The Response of a Weakly Stratified Layer to Buoyancy Forcing

JOSEPH PEDLOSKY

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

(Manuscript received 6 February 2008, in final form 2 October 2008)

ABSTRACT

The response of a weakly stratified layer of fluid to a surface cooling distribution is investigated with linear theory in an attempt to clarify recent numerical results concerning the sinking of cooled water in polar ocean boundary currents.

A channel of fluid is forced at the surface by a cooling distribution that varies in the down-channel as well as the cross-channel directions. The resulting geostrophic flow in the central region of the channel impinges on its boundaries, and regions of strong downwelling are observed. For the parameters of the problem investigated, the downwelling occurs in a classical Stewartson layer but the forcing of the layer leads to an unusual relation with the interior flow, which is forced to satisfy the thermal condition on the boundary while the geostrophic normal flow in the interior is brought to rest in the boundary layer.

As a consequence of the layer's dynamics, the resulting long-channel flow exhibits a nonmonotonic approach to the interior flow, and the strongest vertical velocities are limited to the boundary layer whose scale is so small that numerical models resolve the region only with great difficulty. The analytical model presented here is able to reproduce key features of the previous nonlinear numerical calculations.

1. Introduction

The location of the sinking of cooled water in polar regions is one of the fundamental issues that needs clarification for the theory of the ocean's overturning circulation. Recent work on that sinking (e.g., Pedlosky and Spall 2005) has emphasized the enhancement of the sinking in the vicinity of lateral boundaries of the basin where the vorticity produced by stretching can be dissipated by friction. There have been many other studies of the process (e.g., Pedlosky 1968; LaCasce 2004). However, in these earlier studies, the sinking was supposed to occur in boundary regions with significant vertical stratification. The importance of boundary mixing for the meridional overturning circulation has been emphasized by Marotzke (1997) and Marotzke and Scott (1999). In each of these studies the zones of upwelling or sinking have been substantially stratified.

In a recent paper (Spall 2008), the downwelling induced by buoyancy loss in a boundary current was studied in an attempt to describe the process by which cooled water in polar regions sinks. As previous studies have shown, the tendency is for the sinking to take place adjacent to boundaries where the vorticity induced by the stretching of vortex columns by the sinking fluid can be dissipated by friction. The calculation in Spall's study used the full Massachusetts Institute of Technology (MIT) general circulation model (Marshall et al. 1997). A current was introduced at the entrance to a channel and cooling, uniform in the down-channel direction, produced an evolution of the current in that direction such that an along-channel pressure gradient in geostrophic balance drove fluid to the right-hand boundary of the channel where it underwent strong sinking. In contrast to earlier work, the model develops a mixed layer of very weak but nonzero vertical stratification in which the sinking occurs but the lateral temperature gradients in the layer drive a geostrophic flow forcing the downwelling. This contributes to making the sinking region extremely narrow and the narrowness of the sinking region is such that only a few grid points in the calculation represent the boundary layer, so its spatial resolution is marginal. Although it is not thought that this affects the overall strength of the downwelled fluid, it appears conceptually important to resolve the structure of the dynamics with a simple analytical model: that is the goal of the present study.

One of the curious features of the numerical results is the nonmonotonic behavior of the along-channel flow

Corresponding author address: Joseph Pedlosky, Woods Hole Oceanographic Institution, Woods Hole, MA 02543.
E-mail: jpedlosky@whoi.edu

DOI: 10.1175/2008JPO3996.1

near the boundary. The numerical model has a double boundary layer structure in which a broad Prandtl-type boundary layer appears to act to satisfy the no-slip condition on the along-channel flow: yet, as the boundary is approached within this layer, u, the along-channel flow, *increases* before finally being brought to zero in a very narrow region within the no-slip layer. In the discussion that follows a very simple linear model of a weakly stratified fluid, cooled at the upper surface, is employed to discuss, in particular, the inner region of the boundary layer where the overshoot of u occurs and where the strong sinking is found. The use of this linear model is suggested by the relative insensitivity in the numerical model of Spall (2008) to the degree of nonlinearity. Indeed, Spall suggested that the layer was a modified form of the nonhydrostatic Stewartson layer (Stewartson 1957) found in the theory of homogeneous rotating fluids. There is much that is unrealistic in the analytic model and yet its ability to reproduce salient features of the full numerical model implies that those features are robust and not dependent on the nonlinear nature of the original calculation.

