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Positivity

Ratio Tauberian Theorems for Positive Functions and Sequences in Banach Lattices

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Abstract. We prove two ratio Tauberian theorems and deduce two generalized Tauberian theorems for functions and sequences with values in positive cones of Banach lattices. Two counter-examples are given to show that the hypotheses in the ratio Tauberian theorems are essential.

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1. Introduction

Let X be a Banach space, and $u(\cdot):[0,\infty)\to X$ be a locally integrable function. It is well known that the existence of the Cesàro limit $y := \lim_{t\to\infty} t^{-1}\int_0^t u(s)ds$ implies that the Abel limit $\lim_{\lambda\downarrow 0}\lambda\int_0^\infty e^{-\lambda t}u(t)dt$ also exists and equals y. Similarly, if for a sequence $\{x_n\}_{n=0}^\infty \subset X$ the Cesàro limit $y := \lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} x_k$ exists, then the Abel limit $\lim_{r\uparrow 1}(1-r)\sum_{n=0}^\infty r^n x_n = y$. More generally, Sato [8, Theorems 1 and 3] proved the following ratio limit theorem.

Proposition 1.1. *If $\int_0^\infty e^{-\lambda t}u(t)dt$ (resp. $\sum_{n=0}^\infty r^n x_n$) converges absolutely for all $\lambda>0$ (resp. $0<r<1$) and g is a nonnegative function (resp. $\{a_n\}_{n=0}^\infty$ is a sequence of non-negative real numbers), with $\int_0^\infty g(t)dt>0$ (resp. $\sum_{n=0}^\infty a_n>0$), then the existence of the limit*

$$\lim_{t\to\infty}\left(\int_0^t u(s)ds\right)\Big/\left(\int_0^t g(s)ds\right)\left(\textit{resp. } \lim_{n\to\infty}\left(\sum_{k=0}^n x_k\right)\Big/\left(\sum_{k=0}^n a_k\right)\right)=x \quad (1)$$

implies

$$\lim_{\lambda\downarrow 0}\left(\int_0^\infty e^{-\lambda t}u(t)dt\right)\Big/\left(\int_0^\infty e^{-\lambda t}g(t)dt\right)$$
$$\left(\textit{resp.}\ \lim_{r\uparrow 1}\left(\sum_{k=0}^{\infty} r^k x_k\right)\Big/\left(\sum_{k=0}^{\infty} r^k a_k\right)\right) = x. \quad (2)$$

In general, the existence of the Abel limit does not guarantee the existence of the Cesàro limit (see [2, p. 8] or [6] for examples). The Tauberian theorem of Hardy and Littlewood asserts that if $u(\cdot)$ (resp. $\{x_n\}_{n=0}^\infty$) is bounded, or is positive in a Banach lattice, then the existence of the Abel limit also implies the existence of the Cesàro limit, and the two limits coincide (cf. [3], [2, Theorem 3.3], [6]). In view of this Tauberian theorem, one would ask whether (2) implies (1) for a bounded sequence $\{x_n\}_{n=0}^\infty$ in a Banach space or a positive sequence $\{x_n\}_{n=0}^\infty$ in a Banach lattice. As we will see in two examples to be given in Section 5, the answers for both cases are negative. In order to establish a ratio Tauberian theorem for a positive sequence $\{x_n\}$, some suitable conditions on the sequence $\{a_n\}$ are needed.

The purpose of this paper is to formulate ratio Tauberian theorems for positive functions and sequences in Banach lattices. We will prove in Section 3 both continuous version and discrete version of such ratio Tauberian theorems (see Theorem 3.2, Corollary 3.3, Theorem 3.4). From them we also deduce that, for Banach space valued functions (resp. sequences) which are bounded relatively to the positive function g (resp. sequence $\{a_n\}$) in the denominator, the assertions of the ratio Tauberian theorems hold in the sense of weak limit (see Remark (i) after Theorem 3.4). It is unknown whether they hold in the strong sense. From the ratio Tauberian theorems we can deduce generalized Tauberian theorems (Theorems 4.1 and 4.2). For need in the proofs of the ratio Tauberian theorems, we first prove in Section 2 an auxiliary convergence lemma. To show that the positivity of $\{x_n\}$ and the two conditions (D1) and (D2) are essential in Theorem 3.4, two counter-examples will be given in Section 5.

2. A Convergence Lemma

We first prove the following lemma which generalizes Lemma 4.1 of [6] and forms the basis of the whole arguments in the main results in Section 3.

Lemma 2.1. *Suppose X and Y are two Banach lattices and W is a Banach sublattice of Y. Let E be a subset of W and let $\{\mathcal{F}_\alpha\}$ and $\{\mathcal{G}_\alpha\}$ be two nets of positive linear operators from W to X such that the net $\{(\mathcal{F}_\alpha - \mathcal{G}_\alpha)|_{\overline{\mathrm{span}}E}\}$ is uniformly bounded and such that*

$$\lim_\alpha[\mathcal{F}_\alpha(u) - \mathcal{G}_\alpha(u)] = 0 \quad (3)$$

for all $u \in E$. *If a vector* $w \in W$ *has the property that there are two sequences* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *in* $\overline{\text{span}}E$ *such that* $u_n \le w \le v_n$ *for all* $n \ge 1$ *and*

$$\limsup_{\alpha} ||\mathcal{G}_\alpha(v_n - u_n)|| \to 0 \text{ as } n \to \infty, \tag{4}$$

then $\lim_{\alpha}[\mathcal{F}_\alpha(w) - \mathcal{G}_\alpha(w)] = 0$.

