A Novel Data-based Stochastic Distribution Control for Non-Gaussian Stochastic Systems

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***Abstract*—This note presents a novel data-based approach to investigate the non-Gaussian stochastic distribution control problem. As the motivation of this note, the existing methods have been summarised regarding to the drawbacks, for example, neural network weights training for unknown stochastic distribution and so on. To overcome these disadvantages, a new transformation for dynamic probability density function is given by kernel density estimation using interpolation. Based upon this transformation, a representative model has been developed while the stochastic distribution control problem has been transformed into an optimisation problem. Then, data-based direct optimisation and identification-based indirect optimisation have been proposed. In addition, the convergences of the presented algorithms are analysed and the effectiveness of these algorithms has been evaluated by numerical examples. In summary, the contributions of this note are as follows: 1) a new data-based probability density function transformation is given; 2) the optimisation algorithms are given based on the presented model; and 3) a new research framework is demonstrated as the potential extensions to the existing stochastic distribution control.**

***Index Terms*—Non-Gaussian stochastic systems, probability density function control, kernel density estimation**

I. Introduction

The stochastic distribution control (SDC) problem has been presented by Wang in 1999. In the past two decades, this research topic has been developed rapidly because of the wide industrial applications, such as paper and board making process [1], semiconductor processes [2], etc. Meanwhile, a lot of results have been obtained which can be shown by the following timeline chart. In particular, the B-spline neural network approach has been presented in [3] where the probability density function (PDF) can be represented by the weighting vector of neural network. However, the weights cannot be guaranteed as positive. Then, [4] extended this approach using square-root transformation to ensure the positive weights. Moreover, the rational weighting vector and pseudo-ARMAX model have been further proposed in [4]. Note that the neural network training is essential in order to obtain the weighting vector which means it is difficult to obtain the proper weighs following this approach without weights training. After that, the PDF evolution and LMI-based optimisation approaches were given in [5] where the PDF of the system can be obtained analytically by formula derivation. Although the direct entropy optimisation has been summarised in [6] to avoid the PDF transformation, it has been shown that entropy criterion can be used to make the PDF sharper. However, it is difficult to fully control the shape in order to track the desired PDF using entropy optimisation. $\star$ shown in Fig. 1 denotes the main contribution of this note to eliminate the neural network training for a class of nonlinear systems subjected to the random noises with unknown distributions, which forms the motivation of this note.

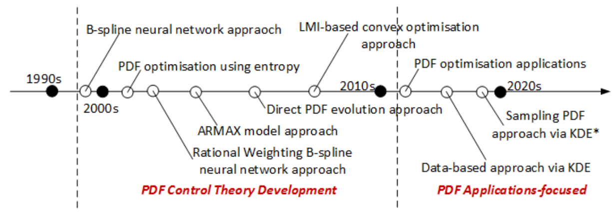

Fig. 1. The development of the probability density function control: theory and applications

Basically, the mentioned disadvantages can be overcame if we can find another transformed vector replacing the neural network weighting vector. Motivated by kernel density estimation (KDE) [7] and interpolation [8], a new representative approach is presented in this note. In particular, the probability density function can be estimated by KDE with the collected sampling data of random variable. In addition, the points of the random variable can be pre-specified in sampling space. Then we can substitute the fixed-value point into the estimated KDE, while a vector consisting of positive real numbers can be obtained to represent current PDF, we call this vector as probability density state vector. Note that this vector is changed along the estimated PDF. In other words, the dynamics of this vector can be used to represent the dynamics of the PDF and even the dynamics of the investigated stochastic system. For example, a Gaussian distribution and a Gamma distribution can be represented by probability density state vector shown in Fig.2 using the sampling operation.

Comparing with the neural network weighting vector, the new positive state vector can be obtained simply without training. Following the presented approach, the desired dif-

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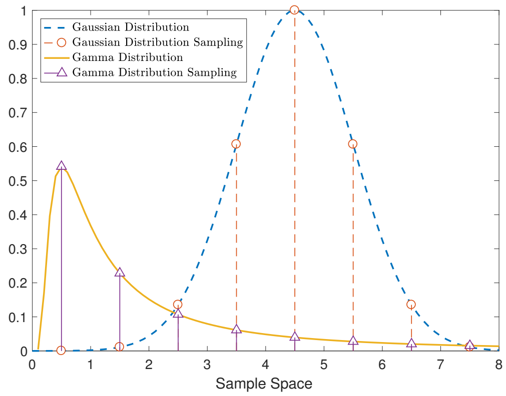

Fig. 2. The probability density function representation using sampling. Note that the sampling difference describes the PDF difference with the fixed points in sample space.

ferentiable PDF can also be converted into a reference vector, then the error between the system output PDF and the desired differentiable PDF can be rewritten as Euclidean vector distance. Therefore, the SDC can be achieved directly by minimising the Euclidean vector distance using gradient descent optimisation. Note that only the PDF defined on a bounded interval has been considered in this note. To describe the dynamics of the PDF, the probability density state vector can further expressed considering system dynamics through a state space model. Once the coefficient matrices of this state space model are obtained by parameter identification [9], the model-based optimisation can be developed as an indirect approach for SDC objective.

In this note, the PDF transformation is presented firstly by KDE and the performance criterion is given via vector distance which leads to the direct optimisation. Based upon the probability density state vector, a state space model can be identified using least square regression which results in the indirect optimisation following quadratic optimisation. Moreover, the convergences of the presented algorithms have been analysed and the SDC problem of multi-variable stochastic system is also discussed as an extension. Generally speaking, the presented algorithms can be considered as an extended framework to the existing SDC algorithms.

