$ makes $(p_1)_I : \mathbb{E}_2(I) \to \mathbb{E}_1(I)$ into a comprehension category. The second condition in the proposition says that the canonical comparison arrows $\{f^*X\} \to f^*\{X\}$ in the diagram above are isomorphisms, but this is clearly equivalent to $\chi_1(f^X)$ being p_0-cartesian, i.e. the condition that χ_1 preserves cartesian-over-cartesian arrows. □

Example 5.2.3. The notion of fibred comprehension category reduces to the usual notion of comprehension category if we take p_0 to be a functor of the form $\mathbb{B} \to \mathbb{1}$, where $\mathbb{1}$ is the terminal category.

Our goal for the rest of this chapter is to construct our main example of a fibred comprehension category. This is the one that comes from an ordinary comprehension category: the indexing base is just the category of contexts, and for each context we consider the comprehension category of extensions of that context and types over each extension. Dependent sums are crucial to the implementation of a comprehension functor in this construction. However, plain comprehension categories with no extra structure or properties beyond dependent sums do not seem to be enough. Hence, we will eventually assume that all comprehension categories involved are *full*.

5.3 Full fibred comprehension categories

Definition 5.3.1. A fibred comprehension category (p_0, p_1, χ_1) is *full* if the functor $\chi_1 : \mathbb{E}_2 \to \mathsf{cod}_{p_0}$ is full and faithful.

As with Definition 5.2.1, this is a 'total' definition which should have a 'fibrewise' counterpart to which it is equivalent.

Proposition 5.3.2. *A fibred comprehension category* (p_0, p_1, χ_1) *is full if and only if for every* $I \in \mathbb{E}_0$ *the comprehension category* $(p_0(I), p_1(I), (\chi_1)_I)$ *is full.*

Proof. The "only if" direction is obvious. For the converse, suppose that each fibre comprehension category is full. This means that χ_1 is full and faithful with respect to $p_0 \circ p_1$-vertical maps going to $p_0 \circ \mathsf{cod}_{p_0}$-vertical maps. It remains to show that χ_1 is full and faithful with respect to $p_0 \circ p_1$-cartesian arrows. However, between any two objects of $\mathbb{E}_2$ there is at most one $p_1 \circ p_0$-cartesian arrow *up to unique* $p_1 \circ p_0$*-vertical isomorphism.* As we know that χ_1 is full and faithful with respect to this latter sort of morphism, the result follows. □

At this point we observe that we can define 'fibred model of type theory' in terms of display map categories, as follows.

Definition 5.3.3. A *fibred display map category* consists of a fibration $p : \mathbb{E} \to \mathbb{B}$ together with for each $I \in \mathbb{B}$ a class of morphisms $\mathcal{E}_I \subseteq \operatorname{Mor} \mathbb{E}(I)$ making $(\mathbb{E}(I), \mathcal{E}_I)$ a display map category such that reindexing along maps in $\mathbb{B}$ preserves the classes of display maps and preserves pullbacks of display maps.

5.4 Fibred unit types

We will generally say that a fibred comprehension category has a certain type constructor if each of its fibre comprehension categories has that type constructor and it is preserved by reindexing.

Definition 5.4.1. A fibred comprehension category (p_0, p_1, χ_1) *has unit types* if $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$ has fibred terminal objects and χ_1 sends them to isomorphisms.

Proposition 5.4.2. *A fibred comprehension category* (p_0, p_1, χ_1) *has unit types if and only if each fibre comprehension category has unit types and these are preserved by the reindexing action of* $\mathbb{E}_0$*.*

Proof. Both hypotheses assert in particular that every fibre of $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$ has a terminal object sent by comprehension to an isomorphism. The former asserts in addition that these are stable under reindexing along arbitrary arrows of $\mathbb{E}_1$. The latter asserts in addition that these are stable under reindexing along p_0-vertical arrows of $\mathbb{E}_1$ and along arbitrary arrows of $\mathbb{E}_0$ (i.e. along p_0-cartesian arrows). □

5.5 Fibred Ehrhard comprehension

The following generalizes Definition 1.4.1.

Definition 5.5.1. Recall that a fibred adjunction, over $\mathbb{C}$ say, is an adjunction between the total categories of two fibrations with base $\mathbb{C}$ consisting of two functors which are morphisms of fibrations such that the unit and counit have vertical components. A fibred fibration

$$\mathbb{E}_2 \xrightarrow{p_1} \mathbb{E}_1 \xrightarrow{p_0} \mathbb{E}_0$$

has *fibred Ehrhard comprehension* if p_1 has a fibred right adjoint section $\top : p_0 \to p_0 \circ p_1$ which itself has a fibred right adjoint $\mathcal{Q}_0 : p_0 \circ p_1 \to p_0$.

Remark 5.5.2. Although we have used the language of fibred adjunctions, this definition is equivalent to one where we assume that each of the fibre fibrations $(p_1)_I : (\mathbb{E}_2)_I \to (\mathbb{E}_1)_I$ has Ehrhard comprehension and moreover the Ehrhard comprehensions are suitably preserved by the reindexing functors. The equivalence follows from standard facts about fibred adjunctions.

A fibred fibration with fibred Ehrhard comprehension gives rise to a fibred comprehension category as follows. Since the counit $\epsilon_X : \top_{\mathcal{Q}_0(X)} \to X$ of the adjunction $\top \dashv \mathcal{Q}_0$ is $p_0 \circ p_1$-vertical, $X \mapsto p_1(\epsilon_X)$ is the object part of a functor $\chi_1 : \mathbb{E}_2 \to (p_0)^{\to}$, easily seen to satisfy $\mathsf{cod}_{p_0} \circ \chi_1 = p_1$. The proof that this χ_1 preserves cartesian arrows is similar to the usual situation for fibrations with Ehrhard comprehension.

Observe that just as in the usual non-fibred case, having Ehrhard comprehension is a mere property of a fibred fibration, and being induced by an Ehrhard comprehension is a mere property of a fibred comprehension category. Moreover, the results of §1.4 generalize easily to the fibred case, and we see that for full fibred comprehension categories, being induced by a fibred Ehrhard comprehension is equivalent to having unit types.

5.6 Dependent sum as comprehension

Definition 5.6.1. The *fibred fibration of second-level types* of a comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ is the fibred fibration

$$\mathbb{E}_2 \xrightarrow{p_1} \mathbb{E}_1 \xrightarrow{p_0} \mathbb{E}_0$$

given by $\mathbb{E}_0 = \mathbb{B}$, $\mathbb{E}_1 = \mathbb{E}$, $p_0 = p$ and the fibration $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$ is given by the pullback of p along $\mathrm{dom} \circ \chi : \mathbb{E} \to \mathbb{B}$.

$$\begin{array}{ccc} \mathbb{E}_2 & \xrightarrow{d_1} & \mathbb{E}_1 = \mathbb{E} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{E}_1 = \mathbb{E} & \xrightarrow[\mathrm{dom} \circ \chi]{} & \mathbb{B} \end{array}$$

Remark 5.6.2. For any $\Gamma \in \mathbb{B}$, the restricted fibrations $(p_1)_\Gamma : \mathbb{E}_2(\Gamma) \to \mathbb{E}_1(\Gamma)$ can be described as follows. The base category is $\mathbb{E}(\Gamma)$ and the fibre over $A \in \mathbb{E}(\Gamma)$ is the category $\mathbb{E}(\Gamma.A)$. Reindexing along $f : A \to B$ in $\mathbb{E}(\Gamma)$ is given by p-reindexing along the comprehension $\Gamma.f : \Gamma.A \to \Gamma.B$.

Given $f : \Delta \to \Gamma$, the action of f by reindexing on $\mathbb{E}_2(\Gamma)$ is to send a type $X \in \mathbb{E}(\Gamma.A)$ to its reindexing along $\Delta.f^*A \to \Gamma.A$.

Proposition 5.6.3. *When the comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ has unit types, the fibred fibration of second-level types has fibred terminals.*

Proof. From the description in Remark 5.6.2, it is clear that all of the restricted fibrations have fibred terminals, and also that these fibred terminals are preserved by reindexing along maps in $\mathbb{E}_0$. □

Now we will justify the claim that dependent sums in type theory are a form of second-level comprehension.

Theorem 5.6.4. *Let $(p : \mathbb{E} \to \mathbb{B}, \chi)$ be a full comprehension category with Ehrhard comprehension. Then it admits strong dependent sums if and only if its fibred fibration of second-level types is a full fibred comprehension category with Ehrhard comprehension.*

Proof. Let us suppose that $(p : \mathbb{E} \to \mathbb{B}, \chi)$ admits strong dependent sums. Then we will construct a fibred Ehrhard comprehension for the fibred fibration of second-level types by showing, for any $\Gamma \in \mathbb{B}$, $A \in \mathbb{E}(\Gamma)$, $X \in \mathbb{E}(\Gamma.A)$, that $\Sigma_A X$ as an object of $\mathbb{E}_1(\Gamma)$ has the universal property required of $\mathcal{Q}_0(X)$.

Let $\Delta \in \mathbb{B}$ and let $B \in \mathbb{E}(\Delta)$. Then maps

$$\begin{array}{lc} \mathbb{E}_2 & \top_B \dashrightarrow X \\ \mathbb{E}_1 & B \dashrightarrow A \\ \mathbb{E}_0 & \Delta \dashrightarrow \Gamma \end{array}$$

in $\mathbb{E}_2$ are given by families of maps

$$
\begin{array}{lcccc}
\mathbb{E} & \top_B & \dashrightarrow & X & \\
\mathbb{B} & \{B\} & \dashrightarrow & \{A\} & \\
 & \searrow & & \searrow & \\
 & & \Delta & \dashrightarrow & \Gamma
\end{array}
$$

which, since (p, χ) has Ehrhard comprehension, correspond naturally to diagrams

$$
\begin{array}{cccccc}
\{\top_B\} & \dashrightarrow & \{X\} & & & \\
\cong\searrow & & \searrow & & & \\
 & \{B\} & \dashrightarrow & \{A\} & & \\
 & \searrow & & \searrow & & \\
 & & \Delta & \dashrightarrow & \Gamma &
\end{array}
$$

in $\mathbb{B}$. But as $\{\top_B\} \cong \{B\}$, and the square

$$
\begin{array}{ccc}
\{X\} & \xrightarrow{\{\theta_{\Gamma,A,X}\}} & \{\Sigma_A X\} \\
\downarrow & & \downarrow \\
\{A\} & \longrightarrow & \Gamma
\end{array}
\tag{5.1}
$$

in $\mathbb{B}$ commutes, such diagrams correspond naturally to diagrams

$$
\begin{array}{cccc}
\{B\} & \dashrightarrow & \{\Sigma_A X\} & \\
\searrow & & \searrow & \\
 & \Delta & \dashrightarrow & \Gamma
\end{array}
$$

in $\mathbb{B}$. Since (p, χ) is full and faithful, these then correspond naturally to diagrams

$$
\begin{array}{lccc}
\mathbb{B} & B & \dashrightarrow & \Sigma_A X \\
\mathbb{E} & \Delta & \dashrightarrow & \Gamma
\end{array}
$$

i.e. to maps $(\Delta, B) \to (\Gamma, \Sigma_A X)$ in $\mathbb{E}_1$, as required. It is clear from the description of the bijection that the display map $\Sigma_A X \to A$ is just the expected 'first projection map' corresponding to $\{\Sigma_A X\} \cong \{X\} \to \{A\}$. Hence it is straightforward to check that the comprehension is full.

Conversely, suppose the fibred fibration of second-level types is a full comprehension category with unit types. We will show that for $\Gamma \in \mathbb{B}$, $A \in \mathbb{E}(\Gamma)$ and $X \in \mathbb{E}(\{A\})$, its comprehension $\mathcal{Q}_0(X)$ (in $\mathbb{E}_1(\Gamma)$) has comprehension $\{\mathcal{Q}_0(X)\}$ isomorphic to $\{X\}$ over Γ in $\mathbb{B}$. Hence from fullness we get a map $X \to \mathcal{Q}_0(X)$ in $\mathbb{E}$ lying over $\{A\} \to \Gamma$ whose comprehension is an isomorphism, so that we may apply 1.6.4 to conclude the proof.

Consider diagrams of the form

$$\begin{array}{ccc} & & \{X\} \\ & \nearrow & \downarrow \\ & & \{A\} \\ & & \downarrow \\ C & \longrightarrow & \Gamma \end{array}$$

in $\mathbb{B}$. Trivially, these correspond (naturally in C) to pairs of diagrams

$$\begin{array}{ccc} & & \{A\} \\ & \nearrow & \downarrow \\ C & \longrightarrow & \Gamma \end{array} \qquad \begin{array}{ccc} & & \{X\} \\ & \nearrow & \downarrow \\ C & \longrightarrow & \{A\} \end{array}$$

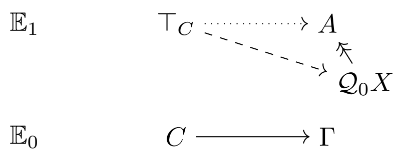
in $\mathbb{B}$, where the map $C \to \{A\}$ on the right is equal to the dashed map $C \to \{A\}$ on the left. Since (p, χ) has Ehrhard comprehension, these correspond to pairs of diagrams

$$\begin{array}{lcc} \mathbb{E} & \top_C \dashrightarrow \{A\} & \top_C \dashrightarrow X \\ & & \\ \mathbb{B} & C \longrightarrow \Gamma & C \cong \{\top_C\} \longrightarrow \{A\} \end{array}$$

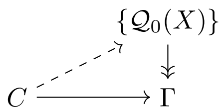
where the map $\{\top_C\} \to \{A\}$ on the right is the comprehension of the map $\top_C \to A$ on the left and the isomorphism $C \cong \{\top_C\}$ is the unit of the adjunction between fibred terminals and Ehrhard comprehension. Since there is a natural isomorphism of fibred terminals $\top_C \cong \top_{\{\top_C\}}$ lying over this latter isomorphism, we can naturally recast these diagrams as a single arrow in $\mathbb{E}_2$.

$$\begin{array}{lc} \mathbb{E}_2 & \top_{\{\top_C\}} \dashrightarrow X \\ & \\ \mathbb{E}_1 & \top_C \dashrightarrow A \\ & \\ \mathbb{E}_0 & C \longrightarrow \Gamma \end{array}$$

Now we can use the Ehrhard comprehension in the fibred fibration of second-level types to write this as an arrow in $\mathbb{E}_1 = \mathbb{E}$:



where the composite $\top_C \to \mathcal{Q}_0X \to A$ is the map $\top_C \to A$ from the previous diagram. Now we use the Ehrhard comprehension in (p, χ) to see that this corresponds naturally to a diagram



in $\mathbb{B}$. Hence $\{X\}$ and $\{\mathcal{Q}_0(X)\}$ are isomorphic as objects in the slice $\mathbb{B}/\Gamma$, as required. $\square$

Remark 5.6.5. Theorem 5.6.4 is phrased for full comprehension categories with Ehrhard comprehension and we leave it open as to whether a similar result is true for more general notions of comprehension. Note that the formulation of such a result for full comprehension categories, say, is not immediate since in general comprehension category structures (not necessarily with Ehrhard comprehension) on a given fibration need not be unique.

Chapter 6

Gluing models of type theory

6.1 Comprehension categories from fibred comprehension categories

A fibred comprehension category may be thought of as a model for a certain kind of 'multi-layered type theory'. We have 'context-types' (objects of the base category $\mathbb{E}_0$), and for each context-type I a set of 'dependent types' over I (objects of $\mathbb{E}_1(I)$), but we do not have a way to extend the context-type I by objects A of $\mathbb{E}_1(I)$. We also have 'second-level dependent types' over each dependent type A (objects of $\mathbb{E}_2(A)$), and this time we do have a way to 'extend' A by objects X of $\mathbb{E}_2(A)$. We saw at the end of the last chapter that, in the fibred fibration of second-level types, this ability to extend first-level types by second-level ones corresponds to dependent sums. We will show that there is a 'comprehension completion' construction on fibred comprehension categories, which freely adds a comprehension to the fibration $p_0 : \mathbb{E}_1 \to \mathbb{E}_0$ in such a way that for any $I \in \mathbb{E}_0$ and $A \in \mathbb{E}_1(A)$, $\widetilde{\mathbb{E}_1}(I.A) \cong \mathbb{E}_2(A)$.

Proposition 6.1.1. *Let (p_0, p_1, χ_1) be a fibred comprehension category. Then there is a comprehension category with underlying fibration $p_1 : \mathbb{E}_2 \to \mathbb{E}_1$ and comprehension given by the composite of χ_1 with the inclusion $(p_0)^{\to} \hookrightarrow \mathbb{E}_1^{\to}$. Moreover this comprehension category is full or has Ehrhard comprehension whenever the original fibred comprehension category is full or has Ehrhard comprehension respectively.*

We actually need to generalize this construction slightly. Rather than treating the base as a mere category, we assume it also has a notion of dependent type (i.e. is the base of a comprehension category) which also give rise to types in the resulting model. In order to give type constructors we will eventually have to assume certain compatibility conditions between the base model of type theory and the fibred one.

The basic situation is as follows: we assume we are given a comprehension category $(p : \mathbb{F} \to \mathbb{B}, \chi)$ and a fibred comprehension category with the same base, $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$. The goal is to produce a comprehension category whose base contains as a subcategory the original category of contexts $\mathbb{B}$, but now the fibre category of types over an old context Γ should contain both the basic dependent types $\mathbb{F}(\Gamma)$ and the new types $\mathbb{E}_1(\Gamma)$.

We will soon find it much more convenient to give the constructions in terms of fibred display map categories rather than fibred comprehension categories, but we will include some of the constructions for the fibred comprehension category case. Hence the basic situation will be one where we are given a display map category $(\mathbb{B}, \mathcal{F})$ and a fibred display map category $p : \mathbb{E} \to \mathbb{B}$ with fibrewise classes of display maps $\mathcal{E}_I \subseteq \operatorname{Mor} \mathbb{E}(I)$.

The basic construction of the gluing comprehension category actually breaks down into two steps. The first is the 'change-of-base along a fibration' construction (see 4.1.11(ii) of [22]) and the second is the 'juxtaposition' construction (see 4.1.11(iii) of [22]).