Proof. For the assumed $w \in W$ and $\{u_n\}, \{v_n\} \in \overline{\text{span}}E$, we have for every $n = 1, 2, \dots$ and for every α

$$\mathcal{G}_\alpha(u_n) \le \mathcal{G}_\alpha(w) \le \mathcal{G}_\alpha(v_n),$$

and so

$$\begin{aligned}\mathcal{G}_\alpha(u_n) - \mathcal{F}_\alpha(v_n) \le \mathcal{G}_\alpha(w) - \mathcal{F}_\alpha(v_n) \le \mathcal{G}_\alpha(w)\\ -\mathcal{F}_\alpha(w) \le \mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(w) \le \mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(u_n).\end{aligned}$$

Therefore we have for every $n = 1, 2, \dots$ and for every α

$$\begin{aligned}&||\mathcal{G}_\alpha(w) - \mathcal{F}_\alpha(w)||\\ &\le ||\mathcal{G}_\alpha(u_n) - \mathcal{F}_\alpha(v_n)|| + ||\mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(u_n)||\\ &\le ||\mathcal{G}_\alpha(u_n - v_n)|| + ||\mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(v_n)|| + ||\mathcal{G}_\alpha(v_n - u_n)|| + ||\mathcal{G}_\alpha(u_n) - \mathcal{F}_\alpha(u_n)||\\ &= 2||\mathcal{G}_\alpha(u_n - v_n)|| + ||\mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(v_n)|| + ||\mathcal{G}_\alpha(u_n) - \mathcal{F}_\alpha(u_n)||.\end{aligned}$$

The uniform boundedness of $\{\mathcal{F}_\alpha - \mathcal{G}_\alpha\}$ on $\overline{\text{span}}E$ implies that (3) holds for all $u \in \overline{\text{span}}E$. It follows from this and (4) that

$$\begin{aligned}&\limsup_{\alpha} ||\mathcal{G}_\alpha(w) - \mathcal{F}_\alpha(w)||\\ &\le \limsup_{\alpha}[2||\mathcal{G}_\alpha(u_n - v_n)|| + ||\mathcal{G}_\alpha(v_n) - \mathcal{F}_\alpha(v_n)|| + ||\mathcal{G}_\alpha(u_n) - \mathcal{F}_\alpha(u_n)||]\\ &\le 2\limsup_{\alpha} ||\mathcal{G}_\alpha(u_n - v_n)|| + 0 + 0 \to 0 \text{ as } n \to \infty.\end{aligned}$$

This shows that $\lim_{\alpha}[\mathcal{F}_\alpha(w) - \mathcal{G}_\alpha(w)] = 0$. This completes the proof. □

For the special case $Y := L^\infty$, Lemma 2.1 leads to the following two corollaries. The following one is also proved directly in [6, Lemma 4.1].

Corollary 2.2. *Let* Ω *be a nonempty Lebesgue measurable subset of a Euclidean space* R^r, $\mathcal{B}(\Omega)$ *be the* σ*-field of all Lebesgue measurable sets in* Ω, *and* m *be Lebesgue measure on* R^r. *Let* X *be a Banach lattice, let* E *be a subset of* $L^\infty(\Omega)$ *which contains* 1, *and let* W *be a Banach sub-lattice of* $L^\infty(\Omega)$ *which contains* E. *Let* $\{\mathcal{F}_\alpha\}$ *be a net of positive linear operators from* W *to* X *and* $\mathcal{F}$ *be a positive linear operator from* W *to* X *such that*

$$\lim_{\alpha} \mathcal{F}_\alpha(u) = \mathcal{F}(u)$$

for all $u \in E$. *If a function* $w \in W$ *has the property that there are two bounded sequences* $\{u_n\}_{n=1}^{\infty}$ *and* $\{v_n\}_{n=1}^{\infty}$ *in* $\overline{\text{span}}E$ *such that*

$$u_n \nearrow w \text{ and } v_n \searrow w \text{ a.e.}[m] \text{ and } \|\mathcal{F}(v_n - u_n)\| \to 0,$$

then $\lim_\alpha \mathcal{F}_\alpha(w) = \mathcal{F}(w)$.

Proof. Since each $\mathcal{F}_\alpha$ is positive and $1 \in E \subset W$, we have

$$-||w||_\infty \mathcal{F}_\alpha(1) = \mathcal{F}_\alpha(-||w||_\infty) \le \mathcal{F}_\alpha(w) \le \mathcal{F}_\alpha(||w||_\infty) = ||w||_\infty \mathcal{F}_\alpha(1),$$

so that $\|F_\alpha(w)\| \le \|F_\alpha(1)\|\|w\|_\infty$ for all $w \in W$, which implies $||\mathcal{F}_\alpha|| = \|\mathcal{F}_\alpha(1)\|$. Similarly, $||\mathcal{F}|| = \|\mathcal{F}(1)\|$. By assumption we have $\lim_\alpha \mathcal{F}_\alpha(1) = \mathcal{F}(1)$, which implies that the operators $\mathcal{F}_\alpha$ are uniformly bounded. The result follows from Lemma 2.1 by taking $Y := L^\infty(\Omega, \mathcal{B}(R), m)$, $\mathcal{G}_\alpha \equiv \mathcal{F}$ for all α. □

The next corollary will be needed in Section 3.

Corollary 2.3. *Let X be a Banach lattice and let E be a subset of $C[0,1]$ such that the linear span* span$E$ *of E is dense in $C[0,1]$. Suppose W is a Banach sub-lattice of $L^\infty[0,1]$ which contains $C[0,1]$. Let $\{\mathcal{F}_\alpha\}$ and $\{\mathcal{G}_\alpha\}$ be two uniformly bounded nets of positive linear operators from W to X such that $\lim_{n\to\infty}[\mathcal{F}_\alpha(u) - \mathcal{G}_\alpha(u)] = 0$ for all $u \in E$. If $w \in W$ has the property that there are two sequences $\{u_n\}$ and $\{v_n\}$ in $C[0,1]$ satisfying $u_n \le w \le v_n$ and $\limsup_\alpha ||\mathcal{G}_\alpha(v_n - u_n)|| \to 0$ as $n \to \infty$, then $\lim_\alpha[\mathcal{F}_\alpha(w) - \mathcal{G}_\alpha(w)] = 0$.*

3. Ratio Tauberian Theorems

We first prove the following proposition which will be used to deduce continuous and discrete ratio Tauberian theorems.