II. Formulation

The following single-input and single-output (SISO) discrete-time stochastic non-linear system is investigated.

$$\begin{aligned} x_{k+1} &= f\left(x_k, u_k\right) + w_k \\ y_k &= h\left(x_k\right) + v_k \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^1$, $y \in \mathbb{R}^1$, $w \in \mathbb{R}^n$ and $v \in \mathbb{R}^1$ stand for the system state, the control input, the system output, the process noise and the measurement noise, respectively. $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^1$ are unknown general non-linear functions. k denotes the sampling index. It is assumed that system (1) is controllable and observable [10].

To analyse the performance of this system, the following Lipschitz condition for the non-linear function should be satisfied.

A1 For any n-dimensional vector x_1, x_2, u_1 and u_2, there exist two positive real numbers L_1 and L_2 such that

$$\begin{aligned} \left\|f\left(x_1, u_1\right) - f\left(x_2, u_2\right)\right\| \le L_1 \left\|x_1 - x_2\right\| \\ + L_2 \left\|u_1 - u_2\right\| \end{aligned} \tag{2}$$

Since the system output is a random process, an extended system output $\gamma_k\left(y, u_k\right)$ should also be considered which denotes the PDF of the system output y_k. The control objective of SDC is to obtain the control input signal u_k to shape $\gamma_k\left(y, u_k\right)$ to any given desired PDF γ_{ref}, even if γ_{ref} is non-Gaussian distribution.

Note that the sampling set of the system output y_k can be obtained with k sampling points, then the output PDF at sampling instant k can be estimated based on the collected data set as follows:

$$\hat{\gamma}_k\left(y, u_k\right) = \frac{1}{k_w \bar{h}} \sum_{i=k-k_w+1}^{k} G\left(\frac{y - y_i}{\bar{h}}\right) \tag{3}$$

where $G\left(\cdot\right)$ and $\bar{h}$ denote Gaussian kernel function and bandwidth, respectively. $k_w \in \mathbb{Z}_+^1$ stands for the sliding window length. Note that the kernel density estimation error is bounded and there exists a size of sliding window, such that the estimation error is arbitrary small. Thus, the estimation error can be eliminated using the robustness of the presented controller design.

It has been shown that the estimated analytical PDF formula has been obtained. However, it cannot be used to design the controller due to the fact that the parameters in this formula are random sampling points. As mentioned in previous section, the pre-specified points of the random variable can be selected as base coordinates while a set of probability density states can be calculated by substituting the selected bases to Eq. (3). In particular, we have

$$z_{i,k} = \hat{\gamma}_k\left(\sigma_i, u_k\right), i = 1, 2, \ldots, m \tag{4}$$

where σ_i stands for the pre-specified coordinate point. $z_{i,k}$ denotes the function value of Eq. (3) in terms of σ_i and m is the number of the pre-specified coordinate point while $m \ge n$.

Since the output PDF can be approximately equivalent to the probability density states, we can further obtain the formula for extended system output $\gamma_k\left(y, u_k\right)$ as follows:

$$\gamma_k\left(y, u_k\right) \sim z_k \tag{5}$$

where m-dimensional vector $z_k = \left[z_1, z_2, \ldots, z_m\right]^T$ denotes the probability density state vector and the complete formulation of the investigated system (1) is obtained.

Basically, high-dimensional probability density states indicate the complete information of PDF, however the computational complexity would also increase. High-order system would lead to complex PDF dynamics, therefore the dimension of the probability density states should be selected higher than the system state vector. In particular, σ_i can be selected as the inflection point of the probability density function and at these inflection points the first order derivative of the output PDF is zero. Namely .

$$\frac{\partial \gamma\left(y, u_k\right)}{\partial \sigma_i} = 0 \tag{6}$$

Note that estimated PDF converges to the desired PDF, thus σ_i can be determined by the inflection point of the desired probability density function, thus dimension m is also determined. Basically, there exists a set of coordinate points, such that the following inequality holds.

$$\sum_{i=1}^{m} \left\| \gamma\left(\sigma_i, u_k\right) - \hat{\gamma}\left(\sigma_i, u_k\right) \right\| < \delta \tag{7}$$

where δ denotes arbitrary small real number. Thus the error can be covered by the robustness of the controller design which has been analysed in [3].

Remark 1: System states reflect the internal information of system output while the probability density states reflect the statistical information of the system output.

Remark 2: In fact this method is to use a finite number of points to represent the instant output PDF. Comparing to the B-spline approach, each point on output PDF can be represented as a zero-order B-spline with very narrow basis function [3].

III. SDC Algorithms

A. Direct Optimisation

Notice that the desired differentiable PDF can be represented by a probability density state vector z_{ref} following the presented approach (3)-(4), the control objective can be rewritten by Euclidean vector distance as the following optimisation problem.

$$\min_{u} \lim_{k\to\infty} \sqrt{\tilde{z}_k^T \tilde{z}_k} \tag{8}$$

where $\tilde{z}_k = z_{ref} - z_k$ stands for the probability density state error.

Based on Eq. (8), the performance criterion can be formulated as follows:

$$J_k = \min_{u} \left\{ \frac{1}{2} \sum_{i=0}^{k-1} \left(\tilde{z}_i^T R_1 \tilde{z}_i + R_2 u_i^2 \right) \right\} \tag{9}$$

where $R_1 \in \mathbb{R}_+^{m\times m}$ and $R_2 \in \mathbb{R}_+^1$ denote the weights of the presented performance criterion J_k while R_1 is symmetric matrix.