Definition 6.1.2. Given a comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ and a cloven fibration $r : \mathbb{C} \to \mathbb{B}$, the *change-of-base comprehension category* is the comprehension category with base $\mathbb{C}$ given as follows. The fibration of types is the fibration $r^*(p)$ given by the pullback of p along r.

$$\begin{array}{ccc} \mathbb{E} \bullet \mathbb{C} & \longrightarrow & \mathbb{E} \\ {\scriptstyle r^*(p)} \downarrow & & \downarrow {\scriptstyle p} \\ \mathbb{C} & \xrightarrow[r]{} & \mathbb{B} \end{array} \tag{6.1}$$

The comprehension $\mathbb{C} \bullet \mathbb{E} \to \mathbb{C}^{\to}$ may be described as the composite

$$\mathbb{E} \bullet \mathbb{C} \xrightarrow{\chi \bullet \mathrm{id}_{\mathbb{C}}} \mathbb{B}^{\to} \bullet \mathbb{C} \to \mathbb{C}^{\to}$$

where the second map sends an object A of $\mathbb{C}$ together with an arrow with codomain $r(A)$ to the chosen r-cartesian lift of that arrow with codomain A. (Consult [22] for the proof that this is a comprehension category.)

Remark 6.1.3. For display map categories, the change-of-base of a display map category $(\mathbb{B}, \mathcal{F})$ along a fibration $r : \mathbb{C} \to \mathbb{B}$ is given by the category $\mathbb{C}$

together with the class $\overline{\mathcal{F}}$ of maps which are r-cartesian over a map in $\mathcal{F}$. It is easy to check that this is indeed a display map category.

Remark 6.1.4. There is a *strict* morphism of comprehension categories

$$\mathbb{E} \bullet \mathbb{C} \to \mathbb{E}.$$

The underlying functors are the two horizontal arrows in diagram (6.1) above and it is straightforward to check that these arrows intertwine the comprehension strictly.

Proposition 6.1.5. *The change-of-base construction is functorial in both lax and colax morphisms of comprehension categories over the same base* $\mathbb{B}$*.*

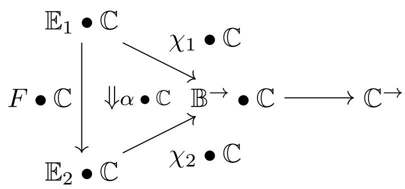
Proof. Suppose that $(F, \alpha) : \mathbb{E}_1 \to \mathbb{E}_2$ is a lax morphism of comprehension categories. Since pullbacks are functorial, there is clearly a functor

$$F \bullet \mathbb{C} : \mathbb{E}_1 \bullet \mathbb{C} \to \mathbb{E}_2 \bullet \mathbb{C}$$

over $\mathbb{C}$. Moreover, since pullback is 2-functorial, there is a 2-cell

$$\alpha \bullet \mathbb{C} : \chi_1 \bullet \mathrm{id}_{\mathbb{C}} \Rightarrow (\chi_2 \bullet \mathrm{id}_{\mathbb{C}}) \circ (F \bullet \mathbb{C}),$$

which on composition with $\mathbb{B}^{\to} \to \mathbb{C}$ gives the required comprehension comparison 2-cell.



The case of a colax morphism is similar. □

Remark 6.1.6. Suppose that $r : \mathbb{C} \to \mathbb{B}$ has fibred terminal objects, so that r has a right adjoint section $\top : \mathbb{B} \to \mathbb{C}$. Then there is a strong morphism of comprehension categories $\mathbb{E} \to \mathbb{E} \bullet \mathbb{C}$ over $\top : \mathbb{B} \to \mathbb{C}$. On types it sends (B, e) (where $B \in \mathbb{B}$ and $e \in \mathbb{E}(B)$) to $(\top_B, e)$, since $\mathbb{E} \bullet \mathbb{C}(\top_B) = \mathbb{E}(B)$. Now the comprehension of $(\top_B, e)$ is a cartesian lift of $\{e\} \to B$ with codomain $\top_B$, hence it is canonically (and naturally) isomorphic to $\top_{\{e\}} \to \top_B$, which is the image of the comprehension of (B, e) under $\top$.

Both of the following propositions are straightforward consequences of the fact that comprehension in $\mathbb{E} \bullet \mathbb{C}$ is given by taking a cartesian lifting of the comprehension in $\mathbb{E}$. They are left as exercises in [22].

Proposition 6.1.7. *The change-of-base construction preserves fullness of comprehension categories.*

Proposition 6.1.8. *The change-of-base construction applied to a comprehension category with Ehrhard comprehension has Ehrhard comprehension.*

Definition 6.1.9. Let $(p : \mathbb{D} \to \mathbb{B}, \chi_D)$ and $(q : \mathbb{E} \to \mathbb{B}, \chi_E)$ be comprehension categories over a common base $\mathbb{B}$. The *juxtaposition comprehension category* $(q \star p : \mathbb{E} \star \mathbb{D} \to \mathbb{B}, \chi_{D\star E})$ is the comprehension category with base $\mathbb{B}$ given as follows. The total category of types $\mathbb{E} \star \mathbb{D}$ is given by the pullback of q along $\mathrm{cod} \circ \chi_D$.

$$\begin{array}{ccccc} \mathbb{E} \star \mathbb{D} & \longrightarrow & \mathbb{E} \bullet \mathbb{B}^{\to} & \longrightarrow & \mathbb{E} \\ {\scriptstyle (\mathrm{cod}\circ\chi_D)^*(q)} \big\downarrow & & \big\downarrow & & \big\downarrow {\scriptstyle q} \\ \mathbb{D} & \xrightarrow[\chi_D]{} & \mathbb{B}^{\to} & \xrightarrow[\mathrm{cod}]{} & \mathbb{B} \end{array}$$

The fibration $q \star p$ is given by the composite of $(\mathrm{cod} \circ \chi_D) * (q)$ with $p : \mathbb{D} \to \mathbb{B}$. The comprehension is given by the composite

$$\mathbb{E} \star \mathbb{D} \to \mathbb{E} \bullet \mathbb{B}^{\to} \xrightarrow{\chi_E \bullet \mathrm{id}_{\mathbb{B}^{\to}}} \mathbb{B}^{\to} \bullet \mathbb{B}^{\to} \to \mathbb{B}^{\to}$$

where the last arrow is given by composition in $\mathbb{B}$. It is easy to check that this does indeed give a comprehension category.

Remark 6.1.10. For display map categories, the juxtaposition $\mathcal{E} \star \mathcal{D}$ of two classes of display maps $\mathcal{D}, \mathcal{E} \subseteq \mathrm{Mor}\, \mathbb{B}$ is simply the class of maps which factorize as a map in $\mathcal{E}$ followed by a map in $\mathcal{D}$

Remark 6.1.11. There is a lax morphism of comprehension categories $\mathbb{E}\star\mathbb{D} \to \mathbb{D}$ over $\mathbb{B}$. The functor $\mathbb{E}\star\mathbb{D} \to \mathbb{D}$ is given by $(\mathrm{cod} \circ \chi_D)^*(q)$, i.e. a context $I \in \mathbb{B}$ with a type $A \in \mathbb{D}(I)$ and a type $X \in \mathbb{E}(\{A\})$ is sent to just $A \in \mathbb{D}(I)$. For any $(I, A, X) \in \mathbb{E}\star\mathbb{D}$ the component of the comprehension comparison 2-cell is just

$$\begin{array}{ccc} \{X\} & \longrightarrow & \{A\} \\ \downarrow & & \big| \\ \{A\} & & \big| \\ \downarrow & & \downarrow \\ \{I\} & = & \{I\} \end{array} .$$

Proposition 6.1.12. *The juxtaposition construction is functorial in each argument with respect to colax morphisms of comprehension categories over the base $\mathbb{B}$.*

Proof. Suppose we have $(F,\alpha) : \mathbb{D}_1 \to \mathbb{D}_2$ and $(G,\beta) : \mathbb{E}_1 \to \mathbb{E}_2$ colax morphisms of comprehension categories. Then there is a morphism $\mathbb{E}_1 \star \mathbb{D}_1 \to \mathbb{E}_2 \star \mathbb{D}_2$ whose functor part sends a type $(B,d,e) \in \mathbb{E}_1 \star \mathbb{D}_1$ (where $B \in \mathbb{B}$, $d \in \mathbb{D}_1(B)$ and $e \in \mathbb{E}(\{d\})$) to $(B, F(d), \overline{G(e)})$, where $\overline{G(e)}$ is the reindexing of $G(e) \in \mathbb{E}(\{d\})$ along $\alpha_d : \{F(d)\} \to \{d\}$. The comprehension comparison 2-cell is given by the top composite in the diagram

$$\begin{array}{ccccc}
\{\overline{G(e)}\} & \longrightarrow & \{G(e)\} & \xrightarrow{\beta_e} & \{e\} \\
\downarrow & & \downarrow & \swarrow & \\
\{F(d)\} & \xrightarrow{\alpha_d} & \{d\} & & \\
& \searrow & & \swarrow & \\
& & B & &
\end{array}$$

where the square is a pullback, being the comprehension of a cartesian arrow. $\square$

Remark 6.1.13. It is easy to check that the juxtaposition of two full comprehension categories is full and that the juxtaposition of two comprehension categories with Ehrhard comprehension has Ehrhard comprehension.

We can now define the gluing model. Though it is plausible that much of the type-theoretic structure that we will construct for the gluing model can be derived in two steps from general theorems about the existence of type constructors in the change-of-base and juxtaposition comprehension categories, we will not attempt this approach here.

Definition 6.1.14. Given a cloven comprehension category $(p : \mathbb{F} \to \mathbb{B}, \chi)$ and a fibred comprehension category over the same base $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$, the *gluing comprehension category* the comprehension category $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ with base $\mathbb{E}_1$ defined as the juxtaposition $\mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1)$ of the comprehension category $\mathbb{E}_2$ on base $\mathbb{E}_1$ with the change-of-base $\mathbb{F} \bullet \mathbb{E}_1$ of $\mathbb{F}$ along $p_0 : \mathbb{E}_1 \to \mathbb{B}$.

We can describe $\mathbb{G}$ more explicitly as follows. An object of $\mathbb{G}$ consists of an object $\Gamma \in \mathbb{E}_1$ together with an object $A \in \mathbb{B}(p_0(\Gamma))$ and an object $X \in \mathbb{E}_2(\chi(A)^*\Gamma)$. A morphism $(\Gamma, A, B) \to (\Gamma', A', B')$ consists of a morphism $f : \Gamma \to \Gamma'$ in $\mathbb{E}_1$ together with a morphism $g : A \to A'$ in $\mathbb{F}$ satisfying $p(g) = p_0(f)$ and a morphism $h : X \to X'$ in $\mathbb{E}_2$ satisfying $p_1(h) = \overline{f}$, where $\overline{g} : \chi(A)^*(\Gamma) \to \chi(A')^*(\Gamma')$ is the unique morphism satisfying $p_0(\overline{f}) = \chi(g)$ and $\chi(A')^{\Gamma'} \circ \overline{f} = f \circ \chi(A)^{\Gamma}$. The functor q is given by $q(\Gamma, A, B) = \Gamma$ and the obvious mapping on morphisms. The comprehension functor sends (Γ, A, B) to the composite

$$\{B\} \xrightarrow{\chi_1(B)} \chi(A)^*(\Gamma) \xrightarrow{\chi(A)^{\Gamma}} \Gamma$$

in $\mathbb{E}_1$. Compare this to the description of the gluing construction for display map categories in Definition 6.1.18 below.

Remark 6.1.15. By composing the two (lax) morphisms of comprehension categories of 6.1.4 and 6.1.11 we get a lax morphism

$$\mathbb{G} = \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) \to \mathbb{F} \bullet \mathbb{E}_1 \to \mathbb{F}.$$

In fact, since the comprehensions of types in $\mathbb{E}_2$ are all p_0-vertical, and the second map is a strict morphism, the composite is also a strict morphism of comprehension categories.

Remark 6.1.16. In the situation where $\mathbb{F}$ has strong unit types, there is a strong morphism of comprehension categories $\mathbb{E}_2 \to \mathbb{G}$ over $\mathbb{E}_1$. Since $\mathbb{F}$ has strong unit types, there is a strong morphism of comprehension categories $\mathbb{1}_{\mathbb{B}} \to \mathbb{F}$ from the trivial comprehension category on $\mathbb{B}$ to $\mathbb{F}$. By the functoriality of change-of-base from 6.1.5, we get a strong morphism

$$\mathbb{1}_{\mathbb{E}_1} \cong \mathbb{1}_{\mathbb{F}} \bullet \mathbb{E}_1 \to \mathbb{F} \bullet \mathbb{E}_1.$$

Hence by the functoriality of juxtaposition from 6.1.12, we get a strong morphism

$$\mathbb{E}_2 \cong \mathbb{E}_2 \star \mathbb{1}_{\mathbb{E}_1} \to \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) = \mathbb{G}.$$

Remark 6.1.17. Suppose that the fibred comprehension category $\mathbb{E}_2$ has strong unit types and that $p_0 : \mathbb{E}_1 \to \mathbb{B}$ has fibred terminal objects. Then there is a strong morphism $\mathbb{F} \to \mathbb{G}$ over the fibred terminals functor $\top : \mathbb{B} \to \mathbb{E}_1$ (the right adjoint section to p_0). We obtain this by composing the functor $\mathbb{F} \to \mathbb{F} \bullet \mathbb{E}_1$ from 6.1.6 with the juxtaposition of the identity on $\mathbb{F} \bullet \mathbb{E}_1$ and the unit types functor $\mathbb{1}_{\mathbb{E}_1} \to \mathbb{E}_2$.

$$\mathbb{F} \to \mathbb{F} \bullet \mathbb{E}_1 \cong \mathbb{1}_{\mathbb{E}_1} \star (\mathbb{F} \bullet \mathbb{E}_1) \to \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) = \mathbb{G}$$

(One checks easily that indeed $\mathbb{F} \bullet \mathbb{E}_1 \cong \mathbb{1}_{\mathbb{E}_1} \star (\mathbb{F} \bullet \mathbb{E}_1)$.)

Note that it is not immediate to translate the gluing construction to display map categories, for the reason that, when $p : \mathbb{E} \to \mathbb{B}$ is a fibred display map category with fibrewise display maps $\mathbb{E}_I \subseteq \mathbb{E}(I)$, it is not in general the case that $\mathcal{E} = \bigcup_I \mathcal{E}_I$ is a class of display maps in $\mathbb{E}$. It is possible to make a mildly weaken the notion of class of display maps to accommodate this example, but we shall not discuss that here. Instead, we state directly what the gluing construction means in terms of display map categories.

Definition 6.1.18. Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category, $p : \mathbb{E} \to \mathbb{B}$ a fibration and for each $I \in \mathbb{B}$ let $\mathcal{E}_I$ be a well-rooted class of display maps in $\mathbb{E}(I)$ such that these fibrewise display maps are preserved by reindexing. Then the *glued display map category* has underlying category $\mathbb{E}$ and class of display maps $\mathcal{G}$ given by $f \in \mathcal{G}$ if and only if $p(f) \in \mathcal{F}$ and the vertical component of f is in $\mathcal{E}_{p(\operatorname{dom} f)}$.

It is easy to verify that $(\mathbb{E}, \mathcal{G})$ is indeed a well-rooted display map category, and that it arises as a change-of-base construction followed by a juxtaposition if we generalize the latter construction to arbitrary classes of maps.

6.2 Type constructors in the gluing model

Before we consider some interesting examples of gluing models in Chapter 7, we will consider in the abstract situation some sufficient and sometimes necessary conditions for the gluing model to have various type constructors. We will mainly work in the setting of a display map category $(\mathbb{B}, \mathcal{F})$ and a fibred display map category $(p : \mathbb{E} \to \mathbb{B}, \mathcal{E})$, with $\mathcal{G}$ the class of display maps in the category $\mathbb{E}$ forming the gluing model. However, we will also cover Σ-types in the setting where $(p : \mathbb{F} \to \mathbb{B}, \chi)$ is a comprehension category, $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ is a fibred comprehension category over the same base $\mathbb{B}$, and $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ is the gluing comprehension category.

For the situation of a fibred display map category, versions of the problem of finding type constructors in the glued display map category have been considered in [38], [46], and [43]. In [38] attention is given to the situations of inverse diagrams and oplax limits. This does not seem to be exactly the same as the situation here, but it is closely related since the situation given by restricting to inverse diagrams with shape the ordinal $\mathbb{2}$ corresponds to the special case of the gluing construction where the fibrewise models in $p : \mathbb{E} \to \mathbb{B}$ are given by pulling back the fibred fundamental fibration of some other model along some functor out of $\mathbb{B}$. In [46] the situation of a display map category fibred over a display map category is considered and some general conditions are given for the existence of Σ-, Id- and Π-types in the total display map category. A separate calculation is performed to see that the polynomial model admits Π-types. In [44], an equivalence is demonstrated between fibrewise structure and total structure.[1] More precisely, it is an equivalence between Σ- and Id-type constructors in the fibres which are *fibrewise* — meaning stable under reindexing — and the same type constructors in the total category $\mathbb{E}$ which are suitably preserved by the

[1] The article [44] is a preprint of [43], and contains a few results omitted from the published version.

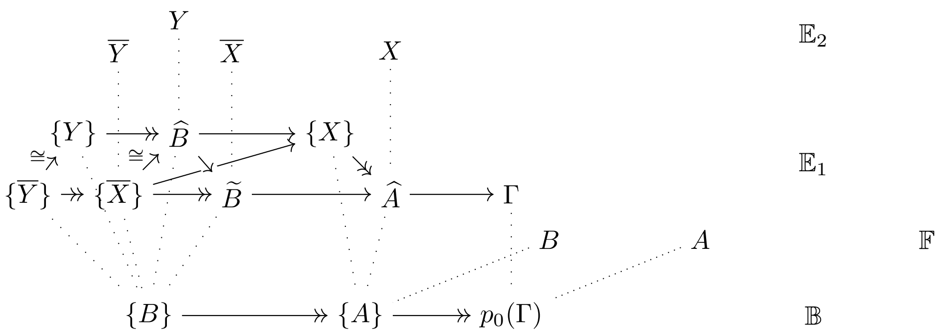
fibration p. Additionally, there is a similar treatment of Π-types, but here the fibrewise notion also involves a completeness condition for the fibration p.

The construction of identity types that we use here is the one given in both [46] and [43]. We will compare the two constructions of identity types given in each of those articles — we will ultimately find it more convenient to employ the one given in [43]. The construction of fibred dependent products given as a necessary and sufficient condition in Theorem 3.21 in [44] (N.B. the same sufficient condition is Proposition 3.14 in [46]) does not apply to our examples here, which are closely related to the polynomial model of [46]. We will introduce the notion of *quasifibred* Π-*type* to give a sufficient condition for the gluing model to have dependent products.