Proposition 3.1. *Let X be a Banach lattice and let $h_s(t) := \begin{cases} 0, & 0 \le t < e^{-s}; \\ t^{-1}, & e^{-s} \le t \le 1 \end{cases}$ for $s > 0$. Let E be a subset of $C[0,1]$ which contains 1 and is such that* span$E$ *is dense in $C[0,1]$ and let W be a Banach sub-lattice of $L^\infty[0,1]$ which contains $C[0,1]$ and the functions h_s, $s > 0$ (for instance, $W = L^\infty[0,1]$). Let $\{G_n\}$ be a uniformly bounded sequence of positive linear functionals on W which satisfies the following conditions for some $s_0 > 0$:*

(A1) $\liminf_{n\to\infty} G_n(h_{s_0}) > 0$;

(A2) $\limsup_{n\to\infty} |G_n(h_{s_0} - h_s)| \to 0$ *as* $s \to s_0$.

If $\{\mathcal{F}_n\}$ is a uniformly bounded sequence of positive linear operators from W into X such that $\lim_{n\to\infty} \frac{\mathcal{F}_n(u)}{G_n(u)} = x$ for some $x \in X$ and all $u \in E$, then $\lim_{n\to\infty} \frac{\mathcal{F}_n(h_{s_0})}{G_n(h_{s_0})} = x$.

Proof. Since $\{\mathcal{F}_n\}$ and $\{G_n\}$ are positive and $1 \in E$, the assumption implies x is a positive element of X. Define positive operators $\mathcal{G}_n : W \to X$ by $\mathcal{G}_n(u) := G_n(u)x, u \in W$. Since $\{G_n\}$ is uniformly bounded, it follows from $\lim_{n\to\infty} \frac{\mathcal{F}_n(u)}{G_n(u)} = x$

for all $u \in E$ that $\|\mathcal{F}_n(u) - \mathcal{G}_n(u)\| \le |G_n(u)| \|\mathcal{F}_n(u) \big/ G_n(u) - x\| \to 0$ for all $u \in E$.

Now, let $\{s_n\}$ and $\{s'_n\}$ be two sequences of positive numbers such that $s_n \nearrow s_0$ and $s'_n \searrow s_0$. We define for every $n = 1, 2, \ldots$ two functions u_n and v_n by

$$v_n(t) := \begin{cases} 0, & 0 \le t < e^{-s'_n}; \\ \frac{e^{s_0}}{e^{-s_0} - e^{-s'_n}}(t - e^{-s'_n}) & e^{-s'_n} \le t \le e^{-s_0} \\ t^{-1}, & e^{-s_0} \le t \le 1 \end{cases}$$

and

$$u_n(t) := \begin{cases} 0, & 0 \le t < e^{-s_0}; \\ \frac{e^{s_n}}{e^{-s_n} - e^{-s_0}}(t - e^{-s_0}) & e^{-s_0} \le t \le e^{-s_n} \\ t^{-1}, & e^{-s_n} \le t \le 1. \end{cases}$$

Then we have for every $n = 1, 2, \ldots$ $u_n, v_n \in C[0,1]$ and

$$h_{s_n} \le u_n \le h_{s_0} \le v_n \le h_{s'_n}.$$

It follows from the positivity of $\mathcal{G}_n$ and condition (A2) that

$$\begin{aligned} \limsup_{n\to\infty} \|\mathcal{G}_n(v_m - u_m)\| &\le \limsup_{n\to\infty} \|\mathcal{G}_n(h_{s'_m} - h_{s_m})\| \\ &\le \limsup_{n\to\infty} |G_n(h_{s'_m} - h_{s_m})| \|x\| \to 0 \end{aligned}$$

as $m \to \infty$. Since $\{\mathcal{G}_n\}$ is uniformly bounded, it follows from Corollary 2.3 that $\|\mathcal{F}_n(h_{s_0}) - G_n(h_{s_0})x\| \to 0$ as $n \to \infty$. This and the assumption (A1) imply that $\lim_{n\to\infty} \frac{\mathcal{F}_n(h_{s_0})}{G_n(h_{s_0})} = x$. This completes the proof. □

Remark. Proposition 3.1 also holds for nets $\{\mathcal{F}_\lambda\}$, $\{G_\lambda\}$ of operators.

We can deduce from Proposition 3.1 the following ratio Tauberian theorem for positive functions.

Theorem 3.2. *Let μ be a positive measure on $[0,\infty)$ satisfying $\mu[0,\infty) > 0$ and the two conditions:*

$$\liminf_{\lambda\downarrow 0} \left(\mu[0, s_0/\lambda]\right) \Big/ \left(\int_0^\infty e^{-\lambda t} d\mu(t)\right) > 0 \text{ for some } s_0 > 0; \tag{C1}$$

$$\left(\mu[0,t]\right) / \left(\mu[0,s]\right) \to 1 \text{ as } t, s \to \infty \text{ with } t/s \to 1. \tag{C2}$$

Let $u(\cdot) : [0,\infty) \to X_+$ be a strongly measurable positive function in a Banach lattice X such that $\int_0^\infty e^{-\lambda t} u(t) dt$ exists for all $\lambda > 0$. Then

$$x = \lim_{\lambda\downarrow 0} \left(\int_0^\infty e^{-\lambda t} u(t) dt\right) \Big/ \left(\int_0^\infty e^{-\lambda t} d\mu(t)\right)$$

exists if and only if

$$x = \lim_{t\to\infty} \frac{1}{\mu[0,t]} \int_0^t u(s) ds.$$

Proof. Conditions (C1) and (C2) are not needed for the "if" part to hold. We omit its proof, which is essentially the same as that of Proposition 1.1. It remains to show the "only if" part. Define $G_\lambda(w) = \frac{\int_0^\infty e^{-\lambda t} w(e^{-\lambda t}) d\mu(t)}{\int_0^\infty e^{-\lambda t} d\mu(t)}$ for $w \in L^\infty[0,1]$. It is clear that G_λ is a positive linear functional on $L^\infty[0,1]$ with $\|G_\lambda\| = G_\alpha(1) = 1$, and $G_\lambda(h_s) = \frac{\mu[0,s/\lambda]}{\int_0^\infty e^{-\lambda t} d\mu(t)}$, $s > 0$. Thus (C1) implies that (A1) of Proposition 3.1 (with $W = L^\infty[0,1]$) holds.

