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ORIGINAL PAPER

Collective motion of inelastic particles between two oscillating walls

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Abstract This study theoretically considers the motion of N identical inelastic particles between two oscillating walls. The particles' average energy increases abruptly at certain critical filling fractions, wherein the system changes into a solid-like phase with particles clustered in their compact form. Molecular dynamics simulations of the system show that the critical filling fraction is a decreasing function of vibration amplitude independent of vibration frequency, which is consistent with previous experimental results. This study considers the entire group of particles as a giant pseudo-particle with an effective size and an effective coefficient of restitution. The N-particles system is then analytically treated as a one-particle problem. The critical filling fraction's dependence on vibration amplitude can be explained as a necessary condition for a stable resonant solution. The fluctuation to the system's mean flow energy is also studied to show the relation between the granular temperature and the system phase.

Keywords Granular system · Collective motion · Phase transition

1 Introduction

Granular systems lose energy due to inter-particle collisions. Mechanical agitation is usually used as an energy supply to keep a granular system active. This kind of excited granular system has already attracted much attention due to its many attractive phenomena, including pattern formation [1,2], segregation [3–5], and phase transition [6–12] which are commonly observed in experiments. The kinetic theory [13] and hydrodynamics model [14,15] are commonly used for statistical analysis of the various behaviors of granular systems. On the other hand, the system as a whole cannot be well understood without careful study of the dynamics of individual particles [16–19]. Especially, study of the interaction between particle and the boundary where the energy is injected [20–22] is particularly important in determining the boundary condition in a continuum approach [23].

Liquid–solid transitions [6] were discovered in the vertical vibrated granular systems, and were observed in recent experiments on a horizontal vibrating system [24–26] consisting of a monolayer of spherical particles rolling on a plate. The plate is driven horizontally by external harmonic force and the particles gain energy by interacting with the substrate and boundary walls, with gravity and air having a negligible effect. When the particles' filling fraction reaches a critical value, the structure and dynamics of the system sharply shift from irregular to regular. Experimental data [26] show that the critical filling fraction is inversely proportional to the vibration amplitude, a counter-intuitive outcome given that larger amplitudes, and thus greater energy, would tend to drive the system into a disordered state. This is actually an example of "Freezing by heating" which was first introduced by Helbing et al. in [27]. They exhibited a new kind of transition with a simple driven particles model that a crystallized state with high energy is obtained by enhancing the noise in a fluid state. Similar phenomenon also occurs in other systems such as colloidal suspension [28] and swirling granules [25]. Here we theoretically investigate the monolayer granular system under horizontal vibration. Based on evidence from molecular dynamic (MD) simulations, this study will show that a highly energetic ordered state can be sustained in the vibrating granular system when a resonance occurs together with inelastic clustering [29–31], and

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the dependence of the critical filling fraction on amplitude can also be explained by the resonant condition.

This report considers N identical particles of a diameter d confined between two oscillating walls. The walls vibrate sinusoidally in phase in the x-direction. The inelastic collisions between the particles, and between the particles and the walls, are characterized by the same coefficient of restitutionε. Friction is neglected in the theoretical analysis and its effects are discussed in the comparison of the theoretical results with the experiments. In the case of $N = 1$, the particle can move in resonance to the vibrating walls with a fractional frequency f/n [32,33], where f is the frequency of the vibrating walls. We solve the analytic solution of resonant velocity for $n = 1$ in Sect. 2. In Sect. 3, we use MD simulations to study the energy and configuration of many-particle systems. We find that inelastic clustering occurs when N is large and ε is smaller than a critical value so that the particles move collectively as a giant particle. We use the resonant conditions for this pseudo particle to determine the critical filling fraction at any given vibrating amplitude. In Sect. 4, we analyze the velocity distribution of the particles. In a gas-like (i.e., dilute) phase, the velocity distribution is well fitted by the superposition of two Gaussian distributions, showing the coexistence of two granular temperatures [34,35]. In a solid-like (i.e., compact) phase, the velocity distribution is a superposition of one Gaussian and one exponential distribution [36]. We discuss how the critical filling fraction depends on the coefficient of restitution in Sect. 5. Section 6 is the conclusion.

2 Stable resonant solutions for a one-particle system

Let the walls oscillate horizontally with velocity $v_p = A\omega \sin \omega t$ (Fig. 1). Suppose a particle is moving towards the left from the right wall at instant t with constant velocity $-v$, $v > 0$. When the particle collides with the wall at instant t', it changes its velocity to v'. By the law of conservation of momentum and the definition of coefficient of restitution, we have:

$$v' = \varepsilon v + (1 + \varepsilon) A\omega \sin \omega t' \tag{1}$$

where the mass of the wall is assumed to be infinite. In the time interval $t' - t$, the particle travels a distance $(L - d) - A \cos \omega t + A \cos \omega t'$, where $A \cos \omega t$ and $A \cos \omega t'$ are the displacements of the right and left wall from their vibrating centers at time t and t', respectively. So we have:

$$v'(t' - t) = (L - d) - A \cos \omega t + A \cos \omega t' \tag{2}$$

When the motion of the particle is in resonance with the vibration, we expect $v' = v \equiv v_R$ from the symmetry of the system. The particle takes a half period of time to travel from the right wall to the left wall and vice versa. Therefore, the