6.3 Dependent sums

Proposition 6.3.1. *Suppose that the comprehension category $\mathbb{F}$ and the fibred comprehension category $\mathbb{E}_2$ admit dependent sums. Then the gluing comprehension category $\mathbb{G}$ of Definition 6.1.14 also admits dependent sums.*

Proof. Let us suppose we have a type (Γ, A, X) in $\mathbb{G}(\Gamma)$ and a type $(\{X\}, B, Y)$ over it. Since the comprehensions of $\mathbb{E}_2$-types are p_0-vertical, we actually have that B is a type in $\mathbb{F}(\{A\})$, and we can represent our input data pictorially as



where $\widehat{A} \to \Gamma$ is the chosen cartesian lift of $\{A\} \to p_0(\Gamma)$ with codomain Γ, $\widetilde{B} \to \widehat{A}$ and $\widetilde{B} \to \{X\}$ are the chosen cartesian lifts of $\{B\} \to \{A\}$ with codomains $\widehat{A}$ and $\{X\}$ respectively, and $\widehat{B} \to \widetilde{B}$ is the map induced by (reindexing) $\{X\} \to \widehat{A}$. The type $\overline{X}$ is a reindexing of $X \in \mathbb{E}_2(\widehat{A})$ along $\widetilde{B} \to \widehat{A}$, hence the fact that $\{\overline{X}\} \to \{X\}$ is cartesian induces a canonical isomorphism $\{\overline{X}\} \cong \widehat{B}$. The type $\overline{Y}$ is a reindexing of $Y \in \mathbb{E}_2(\widehat{B})$ along this isomorphism. The comprehension of (Γ, A, X) is the composite

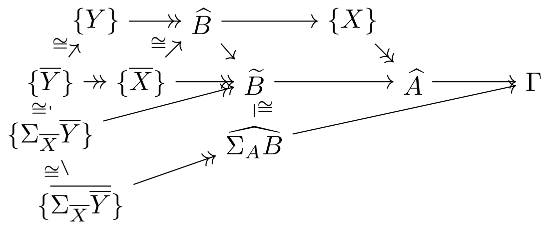
$$\{X\} \to \widehat{A} \to \Gamma$$

and the comprehension of $(\{X\}, B, Y)$ is the composite

$$\{Y\} \to \widehat{B} \to \{X\}$$

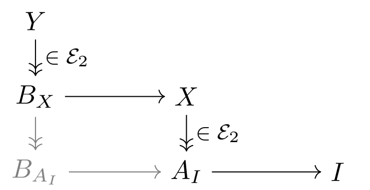
and we see from the diagram that the composite of these composites is isomorphic to the composite from $\{\overline{Y}\}$ to Γ.

Hence we are guided to what the sum should be. We take the sum $\Sigma_A B$ in $\mathbb{F}$, which comes with a canonical isomorphism $\{B\} \cong \{\Sigma_A B\}$. Then we take the sum $\Sigma_{\overline{X}}\overline{Y}$ over $\widetilde{B}$ in $\mathbb{E}_2$, and reindex to $\overline{\Sigma_{\overline{X}}\overline{Y}}$ in $\mathbb{E}_2(\widehat{\Sigma_A B})$ where $\widehat{\Sigma_A B} \to \widetilde{B}$ is a chosen cartesian arrow lifting the isomorphism $\{\Sigma_A B\} \cong \{B\}$. If we add in the comprehension of this type to our diagram above,



we see that the comprehension of $(\Gamma, \Sigma_A B, \overline{\Sigma_{\overline{X}}\overline{Y}})$ is isomorphic over Γ to the composite of the comprehension of our two starting types. In the special case where $\mathbb{F}$ and $\mathbb{E}_2$ are full (fibred) comprehension categories, $\mathbb{G}$ is also a full comprehension category and so by 1.6.4 what we have is enough to conclude that $\mathbb{G}$ has dependent sums. In the general case, it is straightforward but notationally awkward to give the canonical map $Y \to \overline{\Sigma_{\overline{X}}\overline{Y}}$ lying over $\widehat{B} \to \widehat{\Sigma_A B}$ and to show that this gives a q-cocartesian arrow which is stable under reindexing. $\square$

Remark 6.3.2. The proof appears to be simpler in the display map category case. It reduces to showing that one can write an 'alternating composite' of display maps



$$B \xrightarrow{\in \mathcal{F}} A \xrightarrow{\in \mathcal{F}} p(I)$$

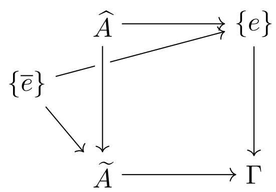
in the form required for a glued display map. We do this by considering the cartesian lift $B_{A_I} \to A_I$ of $B \twoheadrightarrow A$ with codomain A_I and the induced map $B_X \to B_{A_I}$, which is a reindexing of $X \twoheadrightarrow A_I$ and hence a display map. We conclude by simply composing the two $\mathcal{E}_2$-display maps and also the two maps cartesian over $\mathcal{F}$-display maps.

Remark 6.3.3. We can encode the seemingly simpler proof from 6.3.2 for the *full* comprehension category case using something that looks a bit like a distributive law (see [3]) of the comprehension categories $\mathbb{E}_2$ and $\mathbb{F} \bullet \mathbb{E}_1$ over the common base $\mathbb{E}_1$. Observe that, even without fullness, there is a strong morphism of comprehension categories

$$(\mathbb{F} \bullet \mathbb{E}_1) \star \mathbb{E}_2 \to \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1)$$

given as follows. A triple consisting of a context $\Gamma \in \mathbb{E}_1$, a type $e \in \mathbb{E}_2(\Gamma)$, and a type $A \in \mathbb{F}(p_0(\{e\})) = \mathbb{F}(p_0(\Gamma))$ is sent to the triple with context Γ, the type $A \in \mathbb{F}(p_0(\Gamma))$, and the type $\overline{e} \in \mathbb{E}_2(\widetilde{A})$ where $\widetilde{A} \to \Gamma$ is the chosen cartesian lift of $A \twoheadrightarrow p_0(\Gamma)$ and $\overline{e}$ is the reindexing of e along the $\widetilde{A} \to \Gamma$.

Now if $\widehat{A} \to \{e\}$ is a chosen cartesian lift of $\{A\} \to p_0(\Gamma)$ then the composite $\widehat{A} \to \{e\} \to \Gamma$ is the comprehension of (Γ, e, A), and the composite $\{\overline{e}\} \to \widetilde{A} \to \Gamma$ is the comprehension of $(\Gamma, A, \overline{e})$. These are connected by a canonical isomorphism coming from the fact that the diagram



contains two cartesian squares.

Now, observe that a *full* comprehension category $(p : \mathbb{E} \to \mathbb{B}, \chi)$ has dependent sums if and only if there exists a strong morphism of comprehension categories

$$\mathbb{E} \star \mathbb{E} \to \mathbb{E}$$

over $\mathbb{B}$ (this follows from 1.6.4). Hence, suppressing the associativity of juxtaposition, the composite

$$\mathbb{G} \star \mathbb{G} \cong \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) \star \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) \to \mathbb{E}_2 \star \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) \star (\mathbb{F} \bullet \mathbb{E}_1) \to \mathbb{E}_2 \star (\mathbb{F} \bullet \mathbb{E}_1) = \mathbb{G}$$

constructs the dependent sums in $\mathbb{G}$ from the dependent sums in $\mathbb{E}_2$ and $\mathbb{F} \bullet \mathbb{E}_1$ (it is easy to check that the latter inherits its dependent sums from those in $\mathbb{F}$.

6.4 Identity types

We will only describe the remaining type constructors in the case of well-rooted display map categories. The conditions for the existence of identity types that we give here are essentially those of Shulman [38] and Uemura [43].

Recall from 1.8.3 that for a well-rooted display map category to support identity types, we must first identify the *acyclic cofibrations* or *left class* with respect to the display maps, secondly we must show that this class is suitably stable under pullback, and thirdly we must demonstrate factorizations of arbitrary morphisms into an acyclic cofibration followed by a display map.

Lemma 6.4.1. *Let $m : I \to J$ be a map in $\mathbb{B}$, and let $n : A \to B$ be a map in $\mathbb{E}$ lying over m. Then the following are equivalent.*

(i) The map n has the left lifting property with respect to morphisms which are cartesian over display maps in $\mathbb{B}$.

(ii) The map m is an acyclic cofibration in $(\mathbb{B}, \mathcal{F})$.

Proof. If $f : X \to Y$ is cartesian over a $\mathcal{F}$-map, then for any lifting problem

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ {\scriptstyle n}\downarrow & \overset{s}{\nearrow} & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & Y \end{array}$$

in $\mathbb{E}$ it is easy to see that solutions s correspond to solutions of the lifting problem

$$\begin{array}{ccc} I & \xrightarrow{p(u)} & p(X) \\ {\scriptstyle m}\downarrow & \nearrow & \downarrow{\scriptstyle p(f)} \\ J & \xrightarrow[p(v)]{} & p(Y) \end{array}$$

in $\mathbb{B}$. ☐

Lemma 6.4.2. *Let $I \in \mathbb{B}$ and $m : A \to B$ a morphism in $\mathbb{E}(I)$, considered as a p-vertical morphism in $\mathbb{E}$. Then the following are equivalent.*

(i) The map m is in ${}^{\square}\mathcal{G}$.

(ii) The map m is an acyclic cofibration in $(\mathbb{E}(I), \mathcal{E}_I)$.

Proof. Observe that fibrewise display maps are stable under reindexing and under fibred pullback. Thus they are stable under pullback in $\mathbb{E}$. Hence, for any vertical map m over I, the left lifting property with respect to fibrewise display maps in $\mathbb{E}$ is equivalent to the left lifting property with respect to display maps in $\mathbb{E}(I)$. Thus we get (i) $\implies$ (ii).

For (ii) $\implies$ (i), we must also check the left lifting property with respect to morphisms which are cartesian over display maps in $\mathbb{B}$, since $\mathcal{G}$ is the closure under composition of these maps and the fibrewise display maps. But since $p(m) = 1_I$ is an acyclic cofibration, this follows from Lemma 6.4.1. □

In general it need not be the case that ${}^{\square}\mathcal{G}$ is given as the class of maps which lie over an acyclic cofibration and whose vertical component is an acyclic cofibration. In [43], necessary and sufficient conditions are given for ${}^{\square}\mathcal{G}$ to be this class and for the total category to have identity types. Two of the conditions are that the base $(\mathbb{B}, \mathcal{F})$ has identity types and that the fibration $p : \mathbb{E} \to \mathbb{B}$ has fibred identity types, i.e. each fibre category $\mathbb{E}(I)$ has identity types and the left classes are stable under reindexing. The last condition is as follows.

Definition 6.4.3. The gluing data satisfies *the acyclic cofibration condition* if, for any acyclic cofibration $i : A \to B$ in the base $\mathbb{B}$ and every display map $f : X \to Y$ in $\mathbb{E}(B)$, every section of the reindexing $i^*(f)$ is the reindexing of some section of f.

Since the acyclic cofibration condition seems more fundamental than the existence of fibrewise identity types, we deviate from the treatment in [43] to investigate the class ${}^{\square}\mathcal{G}$ while putting off the assumption of identity types in our base models for as long as possible, and not using the full set of assumptions until Proposition 6.4.9.

Proposition 6.4.4. *The acyclic cofibration condition condition holds if and only if for any acyclic cofibration $i : A \to B$ and any $X \in \mathbb{E}(B)$ the functor*

$$i^*/X : \mathcal{E}_B/X \to \mathcal{E}_A/i^*(X)$$

is full.

Proof. Let $i : A \to B$ be an acyclic cofibration in $\mathbb{B}$, and let $f : Y \twoheadrightarrow X$ and $g : Z \twoheadrightarrow X$ be two display maps in $\mathbb{E}(B)$. Now maps $Y \to Z$ over X correspond to sections of $f^*(g)$. □

Lemma 6.4.5. *The acyclic cofibration condition holds if and only if every p-cartesian map lying over an acyclic cofibration has the left lifting property with respect to vertical display maps.*

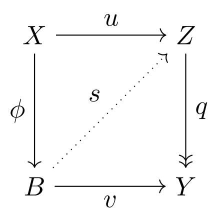
Proof. Since the vertical display maps are stable under reindexing, the two statements are trivial restatements of each other. $\square$

Proposition 6.4.6. *The acyclic cofibration condition holds if and only if* ${}^{\square}\mathcal{G}$ *consists of all maps lying over acyclic cofibrations with vertical component an acyclic cofibration.*

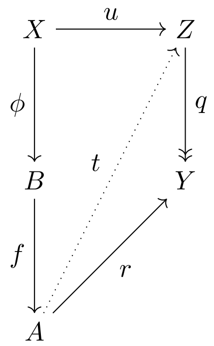
Proof. Supposing the latter statement, then any cartesian map lying over an acyclic cofibration is in ${}^{\square}\mathcal{G}$ so by Lemma 6.4.5 we have the acyclic cofibration condition.

Conversely, by Lemmas 6.4.1 and 6.4.5 we see that ${}^{\square}\mathcal{G}$ contains all cartesian maps lying over acyclic cofibrations, and by Lemma 6.4.2 it contains all vertical maps which are acyclic cofibrations in the respective fibre category. Since left classes are closed under composition, it follows that ${}^{\square}\mathcal{G}$ contains the proposed class. To see that ${}^{\square}\mathcal{G}$ contains nothing more, by 6.4.1 it suffices to check that the vertical component of any member of ${}^{\square}\mathcal{G}$ is an acyclic cofibration in the respective fibre category.

Let $f \circ \phi : X \to A$ be the cartesian-vertical factorization of some member of ${}^{\square}\mathcal{G}$, where $f : B \to A$ is cartesian and $F : X \to B$ is vertical over $p(B) = I$, say. Then by Lemma 6.4.1, we see that f lies over an acyclic cofibration. Hence, by 6.4.5 and 6.4.1, we see that $f \in {}^{\square}\mathcal{G}$ itself. Now, in order to show that ϕ is an acyclic cofibration, suppose that we have a lifting problem in $\mathbb{E}(I)$



where q is in $\mathcal{E}_I$. Then we may equivalently consider it as a lifting problem in $\mathbb{E}$. Since $f : B \to A$ is in ${}^{\square}\mathcal{G}$, it follows that there is a map $r : A \to Y$ such that $r \circ f = v$. Now we can use that fact that $f \circ \phi \in {}^{\square}\mathcal{G}$ to find a solution t to the lifting problem



in $\mathbb{E}$. It is now easy to check that tf is a solution s to the original lifting problem. □

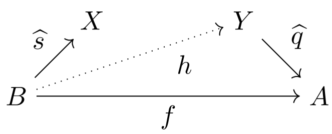
Thus the acyclic cofibration condition determines the left class of $\mathcal{G}$. The class described in Proposition 6.4.6 is called the class of Reedy cofibrations in, for example, [38] and [43]. It remains to check that every arrow admits a $({}^{\square}\mathcal{G}, \mathcal{G})$-factorization, and that ${}^{\square}\mathcal{G}$ is a stable class. For the first, we will need to assume that we have factorizations in each of the fibres and in the base category.

Proposition 6.4.7. *Suppose that every map in $\mathbb{B}$ admits a $({}^{\square}\mathcal{F}, \mathcal{F})$-factorization, and that for each $I \in \mathbb{B}$ every map in $\mathbb{E}(I)$ admits a $({}^{\square}\mathcal{E}_I, \mathcal{E}_I)$-factorization. Then the acyclic cofibration condition implies that every map in $\mathbb{E}$ factorizes as a ${}^{\square}\mathcal{G}$-map followed by a $\mathcal{G}$-map.*

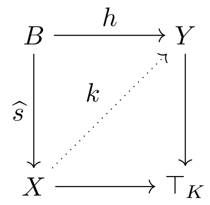
Proof. Let $f : B \to A$ be a map in $\mathbb{E}$. Then there exists a $({}^{\square}\mathcal{F}, \mathcal{F})$-factorization of $p(f) : p(B) \to p(A)$.

$$p(B) \xrightarrow{s} K \xrightarrow{q} p(A)$$

Now s admits a retraction $r : K \to p(B)$, and hence there is a cartesian lift $\widehat{r} : X \to B$ of r with codomain B and a cartesian arrow $\widehat{s} : B \to X$ lying over s with domain B. We also consider a cartesian lift $\widehat{q} : Y \to A$ of q with codomain A, and the induced factorization $h : A \to Y$ of f through $\widehat{q}$.



Now, by 6.4.5, $\widehat{s}$ has the left lifting property with respect to the vertical display map $Y \to \top_K$. Hence there is a map $k : X \to Y$ filling the square



meaning that $k\widehat{s} = h$ and $p(k) = 1_K$. Hence we may factorize k in the fibre category $\mathbb{E}(K)$ into an acyclic cofibration followed by a display map.

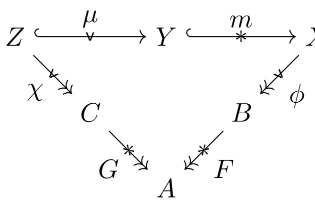
$$X \xrightarrow{x} L \xrightarrow{y} Y$$

Then $\widehat{q} \circ y$ is a $\mathcal{G}$-map. Moreover, x and $\widehat{s}$ are both ${}^{\square}\mathcal{G}$-maps by 6.4.2 and 6.4.5, and since left classes are closed under composition, so is their composite. $\square$

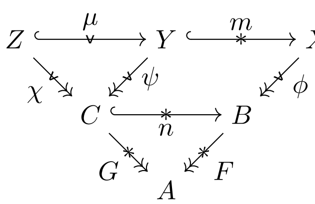
Clearly, we will need to assume stability of left classes for our input data in order to prove stability of ${}^{\square}\mathcal{G}$. In fact we will also assume that the identity types are fibred, i.e. the fibrewise left classes must be stable under reindexing.

Lemma 6.4.8. *Suppose that $(\mathbb{B}, \mathcal{F})$ has identity types, that each fibre category $(\mathbb{E}(I), \mathcal{E}_I)$ has identity types, and that the left classes ${}^{\square}\mathcal{F}$ in each fibre are preserved by reindexing. Then the class of maps in $\mathbb{E}$ which lie over acyclic cofibrations and have vertical component an acyclic cofibration is stable.*

Proof. Beginning with a diagram of the form



where the cartesian arrows are marked with an asterisk and the vertical ones with a 'v', we are required to show that its pullback along a map into A is of the same form. We can complete the interior of the diagram as shown below.