Next, we see that condition (A2) holds. Let $\varepsilon > 0$ be arbitrary. By the assumption, there is a small $\delta > 0$ and large $M > 0$ such that

$$\left| \frac{\mu[0,t]}{\mu[0,s]} - 1 \right| < \varepsilon \text{ for } t, s \text{ such that } s, t > M \text{ and } |t/s - 1| < \delta.$$

For those $s > 0$ which are so close to s_0 that $|\frac{s}{s_0} - 1| < \delta$ and for sufficiently small $\lambda > 0$ we have $s/\lambda, s_0/\lambda > M$ and $|(s/\lambda)/(s_0/\lambda) - 1| = |\frac{s}{s_0} - 1| < \delta$, so that

$$\left| \frac{G_\lambda(h_s)}{G_\lambda(h_{s_0})} - 1 \right| = \left| \frac{\mu[0,s/\lambda]}{\mu[0,s_0/\lambda]} - 1 \right| < \varepsilon.$$

Then $|G_\lambda(h_s) - G_\lambda(h_{s_0})| < \varepsilon G_\lambda(h_{s_0}) \le \varepsilon ||h_{s_0}||_\infty$ for small enough λ. This implies that $\limsup_{\lambda \to 0} |G_\lambda(h_s - h_{s_0})| \le \varepsilon ||h_{s_0}||_\infty$ for s close enough to s_0, i.e., (A2) in Proposition 3.1 holds.

Let $\mathcal{F}_\lambda(w) = \frac{\int_0^\infty e^{-\lambda t} w(e^{-\lambda t}) w(t) dt}{\int_0^\infty e^{-\lambda t} d\mu(t)}$ for $w \in L^\infty[0,1]$. Then $\mathcal{F}_\lambda$ is a positive linear operator from $L^\infty[0,1]$ to X with $\|\mathcal{F}_\lambda\| = \|\mathcal{F}_\lambda(1)\| = \left\| \frac{\int_0^\infty e^{-\lambda t} u(t) dt}{\int_0^\infty e^{-\lambda t} d\mu(t)} \right\|$ (see the proof of Corollary 2.2), which tends to $\|x\|$ as $\lambda \downarrow 0$. It follows that the net $\{\mathcal{F}_\lambda\}_{\lambda \downarrow 0}$ is uniformly bounded.

Let $E := \{t^n; n \ge 0\}$. Since

$$\lim_{\lambda \downarrow 0} \frac{\mathcal{F}_\lambda(t^n)}{G_\lambda(t^n)} = \lim_{\lambda \downarrow 0} \frac{\int_0^\infty e^{-(n+1)\lambda t} u(t) dt}{\int_0^\infty e^{-(n+1)\lambda t} d\mu(t)} = x$$

for all $n \ge 0$, and since span$E$ is the set of all polynomials which is dense in $C[0,1]$, we can apply Proposition 3.1 (with $W = L^\infty[0,1]$) to conclude that

$$\frac{\int_0^{\frac{s_0}{\lambda}} u(t) dt}{\mu[0, s_0/\lambda]} = \frac{\mathcal{F}_\lambda(h_{s_0})}{G_\lambda(h_{s_0})} \to x \text{ as } \lambda \downarrow 0.$$

This proves the "only if" part. □

Remark. (i) If $0 < K := \lim_{\lambda \downarrow 0} \lambda \int_0^\infty e^{-\lambda t} d\mu(t) < \infty$, then by the Tauberian theorem of Hardy and Littlewood, $\lim_{t \to \infty} t^{-1} \int_0^t d\mu(r) = K$, so that conditions (C1) (with any $s_0 > 0$) and (C2) are satisfied. In this case, the conclusion of Theorem 3.2 also follows directly from the Tauberian theorem of Hardy and Littlewood together with Proposition 1.1; furthermore, we note that this also holds for a locally integrable function $u(\cdot) : [0, \infty) \to X$, with X a general Banach space, when $u(\cdot)$ is bounded.

(ii) A similar proof shows that Theorem 3.2 still holds when $t \to \infty$, $s \to \infty$, and $\lambda \downarrow 0$ are replaced by $t \to 0$, $s \to 0$, and $\lambda \to \infty$, respectively.

If g is a nonnegative function, with $\int_0^\infty g(t)dt > 0$, then with the measure μ defined as $\mu(\Omega) := \int_\Omega g(t)dt$, Theorem 3.2 becomes the following ratio Tauberian theorem.