Section 2 describes the basic model. Section 3 outlines the equations for the interior flow outside the Stewartson layer, while section 4 describes the Stewartson layer and the matching condition of the layer to the interior, setting a boundary condition on the interior flow. Section 5 presents the main results for a simple example of the theory. In section 6 some final remarks are made on the overall nature of the problem and its dependence on stratification (or its lack).

2. The model

We consider the flow in a channel of width L and depth D. The fluid in the channel is cooled at the surface at a rate H such that at the upper surface, $z = D$,

$$\kappa_v \frac{\partial T}{\partial z} = H(x, y), \tag{2.1}$$

where T is the temperature *anomaly* above a weak background vertical gradient, that is,

$$T_{\text{total}} = \Delta T_v z/D + T; \tag{2.2}$$

κ_ν is the thermal diffusivity in the vertical direction, and x and y are the long-channel and cross-channel coordinates, respectively.

The independent variables are scaled:

$$(x, y, z) = (Lx', Ly', Dz'). \tag{2.3}$$

The temperature anomaly is scaled, using (2.1):

$$T = \frac{HD}{\kappa_v} T', \tag{2.4}$$

while the horizontal and vertical velocities and the pressure are scaled in expectation of a geostrophic and hydrostatic balance holding over most of the domain; that is,

$$(u, v, w) = \frac{g\alpha HD^2}{f\kappa_v L}\left(u', v', \frac{D}{L}w'\right)$$
$$p = \rho_o \frac{g\alpha HD^2}{\kappa_v} p', \tag{2.5a,b}$$

where α is the coefficient of thermal expansion, g is the acceleration due to gravity, ρ_o is the constant reference density, and f is the constant Coriolis parameter. The Rossby number of the flow,

$$\varepsilon = \frac{g\alpha HD^2}{f^2 \kappa_v L^2}, \tag{2.6}$$

will be assumed small enough so that linear theory will be uniformly applicable. The parameter measuring the background stratification, S, the Burger number, is

$$S = \frac{g\alpha \Delta T_v D}{f^2 L^2}. \tag{2.7}$$

The nondimensional linearized equations of motion for this incompressible fluid on the f plane are

$$\begin{aligned}
u &= -p_y + \frac{E_H}{2}\nabla^2 v + \frac{E_v}{2} v_{zz},\\
-v &= -p_x + \frac{E_H}{2}\nabla^2 u + \frac{E_v}{2} u_{zz},\\
0 &= -p_z + T + \frac{D^2}{L^2}\left[\frac{E_H}{2}\nabla^2 w + \frac{E_v}{2} w_{zz}\right],\\
0 &= u_x + v_y + w_z,\\
wS &= \frac{E_H}{2\sigma_H}\nabla^2 T + \frac{E_v}{2\sigma_v} T_{zz}.
\end{aligned} \tag{2.8a–e}$$

The primes on the dimensionless quantities have been dropped and the Laplacian operator in (2.8) is the horizontal Laplacian,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Subscripts denote differentiation with respect to the independent variable. The parameters,

$$(E_H, E_v) = \frac{2}{fL^2}(\nu_H, \nu_v L^2/D^2), \tag{2.9}$$

and the Prandtl numbers are the ratios

$$\sigma_H = \upsilon_H/\kappa_H, \quad \sigma_\upsilon = \upsilon_\upsilon/\kappa_\upsilon. \tag{2.10a,b}$$

Different viscosity coefficients have been introduced for lateral and vertical momentum mixing but, given the simplicity of the model and the weak stratification that will be assumed, such detailed assumptions about the anisotropy of the mixing are problematic. The theory is qualitatively unchanged if the mixing coefficients are isotropic.

The boundary conditions are

$$\begin{aligned} &T_z = H, \quad z = 1,\\ &T = 0, \quad z = 0,\\ &T_y = u = \upsilon = w = 0, \quad y = 0, 1,\\ &u_z = \upsilon_z = w = 0, \quad z = 0, 1. \end{aligned} \tag{2.11a–d}$$

The thermal conditions represent a nonuniform heating at the upper boundary and a fixed temperature at the lower boundary, which is our substitute in this simple model for a fairly passive fluid layer beneath. The sidewalls are thermally insulated.