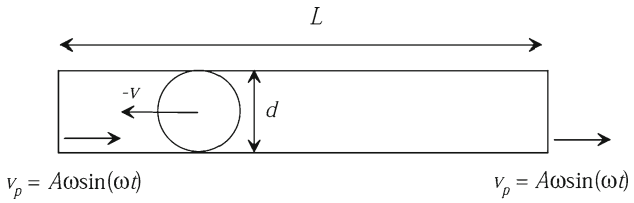

Fig. 1 Schematic diagram of one sphere of diameter d confined in a rectangular container. Two walls in x-direction are at a constant distance L apart and in simple harmonic motion with amplitude A and frequency $\omega/2\pi$

phase difference between two collisions at opposite walls is $\omega t' - \omega t = \pi$. Using variable $\theta = \omega t$, Eqs. (1) and (2) can be solved for v and θ at resonance:

$$v_R = \frac{\pi\omega\,(L - d) \pm 2\omega\sqrt{\left(4\alpha + \pi^2\right) A^2 - \alpha\,(L - d)^2}}{4\alpha + \pi^2} \tag{3}$$

$$\theta_R = \tan^{-1}\left(\frac{2\sqrt{\alpha}\,v_R}{\pi\,v_R - \omega\,(L - d)}\right) \tag{4}$$

where $\alpha = \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^2$. The condition for the existence of real solution v_R is that the value in the square root in Eq. (3) must be positive or zero, which can be expressed as:

$$A/L \geq \sqrt{\frac{\alpha}{4\alpha + \pi^2}}(1 - d/L) \tag{5}$$

The condition says that, for a given length of L, the vibration amplitude A has to be larger than a certain value in order to have a resonant solution. If there is no energy loss in collisions ($\varepsilon = 1$), the right hand side of Eq. (5) becomes zero. That means that resonant solutions can be sustained without any energy supply. Thus, Eq. (5) can be understood physically as the ratio of the input power (proportional to A^2) and the resonant power (proportional to L^2) has to be larger than a certain value in order to have a resonant solution. When there is energy loss in collisions, energy needs to be injected so that the right hand side of Eq. (5) becomes non-zero. If rate of the energy loss is prescribed by ε, doubling the resonant power requires double input-power when $d << L$.

A solution for Eqs. (3) and (4) must be stable to be physically accessible. To determine the stability of the solution, we go back to Eqs. (1) and (2). A small perturbation $(\delta\theta, \delta v)$ to the solution (θ, v) will induce a change $\left(\delta\theta', \delta v'\right)$ for the velocity and phase at the next collision:

$$\begin{pmatrix}\delta\theta' \\ \delta v'\end{pmatrix} = \begin{pmatrix}\varepsilon & -\frac{(1-\varepsilon)\pi}{2A\omega\sin\theta_R} \\ \varepsilon(1+\varepsilon)A\omega\cos\theta_R & \varepsilon - \frac{(1-\varepsilon^2)\pi\cos\theta_R}{2\sin\theta_R}\end{pmatrix}\begin{pmatrix}\delta\theta \\ \delta v\end{pmatrix} \equiv \mathbf{M}\begin{pmatrix}\delta\theta \\ \delta v\end{pmatrix} \tag{6}$$

The eigenvalues of the matrix $\mathbf{M}$ are $\lambda = \left(tr(M) \pm \sqrt{tr(M)^2 - 4\det(M)}\right)/2$, where $tr(M)$ and $\det(M)$ are the