Corollary 3.3. *Let g be a nonnegative function satisfying $\int_0^\infty g(t)dt > 0$ and the two conditions:*

$$\liminf_{\lambda\downarrow 0}\left(\int_0^{\frac{s_0}{\lambda}} g(t)dt\right) \Big/ \left(\int_0^\infty e^{-\lambda t} g(t)dt\right) > 0 \text{ for some } s_0 > 0; \qquad (\mathrm{C1}')$$

$$\left(\int_0^t g(r)dr\right) \Big/ \left(\int_0^s g(r)dr\right) \to 1 \text{ as } t, s \to \infty \text{ with } \frac{t}{s} \to 1. \qquad (\mathrm{C2}')$$

Let $u(\cdot) : [0,\infty) \to X_+$ be a strongly measurable positive function in a Banach lattice X such that $\int_0^\infty e^{-\lambda t} u(t)dt$ exists for all $\lambda > 0$. Then

$$x = \lim_{\lambda\downarrow 0}\left(\int_0^\infty e^{-\lambda t} u(t)dt\right) \Big/ \left(\int_0^\infty e^{-\lambda t} g(t)dt\right)$$

exists if and only if

$$x = \lim_{t\to\infty}\left(\int_0^t u(s)ds\right) \Big/ \left(\int_0^t g(s)ds\right).$$

Remark. If g is a bounded nonnegative function such that

$$K := \liminf_{t\to\infty} \frac{1}{t}\int_0^t g(r)dr > 0, \qquad (5)$$

then conditions (C1′) and (C2′) hold. In fact, we have

$$\liminf_{\lambda\downarrow 0}\left(\int_0^{\frac{s_0}{\lambda}} g(t)dt\right) \Big/ \left(\int_0^\infty e^{-\lambda t} g(t)dt\right) \geq \frac{K}{\|g\|_\infty} s_0 > 0 \text{ for all } s_0 > 0$$

and for sufficiently large s and $t > s$

$$\left|\left(\int_0^t g(r)dr\right) \Big/ \left(\int_0^s g(r)dr\right) - 1\right| \leq \frac{2}{K}\frac{t-s}{s}\|g\|_\infty \to 0 \text{ as } t, s \to \infty \text{ with } \frac{t}{s} \to 1.$$

Let W be the Banach sub-lattice of $L^\infty[0,1]$ consisting of all right-continuous bounded functions $u : [0,1] \to R$ which are also continuous at the point 1. Suppose $\{x_n\}$ is a sequence of positive elements in a Banach lattice X and $\{a_n\}$ is a sequence of positive numbers such that $\sum_{n=0}^\infty r^n a_n < \infty$ for all $0 < r < 1$ and $\sum_{n=0}^\infty a_n > 0$. Define linear operators $\mathcal{F}_r$ and linear functionals G_r on W by

$$\mathcal{F}_r(u) \quad := \quad \left(\sum_{n=0}^\infty r^n u(r^n) x_n\right) \Big/ \left(\sum_{n=0}^\infty r^n a_n\right)$$

and

$$G_r(u) := \left(\sum_{n=0}^{\infty} r^n u(r^n) a_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right)$$

for $u \in W$ and for those $0 < r < 1$ for which the definitions are well-defined. Then

$$\mathcal{F}_r(h_{s_0}) \quad = \quad \left(\sum_{n=0}^{[-s_0/\ln r]} x_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right)$$

and

$$G_r(h_{s_0}) = \left(\sum_{n=0}^{[-s_0/\ln r]} a_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right),$$

where $[-s_0/\ln r]$ denotes the largest integer less than or equal to $-s_0/\ln r$. By arguments similar to those in the proof of Theorem 3.2 we can deduce from Proposition 1.1 and Proposition 3.1 the following ratio Tauberian theorem for positive sequences in Banach lattices.

Theorem 3.4. *Let $\{a_n\}$ be a sequence of nonnegative numbers such that $\sum_{n=0}^{\infty} a_n > 0$. Suppose $\{a_n\}$ satisfies:*

$$\liminf_{r\uparrow 1} \left(\sum_{n=0}^{[-s_0/\ln r]} a_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right) > 0 \textit{ for some } s_0 > 0; \tag{D1}$$

$$\left(\sum_{k=0}^{m} a_k\right) \Big/ \left(\sum_{k=0}^{n} a_k\right) \to 1 \textit{ as } m, n \to \infty \textit{ with } \frac{m}{n} \to 1. \tag{D2}$$

Let $\{x_n\}$ be a sequence of positive elements in a Banach lattice X such that $\sum_{n=0}^{\infty} r^n x_n$ exists for all $0 < r < 1$. Then $x = \lim_{r\uparrow 1} \left(\sum_{n=0}^{\infty} r^n x_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right)$ exists if and only if $x = \lim_{n\to\infty} \left(\sum_{k=0}^{n} x_k\right) \Big/ \left(\sum_{k=0}^{n} a_k\right)$.

Remarks. (i) Let $\{a_n\}$ be as in Theorem 3.4, and $\{x_n\}$ be a sequence in a *general* Banach space X such that $\sum_{n=0}^{\infty} r^n x_n$ exists for all $0 < r < 1$ and

$\sup_{n\geq 0}\|x_n\|/a_n < \infty$ (we let $0/0 = 1$). Then the following weak-type result holds:

$$x = \text{weak-}\lim_{r\uparrow 1}\left(\sum_{n=0}^{\infty} r^n x_n\right)\Big/\left(\sum_{n=0}^{\infty} r^n a_n\right)$$
$$\Longrightarrow\ x = \text{weak-}\lim_{n\to\infty}\left(\sum_{k=0}^{n} x_k\right)\Big/\left(\sum_{k=0}^{n} a_k\right).$$

For, if $\{x_n\}$ is a sequence of real numbers, then, considering the sequence $\{x_n + Ca_n\}$ for some large enough constant $C > 0$, we may assume that $x_n + Ca_n \geq 0$ for all $n \geq 0$. Since we have that

$$x = \text{weak-}\lim_{r\uparrow 1}\left(\sum_{n=0}^{\infty} r^n x_n\right)\Big/\left(\sum_{n=0}^{\infty} r^n a_n\right)$$

exists if and only if

$$x + C = \text{weak-}\lim_{r\uparrow 1}\left(\sum_{n=0}^{\infty} r^n (x_n + Ca_n)\right)\Big/\left(\sum_{n=0}^{\infty} r^n a_n\right),$$

it follows from Theorem 3.4 that the assertion holds. Since for a sequence $\{b_n\}$ of complex numbers we have $\lim_{n\to\infty} b_n$ exists if and only if both $\lim_{n\to\infty}\mathrm{Re}(b_n)$ and $\lim_{n\to\infty}\mathrm{Im}(b_n)$ exist, from the above real case together with a standard argument the general case follows easily. The same assertion also holds for the continuous case.

