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Fuzzy Modeling and Control for Conical Magnetic Bearings Using Linear Matrix Inequality

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Abstract. A general nonlinear model with six degree-of-freedom rotor dynamics and electromagnetic force equations for conical magnetic bearings is developed. For simplicity, a T–S (Takagi–Sugeno) fuzzy model for the nonlinear magnetic bearings assumed no rotor eccentricity is first derived, and a fuzzy control design based on the T–S fuzzy model is then proposed for the high speed and high accuracy control of the complex magnetic bearing systems. The suggested fuzzy control design approach for nonlinear magnetic bearings can be cast into a linear matrix inequality (LMI) problem via robust performance analysis, and the LMI problem can be solved efficiently using the convex optimization techniques. Computer simulations are presented for illustrating the performance of the control strategy considering simultaneous rotor rotation tracking and gap deviations regulation.

Key words: magnetic bearing, rotor dynamics, T–S fuzzy model, fuzzy control, linear matrix inequality, robust performance.

1. Introduction

Due to the noncontact nature of a magnetic suspension, motion resolution of the suspended object either in translation or in high-speed rotation is limited only by the actuators, sensors, and servo system used [33, 38]. The active control problem of a magnetic bearing is complicated due to its inherent nonlinearities associated with the electromechanical dynamics, e.g., gap nonlinearity, gyroscopic effects, and mass unbalance induced vibration. Traditional decentralized PID control used for magnetic bearings still has some limits to fully exploit the possible active potentials for permitting a much higher degree of control of rotor vibration, positioning, and alignment [1]. Many of the previous active magnetic bearing (AMB) control techniques are based on the approximately linearized model. For example, LQR method is used in [22, 40, 26], LQG/LTR and quantitative feedback theory (QFT) are considered in [29], linear Q-parameterization theory is used in [24, 23], and gain scheduled H_∞ control is proposed in [21].

Since an MIMO magnetic bearing system is highly nonlinear, nonlinear control techniques are natural choices that can provide more complete consideration of

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the gyroscopic effects and rotor unbalance response [25, 8] based on the complex nonlinear (or simplified linear) model, and even permit greater use of available clearance during operation [15]. Feedback linearizations are used in [34, 19, 3] for globally exactly linearized control of nonlinear bearing systems. Sliding mode controls are considered in [34, 27, 30] for better robustness with respect to uncertainty. A backstepping-type control is proposed in [5] for a planar rotor disk AMB system to yield global exponential position tracking over the entire clearance. An adaptive autocentering method for reducing the control response to imbalance is proposed in [20], where on-line estimation of the center of mass position and velocity and incorporation of these into a feedback control are considered. Recently, some nontraditional (model free) nonlinear methods are considered for robust and/or learning control of AMB systems, e.g., fuzzy controls in [19, 10, 18], and neural network controls in [6, 13]. Since the difficulty of obtaining an accurate model for a complex AMB system, Schroder et al. [28] propose an on-line tuning control for enhancing the AMB's robustness with respect to disturbance and uncertainty.

Recently, fuzzy model-based control for nonlinear systems via linear matrix inequality (LMI) has found great interests, e.g., [36, 39, 31, 32, 11, 17]. Through a suitable fuzzy partitioning of the whole operation region of a complex nonlinear system, the T–S fuzzy model comprised of a family of local linear models can be constructed for approximating the nonlinear dynamics. A stable fuzzy controller composed of local controllers for each fuzzy subregion can then be constructed based on the T–S fuzzy model. In practice, the T–S fuzzy controller is much easier to implement than the conventional nonlinear controllers that are synthesized based on the plant's original complex nonlinear model derived via first principles. The stability and control design issues using T–S fuzzy model via LMI are discussed in [36, 39]. Practical predictive functional control based on fuzzy model is considered in [31, 32]. Since the dynamics model of a multiple degree-of-freedom (DOF) AMB system is highly complex, the T–S fuzzy model-based control is very promising for the AMB applications [9].

In this paper, we consider the T–S fuzzy modeling and control for the general six-DOF conical magnetic bearings [24, 19, 16]. Based on some simplifying assumptions, a T–S fuzzy model with very small number of local models can be analytically constructed for the complex AMB systems. The parallel distributed compensation (PDC) approach [39, 36] is then adopted for synthesizing a stable fuzzy control for the high speed and high accuracy control of the six-DOF AMB systems, based on the derived T–S fuzzy model. Through the robust performance analysis for disturbance rejection the control problem is translated into a linear matrix inequality problem. By considering it as a generalized eigenvalue minimization problem (GEMP), the required common Lyapunov matrix and the local feedback gain matrices can be determined. The performance of the suggested control strategy can be well tested via computer simulations.

The paper is organized as follows: Section 2 presents the dynamics model derivation of a conical magnetic bearing system. A T–S fuzzy model for magnetic bearing systems is proposed in Section 3.1. In Section 3.2, an LMI-based fuzzy control design using the derived T–S fuzzy model is suggested, and the LMI conditions are found. Representative simulation results are shown in Section 4. Finally, conclusions are made in Section 5.

2. Modeling of Six-DOF Magnetic Bearings

In this section, three-dimensional dynamics equations for a rotating rigid shaft with eccentricity will be first developed. Then the electromagnetic system model for the magnetic bearings with conical air gaps at two sides of the rotor (refer to Figure 1) will be formulated for constructing the complete mathematical model of the AMB system. Since conical gaps are considered, no axial electromagnets are needed and two pairs of electromagnets at each side of the rotor are sufficient for regulating the floating rotor with eccentricity.

2.1. SIX DEGREE-OF-FREEDOM ROTOR DYNAMICS MODEL

Consider the rigid rotor with eccentricity $\mathbf{e} = [e_1\ e_2\ e_3]^{\mathrm{T}}$ shown in Figure 1, where G_{c} and G are the centers of mass and geometry of the whole rotor, respectively. Let O be the geometric center of the magnetic bearing's stators. The operation of the magnetic bearing system is to keep the geometric centers of the rotor and the stator consistent with each other under arbitrary desired rotor-rotating motion, driven by the motor's electromagnetic torque T_m.

Each electromagnet in the left- and right-side stators generates its electromagnetic force f_i, $i = l_1, l_2, l_3, l_4$, and r_1, r_2, r_3, r_4, which has radial and axial compo-

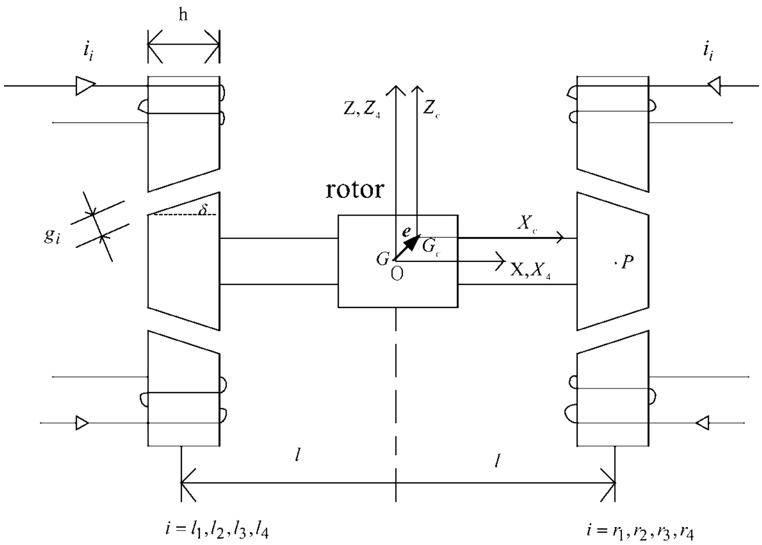

Figure 1. Scheme of a conical magnetic bearing system.

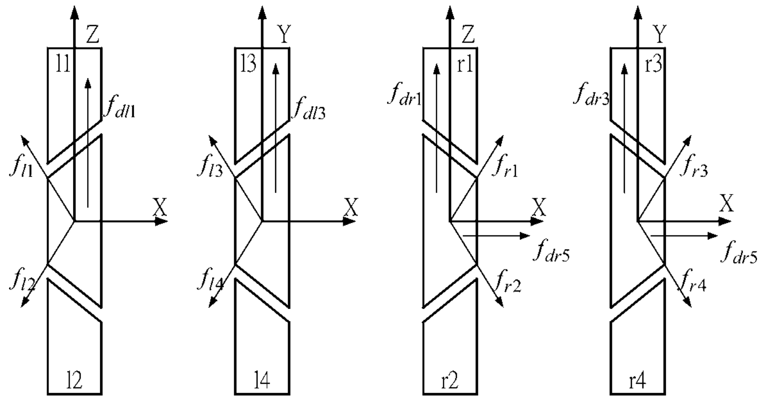

Figure 2. Magnetic and disturbance forces.

nents to control the radial and axial movements of the rotor, respectively. To consider the effects of load disturbance, five external disturbance forces: f_{d1}, f_{d2}, f_{d3}, f_{d4}, and f_{d5}, shown in Figure 2, are included in the derivation of the dynamics model.