The last condition in (2.11) expresses the condition of no stress at the top and bottom of the fluid layer, which seems appropriate as a model of a heated mixed layer not in contact with rigid horizontal boundaries. That condition, through the use of Ekman layers at the top and the bottom of the layer, is translated into a condition on the vertical velocity at the edge of the very thin Ekman layers:

$$w = \mp \frac{E_\upsilon}{2}\frac{\partial}{\partial z}[\upsilon_x - u_y], \quad z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.12}$$

where $z = 0$ and 1 are understood to be at the edge of the Ekman layer.

We will be particularly interested in the limit where S is small but will insist that the temperature anomaly is small enough to maintain a stable stratification consistent with our linearization. Our interest will be focused on the region between the Ekman layers and, in particular, on the boundary layers on the sides of the channel where we anticipate the major vertical motion will occur. We will consider the parameter limit of weak stratification expressed by [see (4.4) below]

$$\sigma_H S \ll E_H^{2/3}(D/L)^{2/3}. \tag{2.13}$$

3. The interior

In the fluid interior the scales for the variables introduced in the last section are presumed to give an accurate measure of the relative importance of the individual terms in the equations of motion. That being the case, the horizontal momentum equations reduce to geostrophic balance and the vertical momentum equation is simply the hydrostatic balance. Denoting the interior dependent variables with a subscript I,

$$u_I = -p_{Iy}, \quad \upsilon_I = p_{Ix}, \quad T_I = p_{Iz}. \tag{3.1a–c}$$

It follows that the horizontal velocity is nondivergent in the interior so that the vertical velocity must be independent of z. From the boundary conditions (2.12) it follows that w_I is $O(E_\upsilon)$ at $z = 0,1$ and so must be of that order for all z. If $S \ll 1$, and assuming that σ_υ is $O(1)$, it follows that the vertical advection of temperature is negligible in the thermal equation (2.8e), which then becomes

$$\frac{E_H}{2\sigma_H}\nabla^2 T_I + \frac{E_\upsilon}{2\sigma_\upsilon}T_{Izz} = 0. \tag{3.2}$$

Once the solution for (3.2) is found, the horizontal velocities can be determined up to a barotropic (z independent) constant of integration from the thermal wind relation. The vertical velocity in the interior is very weak, of order E_H, E_υ (which for simplicity we will assume are of the same order). However, it is not possible to determine the solution of (3.2) until boundary conditions are specified on the sidewalls. At the horizontal boundaries the interior temperature must satisfy the conditions (2.11a,b). To find the appropriate boundary conditions for (3.2) and to determine the barotropic component of the interior flow, it is necessary to examine the boundary layers at y = 0 and 1.

4. The sidewall boundary layer

For a very weakly, nearly homogeneous fluid, the structure of the sidewall boundary layers in linear theory has been described by Stewartson (1957) (see also Greenspan 1968), although the theory needs some alteration to deal with the differing mixing coefficients in the vertical and horizontal directions and the smallness of D/L. Fundamentally though the basic ideas are not significantly altered. In the original theory there are two possible boundary layers; an outer layer with thickness that depends on the quarter power of the friction and which act to satisfy the no-slip condition. That $E_H^{1/4}$ layer depends on vorticity dissipation in Ekman layers on solid horizontal surfaces bounding the fluid on at least one horizontal boundary. We have chosen to examine a layer satisfying a no-stress condition on $z = 0$ and 1: It is easy to show, with the application of (2.12), that this outer layer is no longer possible. That will present a strong constraint on the interior flow. The inner boundary layer, in our notation, has a thickness,

$$\delta_b = (E_H D/L)^{1/3} \tag{4.1}$$

or, in *dimensional* units,

$$\delta_{*b} = \left(\frac{2\upsilon_H}{f}\right)^{1/3} D^{1/3},$$

and is, as expected, independent of the overall channel width L.

We define corrections to the interior fields in the boundary layer near $y = 0$ as

$$u_b = U\bar{u}, \quad \upsilon_{gb} = U\delta_b\bar{\upsilon}_g, \quad \upsilon_{ab} = U\frac{\delta_b}{(D/L)}\bar{\upsilon}_a,$$
$$w_b = U\bar{w}/(D/L), \quad T_b = U\frac{(\sigma_H S)}{\delta_b}\bar{T}, \tag{4.2a–e}$$

where U is an unknown scaling constant for all variables and υ_g is the geostrophically balanced part of the correction to υ in the boundary layer while υ_a is the ageostrophic part. All correction variables are functions of the stretched y variable, $\eta = y/\delta_b$ and must vanish for large η. To lowest order in the small parameter $\sigma_H S/E_H^{2/3}(D/L)^{2/3}$, the correction functions satisfy

$$\bar{u} = -\bar{p}_\eta, \quad \bar{\upsilon}_g = \bar{p}_x, \quad \bar{\upsilon}_a = -\frac{1}{2}\bar{u}_{\eta\eta}, \quad 0 = -\bar{p}_z + \frac{1}{2}\bar{w}_{\eta\eta}$$
$$\bar{w}_z = -\bar{\upsilon}_{a\eta}, \quad \bar{w} = \frac{1}{2}\bar{T}_{\eta\eta}. \tag{4.3a–f}$$