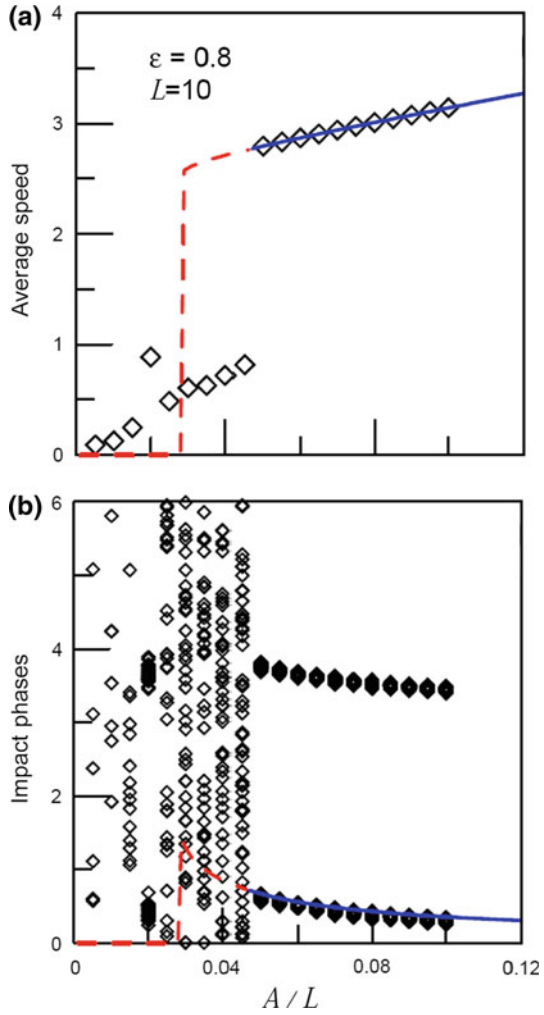

Fig. 2 **a** Average speed and **b** impact phase at two walls of the particle moving between two oscillating walls. The parameters used in simulation are $L = 10, \varepsilon = 0.8$, and $d = 2$. Results of MD simulations (*diamond*) are consistent with the analytic predictions for unstable (*broken line*) and stable (*solid line*) solutions

trace and the determinant of the matrix **M**. A stable solution exists only when $|\lambda| < 1$ [37]. Substituting the solutions for v and θ given in Eqs. (3) and (4) into the expression of λ, we found that the solution with lower sign in Eq. (3) is unstable while the other is stable if the following condition is satisfied:

$$A/L \geq e(\varepsilon)(1 - d/L) \tag{7}$$

where the coefficient $e(\varepsilon)$ is a function of ε only, independent of the driving frequency ω. One can show that $e(\varepsilon)$ is always larger than the factor $\sqrt{\frac{\alpha}{4\alpha+\pi^2}}$ shown on the right hand side of Eq. (5), which means that producing a stable solution requires a larger A than that for the existence of a solution. We plot the resonant solutions, v_R and θ_R as a function of A/L for the system with $d/L = 0.2$ and $\varepsilon = 0.8$ in Fig. 2, where the stable and unstable solutions are respectively indicated by the solid and broken lines.

3 *N*-particle system

For many (N)-particle systems we used molecular dynamics (MD) (see Appendix) to calculate the x-component velocity of the center of mass (CM) $v_{CM} = \frac{1}{N}\sum_{i=1}^{N} v_{xi}$ and determine the impact phase θ_{CM} between the group and the walls by monitoring the instant when $v_{CM} - v_p$ changes its sign. For all the simulations reported in this paper, we fixed the frequency at $f = 1/2\pi$. We ran the simulation for 200 vibration periods and used the particle trajectories in the last 100 periods to calculate the time average impact phase $\langle\theta_{CM}\rangle$ and the mean flow energy $\langle E_{CM}\rangle = \frac{1}{100T}\int_{100T}^{200T} v_{CM}^2 dt$, where $T = 1/f$ is the period of driving force.

We first simulate the one-particle ($N = 1$) case and compare the results with the analytic results of the last section. Figure 2 shows the simulation results of the average speed and the impact phase as a function of relative amplitude A/L, together with the analytic results given by Eqs. (3) and (4). The length of the container and the coefficient of restitution are $L = 5d$ and $\varepsilon = 0.8$, respectively. According to Eq. (5), resonant solutions exist for $A/L \geq 0.0283$. However, a solution is not stable unless $A/L \geq 0.0464$ according to Eq. (7), where $e(\varepsilon = 0.8) = 0.0581$. Therefore, for an A/L below 0.0464, the CM of the particles hits the two walls at random phases so that the average speed is low. Starting from $A/L = 0.0464$, the average speed jumps to a higher value, and the impact phase at each wall converges to a constant value as predicted by the theoretical stable solution.

We repeat the simulation for the two-dimensional case with $L \times W = 20d \times 10d$ at a different amplitude A and a different number N of particles. The results show that the system transits from a gas-like state to a solid-like state at a critical number of particles for a given amplitude. We plot $\langle E_{CM}\rangle$ and $\langle\theta_{CM}\rangle$ as a function of the filling fraction $\phi = N\pi d^2/4LW$ at $A = 2.5d, 3d$ and $3.5d$ in Fig. 3. Two distinct regions separated at a critical filling fraction ϕ_c. For $\phi < \phi_c$, $\langle E_{CM}\rangle$ decreases rapidly when the filling fraction increases. $\langle\theta_{CM}\rangle$ is random and thus yields an average value with a large fluctuation. For $\phi \geq \phi_c$, $\langle E_{CM}\rangle$ increases to a local maximum and then decreases gradually to the average energy $A^2\omega^2/2$ of the external drive and $\langle\theta_{CM}\rangle$ converges to an average value with a small fluctuation similar to the resonant case of the one particle system. The differences of these two states can be further revealed from the configuration of the particles and the motion of the center of mass. For this, we calculate the x-component position of the center of mass $x_{CM}(t)$ as a function of time to study the mean flow motion of the system and the pair correlation function $g(r)$