To minimise J_k, the linear control law can be designed as

$$u_k = K_k z_k \tag{10}$$

where $K \in \mathbb{R}^{1\times m}$ stands for the feedback gain for probability density state z.

Substituting Eq. (10) into Eq. (9), the performance criterion can be rewritten as

$$\begin{aligned} J_k &= \min_{K} \left\{ \frac{1}{2} \sum_{i=0}^{k-1} \left(\tilde{z}_k^T R_1 \tilde{z}_k + R_2 z_k^T K_k^T K_k z_k \right) \right\} \\ &= \min_{K} \left\{ \frac{1}{2} \sum_{i=0}^{k-1} \left(z_k^T \left(R_1 + R_2 K_k^T K_k \right) z_k - 2 z_k^T R_1 z_{ref} \right) \right\} \\ &+ \frac{1}{2} \sum_{i=0}^{k-1} z_{ref}^T R_1 z_{ref} \end{aligned} \tag{11}$$

Note that the last term can be ignored in optimisation because z_{ref} is constant-valued vector which is independent to K.

Since u_k will drive the dynamics of y_k by Eq. (1) with nonlinear dynamics, the solution of K_k cannot be obtained directly even with a quadratic performance criterion. As an example of using the presented PDF representation, the gradient descent algorithm is adopted to search the optimum for the gain K_k, in particular, we have

$$K_{k+1}^T = K_k^T - \varepsilon_k \left. \frac{\partial J_k}{\partial K^T} \right|_{K^T = K_k^T} \tag{12}$$

where $\varepsilon_k \in \mathbb{R}_+^{m\times m}$ is a positive real square matrix which denotes the searching rate. In addition, the selection of ε_k would affect the convergence of the optimisation and the performance of the investigated system, which means that the dynamics of the investigated system has been reflected by the selection of ε_k. Note that the convexity of the performance criterion (9) can be guaranteed when R_2 has been pre-specified as a large real number.

As a summary, the direct optimisation algorithm can be expressed by the following block diagram.

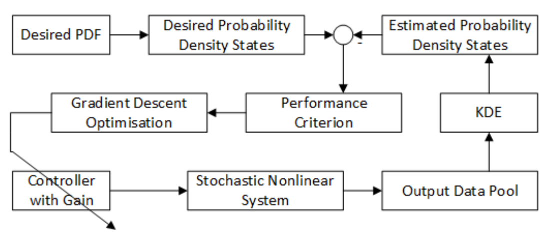

Fig. 3. Block diagram for the data-based PDF control via direct optimisation.

Remark 3: Using gradient descent approach, the optimum of the control law can be achieved by selecting the proper searching step ε_k, which is related to the upper bound $\bar{p}$. Thus the convergence criterion is determined by ε_k and the properties of the nonlinear function $f(\cdot)$.

B. Probability Density State Model

The direct optimisation algorithm cannot reflect the dynamics of the system output PDF which is covered by the system dynamics. More existing controller design methods can be adopted if the dynamics of the system output PDF can be modelled. In other words, a model should be developed to describe the relationship between the probability density states z and control input u.

The dynamics of the output PDF for dynamic system is governed by Kolmogorov forward equation [11]. Since we use probability density states to represent the PDF, the Kolmogorov forward equation is simplified from a partial differential equation to an ordinary differential equation. To further simplify the system design, the following model can be obtained inspired by dynamic linearisation.

$$z_{k+1} = F_k z_k + G_k u_k + \Delta_k \tag{13}$$

where $F_k \in \mathbb{R}^{m\times m}$ and $G_k \in \mathbb{R}^{m\times 1}$ are unknown coefficient matrices. $i = k-1, k-2, \ldots, k-\bar{m}$ and $i = k-1, k-$

$2, \ldots, k-\bar{n}$ are sampling shifting indexes, while $\bar{m}$ and $\bar{n}$ are pre-selected positive integers. Since it is assumed that system (1) is controllable, this induced system is also controllable [10]. Note that z is obtained from KDE while the randomness of the data-based estimation cannot be ignored, therefore the Δ_k has been formulated as a compensative term based on linearisation, where we have

$$\Delta_k = \sum_{i=k-1}^{k-\bar{m}} F_i z_i + \sum_{j=k-1}^{k-\bar{n}} G_j u_j + e_k \tag{14}$$

where $F_i \in \mathbb{R}^{m\times m}$ and $G_i \in \mathbb{R}^{m\times 1}$ are coefficient matrices while Eq. (13) can be further restated as follows:

$$[z_{k+1}, z_k, \cdots] = \Theta\left[\Phi_k, \Phi_{k-1}, \cdots\right] + e_k \tag{15}$$

where

$$\Theta = [F_k, F_{k-1}, \ldots, F_{k-\bar{m}}, G_k, G_{k-1}, \ldots, G_{k-\bar{n}}] \tag{16}$$

$$\Phi_k = \left[z_k^T, z_{k-1}^T, \ldots, z_{k-\bar{m}}^T, u_k, u_{k-1}, \ldots, u_{k-\bar{n}}\right]^T \tag{17}$$

and e_k denotes the identification error.