(ii) If the sequence $\{a_n\}$ in Theorem 3.4 satisfies $\sum_{n=0}^{\infty} a_n < \infty$, then clearly conditions (D1) (with any $s_0 > 0$) and (D2) are satisfied automatically, and one can directly see that both limits in the conclusion of Theorem 3.4 are equal to $(\sum_{n=0}^{\infty} x_n)/(\sum_{n=0}^{\infty} a_n)$, when either $\{x_n\}$ is a positive sequence in a Banach lattice such that $\sum_{n=0}^{\infty} x_n$ converges, or when $\{x_n\}$ is a sequence in a general Banach space satisfying $\sup_{n\geq 0}\|x_n\|/a_n < \infty$.

(iii) Discrete analogs of Remark (i) after Theorem 3.2 and the Remark after Corollary 3.3 also hold. In particular, here is an example of sequence $\{a_n\}$ of nonnegative real numbers which is bounded and satisfies the discrete counterpart of condition (5), and hence satisfies conditions (D1) and (D2).

Example 1. Let $\{k_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of nonnegative integers with positive lower density (i.e., $\sup_{j\geq 1} k_j/j < \infty$). Define

$$a_n = \begin{cases} 1 & \text{if } n = k_j \text{ for some } j \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, putting $K = \liminf_{j\to\infty} j/k_j$ (> 0), we have

$$\liminf_{n\to\infty}\frac{1}{n+1}\sum_{j=0}^{n} a_j = \liminf_{j\to\infty}\frac{j}{k_j} = K > 0.$$

Then using the same argument as in the Remark after Corollary 3.3 together with the fact that $\lim_{r\uparrow 1}(1-r)([-s/\log r]+1) = s$ one can verify conditions (D1) and (D2). Hence Theorem 3.4 holds for this sequence $\{a_n\}$. It is interesting to note that there are many papers treating mean and pointwise ergodic theorems for sequences with positive lower density (e.g., see [1] and [4]; see also Chapter 8 of [5]).

4. Generalized Tauberian Theorems

Next, we deduce from Theorem 3.2 the following generalized Tauberian theorem for positive functions.

Theorem 4.1. *Let $\gamma \geq 0$ and $u(\cdot) : [0,\infty) \to X_+$ be a strongly measurable positive function such that $\int_0^\infty e^{-\lambda t}u(t)dt$ exists for all $\lambda>0$. Then $x = \lim\limits_{\lambda\downarrow 0} \lambda^\gamma \int_0^\infty e^{-\lambda t}u(t)dt$ exists if and only if $x = \lim\limits_{t\to\infty} \frac{\Gamma(\gamma+1)}{t^\gamma}\int_0^t u(s)ds$. The assertion also holds when $t \to \infty$ and $\lambda \downarrow 0$ are replaced by $t \to 0$ and $\lambda \to \infty$, respectively.*

Proof. For the case $\gamma > 0$ let μ be defined as $\mu(\Omega) := \int_\Omega g(t)dt$ with $g(t) := \begin{cases} t^{\gamma-1}/\Gamma(\gamma), & t > 0; \\ 0, & t = 0 \end{cases}$. Then $\int_0^\infty e^{-\lambda t}d\mu(t) = 1/\lambda^\gamma$ and $\mu[0,t] = t^\gamma/\Gamma(\gamma+1)$, and (C2) is satisfied. Since

$$\left((\frac{s}{\lambda})^\gamma/\Gamma(\gamma+1)\right) \Big/ (1/\lambda^\gamma) = (\Gamma(\gamma+1))^{-1}s^\gamma$$

for all $\lambda > 0$, clearly condition (C1) also holds for all $s_0 > 0$. For the case $\gamma = 0$, let μ be the Dirac measure at 0, which is the measure such that $\mu(\{0\}) = 1$ and $\mu(0,\infty) = 0$. Then $\int_0^\infty e^{-\lambda t}d\mu(t) = e^{-\lambda 0} = 1$ for all $\lambda > 0$ and $\mu[0,t] = 1$ for all $t > 0$. Clearly both conditions (C1) and (C2) hold. Thus the assertion follows from Theorem 3.2. □

Now we deduce from Theorem 3.4 the following generalized Tauberian theorem for positive sequences.

Theorem 4.2. *Let $\gamma \geq 0$ and let $\{x_n\} \subset X_+$ be a sequence of positive elements in a Banach lattice X such that $\sum\limits_{n=0}^\infty r^n x_n$ exists for all $0 < r < 1$. Then $x = \lim\limits_{r\uparrow 1}(1-r)^\gamma \sum\limits_{n=0}^\infty r^n x_n$ exists if and only if $x = \lim\limits_{n\to\infty} \frac{\Gamma(\gamma+1)}{(n+1)^\gamma}\sum\limits_{k=0}^n x_k$.*

Proof. For the case $\gamma = 0$, we can take $a_0 = 1$ and $a_n = 0$ for all $n = 1, 2, \ldots$. *Then* $\sum\limits_{k=0}^\infty r^k a_n = \sum\limits_{k=0}^n a_k = 1$ for all $n = 0, 1, 2, \ldots$. Hence both (D1) and (D2) are satisfied and Theorem 3.4 applies.

For the case $\gamma > 0$, we can take $a_n := \binom{-\gamma}{n}(-1)^n$. Then $\sum\limits_{n=0}^\infty r^n a_n = (1-r)^{-\gamma}$ for $0 < r < 1$ and $\sum\limits_{k=0}^n a_k = \sum\limits_{k=0}^n \binom{-\gamma}{k}(-1)^k = \sum\limits_{k=0}^n \binom{k+\gamma-1}{k} = \binom{n+\gamma}{n} = \frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\Gamma(n+1)}$.