$$g(r) = \frac{2LW}{aN(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{i-1} H(r_{ij} - r)H(r + \Delta - r_{ij}) \tag{8}$$

to describe the structure of the particles. The term $a = \pi\,(2r + \Delta)\,\Delta$ is the area of a ring with an inner radius of r and a width $\Delta = 0.1H(x)$. is the Heaviside function and r_{ij} is the distance between particles i and j. In Fig. 4, we plot $\langle g(r)\rangle$ and $x_{CM}(t)$ for three different filling fractions at $A = 3d$ with $\varepsilon = 0.8$. At $\phi = 0.079, x_{CM}(t)$ oscillates back and forth with irregular amplitude at the same frequency as the driving walls. The corresponding $\langle g(r)\rangle$ shows no obvious structural peaks, and we refer to it a gas-like state. At $\phi = 0.314$, the peaks of $\langle g(r)\rangle$ at $r/d = 1$ and 2 appear (This can be considered as an amorphous phase similar to glass state in colloidal suspension. However, the transition is not as conspicuous as the liquid–solid transition. Here we simply classified the system into two states only.), indicating that the number of direct contacts between the particles has increased. As a result, more energy is dissipated due to inelastic collisions, causing $x_{CM}(t)$ to fluctuate with a small amplitude near the middle of the two walls. At $\phi = 0.511 > \phi_c, x_{CM}(t)$ oscillates periodically with a uniform amplitude implying that the particles move collectively in resonance to the driving walls. In the mean time, another distinct peak of $\langle g(r)\rangle$ occurs at $r/d \sim 1.73$, implying a hexagonal packing. We classify this state as a solid-like state. The critical filling fraction ϕ_c we define here indicates the transition point of the two states: the chaotic motion with loose packing and the collective motion with close packing.

In the solid-like state, the compact cluster of particles can be approximated by a giant pseudo particle moving in resonance to the external drive with a velocity v_R given by Eq. (3) except that the size d and the restitute coefficient ε have to be replaced respectively by the effective values d_{eff} and ε_{eff} for the pseudo particle. We can write $d_{eff} = \delta L\phi$, where the geometric factor δ has a value of around 1. In Fig. 3, we plot $v_R^2/A\omega^2$, with $\varepsilon_{eff} = 0$ [33], as a function of ϕ for a solid-like state in solid lines. This fits the simulation result of $< E_{CM} > /A^2\omega^2$ very well. The condition for the existence of real value v_R now reads

$$\delta\phi \geq 1 - \sqrt{4 + \pi^2}A/L \tag{9}$$

The critical filling fractions ϕ_c obtained from the equality in Eq. (9) agree very well (Fig. 5) with the simulation results obtained from Fig. 3.

Quantitatively, our theoretical values of ϕ_c are lower than the experimental values [25,26]. The difference comes from our neglect of friction between the particles and the bottom of the container which dissipates the energy of the particles. For a given ϕ, more energy is needed from the external drive to compensate for energy loss through friction so that E_{CM} can gain enough energy to meet the resonant condition. Thus, the phase transition occurs at a larger A. On the other hand, for a given A, E_{CM} is smaller than the value when friction is neglected. To satisfy the resonant condition, more particles are needed to reduce the distance traveled by CM from one wall to the other. Thus phase transition occurs at a larger value of ϕ_c. When friction was considered in the simulation, the experimental values of ϕ_c could be fitted nicely by selecting a suitable value for the coefficient of Coulomb's friction.

4 Granular temperature

The total energy of the particles is the sum of the mean flow energy E_{CM} and the fluctuation energy defined as $\frac{1}{N}\sum_{i=1}^{N}(v_i - v_{CM})^2$ which is usually used for defining granular temperature T_g:

$$T_g = \frac{1}{N}\sum_{i=1}^{N} u_i^2, u_i = v_i - v_{CM} \tag{10}$$

We have seen in the previous section that E_{CM} of the solid-like state in the two-dimensional case can be obtained approximately by the completely inelastic particle model and the model provides a good prediction to the critical filling fraction. Here we will see that T_g also behaves differently in the gas- and solid-like states. In Fig. 6 we plot the average granular temperature $< T_g >$, scaled by $A^2\omega^2$, as a function of ϕ. We see that $< T_g >$, like $< E_{CM} >$ as shown in Fig. 3, changes abruptly at a critical filling fraction ϕ_c. Surprisingly, $< T_g >$ increases when the system alters its state from gas-like to solid-like, unlike ordinary fluidization [6] which has a low value for T_g in the solid-like state. It has been shown [38] that for a dilute granular gas with a Maxwellian velocity distribution, the granular temperature can be determined by equating the rate of energy dissipated and the power input to yield $T_g \sim A^2\phi^{-1}$. For our system, $< T_g >$ is proportional to A^2 and a decreasing function of ϕ by a power of -1.8 ± 0.2. We plot the velocity distribution $P(u)$ for $A = 3d$ and $\varepsilon = 0.8$ in Fig. 7. These distribution curves can be well fitted by the superposition of two Gaussian distributions in the gas-like state [32,33] while, in the solid-like state, they are more like the superposition of an exponential and a Gaussian distribution [34]:

$$P(u) = \left\{ \begin{array}{ll} \frac{N_1}{N}e^{-u^2/2T_1} + \frac{N_2}{N}e^{-u^2/2T_2}, & \mathrm{gas-like} \\ \frac{N_1}{N}e^{-|u|/\sqrt{T_1/2}} + \frac{N_2}{N}e^{-u^2/2T_2}, & \mathrm{solid-like} \end{array} \right\}. \tag{11}$$