Notice that probability density state vectors for any sampling instant are measurable, therefore the vector Φ_i is known. Thus the parameter matrix Θ can be simply identified via least square algorithm or recursive least square algorithm [9]. In addition, the dynamics of the system output PDF have been presented by F and G which implies that Eq. (13) is an equivalent model to Eq. (1) in terms of the system output PDF.

Remark 4: Although the coefficient matrices are time-variant, these matrices converge to constant-based matrices when the probability density states are close to the equilibriums where identification error e_k is bounded.

C. Indirect Optimisation

Following the modelling process, the dynamics can be identified via data approach. Based upon this new description (13), the indirect optimisation can be obtained using minimum principle [12] and performance criterion (9) while the complete optimisation formulation can be given by

$$\begin{aligned} J_k &= \min_u \left\{ \frac{1}{2} \sum_{i=0}^{k-1} \left(\tilde{z}_i^T R_1 \tilde{z}_i + R_2 u_i^2\right) \right\} \\ s.t. z_{k+1} &= F_k z_k + G_k u_k + \Delta_k \end{aligned} \tag{18}$$

Suppose that the control input is designed with the following structure.

$$u_k = K_k z_k + k_{c,k} \tag{19}$$

Based on Eq. (18), Hamiltonian can be formed as follows:

$$\begin{aligned} H &= \frac{1}{2}\tilde{z}_k^T R_1 \tilde{z}_k + \frac{1}{2} R_2 u_k^2 \\ &\quad + \lambda_{k+1}\left(F_k z_k + G_k u_k + \Delta_k\right) \end{aligned} \tag{20}$$

Using the minimum principle, we have

$$\begin{aligned} \lambda_k^T &= \frac{\partial H}{\partial z_k} \\ &= -R_1 \tilde{z}_k + R_1\left(K_k z_k + k_{c,k}\right) K_k^T \\ &\quad + \left(F_k + G_k K_k\right) \lambda_{k+1}^T \end{aligned} \tag{21}$$

and

$$\frac{\partial H}{\partial u_k} = R_2 u_k + G_k^T \lambda_{k+1}^T = 0 \tag{22}$$

which results in

$$u_k = -R_2 G_k^T \lambda_{k+1}^T \tag{23}$$

Suppose that λ has the similar structure as u which leads to

$$\lambda_k^T = P_k z_k + Q_k \tag{24}$$

Substituting Eq. (24) into Eq. (23), we have

$$\begin{aligned} u_k &= -R_2 G_k^T\left(P_{k+1} z_{k+1} + Q_{k+1}\right) \\ &= -R_2 G_k^T\left(P_{k+1}\left(F_k z_k + G_k u_k + \Delta_k\right) + Q_{k+1}\right) \\ &= -R_2 G_k^T P_{k+1} F_k z_k - R_2 G_k^T P_{k+1} G_k u_k \\ &\quad - R_2 G_k^T P_{k+1} \Delta_k - R_2 G_k^T Q_{k+1} \end{aligned} \tag{25}$$

Rearranging Eq. (25), we can obtain

$$\begin{aligned} u_k &= -\left(1 + R_2 G_k^T P_{k+1} G_k\right)^{-1} R_2 G_k^T P_{k+1} F_k z_k \\ &\quad - \left(1 + R_2 G_k^T P_{k+1} G_k\right)^{-1}\left(R_2 G_k^T P_{k+1}\Delta_k + R_2 G_k^T Q_{k+1}\right) \end{aligned} \tag{26}$$

Comparing with the control law (19), we have

$$K_k = -\left(1 + R_2 G_k^T P_{k+1} G_k\right)^{-1} R_2 G_k^T P_{k+1} F \tag{27}$$

and

$$\begin{aligned} k_{c,k} &= -\left(1 + R_2 G_k^T P_{k+1} G_k\right)^{-1} \\ &\quad \times \left(R_2 G_k^T P_{k+1}\Delta_k + R_2 G_k^T Q_{k+1}\right) \end{aligned} \tag{28}$$

The controller design can be completed once P_{k+1} and Q_{k+1} are determined, then we can further substitute Eq. (24) into Eq. (21) as follows.

$$\begin{aligned} & P_k z_k + Q_k \\ &= -R_1 \tilde{z}_k + R_1\left(K_k z_k + k_{c,k}\right) K_k^T \\ &+ \left(F_k + G_k K_k\right)\left(P_{k+1} z_{k+1} + Q_{k+1}\right) \\ &= -R_1\left(z_{ref} - z_k\right) + R_1 K_k^T\left(K_k z_k + k_{c,k}\right) \\ &+ \left(F_k + G_k K_k\right) \\ &\times \left(P_{k+1} F_k z_k + P_{k+1} G_k u_k + P_{k+1}\Delta_k + Q_{k+1}\right) \\ &= -R_1\left(z_{ref} - z_k\right) + R_1 K_k^T\left(K_k z_k + k_{c,k}\right) \\ &+ \left(F_k + G_k K_k\right) \\ &\times \left(P_{k+1} F_k z_k + P_{k+1} G_k K_k z_k \right. \\ &\left. + P_{k+1} G_k k_{c,k} + P_{k+1}\Delta_k + Q_{k+1}\right) \\ &= \left(R_1 + R_1 K_k^T K_k + \left(F_k + G_k K_k\right) P_{k+1}\left(F_k + G_k K_k\right)\right) z_k \\ &- R_1 z_{ref} + R_1 K_k^T k_{c,k} \\ &+ \left(F_k + G_k K_k\right)\left(P_{k+1} G_k k_{c,k} + P_{k+1}\Delta_k + Q_{k+1}\right) \end{aligned} \tag{29}$$

which results in

$$P_k = R_1 + R_1 K_k^T K_k + \left(F_k + G_k K_k\right) P_{k+1}\left(F_k + G_k K_k\right) \tag{30}$$

and

$$\begin{aligned} Q_k &= R_1 K_k^T k_{c,k} - R_1 z_{ref} \\ &\quad + \left(F_k + G_k K_k\right)\left(P_{k+1} G_k k_{c,k} + P_{k+1}\Delta_k + Q_{k+1}\right) \end{aligned} \tag{31}$$