For a cartesian arrow $f : \overline{A} \to A$, pulling back along f in $\mathbb{E}$ amounts to calculating the pullback of the triangle $p(F) \circ p(n) = p(G)$ in $\mathbb{B}$ and then computing some reindexings — it is easy to check that the composite $m \circ \mu$ goes to an arrow of the required form using the fact that fibrewise acyclic cofibrations are stable under reindexing. For a vertical arrow $\phi : A' \to A$, pulling back along ϕ in $\mathbb{E}$ amounts to calculating some reindexings of ϕ along $p(F)$, $p(n)$ and $p(G)$, and then computing some fibrewise pullbacks — again it is easy to check that the composite $m \circ \mu$ goes to an arrow of the required form using the fact that in each fibre the acyclic cofibrations are a stable class. $\square$

The following is now immediate.

Proposition 6.4.9. *Assume the acyclic cofibration condition and suppose that* $(\mathbb{B}, \mathcal{F})$ *has identity types, that each fibre category* $(\mathbb{E}(I), \mathcal{E}_I)$ *has identity types, and that the left classes* ${}^{\square}\mathcal{E}_I$ *in each fibre are preserved by reindexing. Then* $(\mathbb{E}, \mathcal{G})$ *has identity types.*

Remark 6.4.10. It is worth comparing the treatment here, based on [43], with that in [46]. In [46] the acyclic cofibration condition is replaced by the following: *the terminal objects functor preserves left morphisms* if for any ${}^{\square}\mathcal{F}$-map $m : A \to B$ in $\mathbb{B}$ the induced (cartesian) map $\top_A \to \top_B$ in $\mathbb{E}$ is in ${}^{\square}\mathcal{G}$. Since by 6.4.1 any such map already has the left lifting property with respect to cartesian morphisms lying over a display map, this is clearly a special case of the acyclic cofibration condition. The construction in [46] then proceeds by verifying the full acyclic cofibration condition in the situation where $(\mathbb{E}, \mathcal{G})$ has dependent products, for in this situation the ${}^{\square}\mathcal{G}$-maps are stable under pullback along $\mathcal{G}$-maps. Every cartesian morphism $X \to Y$ is a pullback of one $\top_{p(X)} \to \top_{p(Y)}$, and hence if the terminal objects functor preserves left morphisms and $\mathcal{G}$ models dependent products, then every cartesian morphism lying over an acyclic cofibration is in ${}^{\square}\mathcal{G}$.

6.5 Dependent products

The following was shown by Hermida as Corollary 4.12 in [19].

Theorem 6.5.1 ([19])**.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *be a fibration where* $\mathcal{B}$ *is a cartesian closed category,* p *has simple products, and each fibre of* p *is a cartesian closed category with finite products and exponentials preserved by reindexing. Then* $\mathcal{E}$ *is a cartesian closed category and* p *strictly preserves the cartesian closed structure.*

In fact, the converse is true. Proposition 3.14 in [46] generalizes 6.5.1 to the gluing construction, giving one direction of the following theorem. The converse is given Theorem 3.21 in [44].

Theorem 6.5.2. *Suppose that* $(\mathbb{B}, \mathcal{F})$ *has dependent products. Then* $(\mathbb{E}, \mathcal{G})$ *has dependent products and* $p : \mathbb{E} \to \mathbb{B}$ *preserves them if and only if for each* $I \in \mathbb{B}$ *the fibre category* $(\mathbb{E}(I), \mathcal{E}_I)$ *has dependent products which are stable under reindexing and the fibration* $p : \mathbb{E} \to \mathbb{B}$ *has* $\mathcal{F}$*-products which preserve the fibrewise display maps.*

In the polynomial model of [46], the dependent products in $(\mathbb{E}, \mathcal{G})$ are not preserved by the fibration p. We will give the following generalization of 6.5.2 which covers the construction of the Polynomial model and the examples of the present work. The statement of the Theorem uses the notation $\overline{\mathcal{F}}$ from Remark 6.1.3 for the class of p-cartesian maps f with $p(f) \in \mathcal{F}$, and also refers to concepts defined below in Definitions 6.5.4 and 6.5.9.

Theorem 6.5.3. *Suppose that* $(\mathbb{B}, \mathcal{F})$ *has dependent products and dependent sums. Then* $(\mathbb{E}, \mathcal{G})$ *has dependent products such that* p *sends* $\overline{\mathcal{F}}$*-dependent products to* $\mathcal{F}$*-dependent products if and only if* p *has quasifibred dependent products and* $\mathcal{F}$*-products that preserve the fibrewise display maps.*

It is worth investigating 6.5.2 first, in order to see which parts of the proof can be salvaged. To do this we will introduce terminology for discussing display map categories where only certain dependent products exist.

Definition 6.5.4. Let $(\mathbb{B}, \mathcal{F})$ be a display map category and let $\mathcal{D}$ and $\mathcal{E}$ be subclasses of $\mathcal{F}$. Then $\mathbb{B}$ has *$\mathcal{D}$-products of $\mathcal{E}$-maps* if for any $f : B \to A$ in $\mathcal{D}$ and any $g : C \to B$ in $\mathcal{E}$ the pullback functor

$$f^* : \mathbb{B}/A \to \mathbb{B}/B$$

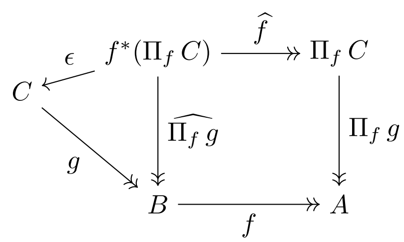
admits a coreflection of g and this coreflection is an object of the subcategory $\mathcal{F}/A$ of $\mathbb{B}/A$. In the case that $\mathcal{E} = \mathcal{F}$ and $(\mathbb{B}, \mathcal{F})$ has $\mathcal{D}$-products of $\mathcal{F}$-maps, we say that $(\mathbb{B}, \mathcal{F})$ has $\mathcal{D}$-dependent products.

By a similar observation to 1.5.5 this is really the existence of a partial right adjoint satisfying a pullback-stability (Beck-Chevalley) condition. One direction of the following proposition appears as Lemma 3.13 in [46].

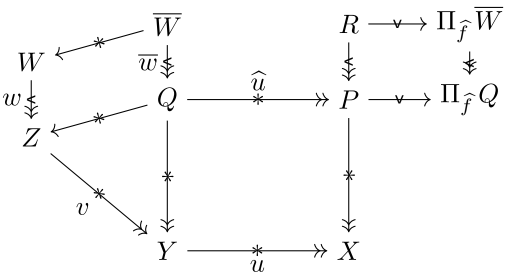
Proposition 6.5.5. *Suppose that* $(\mathbb{B}, \mathcal{F})$ *has* $\mathcal{F}$*-dependent products. Then* $(\mathbb{E}, \mathcal{G})$ *has* $\overline{\mathcal{F}}$*-dependent products which are sent to* $\mathcal{F}$*-dependent products by* p *if and only if* p *has* $\mathcal{F}$*-products which preserve fibrewise display maps.*

Proof. Supposing that $(\mathbb{E}, \mathcal{G})$ has $\overline{\mathcal{F}}$-dependent products which are sent to $\mathcal{F}$-dependent products by p, we can define an $\mathcal{F}$-product as follows. For $f : B \twoheadrightarrow A$ an $\mathcal{E}$-map, and $X \in \mathbb{E}(B)$, define $\Pi_f(X)$ to be the domain of the dependent product of the vertical display map $X \twoheadrightarrow \top_B$ along the cartesian map $\top_B \to \top_A$ lying over f. This dependent product is indeed a vertical map, since it was assumed that $\overline{\mathcal{F}}$-dependent products are sent to $\mathcal{F}$-dependent products by p. Hence it is easy to check that this is indeed a functor and a right adjoint to reindexing, and that it preserves display maps. The Beck-Chevalley condition follows from the Beck-Chevalley condition for the $\overline{\mathcal{F}}$-dependent products.

Conversely, let $u : Y \to X$ be a cartesian morphism lying over a display map $f : B \twoheadrightarrow A$ and let $v : Z \to Y$ be a cartesian morphism lying over a display map $g : C \twoheadrightarrow B$ and $w : W \twoheadrightarrow Z$ an $\mathcal{E}_C$-map, so that vw is a general member of $\mathcal{G}$. Consider the diagram



in $\mathbb{B}$ formed by taking the dependent product $\Pi_f g$, pulling it back along f and filling in the counit $\epsilon : f^*(\Pi_f C) \to C$. We fill in the picture in $\mathbb{E}$ by taking cartesian lifts, as in the diagram below.



We have also added into the diagram $\overline{w} : \overline{W} \twoheadrightarrow Q$, which is the reindexing of w along $\epsilon : f^*(\Pi_f C) \to C$, $\Pi_{\widehat{f}} \overline{W} \twoheadrightarrow \Pi_{\widehat{f}} Q$ which is the fibred $\mathcal{F}$-product of $\overline{w}$ along $\widehat{f}$, and $R \twoheadrightarrow P$, which is the pullback of $\Pi_{\widehat{f}} \overline{W} \twoheadrightarrow \Pi_{\widehat{f}} Q$ along the unit of the adjunction $\widehat{f}^* \dashv \Pi_{\widehat{f}}$. It is now straightforward to check that the composite $R \twoheadrightarrow P \twoheadrightarrow X$ is a dependent product of vw along u. The Beck-Chevalley condition is an easy check using the Beck-Chevalley condition for the $\mathcal{F}$-products. Finally, note that p does indeed send the dependent product of vw

along u to the dependent product of g along f, since $R \twoheadrightarrow P$ is vertical. $\square$

Proposition 6.5.5 is useful because, given a composite of display maps gf, the dependent product functor along gf exists and factorizes (up to isomorphism) as a dependent product functor along f followed by a dependent product functor along g, whenever these latter two functors exist. Thus we have reduced the problem of finding dependent products of $\mathcal{G}$-maps along arbitrary $\mathcal{G}$-maps to just finding them along maps in $\mathcal{E} = \bigcup_{I \in \mathbb{B}} \mathcal{E}_I$. We can reduce the problem further still, by making use of the restriction of the class of display maps *of which* we take dependent products (as opposed to just *along which*).

Observe that the following does not require any specific assumptions on the gluing data.

Lemma 6.5.6. *For any vertical display map $\phi : Y \twoheadrightarrow X$ over $A \in \mathbb{E}(I)$, the pullback functor between restricted slice categories $\phi^* : \overline{\mathcal{F}}/X \to \overline{\mathcal{F}}/Y$ is an equivalence of categories. In particular, such restricted pullback functors have right adjoints (satisfying a Beck-Chevalley condition).*

Proof. Pullbacks of vertical maps with cartesian maps are given by reindexing. More precisely, in our present scenario we are given $\phi : Y \twoheadrightarrow X$ a vertical display map and $u : Z \to X$ cartesian over a display map $f : C \to A$, say. Then the pullback is the square

$$\begin{array}{ccc} W & \xrightarrow{\widehat{u}} & Y \\ {\scriptstyle \widehat{\phi}} \big\downarrow & & \big\downarrow {\scriptstyle \phi} \\ Z & \xrightarrow[u]{} & X \end{array}$$

where $\widehat{u}$ is a cartesian lift of f with codomain Y, and $\widehat{\phi}$ is the unique factorization of $\phi\widehat{u}$ through u.

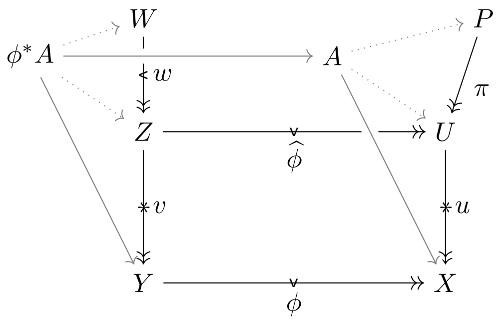
Now for any $Z \in \mathbb{E}$ the functor $p/Z : \overline{\mathcal{F}}/Z \to \mathcal{F}/p(Z)$, which simply applies p, is an equivalence. Since the restricted pullback functor fits into the equivalence $(p/Y) \circ \phi^* \simeq p/X$, we see that $\phi^* : \overline{\mathcal{F}}/X \to \overline{\mathcal{F}}/Y$ itself is an equivalence. The remaining claims are immediate. $\square$

Proposition 6.5.7. *Suppose that $(\mathbb{B}, \mathcal{F})$ has dependent sums. Then $(\mathbb{E}, \mathcal{G})$ has $\mathcal{E}$-products of $\mathcal{G}$-maps if and only if it has $\mathcal{E}$-products of $\mathcal{E}$-maps.*

Proof. The 'only if' direction is trivial since $\mathcal{E} \subseteq \mathcal{G}$. Let us consider the converse. Let $\phi : Y \twoheadrightarrow X$ be a vertical display map and

$$W \xrightarrow{w} Z \xrightarrow{v} Y$$

a map in $\mathcal{G}$. As in the proof of 6.5.6, v is actually the pullback along ϕ of some cartesian map $u : U \to X$. Writing $\widehat{\phi} : W \to U$ for the pullback of ϕ along u, let $\pi : P \to U$ be dependent product of the $\mathcal{E}$-map w along the $\mathcal{E}$-map $\widehat{\phi}$.



Now since $\mathcal{F}$ is closed under composition, the composite $u\pi$ is in $\mathcal{G}$. In order to verify that $u\pi$ is a dependent product of vw along ϕ, we suppose we have a test map $A \to X$, write $\phi^* A \to Y$ for its pullback along ϕ. Whenever there is a map $A \to P$ over X, we get a factorization of $A \to X$ through u by composition with π, and similarly if there is a map $\phi^* A \to W$, then we get a factorization of $\phi^* A \to Y$ through v by composition with w. Observe that the argument of Lemma 6.5.6 can be adapted to show that pullback along ϕ gives a bijection (natural in A) between maps $A \to U$ over X and maps $\phi^* A \to Z$ over Y, and of course this bijection is given by pulling back along $\widehat{\phi}$. Hence we get the following chain of bijections natural in A:

$$\{\text{A map } \phi^* A \to Y \text{ together with a map } \phi^* A \to W \text{ over } Y\}$$
$$\cong$$
$$\{\text{A map } \phi^* A \to Z \text{ together with a map } \phi^* A \to W \text{ over } Z\}$$
$$\cong$$
$$\{\text{A map } A \to U \text{ together with a map } A \to P \text{ over } U\}$$
$$\cong$$
$$\{\text{A map } A \to X \text{ together with a map } A \to P \text{ over } X\}.$$

Hence $u\pi$ is indeed a dependent product of vw along ϕ. $\square$

The most natural way to ask for $\mathcal{E}$-products of $\mathcal{E}$-maps would appear to be to ask for the fibre categories of $p : \mathbb{E} \to \mathbb{B}$ to support dependent products. Let us verify that this is indeed sensible.

Proposition 6.5.8. *The display map category $(\mathbb{E}, \mathcal{G})$ has $\mathcal{E}$-products of $\mathcal{E}$-maps and these products are again in $\mathcal{E}$ if and only if for each $I \in \mathbb{B}$ the fibre display* *map category* $(\mathbb{E}(I), \mathcal{E}_I)$ *has dependent products and reindexing functors preserve dependent products.*

Proof. The forward direction is trivial, using the fact that pullbacks of (vertical) display maps along (vertical) morphisms are preserved by the inclusions of the fibre categories, and that reindexing between the fibre categories corresponds to pulling back along a cartesian morphism.

Conversely, we need to verify that the dependent product in $\mathbb{E}(I)$ of two $\mathcal{E}_I$-maps has the correct universal property for arbitrary test maps, not just vertical ones. But it is easy to see that if the dependent products in each fibre category $\mathbb{E}(I)$ have the correct property for arbitrary vertical test maps, then having the correct property for arbitrary test maps in $\mathbb{E}$ is equivalent to stability under reindexing of the dependent products. □

Now we may complete the proof of the first theorem of dependent products.

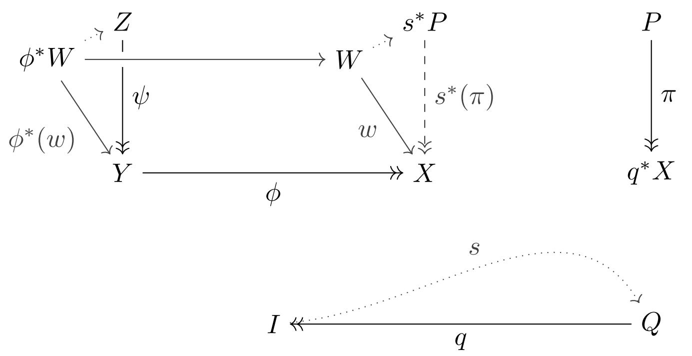
Proof of Theorem 6.5.2. The theorem is immediate from the preceding sequence of propositions, observing that the hypothesis of dependent sums in $(\mathbb{B}, \mathcal{F})$ used in Proposition 6.5.7 may be left out since, in the notation of the proof of that proposition, we know that π is vertical so we already have $u\pi \in \mathcal{G}$. □

Now we may move on to the proof of Theorem 6.5.3. First, we must define one of its terms, which is the condition required for $(\mathbb{E}, \mathcal{G})$ to have $\mathcal{E}$-products of $\mathcal{E}$-maps without $p : \mathbb{E} \to \mathbb{B}$ having fibrewise dependent products.

Definition 6.5.9. The fibration $p : \mathbb{E} \to \mathbb{B}$ has *quasifibred dependent products* if for any $I \in \mathbb{B}$, and any composable pair of display maps

$$Z \xrightarrow{\psi} Y \xrightarrow{\phi} X$$

in $\mathbb{E}(I)$ there exists a display map $q : Q \twoheadrightarrow I$ in $\mathbb{B}$ and a display map $\pi : P \to q^*(X)$ in $\mathbb{E}(Q)$ together with, for every map $w : W \to X$ in $\mathbb{E}(I)$ a bijection, natural in W, between the set of maps $\phi^*(w) \to \psi$ in the slice $\mathbb{E}(I)/Z$ and the set of pairs whose first component is a section $s : I \to Q$ of the display map q and whose second component is a map $w \to s^*(\pi)$ in the slice $\mathbb{E}(I)/Y$. Moreover, this data must be stable under reindexing, in the sense that for any map $h : I' \to I$, the pullback $\widehat{q}$ of q along h together with $\widehat{h}^*(\pi)$ (where $\widehat{h}$ is the pullback of h along q) has the same property for $h^*(\phi)$ and $h^*(\psi)$, and the correspondence is preserved by the reindexing and pulling back (as appropriate) along h.