Figure 8 shows the fitted values for temperatures T_1 and T_2, and particle portions N_1/N and N_2/N as a function of ϕ.$N_1 > N_2$ and $T_1 < T_2$, that is, a lesser portion of particles (N_2/N) has a larger temperature (T_2), while a larger portion of particles (N_1/N) has a smaller temperature (T_1). Because the two particle density domains have different temperatures in gas-like state, the average temperature T_g as a function of ϕ does not simply follow the power law ϕ^{-1}, but rather $\phi^{-1.8}$

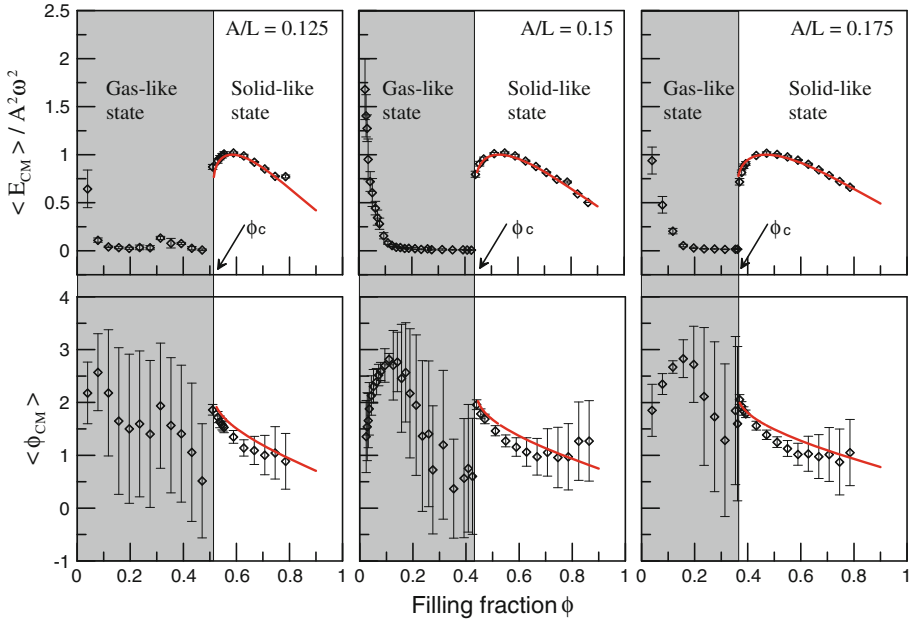

Fig. 3 Time average values of mean flow energy $\langle E_{CM} \rangle$ and impact phase $\langle \theta_{CM} \rangle$ as a function of filling fraction at $A = 2.5d, 3d$ and $3.5d$. The frequency is $f = 1/2\pi$ and the coefficient of restitution is $\varepsilon = 0.8$