Note that K_k is a function of P_{k+1} and $k_{c,k}$ is a functions of P_{k+1} and Q_{k+1}, then P_{k+1} is solvable for each sampling instant by Eq. (30). Next, Q_{k+1} is also solvable by Eq. (31) with P_{k+1}. Substituting P_{k+1} and Q_{k+1} to obtain the value of K_k and $k_{c,k}$ which means the control law is implementable.

Comparing with the direct optimisation block diagram, another block diagram is given for the indirect optimisation approach.

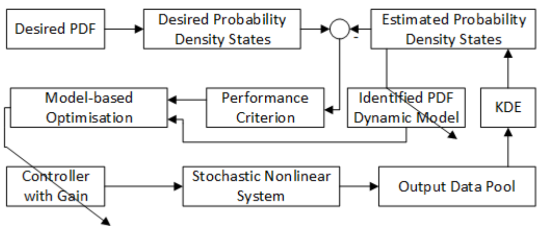

Fig. 4. Block diagram for the data-based output PDF control via indirect optimisation while the PDF dynamics have been represented by an identified model with probability density states.

IV. Performance Analysis

To summarise the presented algorithms, the following theorem is given and the proof has been shown in the previous section.

Theorem 1: For the investigated system model (1), the output PDF tracking can be achieved using the control law (26) with the parameter design (30-31), where the performance criterion (18) can be minimised.

Since the optimisation results in a bounded value of the performance criterion (9), therefore, the distance between the desired PDF and the actual PDF is also bounded which shows that the system output is also bounded and the following theorem is obtained. However, the convergence of the PDF tracking does not imply that the system output is convergence to a constant. If the desired PDF is given with multi-peak shape, the PDF tracking will be reflected as multiple data clusters within the sliding window of KDE. In other words, the value of the system output in the sliding window would randomly jump from one cluster to another. As a compensation of Theorem 1, the following theorem is given to analyse the increment of the system output which is also bounded.

Theorem 2: For the investigated system model (1), both the direct and indirect optimisation algorithms with control laws (10) and (26) achieve the system output PDF tracking convergence in mean-norm sense with bounded errors if the inequality $L_1 + L_2\bar{p} < 1$ meets.

Proof: Theoretically, the control law for direct optimisation and indirect optimisation can be uniformly expressed by the following control law.

$$u_k = \begin{bmatrix} K_k & k_c \end{bmatrix} \begin{bmatrix} z_k \\ 1 \end{bmatrix} \tag{32}$$

where $k_c = 0$ in Eq. (10).

Assume that there always exists a constant matrix Λ_k such that the following equation holds for each k.

$$\begin{bmatrix} z_k \\ 1 \end{bmatrix} = \Lambda_k x_k \tag{33}$$

Then the control law can be rewritten as

$$u_k = \bar{K}_k x_k \tag{34}$$

where $\bar{K}_k = \begin{bmatrix} K_k & k_c \end{bmatrix} \Lambda_k$. Since $\bar{K}_k$ is bounded, suppose that there exists a positive real number $\bar{p}$ as the upper bounded to meet the following equation.

$$\left\|\bar{K}_k\right\| \leq \bar{p}, \forall k = 0, 1, 2, \ldots$$

Denoting $\Delta x_{k+1} = x_{k+1} - x_k$ and $\Delta w_{k+1} = w_{k+1} - w_k$, we have

$$\Delta x_{k+1} = \Delta w_{k+1} + f(x_k, u_k) - f(x_{k-1}, u_{k-1}) \tag{35}$$

Based upon the assumption (2), the following inequality can be obtained.

$$\begin{aligned} \|\Delta x_{k+1}\| &\leq \|\Delta w_{k+1}\| + \|f(x_k, u_k) - f(x_{k-1}, u_{k-1})\| \\ &\leq \|\Delta w_{k+1}\| + L_1 \|\Delta x_k\| \\ &+ L_2 \left\|\bar{K}_k x_k - \bar{K}_{k-1} x_{k-1}\right\| \\ &\leq L_1 \|\Delta x_k\| + \|\Delta w_{k+1}\| + L_2 \|\bar{p} \Delta x_k\| \end{aligned} \tag{36}$$

Using the mathematical expectation operation $E\{\cdot\}$, we have

$$E\{\|\Delta x_{k+1}\|\} \leq (L_1 + L_2\bar{p}) E\{\|\Delta x_k\|\} + \bar{q} \tag{37}$$

where $\bar{q}$ denotes the upper bound of $E\{\|\Delta w_{k+1}\|\}$. Thus, the bounded increment of the system states can be achieved if the coefficient $L_1 + L_2\bar{p} < 1$. Although the constant L_1 and L_2 depend on the property of the non-linear functions $f(\cdot)$ and $g(\cdot)$, $\bar{p}$ would be changed by selecting the parameters of the controllers. The gain K_k for direct optimisation can be governed by the searching rate, meanwhile, R_1 and R_2 for indirect optimisation can be used to obtain the suitable K_k. In particular, Eq. (30) is a Lyapunov function while $P_k > 0$ if k goes to infinity. Then, in Eq. (27), $1 + R_2 G_k^T P_{k+1} G_k > 1$ implies that the condition in Theorem 2 is achievable. ■