In order for the buoyancy force to be negligible in the vertical equation of motion, the condition

$$\sigma_H S \ll E_H^{2/3}(D/L)^{2/3}, \tag{4.4}$$

which defines the parameter restriction of this study. A similar condition (but without the aspect ratio factor) was found by Barcilon and Pedlosky (1967). This condition is equivalent to the condition, in dimensional terms, that the boundary layer thickness

$$\delta_{*b} = \left(\frac{2\upsilon_H}{f}\right)^{1/3} D^{1/3}$$

is much greater than the deformation radius, ND/f, multiplied by $\sigma_H^{1/2}$, the square root of the Prandtl number.

A single equation for the pressure correction function can be found; that is,

$$\bar{p}_{\eta\eta\eta\eta\eta\eta} + 4\bar{p}_{zz} = 0. \tag{4.5}$$

The boundary conditions for (4.5) can be obtained from the Ekman compatibility conditions (2.12). It is straightforward to show that these conditions imply that $\bar{w}$ must be zero on $z = 0,1$, from which it follows that $\bar{p}_z = 0$ at those points. This implies that the solution for $\bar{p}$ can be written as a cosine series; that is,

$$\bar{p} = \mathrm{Re}\sum_{j=1}\cos(j\pi z)[A_{1j}e^{-\gamma_j\eta} + A_{2j}e^{-\gamma_j(1/2+i\sqrt{3}/2)\eta} + A_{3j}e^{-\gamma_j(1/2-i\sqrt{3}/2)\eta}],$$
$$\gamma_j = (2j\pi)^{1/3} \tag{4.6}$$

from which, with (4.3), all other correction functions can be found: they are given in appendix A. Note that $\bar{p}$, and hence the horizontal velocities, have a zero vertical average, so the interior horizontal velocities must also have zero vertical average at the channel boundaries. Thus, the structure of the Stewartson layer imposes a strong condition on the interior flow.

With the solutions to the boundary layer equations, we are now in a position to carry out the matching procedure at $y = 0$. A similar process will occur at $y = 1$ but those details can be skipped.

The matching conditions become, at $y = 0$,

$$u_I + U\bar{u} = 0,$$
$$\upsilon_I + U\left(\frac{\delta_b}{D/L}\right)\left(\bar{\upsilon}_a + \frac{D}{L}\upsilon_g\right) = 0,$$
$$w_I + U\frac{L}{D}\bar{w} = 0,$$
$$T_{Iy} + U\frac{\sigma_H S}{E_H^{2/3}(D/L)^{2/3}}\bar{T}_\eta = 0. \tag{4.7a–d}$$

Here u_I, υ_I, and T_I are $O(1)$ and w_I is $O(E_H)$. In the classical Stewartson layer problems involving a homogeneous fluid, the interior velocity normal to the boundary is zero or, if there is a geostrophic, order one υ_I, then the geostrophically balanced υ_I must satisfy the zero conditions on its own. This, however, cannot be the case here. Since the parameter,

$$\frac{\sigma_H S}{E_H^{2/3}(D/L)^{2/3}},$$

measuring the contribution of the boundary layer to the temperature gradient at the wall is, by hypothesis, small, the *interior temperature gradient* must, to lowest order, satisfy the *insulating condition* on $y = 0$. It is then impossible for the interior to satisfy *that* condition *and* the condition on v. We are forced to the conclusion, then, that we must choose U, the scale for the boundary layer correction to achieve that balance; that is,

$$U = \frac{D/L}{\delta_b} \gg 1. \qquad (4.8)$$

This unusual constraint on the interior can be easily understood in the limit as S goes to zero. Then, the thermal equation is decoupled from the vertical advection and the temperature satisfies a form of Laplace's equation. It is obvious in this limit that the temperature field must directly satisfy the insulating condition directly with the interior variables.