Let frame *OXYZ* ({0}) be the inertial reference frame fixed at stator's geometric center O; frame $Gx_4y_4z_4$ ({4}) be the moving frame attached to the geometric center G of the floating and rotating rotor; and frame $G_cx_cy_cz_c$ ({C}) be the frame attached to the center of mass G_c and parallel to frame {4}. Then the three-dimensional translation and rotation of the rotor can be described by the motion of frame $Gx_4y_4z_4$ by defining three intermediate moving frames: $Gx_1y_1z_1$ ({1}), $Gx_2y_2z_2$ ({2}), and $Gx_3y_3z_3$ ({3}). Frame $Gx_1y_1z_1$ is assumed attached to the geometric center G of the rotor with its x_1, y_1, and z_1 axes parallel to the x, y, and z axes of frame {0}, respectively, and obtained by assuming that the rotor is translated from O to the current position (x_0, y_0, z_0) with respect to frame *OXYZ*. Frame $Gx_2y_2z_2$ is obtained by rotating frame $Gx_1y_1z_1$ about y_1 axis with angle φ, frame $Gx_3y_3z_3$ is obtained by rotating frame $Gx_2y_2z_2$ about z_2 axis with angle θ, and frame $Gx_4y_4z_4$ is defined by a rotation ϕ of $Gx_3y_3z_3$ about x_3 axis (rotor axis). Thus the position and orientation of the floating and rotating rotor with respect to the inertial frame {0} can be expressed as [4]

$$ {}^0_4\mathbf{T} = {}^0_1\mathbf{T}\,{}^1_2\mathbf{T}\,{}^2_3\mathbf{T}\,{}^3_4\mathbf{T} = \begin{bmatrix} {}^0_4\mathbf{R} & \mathbf{r}_G \\ \mathbf{0} & 1 \end{bmatrix} \tag{1} $$

where ${}^0_4\mathbf{R} = [r_{ij}]_{3\times 3}$ is the rotation matrix, and $\mathbf{r}_G = [x_0\ y_0\ z_0]^{\mathrm{T}}$.

The position vector ${}^0\mathbf{G}_c$ of the center of mass of the rotor with eccentricity ${}^4\mathbf{e} = [e_1\ e_2\ e_3]^{\mathrm{T}}$ can be expressed as

$$ {}^0\mathbf{G}_c = {}^0\mathbf{G} + {}^0_4\mathbf{R}\,{}^4\mathbf{e} \tag{2} $$

where ${}^0\mathbf{G} = [x_0\ y_0\ z_0]$ is the position vector of the geometric center G of the rotor. Its velocity vector in terms of {0} is thus ${}^0\boldsymbol{v}_c = [v_x\ v_y\ v_z]^{\mathrm{T}} = {}^0\dot{\mathbf{G}}_c$. The angular velocity vector of the rotor can be expressed as $\boldsymbol{\omega} = \dot{\varphi}\hat{\mathbf{j}}_1 + \dot{\theta}\hat{\mathbf{k}}_2 + \dot{\phi}\hat{\mathbf{i}}_3$, where $\hat{\mathbf{j}}_1 = \hat{\mathbf{j}}$, $\hat{\mathbf{k}}_2 = \sin\varphi\hat{\mathbf{i}} + \cos\varphi\hat{\mathbf{k}}$, $\hat{\mathbf{i}}_3 = \cos\theta\cos\varphi\hat{\mathbf{i}} - \cos\theta\cos\varphi\hat{\mathbf{k}} + \sin\theta\hat{\mathbf{j}}$, and $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors of the x, y, and z axes of {0}, respectively. Hence, the angular velocity of the rotor in terms of {0} is

$$^0\boldsymbol{\omega} = [\sin\varphi\dot{\theta} + \cos\varphi\cos\theta\dot{\phi}\ \ \dot{\varphi} + \sin\theta\dot{\phi}\ \ \cos\varphi\dot{\theta} - \sin\varphi\cos\theta\dot{\phi}]^{\mathrm{T}}. \tag{3}$$

The translational and rotational kinetic energy of the whole rotor can then be expressed, respectively, as [35]

$$T_m = \frac{1}{2}M_{\mathrm{c}}\left(v_x^2 + v_y^2 + v_z^2\right), \tag{4}$$

$$T_r = \frac{1}{2}{}^{\mathrm{c}}\boldsymbol{\omega}^{\mathrm{T}}\,\mathbf{I}_{\mathrm{c}}\,{}^{\mathrm{c}}\boldsymbol{\omega} = \frac{1}{2}{}^{0}\boldsymbol{\omega}^{\mathrm{T}}\,\mathbf{I}_0\,{}^{0}\boldsymbol{\omega} \tag{5}$$

where M_{c} is the total mass of the rotor, ${}^{\mathrm{c}}\boldsymbol{\omega}$ is the angular velocity of the rotor in terms of {C}, and ${}^{\mathrm{c}}\boldsymbol{\omega} = {}^4\boldsymbol{\omega} = {}^4_0\mathbf{R}\,{}^0\boldsymbol{\omega}$. Here $\mathbf{I}_{\mathrm{c}}$ and $\mathbf{I}_0 (= {}^0_{\mathrm{c}}\mathbf{R}\,\mathbf{I}_{\mathrm{c}}\,{}^0_{\mathrm{c}}\mathbf{R}^{\mathrm{T}})$ are the inertia tensors of the rotor with respect to {C} and {0}, respectively. Since the eccentricity is usually very small, the products of inertia with respect to {C} are negligible. Thus, $\mathbf{I}_{\mathrm{c}} = diag[I_{\mathrm{c}xx}, I_{\mathrm{c}yy}, I_{\mathrm{c}zz}]$, where $I_{\mathrm{c}xx}$, $I_{\mathrm{c}yy}$, and $I_{\mathrm{c}zz}$ are the mass moments of inertia of the rotor about the x-, y-, and z-axes of {C}, respectively. The total kinetic energy T and total potential energy U of the rotor are

$$T = T_m + T_r, \tag{6}$$

$$U = -M_{\mathrm{c}}\mathbf{g}^{\mathrm{T}}\,{}^0\mathbf{p}_{\mathrm{c}}, \tag{7}$$

where $\mathbf{g} = [0\ 0\ {-g_{\mathrm{c}}}]^{\mathrm{T}}$, and g_{c} is the gravity constant.

By substituting the total kinetic and potential energies into the Lagrange equation [4, 35]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial\dot{q}_i}(T - U)\right] - \frac{\partial}{\partial q_i}(T - U) = Q_{q_i} \tag{8}$$

where q_i, $i = 1, 2, \ldots, 6$, are the generalized coordinates of the rotor system, and $\mathbf{q} = [q_1, q_2, \ldots, q_6]^{\mathrm{T}} = [x_0, y_0, z_0, \varphi, \theta, \phi]^{\mathrm{T}}$; $Q_{q_i}, i = 1, 2, \ldots, 6$, are the generalized forces/torques, we can obtain the dynamics model of the rotor with eccentricity and disturbance forces as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{F}_{\mathbf{g}} = \mathbf{Q} \tag{9}$$

where $\mathbf{M} = [m_{ij}]_{6\times6}$ is the symmetric positive-definite inertia matrix; $\mathbf{C} = [c_{ij}]_{6\times6}$ is the coefficient matrix for the velocity-dependent terms; $\mathbf{F}_{\mathbf{g}} = [f_{g_1}\ \ldots\ f_{g_6}]^{\mathrm{T}}$ is the gravity forces/torques vector; and $\mathbf{Q} = [Q_{x_0}, Q_{y_0}, Q_{z_0}, Q_{\varphi}, Q_{\theta}, Q_{\phi}]^{\mathrm{T}}$ is the generalized forces/torques vector. Since $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{F}_{\mathbf{g}}$ are lengthy, they are not included here.

The generalized forces/torques can be derived by

$$Q_{q_i} = {}^0\mathbf{F}_r^{\mathrm{T}} \frac{\partial {}^0\mathbf{r}_r}{\partial q_i} + {}^0\mathbf{F}_l^{\mathrm{T}} \frac{\partial {}^0\mathbf{r}_l}{\partial q_i} + (T_m - T_l)\frac{\partial \phi}{\partial q_i}, \quad i = 1, 2, \dots, 6, \tag{10}$$

where ${}^0\mathbf{F}_r$ and ${}^0\mathbf{F}_l$ are the resultant forces acting at the right- and left-side geometric centers of the bearings, respectively [12] (refer to Figures 1 and 2); ${}^0\mathbf{r}_r$ and ${}^0\mathbf{r}_l$ are the position vectors of the right- and left-side geometric centers, respectively; T_m and T_l are the motor driving and disturbance toques, respectively; and

$$\begin{aligned}
{}^0\mathbf{F}_r &= \big[(f_{r1} + f_{r2} + f_{r3} + f_{r4})\sin\delta + f_{dr5}, (f_{r3} - f_{r4})\cos\delta \\
&\quad + f_{dr3}, (f_{r1} - f_{r2})\cos\delta + f_{dr1}\big]^{\mathrm{T}}, \\
{}^0\mathbf{F}_\mathbf{l} &= \big[-(f_{l1} + f_{l2} + f_{l3} + f_{l4})\sin\delta, (f_{l3} - f_{l4})\cos\delta + f_{dl3}, \\
&\qquad (f_{l1} - f_{l2})\cos\delta + f_{dl1}\big]^{\mathrm{T}}, \\
{}^0\mathbf{r}_r &= [x_0 + lr_{11}, y_0 + lr_{21}, z_0 + lr_{31}]^{\mathrm{T}}, \\
{}^0\mathbf{r}_l &= [x_0 - lr_{11}, y_0 - lr_{21}, z_0 - lr_{31}]
\end{aligned}$$

where f_{dr1}, f_{dr3}, f_{dr5} and f_{dl1}, f_{dl3} are the disturbance forces shown in Figure 2.