V. Numerical Examples

To validate the presented algorithms, firstly we consider the following simple stochastic system.

$$\begin{aligned} x_{k+1} &= x_k \sin(x_k) + u_k + w_k \\ y_k &= 0.5x_k + v_k \end{aligned}$$

where w and v are Gaussian noises with zero mean value while the variances for w and v are setup as 1 and 0.1, respectively. Due to the non-linearity of the system, the system output y becomes a non-Gaussian variable.

Following the presented description of the probability density states, we pre-specified the target value for $z_{ref} = [0.0005, 0.2353, 0.1570, 0.0027]$ subjected to the pre-selected points $-5, -1.6667, 1.6667$ and 5 in sample space. Basically, the reference is pre-specified using KDE and the reference non-Gaussian distribution is Gamma distribution with $a = 1$ and $b = 2$. To track the reference probability density state vector z_{ref}, the direct approach can be adopted with the weights $R_1 = 2$ and $R_2 = 1$ in the performance criterion while the searching rate $\varepsilon = diag\{8e-6, 9.6e-6, 1.2e-5, 1.44e-5\}$. Alternatively, the indirect approach can be used while the

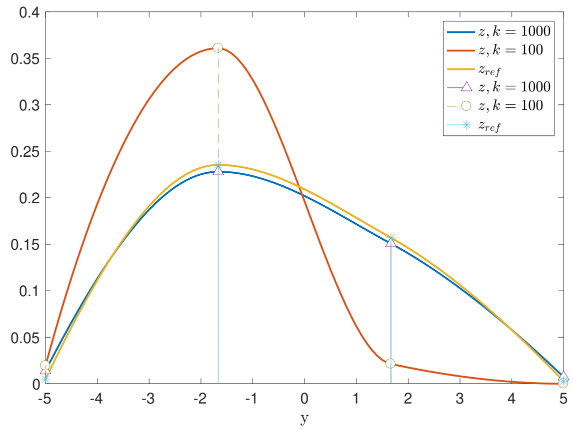

Fig. 5. The control performance of the probability density state vector z using direct approach.

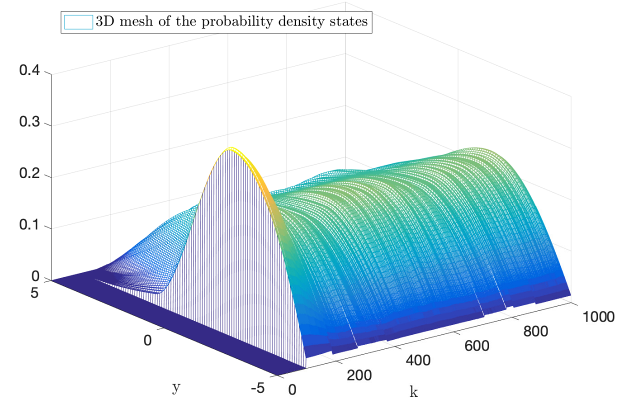

Fig. 6. 3D mesh of the probability density states z using direct approach along the sampling index k.

dynamics of output PDF can be represented by the identified model using least square regression.

The simulation results can be shown by the following figures. In particular, the direct optimisation performance of the probability density state z has been demonstrated by Figs. 5, in which the points are connected using cubic interpolation in order to demonstrate the shape of the probability density function regarding to the probability density states. The original reference PDF is a continuous differentiable function. Note that the control law for direct approach and indirect approach can be generalised to the equivalent form based on the discussion in the previous section. These two algorithms lead to the similar performance thus the results of indirect approach have been omitted. The sliding window length k_w is selected as 30. Before $k = 100$, the control input has been specified as 0 and $0.8sin(k)$ because the KDE needs data to start up. For the purpose of the model identification, the control input should be chosen as a dynamic signal to stimulate the system output which is the reason of using $0.8sin(k)$ as the control input before $k = 100$. Fig. 6 dictates the 3D mesh of z using indirect approach which shows the dynamics of the probability density function while the 3D mesh for direct approach is omitted since it is very similar to Fig. 6. Moreover, the system output and the system input curves are illustrated using Fig. 7, particularly, the system outputs y with both two algorithms are stable and the control inputs u converge to a non-zero constant which matches the presented analysis. In addition, the tracking errors $\tilde{z}$ using the indirect approach has been obtained by Figs. 8 which shows that the presented algorithms can drive the probability density states of the system output close to the desired PDF z_{ref}.

Basically, the performance will be improved if increasing the dimension of the probability density states, however the computational complexity will increase dramatically at the same time and the learning rate matrix would be difficult to select to stabilise the system output. Note that both of the algorithms do not use any information from the investigated system model which implies that the presented algorithms are the pure data-driven control approaches. Comparing the perfor-

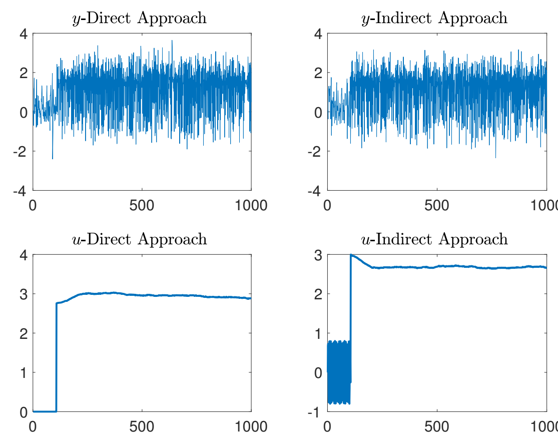

Fig. 7. The performance of the system input u and system output y with direct approach and indirect approach.