Lemma 6.5.10. *The fibration $p : \mathbb{E} \to \mathbb{B}$ has quasifibred dependent products if and only if $(\mathbb{E}, \mathcal{G})$ has $\mathcal{E}$-dependent products of $\mathcal{E}$-maps.*

Proof. It is straightforward to verify that the definitions unfold to the same thing. □

Theorem 6.5.3 now follows immediately.

6.6 Universes

A special case of our main result on universes, Proposition 6.6.3, appears as Proposition 4.3 of [43], based on ideas of Shulman in [38]. This corresponds to the special case of quasifibred universe given below where Ω is taken to be the terminal object of $\mathbb{B}$.

Definition 6.6.1. A *quasifibred universe* in $p : \mathbb{E} \to \mathbb{B}$ consists of an object $\Omega \in \mathbb{B}$ together with an $\mathcal{E}_\Omega$-map $v : \widetilde{\mathcal{V}} \twoheadrightarrow \mathcal{V}$. Then for any $I \in \mathbb{B}$, a display map $\phi : Y \twoheadrightarrow X$ in $(\mathbb{E}(I), \mathcal{E}_I)$ is *v-small* if there exists a morphism $f : I \to \Omega$ in $\mathbb{B}$ such that ϕ arises as a pullback of $f^*(v) : f^*(\widetilde{\mathcal{V}}) \twoheadrightarrow f^*(\mathcal{V})$.

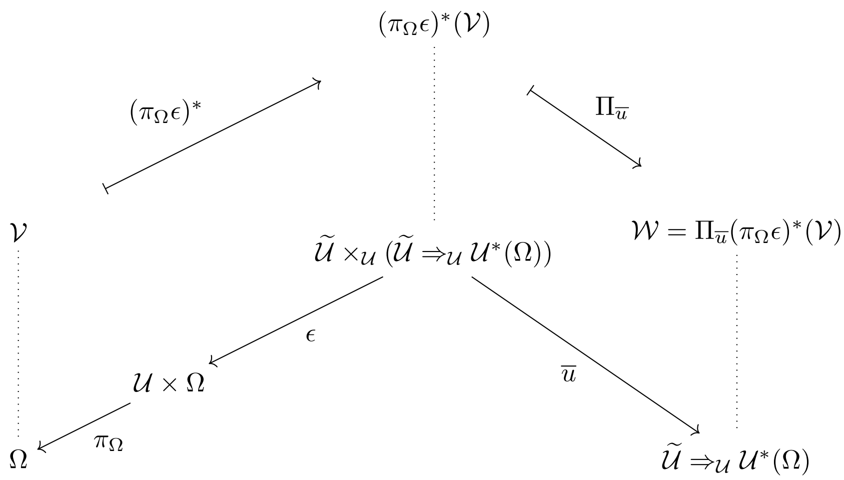
Lemma 6.6.2. *Suppose we are given a quasifibred universe $(\Omega, v : \widetilde{\mathcal{V}} \to \mathcal{V})$ in $p : \mathbb{E} \to \mathbb{B}$ and a universe $u : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $(\mathbb{B}, \mathcal{F})$. Suppose moreover that $(\mathbb{B}, \mathcal{F})$ has dependent products and that the fibration $p : \mathbb{E} \to \mathbb{B}$ has $\mathcal{F}$-products. Then we can construct a universe $w : \widetilde{\mathcal{W}} \to \mathcal{W}$ in $(\mathbb{E}, \mathcal{G})$ for which the w-small display maps are precisely the $\mathcal{G}$-maps which lie over a u-small display map and which have vertical component a v-small display map.*

Proof. The projection down to $\mathbb{B}$ of the base object of the universe is given by $p(\mathcal{W}) = (\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega))$, the fibred exponential over $\mathcal{U}$ from the fibration u to the product projection $\mathcal{U} \times \Omega \to \mathcal{U}$. Maps $I \to p(\mathcal{W})$ give rise (naturally in I)

to a display map $J \twoheadrightarrow I$ together with a map $J \to \Omega$. The component of $\mathcal{W}$ in $\mathbb{E}(p(\mathcal{W}))$ is given as follows. Writing $\overline{u}$ for the pullback of u along the structure map $\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega) \to \mathcal{U}$ as in the square

$$\begin{array}{ccc} \widetilde{\mathcal{U}} \times_{\mathcal{U}} (\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)) & \twoheadrightarrow & \widetilde{\mathcal{U}} \\ {\scriptstyle \overline{u}} \downarrow & & \downarrow {\scriptstyle u} \\ \widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega) & \twoheadrightarrow & \mathcal{U} \end{array}$$

we let $\mathcal{W}$ be the object $\Pi_{\overline{u}}\left((\pi_\Omega \epsilon)^*(\mathcal{V})\right)$, that is to say, the object given by starting with $\mathcal{V} \in \mathbb{E}(\mathcal{V})$, reindexing, and then taking a fibred product, as indicated by the diagram



where ϵ is the evaluation map for the fibred exponential.

Now given $A \in \mathbb{E}(I)$, a map $\phi : A \to \mathcal{W}$ gives, by applying p, a map

$$h : I \to \widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)$$

along which we can take a pullback of $\overline{u}$ as in the square

$$\begin{array}{ccc} J & \xrightarrow{\overline{h}} & \widetilde{\mathcal{U}} \times_{\mathcal{U}} (\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)) \\ {\scriptstyle f} \downarrow & & \downarrow {\scriptstyle \overline{u}} \\ I & \xrightarrow[h]{} & \widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega) \end{array}$$

to get a display map $f : J \twoheadrightarrow I$. We also get a map

$$J \xrightarrow{\overline{h}} \widetilde{\mathcal{U}} \times_{\mathcal{U}} (\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)) \xrightarrow{\epsilon} \mathcal{U} \times \Omega \xrightarrow{\pi_\Omega} \Omega \ .$$

Taking a cartesian lift of $f : J \twoheadrightarrow I$ gives us a cartesian display map $\widetilde{f} : B \twoheadrightarrow A$ (lying over a small display map). Our starting map $\phi : A \to \mathcal{W}$ lies over h, so it is really a map

$$A \to h^*(\mathcal{W}) \cong \Pi_f \, (\pi_\Omega \epsilon \overline{h})^*(\mathcal{V})$$

where the isomorphism comes from the Beck-Chevalley condition for fibred products. Such maps correspond naturally to maps

$$B \cong f^*(A) \to (\pi_\Omega \epsilon \overline{h})^*(\mathcal{V}).$$

But now by pulling back $(\pi_\Omega \epsilon \overline{h})^*(v)$ along this map we get a small vertical display map over B.

It is clear from the construction that all $\mathcal{G}$-maps lying over a small $\mathcal{F}$-map with vertical component a small $\mathcal{E}$-map arise in this way. We can describe the universal fibration $w : \widetilde{\mathcal{W}} \to \mathcal{W}$ explicitly: it lies over

$$\overline{u} : \widetilde{\mathcal{U}} \times_{\mathcal{U}} (\widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)) \to \widetilde{\mathcal{U}} \Rightarrow_{\mathcal{U}} \mathcal{U}^*(\Omega)$$

with codomain $\Pi_{\overline{u}} \, ((\pi_\Omega \epsilon)^*(\mathcal{V}))$ and vertical component the pullback of $v : \widetilde{\mathcal{V}} \to \mathcal{V}$ along the map

$$\overline{u}^* \, \Pi_{\overline{u}} \, ((\pi_\Omega \epsilon)^*(\mathcal{V})) \to (\pi_\Omega \epsilon)^*(\mathcal{V}) \to \mathcal{V}$$

where the first map is the counit of the fibred adjunction $\overline{u} \dashv \Pi_{\overline{u}}$. $\square$

Proposition 6.6.3. *Suppose we are given a quasifibred universe $(\Omega, v : \widetilde{\mathcal{V}} \to \mathcal{V})$ in $p : \mathbb{E} \to \mathbb{B}$ and a universe $u : \widetilde{\mathcal{U}} \to \mathcal{U}$ in $(\mathbb{B}, \mathcal{F})$. Suppose moreover that $(\mathbb{B}, \mathcal{F})$ has dependent products and that the fibration $p : \mathbb{E} \to \mathbb{B}$ has $\mathcal{F}$-products.*

(i) If $\mathcal{F}$-maps and, for each I, the $\mathcal{E}_I$-maps have dependent sums and if the classes of u-small and v-small display maps are closed under dependent sums, then the class of w-small display maps is closed under dependent sums.

(ii) If $(\mathbb{B}, \mathcal{F})$ has identity types and the fibred display map category $(p : \mathbb{E} \to \mathbb{B}, \mathcal{E})$ has fibrewise identity types, and if the classes of u-small and v-small display maps are closed under identity types, then the class of w-small display maps is closed under identity types.

(iii) If $(\mathbb{B}, \mathcal{F})$ has dependent sums and u-small display maps are closed under

dependent sums, if the $\mathcal{F}$-product of a v-small display map along a u-small display map is v-small, and if $p : \mathbb{E} \to \mathbb{B}$ admits quasifibred dependent products in such a way that for any v-small display map the induced operation on $\mathcal{E}$-maps of taking the $\mathcal{E}$-dependent product along it maps v-small display maps to w-small display maps, then the class of w-small display maps is closed under dependent products.

Proof. Each of the claims is proved easily by inspecting the constructions given above. □

Chapter 7

Dialectica models of type theory

In this chapter we apply the results of Chapter 6 to produce some specific examples of models of type theory. Our examples are *Dialectica models of type theory* in the sense that the objects of these models represent propositions of the form generated by Gödel's Dialectica interpretation or a related functional interpretation, such as the variant due to Diller and Nahm.

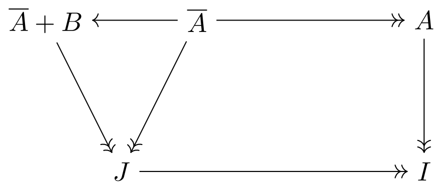
7.1 The polynomial model

The polynomial model was introduced by von Glehn in [46]. We give a full construction of it here, in order to demonstrate how it fits into the framework developed in Chapters 6, the main point being the construction of dependent products using quasifibred dependent products. We also extend the treatment in [46] to include the construction of a universe in the polynomial model.

Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category modelling dependent sums, dependent products and identity types. Moreover, we suppose that $\mathbb{B}$ has extensive finite sums (see 1.7.12). Finally, we also assume that the model admits a universe $u : \widetilde{\mathcal{U}} \to \mathcal{U}$ closed under finite sums, dependent sums, dependent products and identity types.

Definition 7.1.1. *The polynomial model of type theory over* $(\mathbb{B}, \mathcal{F})$ is the model given by the gluing construction of Definition 6.1.18 applied to the fibred display map category described as follows. We take the fibration $p : \mathsf{Poly} \to \mathbb{B}$, where p is the opposite of the codomain fibration $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$, we equip $\mathbb{B}$ with the class $\mathcal{F}$ of display maps, and for each object $I \in \mathbb{B}$ we equip the fibre category $\mathsf{Poly}(I) = (\mathcal{F}/I)^{\mathrm{op}}$ with the class $\mathcal{E}_I$ of product projections.

Indeed, $p : \mathsf{Poly} \to \mathbb{B}$ does have fibred finite products since it is the opposite of a fibration with fibred finite sums, so we do have a fibred display map category. Recall that, by 6.1.18, $(\mathsf{Poly}, \mathcal{G})$ is a display map category with $\mathcal{G}$ being the class of maps which lie over an $\mathcal{F}$-map and have vertical component an $\mathcal{E}$-map (formal dual of coproduct inclusion) as in the diagram



which depicts a display map from left to right, where the square is a pullback.

Proposition 7.1.2. *The polynomial model admits dependent sums.*

Proof. By 6.3.2, it suffices to check that the base and fibrewise models admit dependent sums. The base does by assumption, the fibrewise models do since product projections are closed under composition. □

Proposition 7.1.3. *The polynomial model admits identity types.*

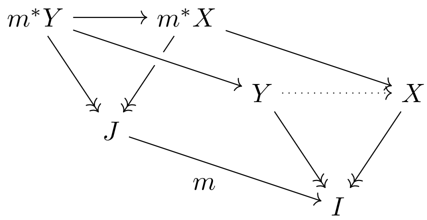
Proof. We use 6.4.9. The base was assumed to have identity types. By 1.8.4, any category with finite products considered as a display map category has identity types where the acyclic cofibrations are precisely the split monomorphisms. Split monomorphisms are clearly preserved by reindexing, so it remains to check the acyclic cofibration condition.

Let $m : J \to I$ be an acyclic cofibration in $\mathbb{B}$ and let $x : X \twoheadrightarrow I$ and $y : Y \twoheadrightarrow I$ be two display maps over I, so that X represents a general object of $\mathsf{Poly}(I)$, and the coproduct inclusion

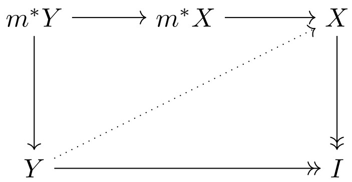
$$i_X : X \hookrightarrow X +_I Y$$

represents the general form of a display map in $\mathsf{Poly}(I)$. We must check that every retraction of $m^*(i_X)$ is the pullback along m of some retraction of i_X. Since m^* preserves fibred coproducts, this amounts to showing that every map $m^*Y \to m^*X$ over J arises from a map $Y \to X$ over I, i.e. that m^* is full.

Suppose we have a map $m^*Y \to m^*X$ over J. Then since the acyclic cofibrations in $\mathbb{B}$ are stable under pullback along display maps, the map $m^*Y \to Y$ in the diagram



is an acyclic cofibration. We may use its left lifting property with respect to $X \twoheadrightarrow I$ to find a dotted arrow making the diagram



commute. A simple application of the pullback lemma shows that this arrow is indeed pulled back to the arrow $m^*Y \to m^*X$. □

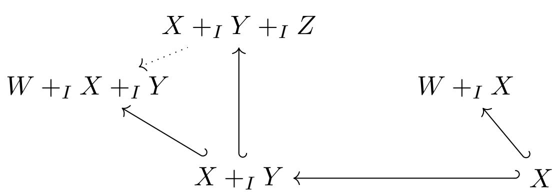
Lemma 7.1.4. *The fibration p : $\mathsf{Poly} \to \mathbb{B}$ has $\mathcal{F}$-products which preserve display maps.*

Proof. The codomain fibration cod : $\mathcal{F} \to \mathbb{B}$ has $\mathcal{F}$-sums given by dependent sum, i.e. composition. Hence its opposite, $p : \mathsf{Poly} \to \mathbb{B}$, has $\mathcal{F}$-products. As they are right adjoints, the $\mathcal{F}$-products preserve the fibrewise product projections. □

Proposition 7.1.5. *The polynomial model admits dependent products.*

Proof. We apply 6.5.3. As the fibration $p : \mathsf{Poly} \to \mathbb{B}$ has $\mathcal{F}$-products by 7.1.4, It remains to check that p has quasifibred dependent products.

Let $I \in \mathbb{B}$ and let $X \twoheadrightarrow I$, $Y \twoheadrightarrow I$ and $Z \twoheadrightarrow I$ be display maps over I. Then a quasifibred product of $X +_I Y +_I Z \hookleftarrow X +_I Y$ along $X +_I Y \hookleftarrow X$ needs to classify, for any $W \twoheadrightarrow I$, the set of maps $Z \to W +_I X +_I Y$ over I.



Let $q : Q \twoheadrightarrow I$ be $Z \Rightarrow_I (Y +_I I) \twoheadrightarrow I$, (where I, or really the identity on I, is used as the unit type in context I). Now let $R \twoheadrightarrow Q$ be $\Sigma_{\pi_Q} \epsilon^*(i_I)$, where i_I is the coproduct inclusion $I \hookrightarrow Y +_I I$, ϵ is the evaluation morphism

$$Z \times_I (Z \Rightarrow_I (Y +_I I)) \to Y +_I I$$

and π_Q is the product projection

$$Z \times_I (Z \Rightarrow_I (Y +_I I)) \to Z \Rightarrow_I (Y +_I I).$$

Then a quasifibred dependent product is given by $q : Q \twoheadrightarrow I$ together with $R +_Q q^*X \hookleftarrow q^*X$. Intuitively, this is because a map $Z \to W +_I X +_I Y$ over I corresponds to a map $Z \to Y +_I I$ over I (section s of q) together with a partially defined function $Z \to W +_I X$ over I whose domain is the preimage of I in the first function (i.e. $s^*(R)$). One may use Lemma 1.7.16 to make this proof precise. □

Now we consider universes in Poly, which were not considered in [46].

Proposition 7.1.6. *The polynomial model admits a universe closed under dependent sums, dependent products and identity types, whose small display maps are precisely those lying over a small display map and with vertical component represented by the inclusion of an object into a binary sum with a small object.*

Proof. We use 6.6.2 to construct the universe itself. This requires us to give a quasifibred universe. Let $\Omega = \mathcal{U}$ and let $v : \widetilde{\mathcal{V}} \to \mathcal{V}$ in $\mathsf{Poly}(\Omega)$ be represented by $0 + \widetilde{\mathcal{U}} \hookleftarrow 0$. It is easy to check that this is indeed quasifibred universe with the correct class of small maps.

To verify its closure under type constructors, we use 6.6.3. By assumption, small display maps in the base are closed under dependent sums, dependent products and identity types. Since the universe is closed under finite sum types, the small fibrewise display maps are closed under dependent sum. The fibrewise identity types are just given by isomorphisms, so the fibrewise display maps are trivially closed under identity types. The fibred $\mathcal{F}$-products are given by dependent sums, which preserve coproducts and smallness, so $\mathcal{F}$-product in $p : \mathsf{Poly} \to \mathbb{B}$ preserve small display maps. Finally, we must check that the quasifibred dependent product of two small fibrewise display maps is small, but this is clear from the construction given in 7.1.5. □

We finish our investigation of the polynomial model by considering the finite sum types.

Proposition 7.1.7. *The polynomial model has extensive finite sums.*

Proof. We use Proposition 1.7.14. The initial object of Poly is the identity $1_0 : 0 \to 0$. It is easy to check that this is indeed a strict initial object. Let

$f : A \twoheadrightarrow I$ and $g : B \twoheadrightarrow J$ be two objects of Poly. Then a coproduct of f and g is given by the sum

$$f + g : A + B \twoheadrightarrow I + J$$

which is indeed a display map since $(\mathbb{B}, \mathcal{F})$ has extensive finite sums. The first coproduct inclusion is

$$\begin{array}{ccc} A & \twoheadrightarrow & A + B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+g} \\ I & \twoheadrightarrow & I + J \end{array}$$

which is a pullback square (since $\mathbb{B}$ is extensive) and hence a display map $f \to f + g$ in Poly. The construction does indeed give us a coproduct, since for $h : C \twoheadrightarrow K$, maps $f + g \to k$ correspond, by extensivity, to maps $I + J \to K$ together with maps $C_I \to A$ over I and $C_J \to B$ over J, where C_I is the pullback of C along $I \to I + J \to K$, and similarly for C_J, as required.