2.2. AIR GAP DEVIATIONS

The air gap widths between the stators of the electromagnets and the rotor can be expressed as

$$g_i = d_{oi} + g_i', \quad i = r1, \dots, r4, l1, \dots, l4, \tag{11}$$

where d_{0i} is the steady-state gap width and g_i' is the gap deviation from d_{0i}. Refer to Figures 1 and 2, the position vector of the geometric center P of the right-side rotor in terms of $\{0\}$ can be represented as ${}^0\mathbf{P} = {}^0_4\mathbf{T}\,{}^4\mathbf{P}$, where ${}^4\mathbf{P} = [l\ 0\ 0\ 1]^{\mathrm{T}}$. The gap deviations g_{r1}', g_{r3}', g_{l1}' and g_{l3}' can be computed as follows:

$$g_{r1}' = [0\ 0\ 1\ 1]\big([l\ 0\ 0\ 1]^{\mathrm{T}} - {}^0\mathbf{P}\big)\cos\delta \approx (l\sin\varphi - z_0)\cos\delta, \tag{12}$$

$$g_{r3}' = [0\ 1\ 0\ 1]\big([l\ 0\ 0\ 1]^{\mathrm{T}} - {}^0\mathbf{P}\big)\cos\delta \approx (-l\sin\theta - y_0)\cos\delta \tag{13}$$

where δ is the conical angle of the bearings; and φ and θ are assumed very small. Similarly,

$$\begin{aligned}
g_{l1}' &= [0\ 0\ 1\ 1]\cdot\big([-l\ 0\ 0\ 1]^{\mathrm{T}} - {}^0_4\mathbf{T}\cdot[-l\ 0\ 0\ 1]^{\mathrm{T}}\big)\cos\delta \\
&\approx (-l\sin\varphi - z_0)\cos\delta, \qquad (14) \\
g_{l3}' &= [0\ 1\ 0\ 1]\cdot\big([-l\ 0\ 0\ 1]^{\mathrm{T}} - {}^0_4\mathbf{T}\cdot[-l\ 0\ 0\ 1]^{\mathrm{T}}\big)\cos\delta \\
&\approx (l\sin\theta - y_0)\cos\delta. \qquad (15)
\end{aligned}$$

Since the rotor is assumed rigid, we have

$$g_{r2}' = -g_{r1}', \qquad g_{r4}' = -g_{r3}', \qquad g_{l2}' = -g_{l1}', \qquad g_{l4}' = -g_{l3}'. \tag{16}$$

2.3. ELECTROMAGNETIC EQUATIONS

For simplicity and without loss of generality, we consider the electromagnetic subsystem with following assumptions [24, 19]:

(1) reluctances of the iron cores are negligible compared with the air gap reluctances;
(2) laminated cores are used and thus the effects of eddy currents are negligible;
(3) speed electromotive forces are very small and negligible;
(4) all electromagnets have identical geometries and properties.

Assume that current power sources are used for driving the electromagnets, then the magnetic circuit equation describing the relation between input current i_i, air gap flux Φ_i, and air gap width g_i for each electromagnet can be expressed as

$$\Phi_i = \frac{\mu_0 A_g N i_i}{g_i}, \quad i = r1, \ldots, r4, l1, \ldots, l4, \tag{17}$$

where N is the number of turns in each magnetic coil; A is the effective area under each magnetic pole; and μ_0 is the permeability of free space. The electromagnetic force f_i generated by the ith electromagnet can then be expressed as

$$f_i = k\Phi_i^2\left(1 + \frac{2g_i}{\pi h}\right), \quad i = r1, \ldots, r4, l1, \ldots, l4, \tag{18}$$

where k is the proportional constant; Φ_i is the air gap flux; g_i is the air gap width; and h is the pole width shown in Figure 1.

3. Fuzzy Model-Based Control Design for Conical Magnetic Bearings

Recently there has been found great interest in fuzzy model-based control for nonlinear systems using the LMI approach. The factors that make the LMI techniques so appealing for control design are that a variety of design specifications and constraints can be recast into the forms of LMIs, and the problem in terms of LMIs can be solved efficiently by convex optimization algorithms [7]. In the procedure of designing a fuzzy model-based control, a suitable T–S fuzzy model must be first constructed for the complex nonlinear systems. Tanaka and Wang [36], and Kirirkidis [14] have shown that T–S fuzzy models are universal approximators for nonlinear systems.

Conventional nonlinear model-based control for complex magnetic bearing systems is rather difficult, and the derived nonlinear control law usually needs complex computation for implementation. The fuzzy model control has the advantage of making the controller much simpler and easier to implement, however, how to construct a simple T–S fuzzy model for the nonlinear AMB system is still challenging. For simplicity, in this study we consider only the six-DOF magnetic bearing system with negligible eccentricity (the case with eccentricity based on this result is to be considered in another study), and an analytically derived T–S fuzzy model with small number of local models is presented. Then the PDC approach [39] can be easily applied for synthesizing a stable fuzzy control that is promising in the sense of easy implementation. In the followings, we will show the procedure for constructing a T–S fuzzy model for the nonlinear AMB systems, and for deriving a stable fuzzy control via LMIs based on the T–S fuzzy model.

3.1. DERIVATION OF T–S FUZZY MODEL FOR CONICAL AMB SYSTEMS

Since the dynamics model of a six-DOF magnetic bearing system is highly complex, the task for constructing a tractable T–S fuzzy model seems difficult. However, we can consider the model of the rotor subsystem from Equation (9) excluding the eccentricity,

$$\mathbf{M}_0\ddot{\mathbf{q}} + \mathbf{C}_0(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{F}_{0\mathbf{g}} = \mathbf{Q}_0 \tag{19}$$

for designing a simple control law, where the subscript “0” means the case with no eccentricity. That is, $\mathbf{M}_0$, $\mathbf{C}_0$, and $\mathbf{F}_{0\mathbf{g}}$ are obtained by substituting zero eccentricity into $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{F}_\mathbf{g}$ in (9), respectively. The state variables are defined as: $x_1 = x_0$, $x_2 = y_0, x_3 = z_0, x_4 = \varphi, x_5 = \theta, x_6 = \phi, x_7 = \dot{x}_0, x_8 = \dot{y}_0, x_9 = \dot{z}_0, x_{10} = \dot{\varphi}$, $x_{11} = \dot{\theta}, x_{12} = \dot{\phi}$.

Tanaka and Wang propose the idea of “sector nonlinearity”, “local approximation”, or a combination of them to construct fuzzy models from given nonlinear dynamical models [36]. In this section, we first use the concept of local approximation, and assume that φ and θ are very small, thus $\sin\varphi \approx \varphi$, $\sin\theta \approx \theta$, $\cos\varphi \approx 1$, $\cos\theta \approx 1$, $\sin^2\varphi \approx 0$, and $\sin^2\theta \approx 0$. Then Equation (19) can be simplified as

$$\begin{aligned}
&M_c\ddot{x}_0 = (f_{r1} + f_{r2} + f_{r3} + f_{r4})\sin\delta - (f_{l1} + f_{l2} + f_{l3} + f_{l4})\sin\delta + f_{dr5},\\
&M_c\ddot{y}_0 = (f_{r3} - f_{r4} + f_{l3} - f_{l4})\cos\delta + f_{dr3} + f_{dl3},\\
&M_c\ddot{z}_0 + M_c g_c = (f_{r1} - f_{r2} + f_{l1} - f_{l2})\cos\delta + f_{dr1} + f_{dl1},
\end{aligned} \tag{20}$$

$$\begin{aligned}
&I_d\ddot{\varphi} + I_{cxx}\theta\ddot{\phi} + 2(I_{cxx} - I_d)\theta\dot{\varphi}\dot{\theta} + 2I_d\theta\sin\phi\cos\phi\dot{\theta}^2 + I_{cxx}\dot{\theta}\dot{\phi}\\
&\quad = (-f_{r1} + f_{r2} + f_{l1} - f_{l2})l\cos\delta + (-f_{dr1} + f_{dl1})l,\\
&I_d\ddot{\theta} + (-I_{cxx} + I_d)\theta\dot{\varphi}^2 - I_{cxx}\dot{\varphi}\dot{\phi}\\
&\quad = (f_{r3} - f_{r4} - f_{l3} + f_{l4})l\cos\delta + (f_{dr3} - f_{dl3})l,\\
&I_{cxx}\theta\ddot{\varphi} + I_{cxx}\ddot{\phi} + I_{cxx}\dot{\varphi}\dot{\theta} + I_d\sin\phi\cos\phi\dot{\theta}^2 = T_m - T_l
\end{aligned} \tag{21}$$

where $I_d = I_{cyy} = I_{czz}$. With further assumptions of $\theta\dot{\varphi}\dot{\theta}, \dot{\theta}^2, \theta\dot{\varphi}^2$ and $\dot{\varphi}\dot{\theta}$ being small compared to the other terms, Equation (21) can be rewritten as

$$\begin{aligned}
&I_d\ddot{\varphi} + I_{cxx}\theta\ddot{\phi} + I_{cxx}\dot{\theta}\dot{\phi} = (-f_{r1} + f_{r2} + f_{l1} - f_{l2})l\cos\delta + (-f_{dr1} + f_{dl1})l,\\
&I_d\ddot{\theta} - I_{cxx}\dot{\varphi}\dot{\phi} = (f_{r3} - f_{r4} - f_{l3} + f_{l4})l\cos\delta + (f_{dr3} - f_{dl3})l,\\
&I_{cxx}\theta\ddot{\varphi} + I_{cxx}\ddot{\phi} = T_m - T_l.
\end{aligned} \tag{21$'$}$$

The simplified dynamic equations (20) and (21′) can be expressed in the state variable form as