By Stirling's formula: $\lim_{t\to\infty} \Gamma(t+1)/((\frac{t}{e})^t\sqrt{2\pi t}) = 1$ (see [7, p. 194]) and the fact that $\lim_{r\uparrow 1}(\ln r)/(r-1) = 1$, we have

$$\left(\frac{\Gamma([\frac{-s}{\ln r}]+\gamma+1)}{\Gamma(\gamma+1)\Gamma([\frac{-s}{\ln r}]+1)}\right) \Big/ \left(\frac{1}{(1-r)^\gamma}\right) = \frac{\Gamma([\frac{-s}{\ln r}]+\gamma+1)(1-r)^\gamma}{\Gamma(\gamma+1)\Gamma([\frac{-s}{\ln r}]+1)}$$

$$\sim \frac{1}{\Gamma(\gamma+1)} \frac{\left(\frac{[\frac{-s}{\ln r}]+\gamma}{e}\right)^{[\frac{-s}{\ln r}]+\gamma}\sqrt{2\pi([\frac{-s}{\ln r}]+\gamma)}}{\left(\frac{[\frac{-s}{\ln r}]}{e}\right)^{[\frac{-s}{\ln r}]}\sqrt{2\pi[\frac{-s}{\ln r}]}}(1-r)^\gamma$$

$$= \frac{1}{\Gamma(\gamma+1)} \left(\frac{[\frac{-s}{\ln r}]+\gamma}{[\frac{-s}{\ln r}]}\right)^{[\frac{-s}{\ln r}]+1/2} e^{-\gamma} \left(\left(\left[\frac{-s}{\ln r}\right]+\gamma\right)(1-r)\right)^\gamma$$

$$\sim \frac{1}{\Gamma(\gamma+1)} \left(\left(\left[\frac{-s}{r-1}\right]+\gamma\right)(1-r)\right)^\gamma \to \frac{s^\gamma}{\Gamma(\gamma+1)}$$

as $r \uparrow 1$. Here $a(r) \sim b(r)$ as $r \uparrow 1$ means that the ratios $a(r)/b(r)$ and $b(r)/a(r)$ are both bounded in some interval $(\delta, 1)$. This implies that (D1) holds for all $s_0 > 0$.

Next we check (D2). In fact, we have

$$\lim_{m,n\to\infty} \left(\frac{\Gamma(m+\gamma+1)}{\Gamma(\gamma+1)\Gamma(m+1)}\right) \Big/ \left(\frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\Gamma(n+1)}\right)$$

$$= \lim_{m,n\to\infty} \left(\frac{(\frac{m+\gamma}{e})^{m+\gamma}\sqrt{2\pi(m+\gamma)}}{(\frac{m}{e})^m\sqrt{2\pi m}}\right) \Big/ \left(\frac{(\frac{n+\gamma}{e})^{n+\gamma}\sqrt{2\pi(n+\gamma)}}{(\frac{n}{e})^n\sqrt{2\pi n}}\right)$$

$$= \lim_{m,n\to\infty} \left[\left(\frac{m+\gamma}{m}\right)^{m+1/2} \Big/ \left(\frac{n+\gamma}{n}\right)^{n+1/2}\right] \left(\frac{m+\gamma}{n+\gamma}\right)^\gamma$$

$$= \lim_{m,n\to\infty} \left(\frac{m+\gamma}{n+\gamma}\right)^\gamma,$$

which is equal to 1 if $m/n \to 1$.

Thus Theorem 3.4 applies. Since

$$\lim_{n\to\infty} \left(\frac{\Gamma(n+\gamma+1)}{\Gamma(\gamma+1)\Gamma(n+1)}\right) \Big/ \left(\frac{(n+1)^\gamma}{\Gamma(\gamma+1)}\right) = \lim_{n\to\infty} \frac{\Gamma(n+\gamma+1)}{\Gamma(n+1)(n+1)^\gamma}$$

$$= \lim_{n\to\infty} \frac{(\frac{n+\gamma}{e})^{n+\gamma}\sqrt{2\pi(n+\gamma)}}{(\frac{n}{e})^n\sqrt{2\pi n}(n+1)^\gamma}$$

$$= \lim_{n\to\infty} \left(\frac{n+\gamma}{n}\right)^{n+1/2} e^{-\gamma} \left(\frac{n+\gamma}{n+1}\right)^\gamma = 1,$$

the conclusion of Theorem 3.4 reduces to the assertion. □

Remark. In the above, we deduce Theorems 4.1 and 4.2 from Theorems 3.2 and 3.4, respectively, and the latter are deduced from Proposition 1.1 and Proposition 3.1, which in turn follow from Corollary 2.3. One can also deduce Theorems 4.1 and 4.2 more directly from Corollary 2.2. We refer to [6, Propositions 4.2 and 4.4] for such different approaches.

5. Counter-examples

In this section, we show by examples that the assumption of positivity of the sequence $\{x_n\}$ and conditions (D1) and (D2) are essential in Theorem 3.4.

Example 2. Let $a_0 = 1$ and $a_n = n^{-1} - (n+1)^{-1}$ for $n \geq 1$. Since $\sum_{k=0}^{\infty} a_k = 2 < \infty$, (D1) and (D2) hold automatically (see Remark (ii) after Theorem 3.4).

Thus we can apply Theorem 3.4 to this sequence $\{a_n\}$, and we have that for any positive sequence $\{x_n\}$ in a Banach lattice, $x = \lim_{r\uparrow 1} \sum_{n=0}^{\infty} r^n x_n$ exists if and only if $x = \lim_{n\to\infty} \sum_{k=0}^{n} x_k$. This is also the case $\gamma = 0$ of Theorem 4.2.