We check that copairing preserves display maps. Suppose that we have two display maps

$$\begin{array}{ccccc} A + X_I & \twoheadleftarrow\!\!\hookleftarrow & X_I & \twoheadrightarrow & X \\ & \searrow & \swarrow & & \downarrow \\ & & I & \twoheadrightarrow & K \end{array}$$

and

$$\begin{array}{ccccc} B + X_J & \twoheadleftarrow\!\!\hookleftarrow & X_J & \twoheadrightarrow & X \\ & \searrow & \swarrow & & \downarrow \\ & & J & \twoheadrightarrow & K \end{array}$$

in Poly. Then their copairing is displayed as

$$\begin{array}{ccccc} (A + B) + (X_I + X_J) & \twoheadleftarrow\!\!\hookleftarrow & X_I + X_J & \twoheadrightarrow & X \\ & \searrow & \swarrow & & \downarrow \\ & & I + J & \twoheadrightarrow & K \end{array}$$

which is indeed in the form of a display map in Poly.

Finally, we check extensivity of Poly. Suppose we have $\mathcal{F}$-maps $f : A \twoheadrightarrow I$,

$g : B \twoheadrightarrow J$, $w : W \twoheadrightarrow K$, $x : X \twoheadrightarrow L$, and $y : Y \twoheadrightarrow M$ considered as objects in Poly and a diagram

$$\begin{array}{ccccc}
(x) & \longrightarrow & (w) & \longleftarrow & (y) \\
\downarrow & & \downarrow & & \downarrow \\
(f) & \twoheadrightarrow & (f+g) & \twoheadleftarrow & (g)
\end{array}$$

in which the bottom row is a coproduct diagram in Poly. It is straightforward, but notationally awkward, exercise to check that in such diagram both squares are pullbacks if and only if the top row is a coproduct diagram. □

Remark 7.1.8. Since the universe in $(\mathbb{B}, \mathcal{F})$ is closed under finite sums, it is easy to see that the small display maps in Poly are closed under addition.

7.2 The Dialectica model

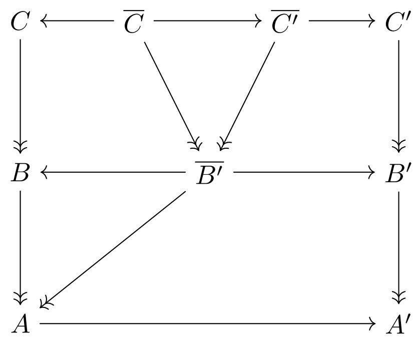
With the techniques developed heretofore, we can extend the polynomial model of [46] to one that more closely matches the Dialectica categories introduced by de Paiva in [11] by including the fibration of predicates. In [11], the fibration of predicates was taken to be the subobject fibration of the base category — here we shall allow the fibration of display maps to play an analogous role.

We assume that $(\mathbb{B}, \mathcal{F})$ is a well-rooted display map category modelling dependent sums, dependent products, identity types and extensive finite sums.

Definition 7.2.1. The *Dialectica fibration* $p : \mathsf{Dial} \to \mathbb{B}$ has as total category the opposite of the fibred opposite fibration of the fibred fibration of second-level types. In other words, an object of Dial is a composable pair of display maps

$$C \xrightarrow{g} B \xrightarrow{f} A$$

and a map $(g, f) \to (g', f')$ is a diagram of the form



where all three interior quadrilaterals are pullback squares. The functor p sends the object displayed above to its codomain A.

The Dialectica fibration has finite products. The terminal object in $\mathsf{Dial}(I)$ is the composable pair

$$0 \twoheadrightarrow I \twoheadrightarrow I$$

and the product of

$$X \xrightarrow{f_1} A \xrightarrow{f_0} I$$

with

$$Y \xrightarrow{g_1} B \xrightarrow{g_0} I$$

is

$$X + Y \xrightarrow{f_1 + g_1} A + B \xrightarrow{[f_0, g_0]} I.$$

One uses the extensive finite sums in $(\mathbb{B}, \mathcal{F})$ to check the universal property. Hence the Dialectica fibration is a fibred display map category in which the fibre categories have product projections as display maps. In particular, we get a well-rooted display map category $(\mathsf{Dial}, \mathcal{G})$.

As in §7.1, we use the results of Chapter 6 to derive the type constructors in the Dialectica model.

Proposition 7.2.2. *The Dialectica model admits dependent sums.*

Proof. Almost identical to the proof of 7.1.2. □

Proposition 7.2.3. *The Dialectica model admits identity types.*

Proof. This is similar to the proof of 7.1.3. It is straightforward to adapt the verification of the acyclic cofibration condition, the rest is identical. □

Proposition 7.2.4. *The Dialectica fibration has $\mathcal{F}$-products which preserve display maps.*

Proof. As in the proof of 7.1.4, the products are given by composition: the dependent sum of

$$B \xrightarrow{f} A \xrightarrow{g} I$$

along $h : I \twoheadrightarrow J$ is

$$B \xrightarrow{f} A \xrightarrow{hg} J.$$

It is easy to check that this preserves the fibred products. □

Proposition 7.2.5. *The Dialectica model admits dependent products.*

Proof. As in 7.1.5 we just need to check that $p : \mathsf{Dial} \to \mathbb{B}$ has quasifibred dependent products. Extending the notation of 7.1.5, suppose we have $A \twoheadrightarrow X$, $B \twoheadrightarrow Y$, and $C \twoheadrightarrow Z$, display maps over the display maps $X \twoheadrightarrow I$, $Y \twoheadrightarrow I$, and $Z \twoheadrightarrow I$.

Then the quasifibred dependent product of $(A + B + C \to X + Y + Z) \to (A + B \to X + Y)$ along $(A + B \to X + Y) \to (A \to X)$ is given by the object Q represented in type theory by

$$Z \Rightarrow Y + 1\Sigma(f : Z \Rightarrow Y + 1)\Pi(z : Z)\Pi(y : \mathbb{1}_Y(f(z)))B(y) \Rightarrow C(z)$$

together with the display map $R \twoheadrightarrow Q$ over it represented in type theory by

$$< f, \_ >: Q \vdash \Sigma(z : Z)\mathbb{1}_1(f(z))$$

together with the display map over that represented by the type

$$q : Q, < z, \_ >: R(q) \vdash C(z).$$

Here we using the notation

$$z : X + Y \vdash \mathbb{1}_X(z)$$

for the coproduct inclusion $X \hookrightarrow X+Y$ and the underscore for unused variables. We omit the verification that this satisfies the definition of quasifibred dependent product. □

7.3 The Diller-Nahm model

We will adapt the construction of the polynomial model to give a model which corresponds to the Diller-Nahm variant of Gödel's Dialectica interpretation. This model is to the Diller-Nahm category of Chapter 4 (and the category $\mathbf{DC}_!$ of [11]) as the polynomial model is to the Dialectica category $\mathbf{DC}$ of [11].

Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category modelling dependent sums, dependent products and identity types. We need to assume that it models strong finite sum types, but we do not need extensive finite sums as we did for the polynomial model. We will also assume that the model admits a universe $u : \widetilde{\mathcal{U}} \to \mathcal{U}$ closed under finite sums, dependent sums, dependent products and identity types.

We also need structure on $(\mathbb{B}, \mathcal{F})$ corresponding the monad M from Chapter 4. In terms of the syntax of type theory, we want a type-forming rule

$$\frac{\Gamma \vdash A \ \ \text{type}}{\Gamma \vdash A^\bullet \ \ \text{type}}((-)^\bullet\text{–}\mathsf{form})$$

together with introduction rule,

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \{a\} : A^\bullet}((-)^\bullet\text{–}\mathsf{int}_\eta)$$

a sort of 'elimination rule',

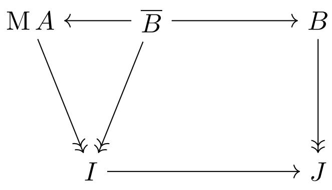
$$\frac{\Gamma, x : A \vdash t : B^\bullet}{\Gamma, s : A^\bullet \vdash \bigcup_{x \in s} t : B^\bullet}$$

and some other rules which allow us to deduce that $(-)^\bullet$ is a monad on the category of types in context Γ. We model this not merely by a monad (M, η, μ) on the category $\mathbb{B}$, but by a fibred monad M on the fibration of types $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$ — by a fibred monad we mean one whose functor part is a cartesian functor and whose unit and multiplication are vertical. For each $I \in \mathbb{B}$, the monad M satisfies $\mathrm{M}\,0_I \cong \top_I$, and also for any pair of display maps $X \twoheadrightarrow I$ and $Y \twoheadrightarrow I$ the canonical map $\mathrm{M}(X +_I Y) \to \mathrm{M}\,X \times_I \mathrm{M}\,Y$ is an isomorphism.

Definition 7.3.1. *The Diller-Nahm model of type theory fibred over* $(\mathbb{B}, \mathcal{F})$ is the fibration $p : \mathsf{Dill}(\mathcal{F}, \mathrm{M}) \to \mathbb{B}$ where p is the opposite of the fibred Kleisli category of M. For each object $I \in \mathbb{B}$ the class $\mathcal{E}_I$ in $\mathsf{Dill}(I) = \mathsf{Dill}(\mathcal{F}, \mathrm{M})(I)$ is the class of product projections.

Note that p does indeed have fibred finite products since $(\mathcal{F})_\mathrm{M}$ has fibred finite sums, since it is the fibrewise Kleisli category of $\mathcal{F}$ which was assumed to have fibred finite sums. Moreover, it follows from 3.3.3 that the fibres of p have finite biproducts (which are clearly stable under reindexing).

Let us make explicit what the maps and display maps are in $\mathsf{Dill}(\mathbb{B}, \mathcal{F})$. A map $(I, f : A \twoheadrightarrow I) \to (J, g : B \twoheadrightarrow J)$ is given by a diagram



or rather an equivalence class of such diagrams. A display map is one for which $A \cong \overline{B} +_J C$ for some $h : C \twoheadrightarrow I$ and the map $\overline{B} \to \mathrm{M}(\overline{B} +_J C)$ is a coproduct inclusion $\overline{B} \hookrightarrow \overline{B} +_J C$ followed by the unit of the monad.

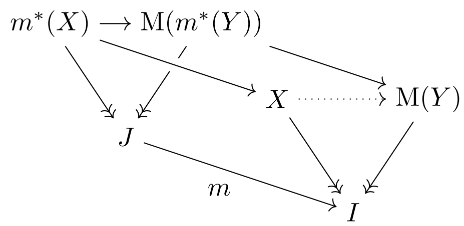
Lemma 7.3.2. *The Diller-Nahm model has* Σ*-types.*

Proof. This is the same as 7.1.2. ☐

Lemma 7.3.3. *The Diller-Nahm model has* Id*-types.*

Proof. As in 7.1.3, $\mathcal{F}$ was assumed to model Id-types, and the fibres of p to have product projections for display maps, and hence Id-types. Thus it remains to verify the acyclic cofibration condition.

Let $X \twoheadrightarrow I$, $Y \twoheadrightarrow I$ be two display maps, and suppose that $m : J \to I$ is an acyclic cofibration in $(\mathbb{B}, \mathcal{F})$. Then we must check that any section of $m^*(X) \oplus_J m^*(Y) \cong m^*(X \oplus_I Y) \to m^*(Y)$ in $(\mathcal{F})^{\mathrm{op}}_{\mathrm{M}}(J)$ is the pullback along m of some section of $X \oplus_I Y \to Y$ in $(\mathcal{F})^{\mathrm{op}}_{\mathrm{M}}(J)$. Unpacking the definition, this means that any map $m^*(X) \to \mathrm{M}\, m^*(Y) \cong m^*(\mathrm{M}\, Y)$ over J must arise as a pullback along m of some map $X \to \mathrm{M}\, Y$ over I.



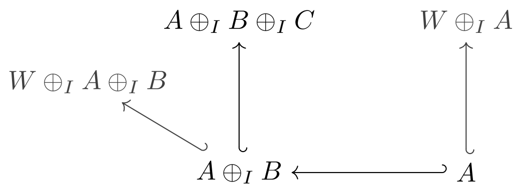
Since acyclic cofibrations in $\mathcal{F}$ are stable under pullback, it follows that $m^*(X) \to X$ is an acyclic cofibration. Hence it has the left-lifting property with respect to the display map $\mathrm{M}(Y) \to I$, and so we get a map $X \to \mathrm{M}(Y)$. Indeed, the original map $m^*(X) \to \mathrm{M}(m^*(Y))$ is a reindexing of this map, by the pullback lemma. ☐

Lemma 7.3.4. *The fibration* $p : \mathsf{Dill} \to \mathbb{B}$ *has* $\mathcal{F}$*-products which preserve display maps.*

Proof. As in 7.1.4, the codomain fibration cod : $\mathcal{F} \to \mathbb{B}$ has $\mathcal{F}$-sums given by dependent sum. Now it is easy to check that $\mathcal{F}$-sums in any fibration induce $\mathcal{F}$-sums in the Kleisli fibration. Hence we get $\mathcal{F}$-products in the opposite, which is $\mathsf{Dill} \to \mathbb{B}$. As they are right adjoints, the $\mathcal{F}$-products preserve the fibrewise product projections. □

Lemma 7.3.5. *The Diller-Nahm model has* Π*-types.*

Proof. By 6.5.3, it suffices to verify that the Diller-Nahm fibred model of type theory has quasifibred Π-types. Let $I \in \mathbb{B}$ and suppose that $A \twoheadrightarrow I$, $B \twoheadrightarrow I$ and $C \twoheadrightarrow$ are three display maps over I, so that we have the following diagram of display maps in the fibre $\mathsf{Dill}(\mathcal{F}, \mathrm{M})(I)$,



where the display maps have been drawn as the underlying coproduct inclusion in the opposite direction. For any other type over A, i.e. coproduct inclusion $A \hookrightarrow W \oplus_I A$, its weakening with respect to $A \oplus_I B$ is the pushout $A \oplus_I B \hookrightarrow W \oplus_I A \oplus_I B$. Now maps

$$W \oplus_I A \oplus_I B \to A \oplus_I B \oplus_I C$$

in the slice $\mathsf{Dill}(\mathcal{F}, \mathrm{M})(I)/A \oplus_I B$ are just given by maps

$$C \to \mathrm{M}(W +_I A +_I B)$$

in the slice $\mathbb{B}/I$. But the fact that $W +_I A +_I B$ is a biproduct of $W +_I A$ and B in $\mathcal{F}_{\mathrm{M}}(I)$ means that these maps correspond naturally to pairs of maps

$$C \to \mathrm{M}(W +_I A), \quad C \to \mathrm{M}\, B$$

in $\mathcal{F}/I$.

Hence we can define the quasifibred Π-type to have component in the base the display map

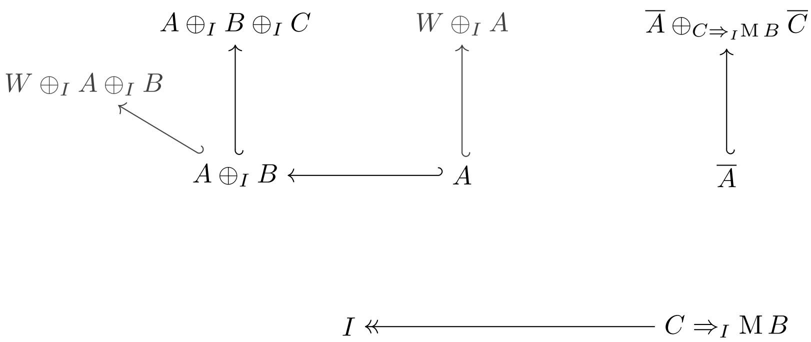
$$I \twoheadleftarrow (C \Rightarrow_I \mathrm{M}\, B)$$

(the fibred exponential in $\mathcal{F}/I$) and over the weakening $\overline{A}$ of A to $(C \Rightarrow_I \mathrm{M}\, B)$

we take the display map given by

$$\overline{A} \hookrightarrow \overline{A} \oplus_{C \Rightarrow_I \mathrm{M}\, B} \overline{C}$$

where $\overline{C}$ is the weakening of C to context $(C \Rightarrow_I \mathrm{M}\, B)$.



Since the reindexing of $\overline{A} \hookrightarrow \overline{A} \oplus_{C \Rightarrow_I \mathrm{M}\, B} \overline{C}$ along any section of $(C \Rightarrow_I \mathrm{M}\, B) \twoheadrightarrow I$ is $A \hookrightarrow A \oplus_I C$, it is easy to verify that this does indeed have the required property of a quasifibred exponential. Moreover, it is easy to verify stability under reindexing. $\square$

Proposition 7.3.6. *The Diller-Nahm model admits a universe closed under dependent sums, dependent products and identity types, whose small display maps are precisely those lying over a small display map and with vertical component represented by the inclusion of an object into a binary sum with a small object.*

Proof. As in 7.1.6, we can use 6.6.2 to construct the universe from a quasifibred universe. Let $\Omega = \mathcal{U}$ and let $v : \widetilde{\mathcal{V}} \to \mathcal{V}$ in $\mathsf{Dill}(\Omega)$ be represented by $\mathrm{M}(0 + \widetilde{\mathcal{U}}) \hookleftarrow 0$, the formal dual of a coproduct inclusion. The details are similar to 7.1.6. $\square$

7.4 Diller-Nahm with predicates

In this section we informally sketch out a 'Diller-Nahm model with predicates'. This adds back in the layer of predicates considered for the Diller-Nahm category in Chapter 4, now in our dependently-typed setting. The relationship to the model of §7.3 is more general than between the models of §7.1 and §7.2: in §7.2 we considered only the special case where the predicates are given by display maps. We do not set out the 'polynomial model with predicates', but the main ideas necessary should be covered in this section.