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}_{\mathrm{c}} + \mathbf{B}_{\mathrm{d}}\mathbf{u}_{\mathrm{d}} \tag{22}$$

where $\mathbf{x}_{12\times 1} = [\mathbf{x}_1^{\mathrm{T}}\ \mathbf{x}_2^{\mathrm{T}}]^{\mathrm{T}} = [\mathbf{x}_1^{\mathrm{T}}\ \dot{\mathbf{x}}_1^{\mathrm{T}}]^{\mathrm{T}}$ is the state vector; $\mathbf{u} = [f_{r1}, f_{r2}, f_{r3}, f_{r4}, f_{l1}, f_{l2}, f_{l3}, f_{l4}, T_m]^{\mathrm{T}}$ is the 9×1 input vector consisting of the magnetic levitation forces generated by the electromagnets and the motor driving torque for rotor rotation; $\mathbf{u}_{\mathrm{d}} = [f_{dr1}, f_{dr2}, f_{dr3}, f_{dl1}, f_{dl3}, T_l]^{\mathrm{T}}$ is the 6×1 disturbance forces/torque vector, here T_l is the load torque; and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{6\times 6} & \mathbf{I} \\ \mathbf{0}_{6\times 6} & \mathbf{A}_{22} \end{bmatrix}, \qquad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{A}'_{22} \end{bmatrix},$$

$$\mathbf{A}'_{22} = \begin{bmatrix} 0 & -I_r x_{12} & 0 \\ I_r x_{12} & 0 & 0 \\ 0 & I_r x_5 x_{12} & 0 \end{bmatrix},$$

$$\mathbf{B} = \left[\mathbf{0}_{9\times 6}, \mathbf{B}_2^{\mathrm{T}}\right]^{\mathrm{T}}, \qquad \mathbf{B}_d = \left[\mathbf{0}_{6\times 6}, \mathbf{B}_{d2}^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\mathbf{B}_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} & 0 \\ 0 & 0 & b_{23} & b_{24} & 0 & 0 & b_{27} & b_{28} & 0 \\ b_{31} & b_{32} & 0 & 0 & b_{35} & b_{36} & 0 & 0 & 0 \\ b_{41} & b_{42} & 0 & 0 & b_{45} & b_{46} & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & 0 & 0 & b_{57} & b_{58} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{69} \end{bmatrix},$$

$$\begin{aligned}
&b_{1j} = \sin\delta/M_{\mathrm{c}}, && j = 1, 2, 3, 4; && b_{1j} = -\sin\delta/M_{\mathrm{c}}, && j = 5, 6, 7, 8,\\
&b_{2j} = \cos\delta/M_{\mathrm{c}}, && j = 3, 7; && b_{2j} = -\cos\delta/M_{\mathrm{c}}, && j = 4, 8,\\
&b_{3j} = \cos\delta/M_{\mathrm{c}}, && j = 1, 5; && b_{3j} = -\cos\delta/M_{\mathrm{c}}, && j = 2, 6,\\
&b_{4j} = -l\cos\delta/I_d, && j = 1, 6; && b_{4j} = l\cos\delta/I_d, && j = 2, 5,\\
&b_{5j} = l\cos\delta/I_d, && j = 3, 8; && b_{5j} = -l\cos\delta/I_d, && j = 4, 7,\\
&b_{69} = 1/I_{\mathrm{c}xx};
\end{aligned}$$

$$\mathbf{B}_{d2} = \begin{bmatrix} 0 & 0 & b_{d,13} & 0 & 0 & 0 \\ 0 & b_{d,22} & 0 & 0 & b_{d,25} & 0 \\ b_{d,31} & 0 & 0 & b_{d,34} & 0 & 0 \\ b_{d,41} & 0 & 0 & b_{d,44} & 0 & 0 \\ 0 & b_{d,52} & 0 & 0 & b_{d,55} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{d,66} \end{bmatrix},$$

$$b_{d,13} = b_{d,22} = b_{d,25} = b_{d,31} = b_{d,34} = 1/M_{\mathrm{c}};$$

$$b_{d,41} = b_{d,55} = -l/I_d; b_{d,44} = b_{d,52} = l/I_d;$$
$$b_{d,66} = -1/I_{cxx};$$

$$\mathbf{G}_c = [\mathbf{0}_{1\times 6} \quad 0 \quad 0 \quad -g_c \quad 0 \quad 0 \quad 0]^T,$$

here M_c is the mass of the rotor; l is one half of the rotor length (Figure 1); and $I_r = I_{cxx}/I_d$.

While $\mathbf{A}(\mathbf{x})$ is still dependent of the state vector, it is a function of x_5 and x_{12} only, and its entries become very simple. Moreover, since $\mathbf{B}$ is only a constant matrix, the derivation of a T–S fuzzy model and then the control design and finding of solution of LMIs become much less difficult than the approaches directly based on the original complex model.

Next, in accordance with the idea of sector nonlinearity [36], we can assume that $x_5 \in [\theta_{\min}, \theta_{\max}]$ and $x_{12} \in [\omega_{\min}, \omega_{\max}]$, for the nonlinear terms in Equation (22). By the definition of $z_1 \equiv x_{12}$ and $z_2 \equiv x_5 \cdot x_{12}$, the maximal and minimal values of $z_1(t)$ and $z_2(t)$ can be deduced as:

$$\begin{aligned} &\max_{x_{12}} z_1(t) = \omega_{\max}, \qquad \min_{x_{12}} z_1(t) = \omega_{\min}, \\ &\max_{x_5,x_{12}} z_2(t) = \beta_{\max}, \qquad \min_{x_5,x_{12}} (z_2) = \beta_{\min}. \end{aligned} \tag{23}$$

Choose $z_1(t)$ and $z_2(t)$ as the antecedent variables of the T–S fuzzy model, and define two fuzzy sets for each variable with membership functions shown in Figures 3 and 4, respectively.

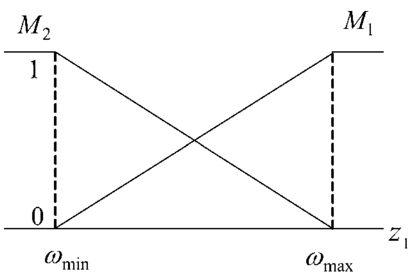

Figure 3. Definition of two membership functions for z_1.

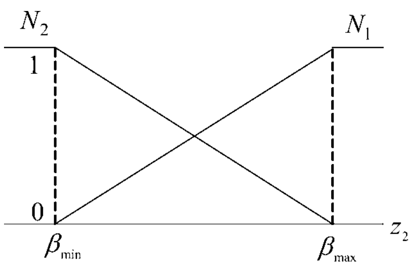

Figure 4. Definition of two membership functions for z_2.

Then a T–S fuzzy model with four fuzzy IF–THEN rules for approximating the nonlinear AMB system can be constructed analytically as follows:

MODEL RULE i.

$$\begin{aligned}&\text{IF } z_1 \text{ is } C_1^i \text{ and } z_2 \text{ is } C_2^i\\&\text{THEN } \dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}_\mathrm{c} + \mathbf{B}_\mathrm{d}\mathbf{u}_\mathrm{d}, \quad i = 1, 2, 3, 4,\end{aligned} \tag{24}$$

where

$$\begin{aligned}C_1^1 &= M_1, & C_2^1 &= N_1; & C_1^2 &= M_1, & C_2^2 &= N_2;\\ C_1^3 &= M_2, & C_2^3 &= N_1; & C_1^4 &= M_2, & C_2^4 &= N_2;\end{aligned}$$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{0}_{6\times 6} & \mathbf{I}_{6\times 6} \\ \mathbf{0}_{6\times 6} & \mathbf{A}_{i,22} \end{bmatrix}, \quad i = 1, 2, 3, 4,$$

and

$$\mathbf{A}_{i,22} = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{A}'_{i,22} \end{bmatrix}, \quad i = 1, 2, 3, 4,$$

$$\mathbf{A}'_{1,22} = \begin{bmatrix} 0 & -I_r\omega_{\max} & 0 \\ I_r\omega_{\max} & 0 & 0 \\ 0 & I_r\beta_{\max} & 0 \end{bmatrix}, \qquad \mathbf{A}'_{2,22} = \begin{bmatrix} 0 & -I_r\omega_{\max} & 0 \\ I_r\omega_{\max} & 0 & 0 \\ 0 & I_r\beta_{\min} & 0 \end{bmatrix},$$

$$\mathbf{A}'_{3,22} = \begin{bmatrix} 0 & -I_r\omega_{\min} & 0 \\ I_r\omega_{\min} & 0 & 0 \\ 0 & I_r\beta_{\max} & 0 \end{bmatrix}, \qquad \mathbf{A}'_{4,22} = \begin{bmatrix} 0 & -I_r\omega_{\min} & 0 \\ I_r\omega_{\min} & 0 & 0 \\ 0 & I_r\beta_{\min} & 0 \end{bmatrix}.$$

The output of the T–S fuzzy model can be inferred as

$$\dot{\mathbf{x}} = \sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{A}_i x + \mathbf{B}\mathbf{u} + \mathbf{G}_\mathrm{c} + \mathbf{B}_\mathrm{d}\mathbf{u}_\mathrm{d} \tag{25}$$

where

$$h_i(\mathbf{z}) = \frac{w_i(\mathbf{z})}{\sum_{i=1}^{4} w_i(\mathbf{z})}, \quad \mathbf{z} = [z_1, z_2]^\mathrm{T}, \tag{26}$$

$$w_i(\mathbf{z}) = C_1^i(z_1)C_2^i(z_2), \quad i = 1, 2, 3, 4 \tag{27}$$

here $C_j^i(z_j)$ is the grade of membership of z_j in fuzzy set C_j^i. By the membership function definitions in Figures 3 and 4, we have

$$w_i(\mathbf{z}) > 0, \quad i = 1, 2, 3, 4, \quad \sum_{i=1}^{4} w_i(\mathbf{z}) > 0 \tag{28}$$

and

$$h_i(\mathbf{z}) > 0, \quad i = 1, 2, 3, 4, \quad \sum_{i=1}^{4} h_i(\mathbf{z}) = 1. \tag{29}$$

Notice that by substituting (27) and (26) into (25) we can obtain the simplified dynamic model (22), hence the above four-rule T–S fuzzy model is sufficient for the AMB rotor subsystem.