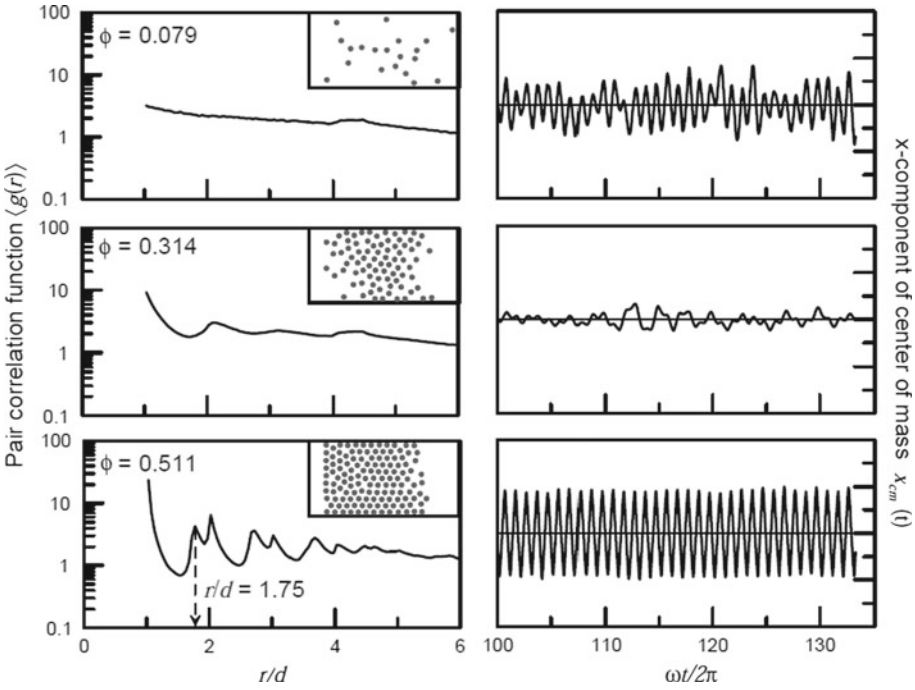

Fig. 4 The time average pair correlation function $\langle g(r) \rangle$ and x-component of CM $x_{cm}(t)$ for three different filling fractions $\phi = 0.079$, 0.314, and 0.511 at $A = 3d$ with $\varepsilon = 0.8$. The *inset* graphs at *right corner* of first column are the corresponding snapshot configurations of particles at $t = 128T$

as seen in Fig. 7. We can calculate T_g analytically using the velocity distribution Eq. (11) to obtain

$$T_g = \frac{T_1 + kT_2}{1 + k} \tag{12}$$

where $k = \frac{N_2}{N_1}\sqrt{\frac{T_2}{T_1}}$ for the gas-like state and $k = \frac{N_2}{N_1}\sqrt{\frac{\pi T_2}{T_1}}$ for the solid-like state. Based on the values for T_1, T_2, N_1/N, and N_2/N given in Fig. 8, we found the value of k is small compared to 1 but kT_2 is always comparable with T_1. It turns out that the numerical value of T_g is roughly two times that of T_1 for both the gas- and solid-like states. It would be interesting to see whether the relation $T_g \approx 2T_1$ has any explanation or is simply a special case in our system.

5 Dependence of the restitution coefficient

Our discussion above is based on simulation results given $\varepsilon = 0.8$. In Fig. 9 we show both the average total energy, $< E >$, and the pair correlation function at $r = d$, $< g(d) >$, as a function of filling fraction at $\varepsilon = 0.8$, 0.85, 0.87, 0.9, 0.95, and 0.98. When $\varepsilon \to 1$, $< g(d) >$ approaches the

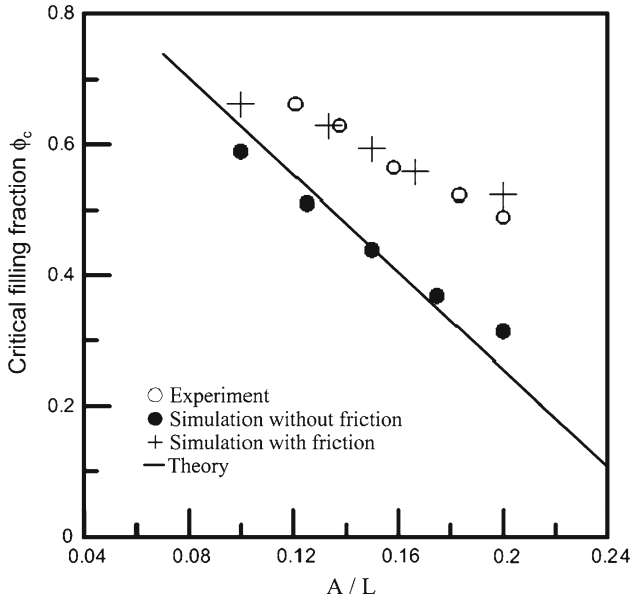

Fig. 5 Dependence of the critical filling fraction on the relative amplitude for 2-dim case (*filled circle*), $L \times W = 20d \times 10d$. The frequency is $f = 1/2\pi$ and the coefficient of restitution is $\varepsilon = 0.8$. The *solid line* is the result of the equality in Eq. (9) with $\varepsilon_{eff} = 0$ and $\delta = 1$. The symbol *circles* are the experimental results of the two-dimensional system reported in Ref. [26] with the same definition of filling fraction as in this report. The symbol *crosses* are simulation results including the effect of Coulomb's friction

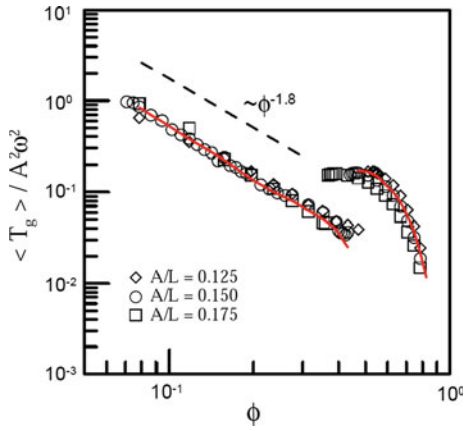

Fig. 6 **a** Time average of fluctuation energy $\langle T_g \rangle$, scaled by $A^2\omega^2$, as a function of filling fraction ϕ at the parameters same as Fig. 3. The *solid lines* are plotted according to Eq. (12) with parameters shown in Fig. 8