We assume that we have a well-rooted display map category $(\mathbb{B}, \mathcal{F})$ modelling Σ, Π, Id, and strong finite sum types, and that we are given a fibred monad

(M, η, μ) on the fibration $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$ which is an additive monad in each fibre. However, now we will also assume that for each $I \in \mathbb{B}$, the monad M_I comes with a display map

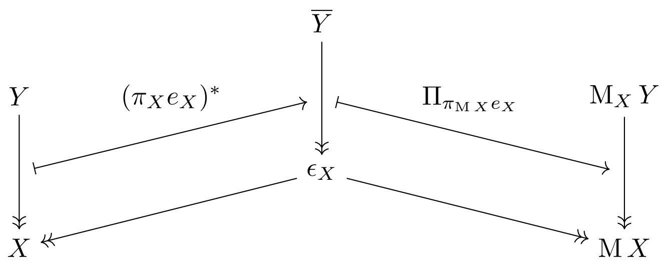
$$\begin{array}{c} \epsilon_X \\ \downarrow e_X \\ X \times_I \mathrm{M}\, X \end{array}$$

for each display map $X \twoheadrightarrow I$, and these display maps are stable (up to isomorphism) under reindexing along maps into I. As a type, ϵ_X should be thought of as $\coprod_{S \in \mathrm{M}\, X} S$, or the 'membership relation' between X and $\mathrm{M}\, X$ where an element of X can be a member of an element of $\mathrm{M}\, X$ in multiple ways. Thus we will assume we are also given dotted maps fitting in the following diagram making both squares pullbacks, with both families of maps stable under reindexing.

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & \epsilon_{I,X} \\ {\scriptstyle (1_X, 1_X)}\downarrow & & \downarrow \\ X \times X & \xrightarrow[1_X \times_I \eta_X]{} & X \times_I \mathrm{M}\, X \end{array} \qquad \begin{array}{ccc} \Sigma_{\mathrm{M}\, X}(\overline{\epsilon_X} \times \overline{\epsilon_{\mathrm{M}\, X}}) & \xrightarrow{\tau_X} & \epsilon_X \\ \downarrow & & \downarrow \\ X \times_I \mathrm{M}\,\mathrm{M}\, X & \xrightarrow[1_X \times_I \mu_X]{} & X \times_I \mathrm{M}\, X \end{array}$$

Intuitively, these express the statements '$\eta(x) = \{x\}$' and '$\mu(F) = \bigcup F$'.

We can use this to extend the fibred monad to a system of predicates above $\mathcal{F}$. The natural case to consider is the one where the predicates are just display maps. That is to say, we take the fibration $\mathcal{F}_2 \to \mathcal{F}$ given by change-of-base of $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$ along $\mathrm{dom} : \mathcal{F} \to \mathbb{B}$. We can extend M to a fibred monad on $\mathcal{F}_2 \to \mathcal{F}$ using the membership predicates. As a functor, this is given by pulling back along $\epsilon_X \to X$ and taking the dependent product along $\epsilon_X \to \mathrm{M}\, X$.



More generally, we could consider a fibred cartesian closed category $p : \mathcal{P} \to \mathbb{B}$ with $\mathcal{F}$-products and $\mathcal{F}$-sums. Then our 'membership predicates' are given by a choice, for each $I \in \mathbb{B}$ and $X \twoheadrightarrow I$, of $\epsilon_{I,X} \in \mathcal{P}(X \times_I \mathrm{M}\, X)$, which is stable under reindexing in I. Then the additional conditions correspond to isomorphisms

$$(a : X) \quad \epsilon_{I,X}(a, \eta_X(a)) \cong \top$$

in $\mathcal{P}(X)$ and

$$(a : X, z : \mathrm{M}\,\mathrm{M}\,X) \quad \epsilon(x, \mu_X(z)) \cong \Sigma_{b:\mathrm{M}(X)}\epsilon_{I,X}(a,b) \times \epsilon_{I,\mathrm{M}(X)}(b,c)$$

in $\mathcal{P}(X \times_I \mathrm{M}\,\mathrm{M}\,X)$, with both isomorphisms stable under reindexing in I. We lift M to the change of base of $\mathcal{P}$ along $\mathrm{dom} : \mathcal{F}_2 \to \mathcal{F}$ by sending $\phi \in \mathcal{P}(X \twoheadrightarrow I)$ to

$$(b : \mathrm{M}_I\,X) \quad \Pi_{a:X}\epsilon(a,b) \Rightarrow \phi(a),$$

which we denote by $\mathrm{M}_{X\twoheadrightarrow I}\,\phi(b)$. From the isomorphisms above we easily construct, for any $\phi \in \mathcal{P}(X)$, isomorphisms

$$(a : X) \quad \mathrm{M}_{X\twoheadrightarrow I}\,\phi(\eta(a)) \cong \phi(a)$$

and

$$\begin{aligned}(c : \mathrm{M}\,\mathrm{M}\,X) \quad \mathrm{M}_{X\twoheadrightarrow I}\,\phi(\mu(c)) &\cong \Pi_{a:X}\Pi_{b:\mathrm{M}\,X}\epsilon(a,b) \times \epsilon(b,c) \Rightarrow \phi(a)\\ &\cong \mathrm{M}_{\mathrm{M}\,X\twoheadrightarrow I}\,\mathrm{M}_{X\twoheadrightarrow I}\,\phi(c)\end{aligned}$$

both reindexing stable in I and natural in ϕ. These tell us that we can define a unit and multiplication for the lifted monad which are cartesian above the original unit and multiplication.

This data allows us to define a version of a fibred model of type theory which combines the Dialectica fibration of 7.2.1 with the first and second Kleisli fibration constructions of 4.2.4.

7.5 The error Dialectica model

The final example will be a dependently-typed version of the category Dial^+ considered in [4]. We might call this the 'error Dialectica' model, since it is to the error monad $(-) + \top$ in a category with binary coproducts and a terminal object what the Diller-Nahm model is to the finite multisets monad. We will be able to construct dependent sums and identity types in a manner which is by now routine. However, the error Dialectica model only has *weak* dependent products, just as the analogous category construction did in [4]. Hence, we conclude our treatment of this example by checking that the results of Chapter 2 apply. For this last part, we will have to restrict our work to the full split comprehension category case, which will involve providing an ad hoc construction of the identity types, since we did not provide a construction of strictly stable identity types in Chapter 6.

Let $(\mathbb{B}, \mathcal{F})$ be a well-rooted display map category modelling dependent sums, dependent products and identity types. Moreover, we suppose that $\mathbb{B}$ has strong finite sum types.

We can use the strong finite sum types and the unit type to get a fibred monad $(-) + 1$ on $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$. This monad is fibred in the same sense as the monad M in section 7.3. For $I \in \mathbb{B}$, the monad on $\mathcal{F}/I$ is the one with functor part given by taking $X \twoheadrightarrow I$ to $X +_I 1_I \twoheadrightarrow I$, where 1_I here is the identity on I, modelling the unit type in context I.

Definition 7.5.1. *The error Dialectica model of type theory fibred over* $(\mathbb{B}, \mathcal{F})$ is the fibration $p : \mathsf{Dial}^+ \to \mathbb{B}$ where Dial^+ is the opposite of the Kleisli fibration $\mathcal{F}_+$ for the monad $(-) + 1$ on $\mathcal{F}$. For an object $I \in \mathbb{B}$, the class $\mathcal{E}$ is the class of product projections in Dial^+/I.

Indeed, $p : \mathsf{Dial}^+ \to \mathbb{B}$ does have fibred finite products, since it is the opposite of the Kleisli fibration $\mathcal{F}_+ \to \mathbb{B}$ which has fibred finite sums, so we do have a fibred display map category. Hence, by 6.1.18, $(\mathsf{Dial}^+, \mathcal{G})$ is a display map category with $\mathcal{G}$ being the class of maps which lie over an $\mathcal{F}$-maps and have vertical component an $\mathcal{E}$-map.

Proposition 7.5.2. *The error Dialectica model admits dependent sums.*

Proof. This is the same argument as 7.1.2. □

Proposition 7.5.3. *The error Dialectica model admits identity types.*

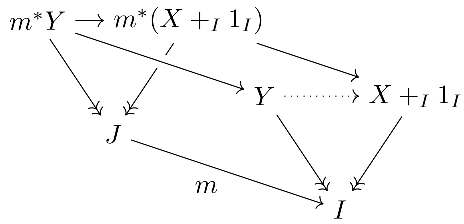
Proof. This is very similar to 7.1.3. The only slight difference is in checking the acyclic cofibration condition.

Let $m : J \to I$ be an acyclic cofibration in $\mathbb{B}$ and let $x : X \twoheadrightarrow I$ and $y : Y \twoheadrightarrow I$ be two display maps over I, so that X represents a general object of $\mathsf{Dial}^+(I)$, and the map

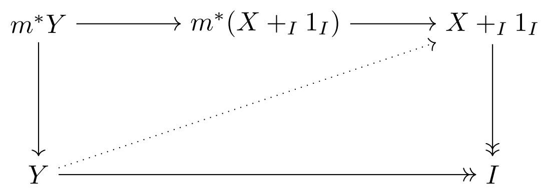
$$X \xrightarrow{i_X} X +_I Y \xrightarrow{\eta_{X+_I Y}} X +_I Y +_I 1_I$$

given by a coproduct inclusion followed by the unit of the monad (which, incidentally, is another coproduct inclusion) represents the general form of a display map in $\mathsf{Dial}^+(I)$. We must check that every retraction of $m^*(i_X)$ is the pullback along m of some retraction of i_X, where the retractions are retractions as Kleisli arrows. Since m^* preserves fibred coproducts, this amounts to showing that every map $m^*Y \to m^*X +_J 1_J \cong m^*(X +_I 1_I)$ over J arises from a map $Y \to X +_I 1_I$ over I, i.e. that m^* is full.

Suppose we have a map $m^*Y \to m^*(X +_I 1_I)$ over J. Then since the acyclic cofibrations in $\mathbb{B}$ are stable, the map $m^*Y \to Y$ in the diagram



is an acyclic cofibration. We may use its left lifting property with respect to $X \twoheadrightarrow I$ to find a dotted arrow making the diagram



commute. A simple application of the pullback lemma shows that this arrow is indeed pulled back to the arrow $m^*Y \to m^*(X +_I 1_I)$. $\square$

Lemma 7.5.4. *The fibration* $p : \mathsf{Dial}^+ \to \mathbb{B}$ *has* $\mathcal{F}$*-products which preserve display maps.*

Proof. As in 7.1.4, the codomain fibration $\mathrm{cod} : \mathcal{F} \to \mathbb{B}$ has $\mathcal{F}$-sums given by dependent sum. Now it is easy to check that $\mathcal{F}$-sums in any fibration induce $\mathcal{F}$-sums in the Kleisli fibration. Hence we get $\mathcal{F}$-products in the opposite, which is $\mathsf{Dial}^+ \to \mathbb{B}$. As they are right adjoints, the $\mathcal{F}$-products preserve the fibrewise product projections. $\square$

Proposition 7.5.5. *The display map category* $(\mathsf{Dial}^+, \mathcal{G})$ *has* $\overline{\mathcal{F}}$*-dependent products of* $\mathcal{G}$*-maps, and* $\mathcal{E}$*-products of* $\overline{\mathcal{F}}$*-maps.*

Proof. For the first statement we use 6.5.5 which requires the existence of fibred products which preserve the fibrewise display maps. This was checked in 7.5.4. The second statement is just 6.5.6. $\square$

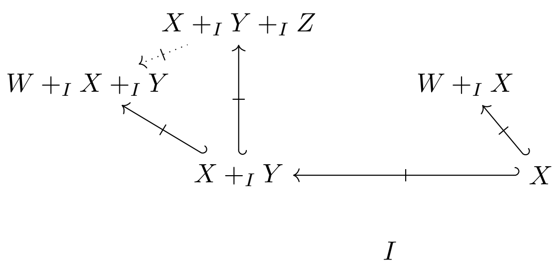
Proposition 6.5.7 tells us that we would have all dependent products if we could find $\mathcal{E}$-dependent product of $\mathcal{E}$-maps in Dial^+. However, this is not to be expected. Instead, we can construct *weak* dependent products of $\mathcal{E}$-maps along $\mathcal{E}$-maps.

To see this, we will show that it admits weak dependent products of $\mathcal{E}$-maps along $\mathcal{E}$-maps. This suffices since an ordinary dependent product is a weak dependent product, and it is easy to see that an operation taking weak dependent products along a composite can be given by giving weak dependent products along each factor.

Lemma 7.5.6. *The error Dialectica model has $\mathcal{E}$-dependent products of $\mathcal{E}$-maps.*

Proof. The proof will go by showing that $p : \mathsf{Dial}^+ \to \mathbb{B}$ has 'weak quasifibred dependent products', which is the obvious weak analogue of the ordinary quasi-fibred dependent products of Definition 6.5.9. It can be found by unfolding the evident definition of weak $\mathcal{E}$-dependent products of $\mathcal{E}$-maps.

Let $I \in \mathbb{B}$ and let $X \twoheadrightarrow I$, $Y \twoheadrightarrow I$. Then a weak quasifibred dependent product of $X +_I Y +_I Z +_I 1_I \hookleftarrow X +_I Y$ along $X +_I Y +_I 1_I \hookleftarrow X$ needs to weakly classify, for any $W \twoheadrightarrow I$, the set of maps $Z \to W +_I X +_I Y +_I 1_I$ over I. Intuitively, a partial map $Z \to W + X + Y$ can be broken down into a partial map $Z \to W + X$ and a partial map $Z \to Y$ (with disjoint domains).



Let $q : Q \twoheadrightarrow I$ be $Z \Rightarrow_I (Y +_I 1_I) \twoheadrightarrow I$. Now let $R \twoheadrightarrow Q$ be the pullback of the coproduct inclusion $W +_I X +_I 1_I \hookleftarrow X$ along q. Hence for any section $s : I \to Q$ of q, the pullback $s^*(R \twoheadrightarrow Q)$ is just $W +_I X +_I 1_I \hookleftarrow X$.

We must show that there is a natural, reindexing-stable retraction of the set of maps

$$f : Z \to W +_I X +_I Y +_I 1_I$$

over I inside the set of pairs

$$g : Z \to Y +_I 1_I$$

over I together with

$$h : Z \to W +_I X +_I 1_I$$

over I. Given such a map f, we construct g as

$$Z \xrightarrow{f} W +_I X +_I Y +_I 1_I \cong Y +_I W +_I X +_I 1_I \xrightarrow{1_Y + !_{W +_I X +_I 1_I}} Y +_I 1_I$$

and we construct h as

$$Z \xrightarrow{f} W +_I X +_I Y +_I 1_I \xrightarrow{1_{W+_I X}+!_{Y+_I 1_I}} W +_I X.$$

Conversely, given such a pair (g, h), we construct f as

$$Z \xrightarrow{(1_Z,g)} Z\times(Y+_I 1_I) \cong Z\times_I Y+_I Z \xrightarrow{\pi_Y+_I h} Y+_I(W+_I X+_I 1_I) \cong W+_I X+_I Y+_I 1_I$$

It it is easy to check that this retraction is natural and reindexing-stable. □

Corollary 7.5.7. *The error Dialectica model has weak dependent products.*

We would now like to apply the results of Chapter 2 to obtain a model of type theory in the Karoubi envelope of Dial^+. We must face up to the issue that our theory of gluing models takes place in the 'weak' world of display map categories, whereas our development of the idempotent completion models takes place in the 'strict' world of split comprehension categories.

One solution is to employ a coherence theorem such as that in [32] to obtain a split model with strictly stable type constructors from our weak model. The issue with this approach is that to apply the coherence theorem in [32] we would require their condition '(LF)' to hold in the gluing model. This condition is closely related to having dependent products, this it is unlikely to hold in the error Dialectica model, which only has weak dependent products. We are either faced with the problem of generalizing [32] or returning to the problem of the Karoubi envelope in the non-split situation.

The alternative is to redevelop the theory of gluing models in terms of split comprehension categories from the beginning. This is analogous to the approach taken in [38] for inverse limits of (split) type-theoretic fibration categories. One advantage of this approach is that it gives us more detailed information about the split model at the end. Since the input to the error Dialectica model is a model of type theory with dependent products then, if necessary, we can apply the coherence theorem of [32] to the input model instead of the output.

We take the latter approach. Actually, for most results it is largely obvious that they can be sharpened to the split case. The case of identity types is the one where we must be careful.

Definition 7.5.8. A *split fibred split comprehension category* is a fibred comprehension category $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ such that p_0 and p_1 are both split fibrations and for each $f : J \to I$ the canonical colax morphism of (split) comprehension categories

$$((p_1)_I : \mathbb{E}_2(I) \to \mathbb{E}_1(I), (\chi_1)_I) \to ((p_1)_J : \mathbb{E}_2(J) \to \mathbb{E}_1(J), (\chi_1)_J)$$

as in Proposition 5.2.2 is moreover a *strict* morphism.

Proposition 7.5.9. *Given a split fibred split comprehension category* $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ *over a split comprehension category* $(p : \mathbb{F} \to \mathbb{B}, \chi)$*, the gluing model* $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ *is a split comprehension category.*

Proof. It is easy to check that the change-of-base along a split fibration of a split comprehension category is a split comprehension category and that the juxtaposition of two split comprehension categories is again split. □

Now we must show that our constructions of dependent sums, dependent products and identity types produce *strictly stable* type constructors when provided with correspondingly strictly stable input data. We restrict to the case of *full* split fibred split comprehension categories over a *full* split comprehension category.

Definition 7.5.10. A (co)lax morphism of comprehension categories between split comprehension categories is itself *split* if it is split as a morphism of fibrations, i.e. if it preserves the chosen cartesian arrows.

Remark 7.5.11. It is easy to check that the canonical strong morphisms

$$((p_1)_I : \mathbb{E}_2(I) \to \mathbb{E}_1(I), (\chi_1)_I) \to ((p_1)_J : \mathbb{E}_2(J) \to \mathbb{E}_1(J), (\chi_1)_J)$$

induced by reindexing along $f : J \to I$ between the fibres of a split fibred split comprehension category are split morphisms. (This is just a statement about a fibration over a fibration).