3.2. FUZZY CONTROL DESIGN VIA LINEAR MATRIX INEQUALITY

In this section we will propose a robust fuzzy control design for the nonlinear AMB system based on the derived T–S fuzzy model. For arbitrary trajectory tracking, first define the tracking error vector as

$$\mathbf{e} = \mathbf{x} - \mathbf{x}_d \tag{30}$$

where

$$\mathbf{x}_d = \left[\mathbf{x}_{d1}^{\mathrm{T}}\, \mathbf{x}_{d2}^{\mathrm{T}}\right]^{\mathrm{T}} = [x_{0\mathrm{d}}, y_{0d}, z_{0d}, \varphi_d, \theta_d, \phi_d, \dot{x}_{0d}, \dot{y}_{0d}, \dot{z}_{0d}, \dot{\varphi}_d, \dot{\theta}_d, \omega_d]^{\mathrm{T}}$$

is the desired trajectory vector. Notice that $\phi_d(t)$ and $\omega_d(t)$ are the desired angular displacement and velocity trajectories of the rotor about its rotating axis, respectively. Differentiating Equation (30), we have

$$\dot{\mathbf{x}} = \dot{\mathbf{e}} + \dot{\mathbf{x}}_d. \tag{31}$$

Substituting (31) into the model rules (24), we can obtain the T–S fuzzy model for the error dynamics as follows:

ERROR RULE i.

$$\begin{aligned}&\text{IF } z_1 \text{ is } C_1^i \text{ and } z_2 \text{ is } C_2^i\\ &\text{THEN } \dot{\mathbf{e}} = \mathbf{A}_i\mathbf{e} + \mathbf{A}_i\mathbf{x}_d + \mathbf{B}\mathbf{u} + \mathbf{G}_\mathrm{c} + \mathbf{B}_\mathrm{d}\mathbf{u}_\mathrm{d} - \dot{\mathbf{x}}_d, \quad i = 1, 2, 3, 4,\end{aligned} \tag{32}$$

and the output of the error T–S fuzzy model is inferred as

$$\dot{\mathbf{e}} = \sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{A}_i\mathbf{e} + \sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{A}_i\mathbf{x}_d + \mathbf{B}\mathbf{u} + \mathbf{G}_\mathrm{c} + \mathbf{B}_\mathrm{d}\mathbf{u}_\mathrm{d} - \dot{\mathbf{x}}_d. \tag{33}$$

In this study, the PDC (Parallel Distributed Compensation) [39, 36] with control rules constructed based on the T–S fuzzy model rules is adopted for the fuzzy control design. Each control rule has a linear state feedback part and a feedforward part to compensate for the effect of gravity, that is,

CONTROL RULE i.

$$\begin{array}{l}\text{IF } z_1 \text{ is } C_1^i \text{ and } z_2 \text{ is } C_2^i \\ \text{THEN } \mathbf{u} = -\mathbf{K}_i\mathbf{e} - \mathbf{B}^{+}\mathbf{A}_i\mathbf{x}_d - \mathbf{B}^{+}(\mathbf{G}_\mathrm{c} - \dot{\mathbf{x}}_d), \quad i = 1, 2, 3, 4\end{array} \tag{34}$$

where $\mathbf{B}^{+} = [\mathbf{0}_{9\times 6}\ \mathbf{B}_2^\mathrm{T}(\mathbf{B_2}\mathbf{B}_2^\mathrm{T})^{-1}]$. So the design objective is to determine the local feedback gains $\mathbf{K}_i$ in the consequent parts of the control rules via LMI.

The output of the PDC controller can be inferred as:

$$\mathbf{u} = -\sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{K}_i\mathbf{e} - \mathbf{B}^{+}\sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{A}_i\mathbf{x}_d - \mathbf{B}^{+}\mathbf{G}_\mathrm{c} + \mathbf{B}^{+}\dot{\mathbf{x}}_d = \mathbf{u}_{dc} + \Delta\mathbf{u} \tag{35}$$

where

$$\mathbf{u}_{dc} = -\mathbf{B}^{+}\mathbf{G}_\mathrm{c} \tag{36}$$

$$\Delta\mathbf{u} \equiv -\sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{K}_i\mathbf{e} - \mathbf{B}^{+}\sum_{i=1}^{4} h_i(\mathbf{z})\mathbf{A}_i\mathbf{x}_d + \mathbf{B}^{+}\dot{\mathbf{x}}_d. \tag{37}$$

By substituting (35) into (33), and since $\mathbf{B}$ is a constant matrix in the T–S fuzzy model and $\sum_{i=1}^{4} h_i(\mathbf{z}) = 1$, the error dynamics for the whole closed-loop system can be derived as

$$\dot{\mathbf{e}} = \sum_{i=1}^{4} h_i(\mathbf{z})(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i)\mathbf{e} + \mathbf{B}_d\mathbf{u}_d. \tag{38}$$

For the consideration of robustness with respect to the disturbance $\mathbf{u}_d$, the following robust performance requirement [37] for the tracking error is to be met:

$$\frac{\int_0^{t_f} \mathbf{e}^\mathrm{T}\,\mathbf{Q}\mathbf{e}\,\mathrm{d}t}{\int_0^{t_f} \mathbf{u}_d^\mathrm{T}\,\mathbf{u}_d\,\mathrm{d}t} < \rho^2 \tag{39}$$

where $\mathbf{Q} = \alpha\mathbf{P}$, $\alpha > 0$, and $\mathbf{P}$ is a symmetric positive definite matrix. Equation (39) means that the effect of $\mathbf{u}_d$ on the error must be attenuated below a prescribed level ρ. To synthesize the fuzzy controller that can reject the external disturbances of the AMB system, we can select a positive definite function

$$V = \mathbf{e}^\mathrm{T}\mathbf{P}\mathbf{e}. \tag{40}$$

The requirement (39) for a prescribed $\rho > 0$ can be shown to be equivalent to the following condition:

$$\dot{V} + \alpha\mathbf{e}^\mathrm{T}\mathbf{P}\mathbf{e} - \rho^2\,\mathbf{u}_d^\mathrm{T}\,\mathbf{u}_d \leqslant 0. \tag{41}$$

By integrating (41) from 0 to t_f with initial condition $\mathbf{e}(0) = \mathbf{0}$, we have

$$V + \int_0^{t_f} \left(\mathbf{e}^\mathrm{T}\mathbf{Q}\mathbf{e} - \rho^2\,\mathbf{u}_d^\mathrm{T}\,\mathbf{u}_d\right)\mathrm{d}t \leqslant 0. \tag{42}$$

Thus,

$$\int_0^{t_f} \left(\mathbf{e}^{\mathrm{T}}\mathbf{Q}\mathbf{e} - \rho^2\,\mathbf{u}_d^{\mathrm{T}}\,\mathbf{u}_d\right)\,\mathrm{d}t \leqslant -V \leqslant 0. \tag{43}$$

Equation (43) implies (39). Therefore if (41) holds, the robust performance requirement can be guaranteed under $\mathbf{u}_d$.

The LMI constraints can be derived from (41). First, rewrite (41) as

$$\dot{\mathbf{e}}^{\mathrm{T}}\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\dot{\mathbf{e}} + \alpha\mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{e} - \rho^2\,\mathbf{u}_d^{\mathrm{T}}\,\mathbf{u}_d \leqslant 0 \tag{44}$$

and substituting (38) into (44), we can obtain

$$\sum_{i=1}^{4} h_i\mathbf{e}^{\mathrm{T}}\left[(\mathbf{A}_{\mathrm{i}} - \mathbf{B}\mathbf{K}_i)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i)\right]\mathbf{e} + \mathbf{u}_d^{\mathrm{T}}\,\mathbf{B}_d^{\mathrm{T}}\,\mathbf{P}\mathbf{e} + \mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{B}_d\mathbf{u}_d + \alpha\mathbf{e}^{\mathrm{T}}\mathbf{P}\mathbf{e} - \rho^2\mathbf{u}_d^{\mathrm{T}}\mathbf{u}_d \leqslant 0 \tag{45}$$

That is,

$$\sum_{i=1}^{4} h_i \begin{bmatrix} \mathbf{e} \\ \mathbf{u}_d \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (\mathbf{A}_{\mathrm{i}} - \mathbf{B}\mathbf{K}_i)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i) + \alpha\mathbf{P} & \mathbf{P}\mathbf{B}_d \\ \mathbf{B}_d^{\mathrm{T}}\mathbf{P} & -\rho^2\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{e} \\ \mathbf{u}_d \end{bmatrix} \leqslant 0. \tag{46}$$

Therefore, if the following constraints are satisfied:

$$\begin{bmatrix} -(\mathbf{A}_{\mathrm{i}} - \mathbf{B}\mathbf{K}_i)^{\mathrm{T}}\mathbf{P} - \mathbf{P}(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i) - \alpha\mathbf{P} & -\mathbf{P}\mathbf{B}_d \\ -\mathbf{B}_d^{\mathrm{T}}\mathbf{P} & \rho^2\mathbf{I} \end{bmatrix} \geqslant 0, \quad i = 1, 2, 3, 4, \tag{47}$$

then Equation (46) holds. Applying the Schur complements for nonstrict inequalities [2], (47) becomes

$$-(\mathbf{A}_{\mathrm{i}} - \mathbf{B}\mathbf{K}_i)^{\mathrm{T}}\mathbf{P} - \mathbf{P}(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i) - \alpha\mathbf{P} - \frac{1}{\rho^2}\mathbf{P}\mathbf{B}_d\mathbf{B}_d^{\mathrm{T}}\mathbf{P} \geqslant 0, \quad i = 1, 2, 3, 4. \tag{48}$$

Conditions (48) can be solved by considering it as a generalized eigenvalue minimization problem (GEMP), that is, to maximize α subject to the following constraints:

1. $\mathbf{P} > \mathbf{0}$,
2. $(\mathbf{A}_{\mathrm{i}} - \mathbf{B}\mathbf{K}_i)^{\mathrm{T}}\mathbf{P} + \mathbf{P}(\mathbf{A}_i - \mathbf{B}\mathbf{K}_i) + \frac{1}{\rho^2}\mathbf{P}\mathbf{B}_d\mathbf{B}_d^{\mathrm{T}}\mathbf{P} \leqslant -\alpha\mathbf{P}, \quad i = 1, 2, 3, 4$

(49)

Since the second inequalities in (49) are not jointly convex in $\mathbf{P}$ and $\mathbf{K}_i$, it is difficult to find a common solution $\boldsymbol{P}$ and the feedback gains $\mathbf{K}_i$. Fortunately, the inequalities can be transferred into linear matrix inequalities by variable transformation.