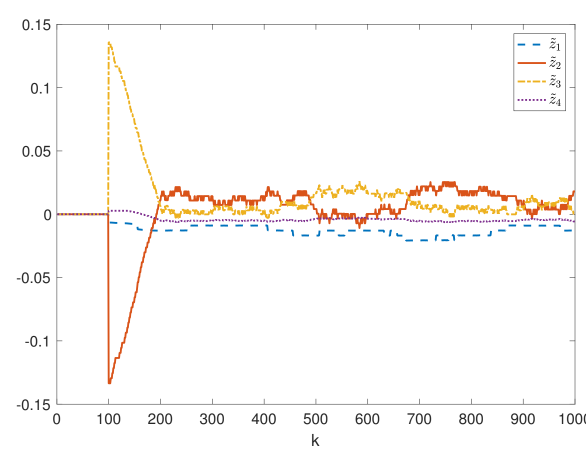

Fig. 8. The error of the probability density states $\tilde{z}$ using indirect approach.

mance to the existing B-spline neural network modelling based approach, the equivalent control results would be obtained with the trained weights. However the algorithm complexity is

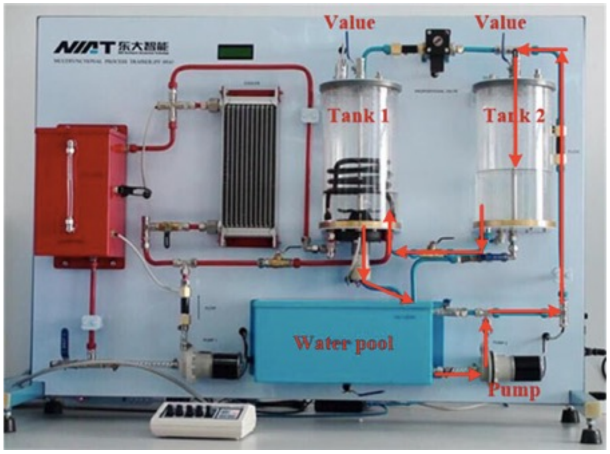

Fig. 9. The structure of the twin tank level process with interconnection.

greater than the presented algorithm which brings the difficulty for practical implementation in real time.

To validate the effectiveness of the presented algorithms with practical application, the following twin tank level process has been modelled as follows, where the system structure has been demonstrated by Fig. 9.

$$
\begin{aligned}
A_1 \frac{dx_1}{dt} &= -c_1 - k_1\sqrt{x_1} + k_0\sqrt{x_2 - x_1} + w_1 \\
A_2 \frac{dx_2}{dt} &= k_4 u_2 - c_2 + c_1 - k_2\sqrt{x_2} - k_0\sqrt{x_2 - x_1} + w_2 \\
y &= x_1 + v
\end{aligned}
$$

where x_1 is the level of tank 1 and x_2 denotes the level of tank . A_1 and A_2 are the cross-sectional area. c_1 and c_2 are constant parameters of the valves and pumps. k_0, k_1,k_2 and k_4 stand for the ratio of the valves. w and v are zero-mean non-Gaussian noises. In particular, the discrete-time model can be obtained using $0.1s$ as the sampling time. Moreover, the parameters are pre-measured as $A_1 = A_2 = 167.4cm^2$, $k_0 = 0.7$, $k_1 = 0.25$, $k_2 = 0$, $k_4 = 0.1$, $c_1 = 0$ and $c_2 = 2.88$.

Using the presented algorithm, we can pre-defined the reference probability density states as $z_{ref} = [0.1, 0.175, 0.12, 0.01]$. Then, the simulation results have been shown by Figs. 10, 11 and 12, where the PDF of the system output y has been adjusted to track the desired PDF along the sampling instant k. Note that the system output y is still bounded and the convergences for both the system output and its PDF tracking have been achieved.

VI. SDC for Multi-variable Stochastic Systems

In this section, we will further discuss how to extend the presented method to multiple-input and multiple-output (MIMO) stochastic systems. Different from the single system output PDF, multi-output will leads to multi-dimensional PDF which is called joint probability density function (JPDF) [13]. Since the JPDF can be projected to each single random sampling space as marginal probability density function (MPDF), the KDE can be used again to estimate the MPDF separately and

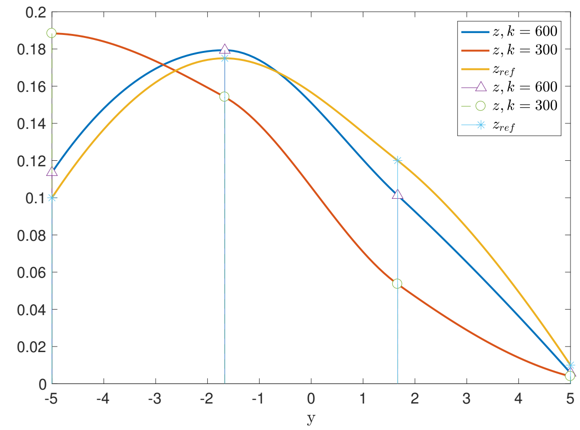

Fig. 10. The structure of the twin tank level process with interconnection.