Carnahan-Starling equation [39]. We see that ϕ_c is insensitive to the exact value of ε. However, the gas- and solid-like states become indistinguishable when ε is larger than the value $\varepsilon_c \approx 0.93$ (The transition point is identified by looking for the point where the second derivative of the energy changes sign.). When $\varepsilon > \varepsilon_c$, $< E >$ becomes a monotonically decreasing function without a local maximum or minimum and the discontinuity of the slope of $g(d)$ disappears. That is, when $\varepsilon > \varepsilon_c$, inelastic clustering does not occur and the granular temperature would be so high that particles would be unable to form a stable cluster. The situation is similar to water condensation where steam will never condense to water beyond a critical temperature regardless of pressure. According to a simplified model called independent collision wave (ICW) approximation reported by Bernu and Mazighi [29], the value of ε_c that causes n_c particles in a one-dimensional system to collapse into a cluster is given by $\varepsilon_c \approx 1 - \pi/n_c$. For a two dimensional system n_c is the number of particles per column width [31]. Here, we use ϕ_c to estimate the value of n_c and then calculate ε_c to obtain a value roughly 30% larger than the predicted ICW approximation, but this deviation is expected. Since our particles are confined in a two-dimensional finite region and energy is steadily injected, the interference of collision waves may not be neglected so that particles in our system cluster more readily than in the ICW approximation.

6 Conclusion

We simulated the motion of N identical inelastic particles between two oscillating walls and studied the properties of

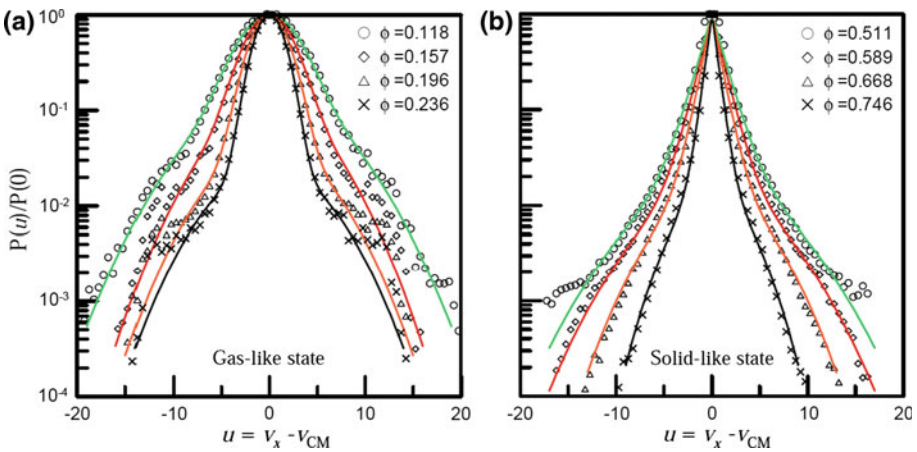

Fig. 7 Velocity distribution function of **a** gas-like state at ϕ =0.118, 0.157, 0.196, and 0.236, and **b** solid-like state at ϕ =0.511, 0.589, 0.668, and 0.746. The *solid lines* are the fitting curves given by Eq. (11) with the parameters values shown in Fig. 8

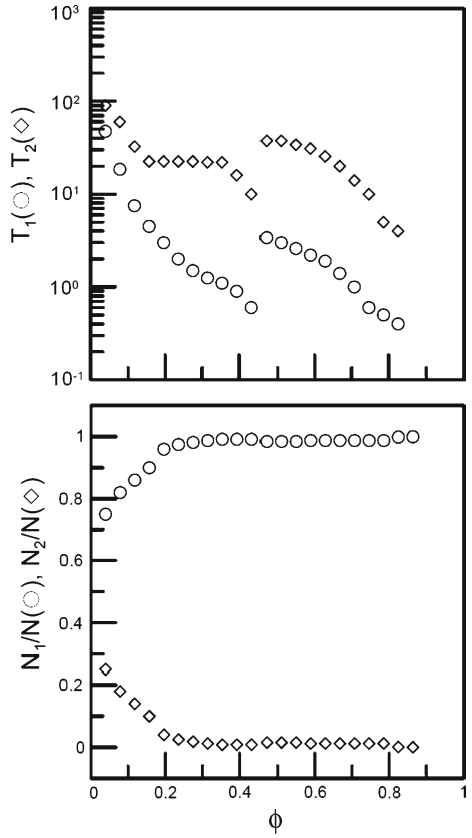

Fig. 8 The temperatures T_1, T_2 and particle portions $N_1/N, N_2/N$ respectively as a function of the filling fraction with these values, Eq. (11) fits the velocity distribution (Fig. 7) approximately