But, if we let $b_n = (-e^{i\theta})^n$ for $n \geq 0$, where $i = \sqrt{-1}$ and $\theta \neq \pi (\mathrm{mod}\ 2\pi)$, then $\sum_{k=0}^{n} b_k = \frac{1-(-e^{i\theta})^{n+1}}{1+e^{i\theta}}$. While

$$\lim_{r\uparrow 1} \frac{\sum_{n=0}^{\infty} r^n b_n}{\sum_{n=0}^{\infty} r^n a_n} = \lim_{r\uparrow 1} \frac{1}{2} \sum_{n=0}^{\infty} r^n (-e^{i\theta})^n = \lim_{r\uparrow 1} \frac{1}{2(1+re^{i\theta})} = \frac{1}{2(1+e^{i\theta})},$$

$\lim_{n\to\infty} \left(\sum_{k=0}^{n} b_k \right) \Big/ \left(\sum_{k=0}^{n} a_k \right)$ does not exist. This sequence $\{b_n\}$ is bounded but not positive and $\sup_{n\geq 0} |b_n|/a_n = \infty$, so this shows that the assertion in Theorem 3.4 (resp. Remark (i) after Theorem 3.4) may fail if $\{x_n\}$ is not a positive sequence (resp. $\sup_{n\geq 0} \|x_n\|/a_n = \infty$).

Example 3. Let $\{a_n\}$ be a sequence of numbers with $\limsup_{n\to\infty} |a_n|^{1/n} \leq 1$, and let $b_0 = 0$ and $b_n = a_{n-1}$ for all $n \geq 1$. Then we have for every $0 < r < 1$

$$\sum_{n=0}^{\infty} r^n a_n = r^{-1} \sum_{n=1}^{\infty} r^n b_n = r^{-1} \sum_{n=0}^{\infty} r^n b_n.$$

Hence

$$\frac{\sum_{n=0}^{\infty} r^n b_n}{\sum_{n=0}^{\infty} r^n a_n} = r \to 1 \text{ as } r \to 1.$$

If we define $a_n := \begin{cases} n & \text{if } n = 2^k \text{ for some } k = 0, 1, \dots; \\ 0 & \text{otherwise,} \end{cases}$ then both $\{a_n\}$ and $\{b_n\}$ are positive sequences, and for every $k = 0, 1, 2, \dots$

$$\sum_{j=0}^{2^k} b_j = \sum_{j=0}^{2^k-1} a_j = \sum_{j=0}^{k-1} 2^j = 2^k - 1,$$

$$\sum_{j=0}^{2^k+1} b_j = \sum_{j=0}^{2^k} a_j = \sum_{j=0}^{k} 2^j = 2^{k+1} - 1,$$

and

$$\sum_{j=0}^{2^k+1} a_j = \sum_{j=0}^{2^k-1} a_j + a_{2^k} + a_{2^k+1} = 2^k - 1 + 2^k = 2^{k+1} - 1.$$

Hence

$$\frac{\sum_{j=0}^{n} b_j}{\sum_{j=0}^{n} a_j} = \begin{cases} \frac{2^k-1}{2^{k+1}-1} & \text{if } n = 2^k; \\ 1 & \text{if } n = 2^k + 1. \end{cases}$$

Thus the limit does not exist as $n \to \infty$. In the following we check that (D1) holds but (D2) does not.

Letting $S(r) = \sum_{n=0}^{\infty} 2^n r^{2^n} = r + 2r^2 + 4r^4 + \dots$ for $0 < r < 1$, we have

$$\frac{r}{1-r} = r + (r^2 + r^3) + (r^4 + \dots + r^7) + \dots < S(r),$$

and

$$\begin{aligned} S(r) &< r + 2\{r^2 + (r^3 + r^4) + (r^5 + \dots + r^8) + \dots\} \\ &= \frac{r}{1-r} + \frac{r^2}{1-r} = \frac{r(1+r)}{1-r}, \end{aligned}$$

so that $S(r) \sim (1-r)^{-1}$ as $r \uparrow 1$.

On the other hand, since $\lim_{r \uparrow 1} (\ln r)/(r-1) = 1$, we have

$$2^{[\ln(\frac{-s}{\ln r})/\ln 2]+1} \sim 2^{\ln(\frac{-s}{\ln r})/\ln 2} = (2^{1/\ln 2})^{\ln(\frac{-s}{\ln r})} = \frac{-s}{\ln r} \sim \frac{s}{1-r} \text{ as } r \uparrow 1.$$

Therefore we have for every $s > 0$

$$\begin{aligned} &\left(\sum_{n=0}^{[-s/\ln r]} a_n\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right) = \left(\sum_{1 \le 2^k \le -s/\ln r} a_{2^k}\right) \Big/ \left(\sum_{n=0}^{\infty} r^n a_n\right) \\ &= \left(\sum_{k=0}^{m-1} 2^k\right) \Big/ \left(\sum_{n=0}^{\infty} 2^n r^{2^n}\right) = (2^m - 1) \Big/ \left(\sum_{n=0}^{\infty} 2^n r^{2^n}\right) \\ &\sim \frac{s}{1-r} \Big/ \frac{1}{1-r} = s \end{aligned}$$

as $r \uparrow 1$, where $m := [\ln(-s/\ln r)/\ln(2)] + 1$. This implies that (D1) holds for all $s_0 > 0$.

If $m = 2^k$ and $n = 2^k - 1$, then $m/n \to 1$ as $k \to \infty$, but we have that

$$\left(\sum_{k=0}^{m} a_k\right) \Big/ \left(\sum_{k=0}^{n} a_k\right) = \frac{2^{k+1} - 1}{2^k - 1} \to 2.$$

Hence condition (D2) of Theorem 3.4 is not satisfied. This example shows that Theorem 3.4 may fail if condition (D2) is not satisfied.

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