It then follows that to lowest order,

$$\bar{u} = 0, \quad \bar{w} = 0, \quad \bar{v}_a + \frac{D}{L}\bar{v}_g = -v_I, \quad \eta = 0. \quad (4.9\text{a–c})$$

Those three conditions determine A_{1j}, A_{2j}, and A_{3j}. Note that this implies that the flow in the boundary layer is forced by the impinging *geostrophic* flow in the interior at the wall, a conclusion that Spall (2008) has found in his numerical study. Thus, the sinking in the boundary layer is forced indirectly by the cooling as it generates a down-channel pressure gradient and a geostrophic flux toward the boundary where the sinking takes place.

The interior problem is then reduced to a solution of (3.2) subject to the condition that $T_{Iy} = 0$ at $y = 0,1$. That determines the baroclinic flow in the interior. The barotropic component of the interior flow, for which T and w are zero, satisfies

$$E_H \nabla^2 \zeta_{Ib} = 0, \quad \zeta_{Ib} = \nabla^2 p_{Ib}, \qquad (4.10\text{a,b})$$

where the subscript b denotes a barotropic pressure field independent of z. The boundary conditions for this flow are that the geostrophic, barotropic velocities must cancel the vertical average of the baroclinic solution obtained from (3.2) so that

$$\int_0^1 (u_I, v_I)\,dz = 0, \quad y = 0,1, \qquad (4.11)$$

since the boundary layer corrections to the horizontal velocity have no vertical average.

5. An example

To illustrate the theory, consider a simple example: a cooling function, as in (2.11 a), of the form

$$H = \mathrm{Re} H_1 e^{ikx} \cos \pi y, \quad 0 \le y \le 1. \qquad (5.1)$$

The form imposes a cooling on one-half of the channel and a heating on the other half. We will focus on the boundary, at $y = 0$, where the cooling will take place (for specified values of x). The form is chosen to make the satisfaction of the thermal conditions on the sidewalls, $T_{Iy} = 0$, very simple. The solution to (3.2) that accomplishes that is

$$T_I = \frac{H_1}{K}\frac{\sinh Kz}{\cosh K} e^{ikx} \cos \pi y,$$
$$K^2 = (k^2 + \pi^2)\frac{\sigma_v E_H}{\sigma_H E_v}, \qquad (5.2\text{a,b})$$

and the real part of the above expression is understood. From the thermal wind relation the horizontal velocities are determined up to a barotropic component; that is,

$$u_I = \pi \frac{H_1}{K^2}\frac{\cosh Kz}{\cosh K} e^{ikx} \sin \pi y + u_{IB}(x,y),$$
$$v_I = ik\frac{H_1}{K^2}\frac{\cosh Kz}{\cosh Kz} e^{ikx} \cos \pi y + v_{IB}(x,y). \qquad (5.3\text{a,b})$$

Note that the thermally driven, baroclinic, part of the down-channel velocity satisfies the no-slip condition on the channel boundaries. Since the barotropic velocities must cancel the vertical mean of the baroclinic velocity on $y = 0,1$ the boundary conditions for the barotropic velocities are

$$\left.\begin{aligned} u_{IB} &= 0 \\ v_{IB} &= -ik\frac{H_1}{K^3}\tanh K \cos \pi y \end{aligned}\right\}, \quad y = 0,1. \qquad (5.4)$$

The barotropic interior velocities are in geostrophic balance and are found from the pressure field that is a solution of (4.10); namely,

$$p_{IB} = e^{ikx}\left[\frac{A}{2k} y \frac{\cosh ky}{\cosh k} + \frac{B}{2k}(y-1)\frac{\cosh k(y-1)}{\cosh k} + E\frac{\sinh ky}{\sinh k} + F\frac{\sinh k(y-1)}{\sinh k}\right].$$
$$u_{IB} = -\frac{\partial p_{IB}}{\partial y}, \quad v_{IB} = \frac{\partial p_{IB}}{\partial x} \qquad (5.5)$$

The coefficients A, B, E, and F are determined by applying (5.4). The result is shown in appendix B.