Defining new variable $\mathbf{X} = \mathbf{P}^{-1}$, and multiplying the inequalities on the left and right by $\mathbf{X}$, we have

$$\mathbf{X}(\mathbf{A}_i - \mathbf{BK}_i)^{\mathrm{T}}\mathbf{PX} + \mathbf{XP}(\mathbf{A}_i - \mathbf{BK}_i)\mathbf{X} + \frac{1}{\rho^2}\mathbf{XPB}_d\mathbf{B}_d^{\mathrm{T}}\mathbf{PX} \leqslant -\alpha\mathbf{XPX},$$
$$i = 1, 2, 3, 4. \tag{50}$$

Equation (50) can be rewritten as

$$\mathbf{XA}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X} - (\mathbf{BM}_i)^{\mathrm{T}} - \mathbf{BM}_i + \frac{1}{\rho^2}\mathbf{B}_d\mathbf{B}_d^{\mathrm{T}} \leqslant -\alpha\mathbf{X}, \quad i = 1, 2, 3, 4, \tag{51}$$

where $\mathbf{M}_i \equiv \mathbf{K}_i\mathbf{X}$. That is, the PDC control design problem can be transformed to the problem of maximizing α subject to the following linear matrix inequality constraints:

$$\begin{aligned} &1. \quad \mathbf{X} > \mathbf{0}, \\ &2. \quad \mathbf{XA}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{X} - (\mathbf{BM}_i)^{\mathrm{T}} - \mathbf{BM}_i + \frac{1}{\rho^2}\mathbf{B}_d\mathbf{B}_d^{\mathrm{T}} \leqslant -\alpha\mathbf{X}, \quad i = 1, 2, 3, 4. \end{aligned} \tag{52}$$

If there exists a common $\mathbf{X}$ and $\mathbf{M}_i$'s satisfying the above LMI constraints, then the common $\mathbf{P}$ and the feedback gains can be obtained as

$$\mathbf{P} = \mathbf{X}^{-1} \quad \text{and} \quad \mathbf{K}_i = \mathbf{M}_i\mathbf{P}, \quad i = 1, 2, 3, 4. \tag{53}$$

There exist methods in the literature for solving the LMI problems, such as the interior point algorithm [2]. The MATLAB software package has incorporated this algorithm into the solvers of LMI control toolbox. In this study, the instruction **gevp** is used to solve the above GEMP problem. Based on the proper choice of suitable α, $\mathbf{P}$ and the feedback gains $\mathbf{K}_i$, $i = 1, 2, 3, 4$, can thus be determined, and the design of the control law (35) is accomplished.

Control law (35), derived based on the T–S fuzzy model for the rotor subsystem, can be used for computing the required control magnetic forces and motor driving torque for the operation of a six-DOF AMB system. In this study, we assume that CSI (current source inverter)-based drives for the electromagnetic subsystems are adopted, so (35) can be directly implemented for evaluating the required driving current commands, based only on the algebraic magnetic force relations:

$$f_i = k\Phi_i^2\left(1 + \frac{2 \cdot g_i}{\pi h}\right), \quad i = r1, \ldots, r4, l1, \ldots, l4, \tag{54}$$

where the air gap flux Φ_i of each magnetic pole is

$$\Phi_i = \frac{\mu_0 A_g N i_i}{g_i}, \quad i = r1, \ldots, r4, l1, \ldots, l4. \tag{55}$$

That is,

$$f_i = k\left(\frac{\mu_0 A_g N}{g_i}\right)^2\left(1 + \frac{2 \cdot g_i}{\pi h}\right) \cdot i_i^2 = K_i i_i^2, \quad i = r1, \ldots, r4, l1, \ldots, l4, \tag{56}$$

where

$$K_i \equiv k\left(\frac{\mu_0 A_g N}{g_i}\right)^2\left(1+\frac{2\cdot g_i}{\pi h}\right)$$

and i_i is the input current through the ith electromagnetic coil.

Since we consider the attractive type AMB, the electromagnetic force must meet the constraints:

$$f_i > 0, \quad i = r1, \ldots, r4, l1, \ldots, l4, \tag{57}$$

in (20) and (21′). $\mathbf{u}_{dc} = [f_{r1,d}, f_{r2,d}, f_{r3,d}, f_{r4,d}, f_{l1,d}, f_{l2,d}, f_{l3,d}, f_{l4_d}, 0]^{\mathrm{T}}$ computed by (36) is used for levitating the rotor and maintaining the balance of the system at steady state. The components of $\mathbf{u}_{dc}$ can be equivalently rescaled as:

$$\mathbf{u}_{d\mathrm{c}} = [2f_{r1,d},\ 0,\ 0,\ 0,\ 2f_{l1,d},\ 0,\ 0,\ 0,\ 0]^{\mathrm{T}}. \tag{58}$$

For meeting the constraints (57), a constant and positive bias force f_0 can be introduced into (58), that is, the dc parts of the suspension control forces can be chosen as

$$\mathbf{u}'_{d\mathrm{c}} = [2f_{r1,d}+f_0,\ f_0,\ f_0,\ f_0,\ 2f_{l1,d}+f_0,\ f_0,\ f_0,\ f_0,\ 0]^{\mathrm{T}}. \tag{59}$$

In summary, the control law (35) used for implementing the derived control strategy is

$$\mathbf{u} = \mathbf{u}'_{d\mathrm{c}} + \Delta\mathbf{u}. \tag{60}$$

4. Simulation and Results

In this section representative computer simulations are used for illustrating the control performance of the proposed T–S fuzzy model control strategy for a conical magnetic bearing with no rotor unbalance. The nominal values of the parameters used in the simulation are chosen as:

$A = 1.532\times10^{-3}\ \mathrm{m}^2$, $h = 0.04$ m, $l = 0.27$ m,
$d_{0i} = 5.5\times10^{-4}$ m, $i = r1, \ldots, r4, l1, \ldots, l4$,
$M_{\mathrm{C}} = 10.2667$ kg, $g_{\mathrm{c}} = 9.81\ \mathrm{m/sec}^2$, $I_{\mathrm{c}xx} = 0.0113\ \mathrm{kg–m}^2$,
$I_{\mathrm{c}yy} = I_{\mathrm{c}zz} = 0.333\ \mathrm{kg–m}^2$, $k = 4.67555760\times10^{8}\ \mathrm{nt/Wb}^2$,
$N = 100$, $\delta = \pi/18$ rad [19, 23].

Setting the operating ranges for x_5 and x_{12} as: $\theta_{\min} = -\pi/100$ rad, $\theta_{\max} = \pi/100$ rad, $\omega_{\min} = -104.72$ rad/s, and $\omega_{\max} = 4188.8$ rad/s, the parameters for the controller are selected via trial and error as follows: $\alpha = 5414$, $\rho = 0.3027\times10^{-12}$, $f_0 = 10$ nt.

The desired trajectory command, using the path generation methods in [4], for rotor angular velocity tracking and gap deviations regulation are selected as

$$\omega_d(t) = \begin{cases} \omega_f \left(\frac{t^3}{t_{f1}^3}\right)\left(10 - 15\frac{t}{t_{f1}} + 6\frac{t^2}{t_{f1}^2}\right), & 0 \leqslant t \leqslant 10, \\ \omega_f, & t > 10, \end{cases}$$

where ω_f is the desired final angular velocity after t_{f1}, and

$$\mathbf{x}_{d1}(t) = \begin{cases} \mathbf{x}_{1,0}\left(1 - 10\frac{t^3}{t_{f2}^3} + 15\frac{t^4}{t_{f2}^4} - 6\frac{t^5}{t_{f2}^5}\right), & 0 \leqslant t \leqslant 0.5, \\ \mathbf{0}, & t > 0.5, \end{cases}$$

where $\mathbf{x}_{1,0}$ is the initial state vector for $\mathbf{x}_1$, and t_{f2} is the time $\mathbf{x}_{d1}$ reaches zero.