Proposition 7.5.12. *Suppose that* $(p : \mathbb{F} \to \mathbb{B}, \chi)$ *has strictly stable dependent sums (see Definition 1.6.7 and that* $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ *has fibrewise strictly stable dependent sums, meaning strictly stable dependent sums in each fibre comprehension category that are strictly preserved by* $\mathbb{B}$*-reindexing. Then* $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ *has strictly stable dependent sums.*

Proof. Observe that a full split comprehension category $(p : \mathbb{F} \to \mathbb{B}, \chi)$ has strictly stable dependent sums if and only if it admits a *split* strong morphism of comprehension categories $\mathbb{F} \star \mathbb{F} \to \mathbb{F}$. Also observe that a full split fibred split comprehension category admits fibred strictly stable dependent sums if and only if its total full split comprehension category $\mathbb{E}_2$ admits a *split* strong morphism of comprehension categories $\mathbb{E}_2 \star \mathbb{E}_2 \to \mathbb{E}_2$. Now we use the fact that reindexings along maps in $\mathbb{B}$ are split strong morphisms of split comprehension categories to see that there is a canonical split strong morphism

$$(\mathbb{E}_1 \bullet \mathbb{F}) \star \mathbb{E}_2 \to \mathbb{E}_2 \star (\mathbb{E}_1 \bullet \mathbb{F})$$

of split comprehension categories over $\mathbb{E}_1$. We can now use the technique of Remark 6.3.3 to deduce the existence of dependent sums in the gluing model. $\square$

For dependent products, we at least ask for strictly stable dependent products in the base. The following corresponds to having $\mathcal{F}$-products which preserve the fibrewise display maps.

Definition 7.5.13. Let $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ be a full split fibred split comprehension category over a full split comprehension category $(p : \mathbb{F} \to \mathbb{B}, \chi)$. Then it has *split* $\mathbb{F}$*-products* if for each $I \in \mathbb{B}$ and $\phi \in \mathbb{F}(I)$ with comprehension $f : A \to I$ we have a functor mapping $\psi \in \mathbb{E}_1(A)$ to $\Pi_\phi(\psi) \in \mathbb{E}_1(I)$ together with a natural transformation $\mathsf{ev}_{I,\phi,\psi} : f^*(\Pi_\phi(\psi)) \to \psi$ making Π_ϕ into a right adjoint to the reindexing $f^* : \mathbb{E}_1(I) \to \mathbb{E}_1(A)$, such that this data is strictly stable under reindexing in I, and moreover for each such I and ϕ the functor Π_ϕ extends to a split strong morphism of split comprehension categories $\mathbb{E}_2(\psi) \to \mathbb{E}_2(\Pi_\phi(\psi))$ and these functors commute strictly with reindexing in I.

It is easy to see that the existence of strictly stable dependent sums in $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ can be broken down into the existence of strictly stable $\overline{\mathbb{F}}$-dependent products and $\mathbb{E}_2$-dependent products, as in §6.5.

Proposition 7.5.14. *Suppose that* $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ *has split* $\mathbb{F}$*-products and that* $(p : \mathbb{F} \to \mathbb{B}, \chi)$ *has strictly stable dependent products. Then* $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ *has strictly stable* $\overline{\mathbb{F}}$*-dependent sums.*

Proof. It is straightforward to check that the proof of Proposition 6.5.5 translates to the split setting and results in strictly stable dependent products. $\square$

Definition 7.5.15. Let $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ be a full split fibred split comprehension category over a full split comprehension category $(p : \mathbb{F} \to \mathbb{B}, \chi)$. Then it has *split quasifibred dependent products* if for each $I \in \mathbb{B}$, $X \in \mathbb{E}_1(I)$, $\phi \in \mathbb{E}_2(X)$ with comprehension $f : Y \twoheadrightarrow X$ in $\mathbb{E}_1(I)$, and $\psi : \mathbb{E}_2(Y)$ we are given choices of a type $\xi \in \mathbb{F}(I)$ with comprehension $q : Q \twoheadrightarrow I$ in $\mathbb{B}$ and a type $\pi \in \mathbb{E}_1(Q)$, which are strictly stable under reindexing in I, and, for each type $\omega \in \mathbb{E}_2(X)$, a bijection which is strictly stable under reindexing in I and natural in ω between the set of maps $f^*(\omega) \to \psi$ in $\mathbb{E}_2(Y)$ and the set whose elements are pairs composed of a section $s : I \to Q$ of $q : Q \twoheadrightarrow I$ together with a map $\omega \to s^*(\pi)$ in $\mathbb{E}_2(X)$.

Proposition 7.5.16. *Suppose that* $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ *has split quasifibred dependent products and that* $(p : \mathbb{F} \to \mathbb{B}, \chi)$ *has strictly stable dependent products and strictly stable dependent sums. Then* $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ *has strictly stable* $\overline{\mathbb{F}}$*-dependent sums.*

Proof. It is straightforward to port the proofs of Proposition 6.5.7 and 6.5.10 to the strictly stable situation. □

For identity types, we need to take a few more steps back. This is because for the gluing model we worked with identity types defined in terms of a factorization system on the category of contexts as in 1.8.3, whereas we defined strictly stable identity types in 1.8.1 in terms of path objects, i.e. factorizations of diagonal morphisms.

It was first shown by Gambino and Garner in [15] that the syntactic category of contexts of a type theory with identity types admits a weak factorization system whose right class is the closure under retracts of the display maps. The identity types are used in a sort of 'mapping space' to construct the factorizations of arbitrary morphisms into a left-map followed by a display map. The argument has been translated to more general models of type theories by Shulman in [38] and Emmenegger in [14]. The natural thing to do here would be to adapt the arguments of [15] to our situation. However, we can save work by considering the particular situation at hand, where the fibrewise models of type theory are all comprehension categories of the form given in Example 1.2.8.

Remark 7.5.17. It is worth remarking that it seems likely that the arguments of [15] can be transported to this situation, along the following lines. Define a *structured left map* in context I to be a map $i : A \to B$ in $\mathbb{E}(I)$ equipped with, for every $f : J \to I$ and every $X \in \mathbb{E}(f^*(B))$, a left map structure on $X^*(f^*(i))$. That it is to say, for each $C \in \mathbb{E}(X^*(f^*(B)))$ for which we have a map $\{X^*(f^*(A))\} \to \{C\}$ making the obvious triangle in $\mathbb{B}$ commute, we get a section of $\{C\} \to \{X^*(f^*(B))\}$, making the obvious pair of diagrams commute. Furthermore, we ask that this filling operation is stable under reindexing (in I) and weakening ('in B'). Hence structured left maps can be reindexing and weakened. The weakening part is not necessary if we have dependent products, but in our example here we do not have dependent products. An *identity factorization* of a morphism $f : A \to B$ in $\mathbb{E}(I)$ is a type $C \in \mathbb{E}(\{B\})$ and a structured left map $A \to \Sigma_B C$ such that the obvious triangle in $\mathbb{B}$ commutes. Having identity types means having a strictly stable choice of identity factorizations for diagonal maps. We would interpret [15] as deriving arbitrary identity factorizations from identity factorizations for diagonals in any split comprehension category with dependent sums. To make this more precise, we define the *comprehension category of contexts* to have base $\mathbb{B}'$ with objects lists of types $(I, A_1, \ldots, A_n)$ with $I \in \mathbb{B}$ and $A_1 \in \mathbb{E}(I)$, $A_2 \in \mathbb{E}(\{A_1\})$, etc. and the evident morphisms. The types $\mathbb{E}'((I, A_1, \ldots, A_n))$ are given as 'dependent contexts', i.e. lists $(B_1, \ldots, B_m)$ with $B_1 \in \mathbb{E}(\{A_n\})$, $B_2 \in \mathbb{E}(\{B_1\})$, etc. and we give this category the evident morphisms. There is an evident comprehension, and this comprehension category has 'strict dependent sums'. We also have a strong morphism of comprehension category $(\mathbb{B}', \mathbb{E}') \to (\mathbb{B}, \mathbb{E})$ given by taking dependent sums until a context is just a single type, and this strong morphism is an equivalence. Now the objects and types of $(\mathbb{B}', \mathbb{E}')$ model the rules for contexts given in [15], so, although that article officially only dealt with the classifying category of some type theory, *mutatis mutandis* we can interpret its arguments as giving us identity factorizations for arbitrary arrows in $(\mathbb{B}', \mathbb{E}')$. The equivalence to $(\mathbb{E}, \mathbb{B})$ gives us identity factorizations there too.

Then clearly this argument translates to the fibred model of type theory case, in that we get arbitrary identity factorizations in the fibrewise models and these are stable under reindexing. We must also formulate the 'structured acyclic cofibration condition' and show that structured left maps are closed under composition. A possible issue with taking this approach here is that as well as checking the details of the above we also need to verify the 'preserving idempotents' condition. We can shortcut this work in our specific case.

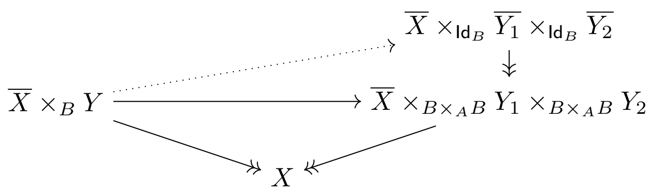
From here on we assume that the fibrewise comprehension categories $\mathbb{E}_2(I) \to \mathbb{E}_1(I)$ are of the form 1.2.8, coming from a strictly stable choice of fibred finite products. Let us described the identity types in $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$.

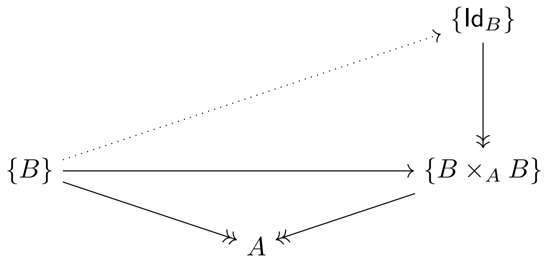
Definition 7.5.18. Suppose that each fibre of $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ is given as in 1.2.8 by a category with finite products. Then we say it satisfies the *split acyclic cofibration condition* (with respect to $(p : \mathbb{F} \to \mathbb{B}, \chi)$) if for every $I \in \mathbb{B}$ and $A \in \mathbb{F}(I)$, the reflexivity map $\mathsf{refl}_A : \{A\} \to \{\mathsf{Id}_A\}$ is equipped with, for each $B, C \in \mathbb{F}(\mathsf{Id}_{I,A})$ and $f : \mathsf{refl}_A^*(B) \to \mathsf{refl}_A^*(C)$ a choice of $g : B \to C$ with $\mathsf{refl}_A^*(g) = f$, and moreover these choices are stable under reindexing in I.

Proposition 7.5.19. *Suppose that $(p : \mathbb{F} \to \mathbb{B}, \chi)$ has strictly stable identity types and that each fibre of $(p_0 : \mathbb{E}_1 \to \mathbb{B}, p_1 : \mathbb{E}_2 \to \mathbb{E}_1, \chi_1)$ is given as in 1.2.8 by a category with finite products and that the split acyclic cofibration condition is satisfied. Then $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ has strictly stable identity types.*

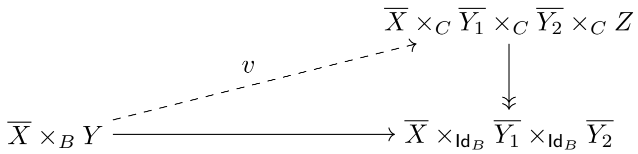
Proof. Let (A, X) be an object of $\mathbb{E}_1$ and let (B, Y) be an object of $\mathbb{G}(A, X)$, thus $B \in \mathbb{F}(A)$ and $Y \in \mathbb{E}_1(A)$. Then to give an identity type for (B, Y) we must give a type over $(B \times_A B, Y_1 \times_{B \times_A B} Y_2)$ where Y_i is the reindexing of Y along (the comprehension of) $\pi_i : B \times_A B \to B$. We give this type as $(\mathsf{Id}_B, \top_{\mathsf{Id}_B})$, where Id_B is the usual identity type of B in context A.

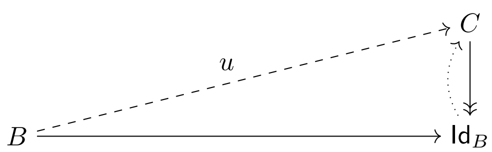
To give a reflexivity morphism we need to give a dotted arrow filling in the diagram





We let its component in the base be refl_B, the reflexivity morphism for B. Now the reindexing $\mathsf{refl}_B^*(\overline{Y_1} \times_{\mathsf{Id}B} \overline{Y_2})$ is just $Y \times_B Y$, so we take the vertical component to the product of $\overline{X}$ with the diagonal map $Y \to Y \times_B Y$. Now suppose we have a type (C, Z) over $(\mathsf{Id}_B, \top_{\mathsf{Id}_B})$ and maps u and v as in the following.





Then using the structure of identity types in the base we obtain a section $s : \mathsf{Id}_B \to C$ of $C \twoheadrightarrow \mathsf{Id}_B$ making the evident triangle commute. To lift the section to $\mathbb{E}_1$, we need to give a map $\overline{X} \times_{\mathsf{Id}_B} \overline{Y_1} \times_{\mathsf{Id}_B} \overline{Y_2} \to s^*(Z)$. By the split acyclic cofibration condition, it suffices to give such a map after reindexing along refl_B, i.e. to give a map $\overline{X} \times_B Y \times_B Y \to v^*(Z)$. We give this map as the composite

$$\overline{X} \times_B Y \times_B Y \xrightarrow{(\pi_1, \pi_2)} \overline{X} \times_B Y \xrightarrow{\overline{u}} \overline{X} \times_B Y \times_B Y \times_B v^*(Z) \to v^*(Z)$$

where $\overline{u}$ is the vertical component of u. It is easy to see that this construction does indeed equip $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ with strictly stable identity types. $\square$

Proposition 7.5.20. *In the situation of Proposition 7.5.19, if* $(p : \mathbb{F} \to \mathbb{B}, \chi)$ *has identity types that preserve idempotents, then the identity types of* $(q : \mathbb{G} \to \mathbb{E}_1, \chi_{\mathbb{G}})$ *preserve idempotents.*

Proof. Continuing with the notation of 7.5.19, suppose we have an idempotent $(b, \beta) : (B, Y) \to (B, Y)$ in $\mathbb{G}(A, X)$, so that we have an idempotent $b : B \to B$ in $\mathbb{F}(A)$ and a map $\beta : \overline{X} \times Y \to b^*(Y)$ in $\mathbb{E}_1(\{B\})$ such that the composite

$$\overline{X} \times Y \xrightarrow{(\pi_{\overline{X}}, \beta)} \overline{X} \times_B b^*(Y) = b^*(\overline{X} \times_B Y) \xrightarrow{b^*(\beta)} (bb)^*(Y) = b^*(Y)$$

is equal to β. Let $K_{A,B}(b) : \mathsf{Id}_B \to \mathsf{Id}_B$ be the idempotent associated to b as in Definition 2.8.2. Then the idempotent we require on $(\mathsf{Id}_B, \overline{Y_1} \times_{\mathsf{Id}_B} \overline{Y_2})$ lies over $K_{A,B}(b)$ and has vertical component

$$\overline{X} \times_{\mathsf{Id}_B} \overline{Y_1} \times_{\mathsf{Id}_B} \overline{Y_2} \to \overline{X} \times_{\mathsf{Id}_B} \overline{Y_1} \times_{\mathsf{Id}_B} X \times_{\mathsf{Id}_B} \overline{Y_2} \xrightarrow{\overline{\beta_1} \times_{\mathsf{Id}_B} \overline{\beta_2}} \overline{Y_1} \times_{\mathsf{Id}_B} \overline{Y_2},$$

i.e. simply the reindexing of the product over $B \times_A B$ of the reindexings β_1 and β_2 of β along $\pi_1, \pi_2 : B \times_A B \rightrightarrows B$ respectively. It is easy to see that this does indeed give an reindexing stable choice of idempotent, and that it makes the diagram (2.3) commute. $\square$

The development above culminates in the following result.

Theorem 7.5.21. *Suppose that* $(p : \mathbb{F} \to \mathbb{B}, \chi)$ *is a full split comprehension category with (strictly stable type constructors modelling) dependent sums, weak dependent products, identity types which preserve idempotents, and strong finite product and coproduct types. Then the idempotent completion of its error Dialectica model (of § 7.5) is a full split comprehension category modelling dependent sums, (strong) dependent products and identity types.*

Further work

To summarize the content of this thesis, we have set out two main theories of building models of type theory: the gluing construction and the idempotent splitting construction. We have employed these to show that de Paiva's Dialectica categories [11] may be generalized to the dependent type theory case, extending the work of von Glehn in [46].

It remains to investigate which type-theoretic principles are validated by these models. In [46] it was shown that the polynomial model does not in general validate functional extensionality in the sense of [42]. To understand the status of this principle in the Diller-Nahm model constructed in §7.3, we need to make more assumptions about the identity types of objects of the form $\mathrm{M}(A)$. The argument of [46] can be adapted to show that function extensionality is not validated in a situation where propositionally distinct elements a, b of A remain propositionally distinct elements $\{a\}$, $\{b\}$ of $\mathrm{M}(A)$. In the situation where $\mathsf{Id}(\mathrm{M}(A))$ is always contractible — for instance, $\mathrm{M}(A)$ is something resembling the free real vector space on the set of points of A together with the natural topology — it seems reasonable to conjecture that the resulting model is equivalent to the starting one, in some suitable sense.

It is natural to seek to iterate these constructions. In [46] it is observed that the twice-iterated polynomial model is in general different to a certain hypothetical model, which model we have constructed here as the Dialectica model in 7.2. We showed in 7.1.7 that the polynomial model admits the correct sort of finite sum types to allow iteration, but we leave thorough comparison of the Dialectica and iterated polynomial models to future work. One may ask when a monad M suitable for the Diller-Nahm can be found both in examples of Dialectica models, to allow iteration, and also in general models. It is worth noting that, although our intuition for the monad M is based on the free commutative monoid monad or finite multisets monad, the axioms (which are closely related to those for a *Girardian comonad* given in [4]), may be satisfied more generally.

Finally, we mention one more avenue for further exploration, which was also subject to speculation in [46]. This is to create a general 'model theory of type theory', by analogy with the connection between topos theory and higher-order intuitionistic type theory [25]. What makes the case of topos theory interesting is the correspondence between Grothendieck toposes over a base and geometric theories defined with respect to that base. Another way to say this is to say that the construction of the category of sheaves over a site, a relatively simple and well-understood process, is a rich source of models of higher-order intuitionistic type theory. It remains to tie down precisely the apparent parallels between the construction of presheaves on a small category and the construction of the polynomial model, the former being a free completion under small colimits and the latter a model carried by the free completion under dependent sums. As we saw in Chapter 6, the gluing construction can be thought of as adding in missing context extensions to a fibred model of type theory. Perhaps the framework we have given here will allow one to formulate a universal property for the gluing construction in terms of our notion of morphism of comprehension categories, exhibiting gluing models as a 'free comprehension completion'.

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