The coefficients for the boundary layer solution (4.6) are found by applying (4.9) now that the interior flow is determined. For the example under consideration, it can be shown that

$$A_{1j} = 0, \quad A_{3j} = -A_{2j} e^{2\pi i/3},$$

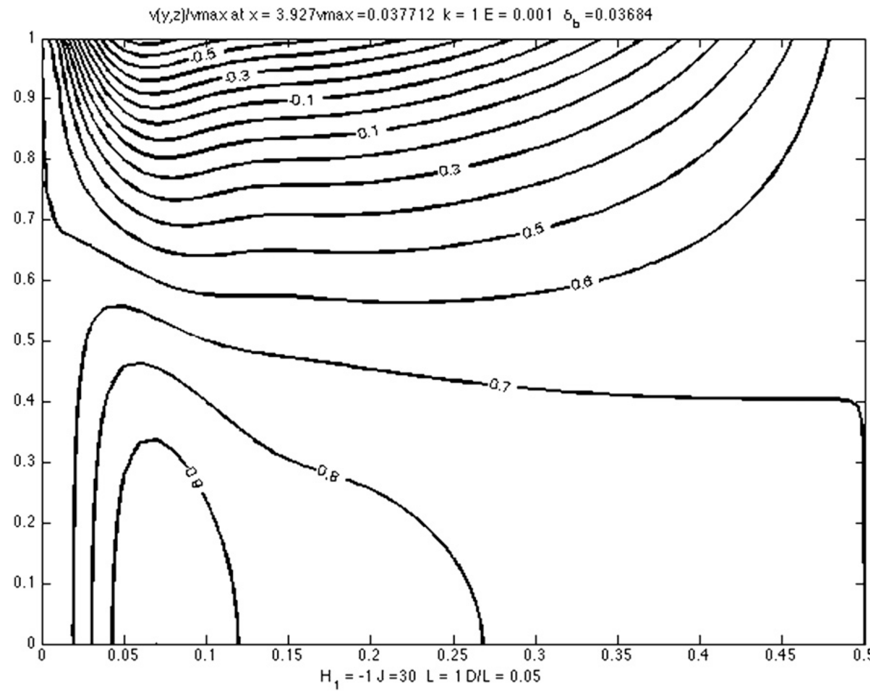

FIG. 1. The cross-channel velocity v in $0 \leq y \leq 0.5$ for $E_H = 0.001$, $D/L = 0.05$, and $H_I = -1$. The contours of v are scaled with its absolute maximum of 0.0377.

and

$$A_{2j} = \frac{2ikH_1(-1)^j \tanh K}{(1 - e^{2\pi i/3})(j^2\pi^2 + K^2)[(ik(D/L) + j\pi)]}, \tag{5.6a–c}$$

which completes the solution. In constructing the solutions discussed below, and shown in the following figures, 30 vertical modes of the boundary layer series (4.6) were retained. Adding more modes did not change the solution.

Figure 1 shows the contours of the velocity in the cross-channel direction at a position, $kx = 1.25\pi$, where the flow in the upper part of the water column is being driven toward the boundary $y = 0$. Half the channel width is shown: The solution for v is antisymmetric about $y = 0.5$. In $0 \leq y \leq 0.5$ flow is being driven toward the boundary $y = 0$ where it sinks in a boundary layer of width $\delta_b = 0.037$. The resulting velocity profile of the zonal velocity shown in the half channel (the zonal velocity is *symmetric* across the channel) at $z = 0.8$ in Fig. 2. Note the monotonic decrease of the zonal velocity as $y = 0$ is approached by the interior solution but, as we approach the boundary layer, the structure of the Stewartson layer produces a local enhancement of the down-channel velocity—just as found in the numerical model of Spall (2008). We can see here that this is due entirely to the damped oscillatory behavior of the layer's structure and the relatively large amplitude of the correction driven by the need of the Stewartson layer to bring the cross-channel velocity to rest. Figure 3 shows the profile of the vertical velocity, again at $z = 0.8$. The vertical velocity is entirely limited to the sidewall boundary layer since the interior velocity, frictionally driven, is extremely small, that is, $O(E_H, E_v)$. As in the numerical model, the sinking is limited to the boundary region where the vorticity production due to vortex tube stretching can be balanced by viscous dissipation.

Although the motion is three-dimensional, the vertical velocity is produced by the ageostrophic component of v in the boundary layer, which satisfies

$$\frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.7}$$

so that a streamfunction for this ageostrophic circulation can be generated (Fig. 4). The contours of that streamfunction in the vertical plane give a clear picture of that part of the flow that leads to the sinking. A very similar structure is seen in Spall's nonlinear numerical calculations where the oscillatory structure of the streamfunction field in y is evident. Note that the ageostrophic velocity in the boundary layer is largely driven by the *geostrophic* velocity in the interior.