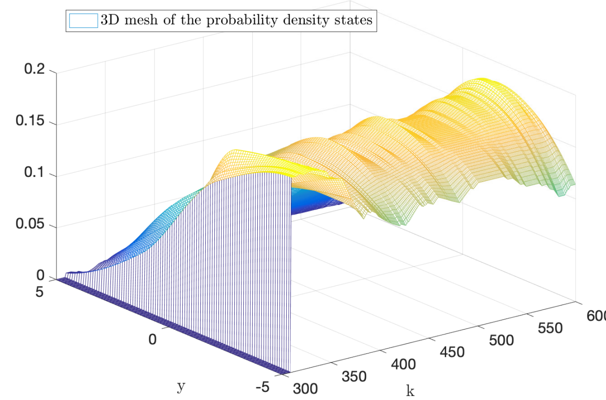

Fig. 11. The structure of the twin tank level process with interconnection.

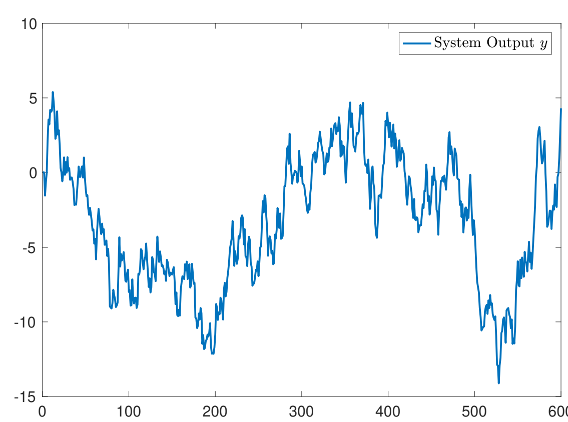

Fig. 12. The output of the twin-tank level process control system around the equilibrium which is $23cm$.

probability states for each system output can be obtained as follows if the investigated system is of s system outputs.

$$
z_{i,k} = \begin{bmatrix} z_{i,1,k} & z_{i,2,k} & \cdots & z_{i,n_i,k} \end{bmatrix}^T \tag{38}
$$

where $i = 1, 2, \dots, s$ denotes the index of the system outputs and n_i stands for the dimension of probability density state vector for i-th system output.

The full information of the multi-variable system output JPDF can be represented using the vectorisation operation, where the probability density state vector can be further expressed by

$$z_k = \left[\begin{array}{cccc} z_{1,k}^T & z_{2,k}^T & \dots & z_{s,k}^T \end{array} \right]^T \tag{39}$$

Once the description is determined, the direct optimisation can be achieved. For indirect approach, the modelling progress can be used similarly. Suppose to the MIMO system is with l-dimensional input, the general model can be obtained as follows:

$$\begin{aligned} \begin{bmatrix} z_{1,k+1} \\ z_{2,k+1} \\ \vdots \\ z_{s,k+1} \end{bmatrix} &= \begin{bmatrix} F_{11,k} & F_{12,k} & \cdots & F_{1s,k} \\ * & F_{22,k} & \cdots & F_{2s,k} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & F_{ss,k} \end{bmatrix} \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{s,k} \end{bmatrix} \\ &+ \begin{bmatrix} G_{11,k} & G_{12,k} & \cdots & G_{1l,k} \\ G_{21,k} & G_{22,k} & \cdots & G_{2l,k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{l1,k} & G_{l2,k} & \cdots & G_{ll,k} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \end{bmatrix} \end{aligned} \tag{40}$$

where $*$ denotes the symmetrical operation. Without loss of the generalities, the time sliding team Δ_k has been ignored for simplification comparing with Eq.(13).

Once the coefficient matrices are identified using data, the indirect optimisation can be achieved. Note that the system output is probabilistic independent if $F_{ij,k} = 0, i \neq j$ which means that this model can also be used to analyse the probabilistic couplings [14] for MIMO stochastic systems. However, this approach has a problem of dimension explosion because the n_i for each system output has to be large enough to reflect the properties of its PDF. This means the coefficient matrices would have a high-dimension and are complicated to obtain the control law. This is a potential future work to SDC problem for multi-variable stochastic systems.

VII. Conclusion

In this paper, a novel transformation for probability density function has been presented. Using the KDE-based probability density states, the system output PDF can be characterised without training neural network. Following this new description, the control objective can be restated as assign the probability density states to the desired vector. To achieve this objective, two data-based control algorithms are presented separately. In particular, the direct optimisation approach is given without modelling the system and the linear form control law has been obtained with optimal gain via gradient descent algorithm. Alternatively, an indirect approach is also presented where the dynamic model has been identified firstly using the calculated probability density states. It has shown that this model builds a link between the PDF and the system control input. Based upon this model, the minimum principle can be adopted to minimise the vector distance based performance criterion. Basically, these two algorithms can be generalised as a linear form uniformly while the performance of the investigated closed-loop system is also analysed. In addition, the multi-variable system output PDF control problem has been discussed while the vectorisation is used following the presented idea. The results of the simulation shows the effectiveness of the presented transformation, description and algorithms. As the potential perspectives of the stochastic distribution control, the probability states controllability, control input saturation, system output delay, etc. should be further investigated as the future works.

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