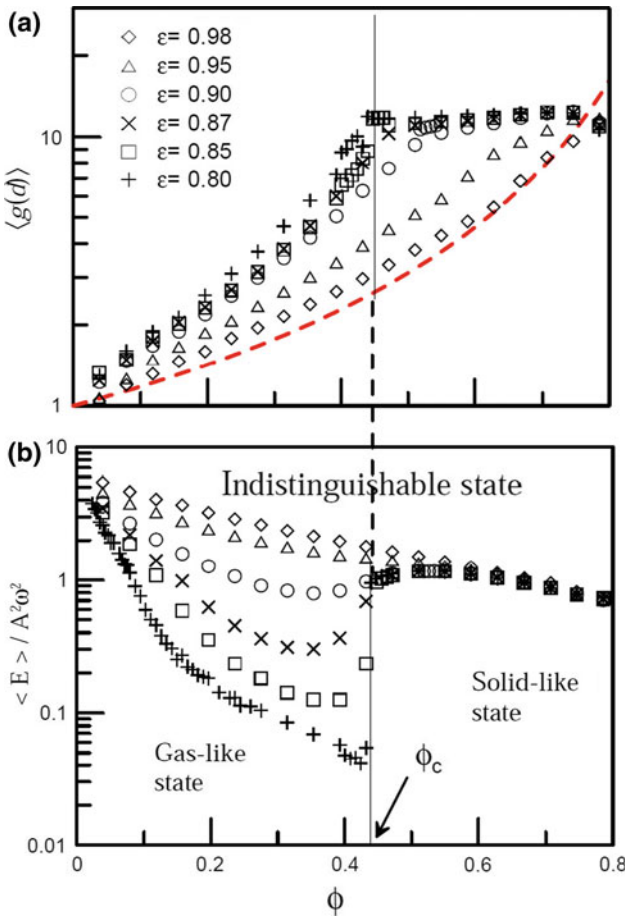

Fig. 9 **a** The pair correlation function at contact $\langle g(d) \rangle$ and **b** the time average energy $\langle E \rangle / A^2\omega^2$ as a function of filling fraction at $\epsilon = 0.8(!), 0.85((), 0.87(\times), 0.9(+), 0.95(1)$, and 0.98($). The *dash line* in **a** is the Carnahan-Starling equation for nearly inelastic disk

the gas- and solid-like phases of the system at different filling fractions, driving amplitudes and coefficients of restitution. The transition requires two conditions: (1) the filling fraction of the particles and the driven amplitude must satisfy the resonant condition given by Eq. (9); (2) the particle's coefficient of restitution has to be small enough that the inelastic clustering can occur at the filling fraction which satisfies the resonant condition. At the transition point, both of the mean flow energy and the fluctuation energy increase dramatically. We found that the mean flow energy of the solid-like state can be well approximated by the resonant energy of a pseudo particle in the completely inelastic model. The resonant condition gives the value of critical filling fractions which is a decreasing function of vibration amplitude, consistent with previous experimental results. We have also found, from an analysis for the fluctuation energy, that the gas-like state of our system can be described by a dilute particle system with two coexistent temperatures.

Appendix: Simulation method

We consider N identical spheres of diameter $d = 2$ and mass $m = 1$ which are confined between two oscillating walls. The two walls are in simple harmonic motion in direction x at amplitude A at frequency $f = 1/2\pi$. Besides the dissipative collisions which are characterized by the coefficient of restitution ε, there is no other way to dissipate energy from the system. Frictions and rotations of the particles are not considered in our simulation. Energy is supplied by the particles striking the walls.

The collision force $\vec{F}_{ij}$ between the ith and jth particles is modeled by the dissipative linear spring force:

$$\vec{F}_{ij} = -\left[\frac{m}{2}\omega_s^2\left(d - |\vec{r}_{ij}|\right) + 2\gamma\left(\vec{u}_{ij}\cdot\vec{r}_{ij}/|\vec{r}_{ij}|\right)\right]\vec{r}_{ij}/|\vec{r}_{ij}|$$

where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$ is the vector along the line connecting the centers of particles i and j, and $\vec{u}_{ij} = \vec{u}_j - \vec{u}_i$ is the relative velocity. When the contact duration of collision t_c and the coefficient of restitution ε are specified, the parameters ω_s and γ can be determined by

$$\gamma = -\frac{\ln\varepsilon}{t_c}, \quad \omega_s = \sqrt{(\pi/t_c)^2 + \gamma^2}$$

We have used $t_c = 3 \times 10^{-3}$ s in all computations.

We integrate the equation of motion:

$$\frac{d\vec{v}_i}{dt} = \begin{cases} \sum \vec{F}_{ij} & \text{if } \left|\vec{r}_i - \vec{r}_j\right| < d_i + d_j \\ 0 & \text{else} \end{cases}$$

$$\frac{d\vec{r}_i}{dt} = \vec{v}_i$$

by Verlet's Method [40] with time step $dt = t_c/30$ which is small enough to promised the convergence of average kinetic energy. Collisions between particles and walls are also treated by the spring force with the same ε.

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