The initial state vector used in the selected simulations is

$$\mathbf{x}(0) = \left[1 \times 10^{-5}, 1 \times 10^{-5}, 1 \times 10^{-5}, \mathbf{0}_{1\times 9}\right]^{\mathrm{T}}.$$

In the simulations $t_{f1} = 10$ s and $t_{f2} = 0.5$ s are set. The simulations are terminated at $t = 12$ s. The feedback gain matrices $\mathbf{K}_i$, $i = 1, 2, 3, 4$, and the common Lyapunov matrix $\mathbf{P}$ are computed using MATLAB's LMI control toolbox. Proper $\mathbf{K}_i$ and $\mathbf{P}$ can be obtained by choosing the results with sufficiently high value of α (Lyapunov function decay-rate scaling factor). They are lengthy and not listed here.

Simulation results for the nominal case with $\omega_f = 4188.8$ rad/s are shown in Figure 5. From Figures 5(a)–(d), we know that the gap deviations can approach zero, and their convergence rates are rather fast. The required levitation control currents for the electromagnetic coils computed by the fuzzy model-based control strategy are shown in Figures 5(e)–(l). The rotor angular velocity response and the tracking error are shown in Figures 5(m) and (n). Figure 5(o) shows the corresponding required motor control torque for the high speed operation of the rotor.

Notice that the fuzzy controller designed is based on the derived T–S fuzzy model and not on the original complex dynamics model, thus, from Figure 5(n) we can see that the tracking error of rotor angular velocity does not approach zero. However, once the rotor speed reaches the final desired value, the error can fall into a narrow bound and be very small (determined by the design parameter ρ).

Figure 6 shows the results for the case that is with parametric uncertainty, and external disturbances during $t \in [6, 8]$ s. In this case, the rotor mass assumed changed to be 12.32 kg and the previous nominal controller parameters are still used for the simulation. From Figures 6(a)–(d) and 6(n) we know that the gap deviations and the rotor angular velocity tracking error remain considerably small in spite of the occurrence of external disturbance and parameter uncertainty. Thus, when ρ is selected as small as possible, and the common $\mathbf{P}$ and the feedback gains

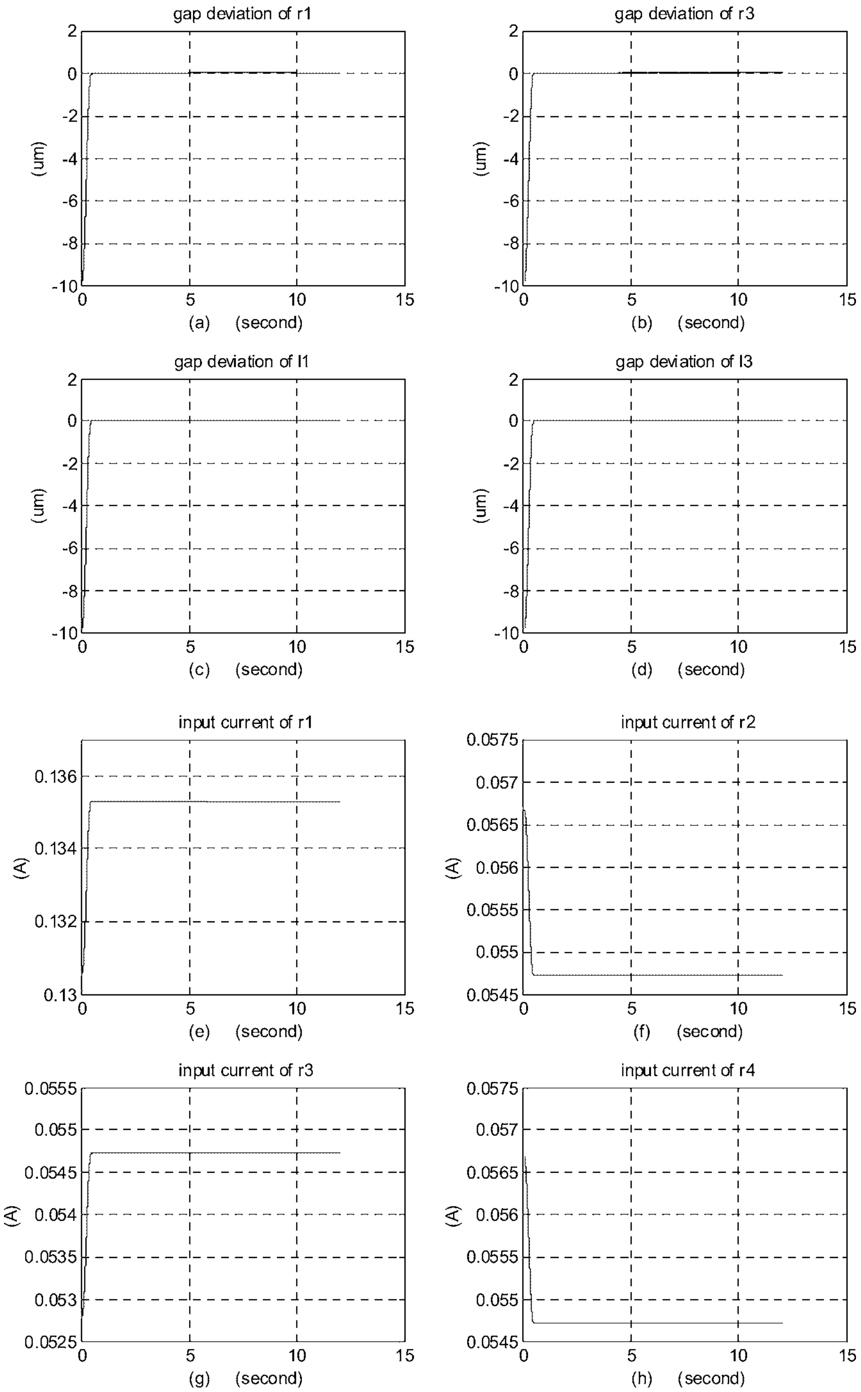

Figure 5. Simulation results for the nominal case with high speed tracking. (a)–(d) gap deviations; (e)–(l) control currents; (m) rotor velocity response; (n) velocity error; (o) motor control torque.

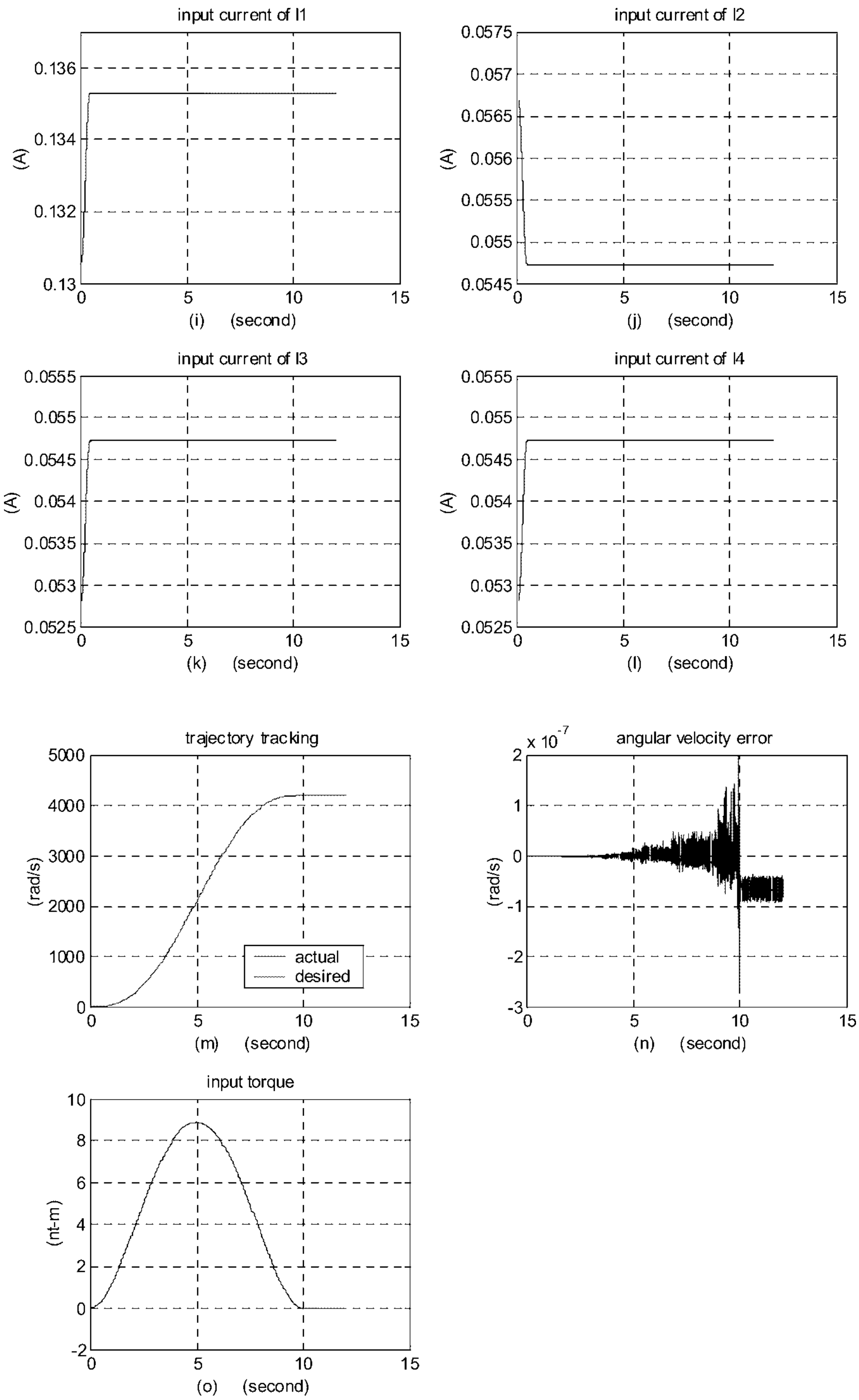

Figure 5. (Continued.)

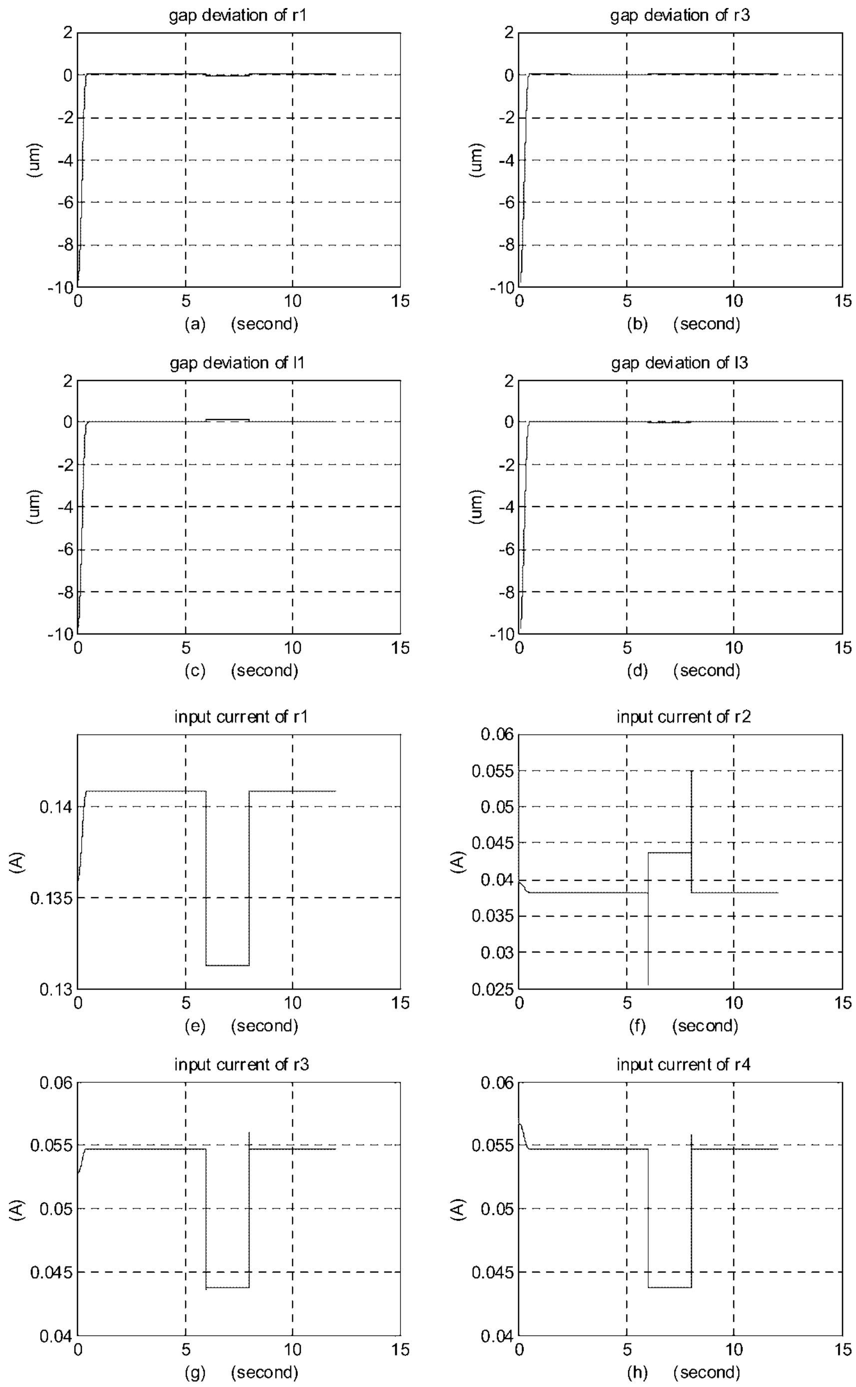

Figure 6. Simulation results for the case with external disturbance vector [10 0 5 0 3 2] at time 6–8 s. (a)–(d) gap deviations; (e)–(l) control currents; (m) rotor velocity response; (n) velocity error; (o) motor control torque.

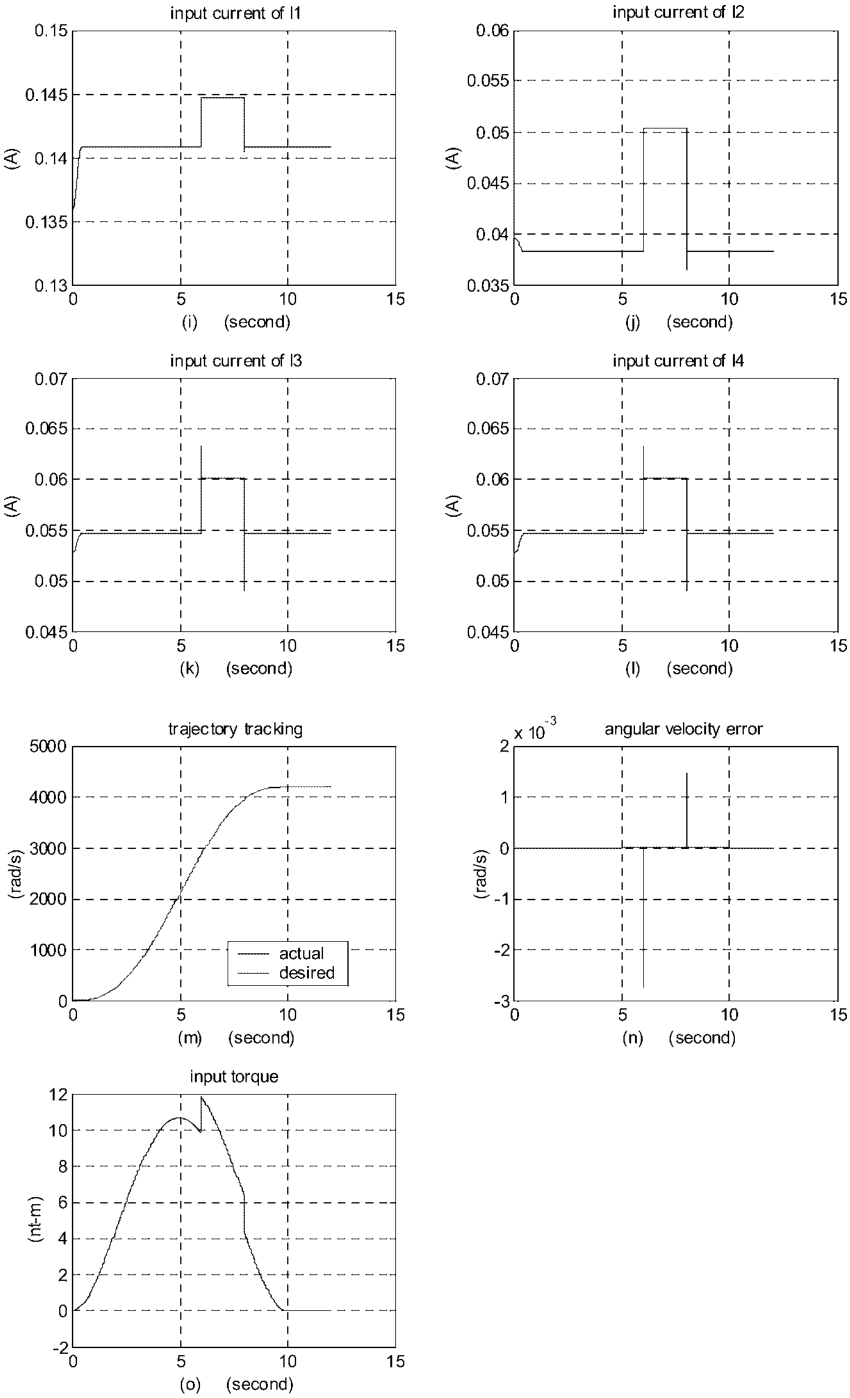

Figure 6. (Continued.)

$\mathbf{K}_i$, $i = 1, 2, 3, 4$, are found with a bigger α, the controller can have excellent capability to reject disturbance and high robustness with respect to mass uncertainty.

5. Conclusions

In this paper, a systematic modeling approach using the Lagrangian method for deriving the general six-DOF rotor dynamics equation for a conical AMB system with rotor eccentricity is first presented. A conical AMB system with rotor eccentricity is highly nonlinear and severely coupled due to the conical shape of the bearings. A procedure for systematically constructing a simple T–S fuzzy model with very small number of rules that can exactly represent the simplified nonlinear AMB rotor subsystem assumed no eccentricity is suggested, and a PDC control design based on the T–S fuzzy model is proposed. Since the number of rules is very small, it is not difficult to find a common Lyapunov matrix $\mathbf{P}$, and no relaxation methods are needed. The feedback gain matrices $\mathbf{K}_i$ and $\mathbf{P}$ can be simultaneously determined by considering the control design problem as a GEMP problem via LMI constraints. Proper $\mathbf{K}_i$ and $\mathbf{P}$ can be obtained by choosing the results with sufficiently high value of the Lyapunov function decay-rate scaling factor α. The derived fuzzy controller is much simpler than those derived using conventional nonlinear control theory based on the complete complex dynamics model. Representative computer simulation results show that the controller is applicable for arbitrary rotor trajectory tracking control within the selected operating range, and has high robustness with respect to disturbance and uncertainty. In this study we only consider the case with rigid rotor and no rotor eccentricity is included in the control design. However, the suggested approach and the derived results might be useful for considering the cases with rotor eccentricity and/or rotor flexibility. These interesting problems need further future studies.

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