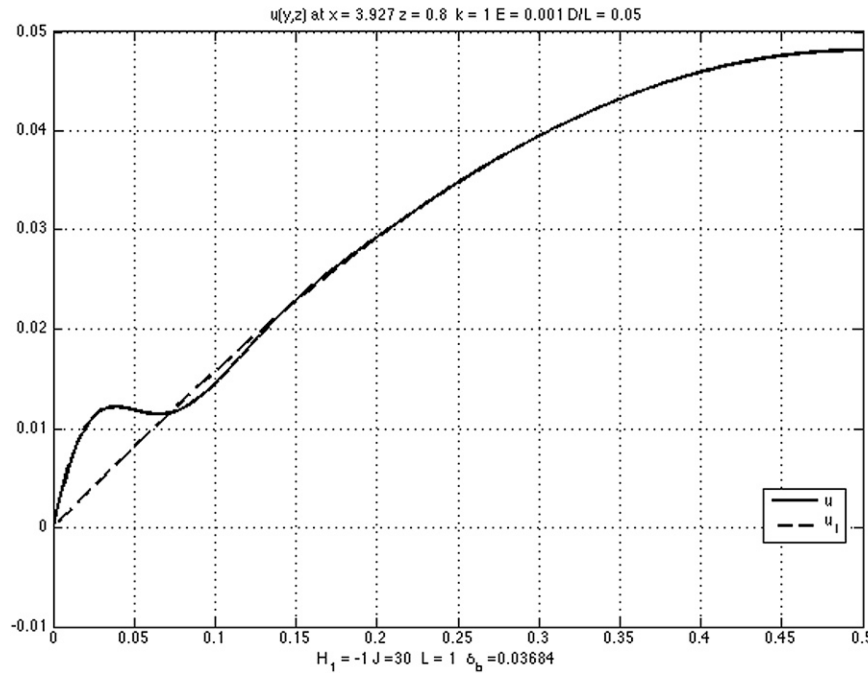

FIG. 2. The profile of the down-channel velocity u at $z = 0.8$ for the same parameters as in Fig. 1.

6. Discussion

A simple linear model has been used to discuss the response of a weakly stratified layer to a nonuniform cooling of the surface. The nonuniformity in the downstream direction is imposed to mimic the downstream variation that would appear naturally if the cooling were uniform but if, as in the nonlinear model of Spall (2008),

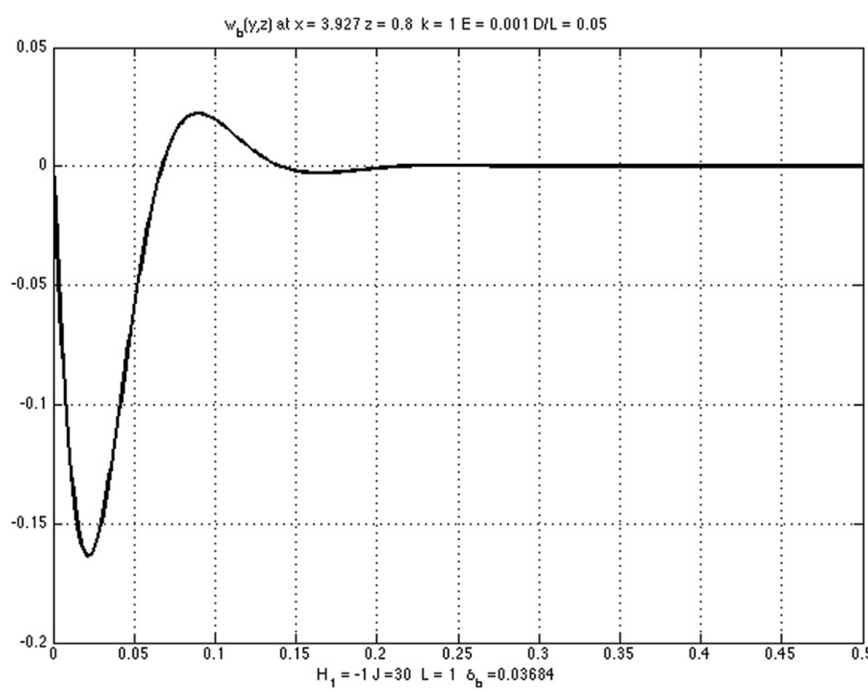

FIG. 3. The vertical velocity profile at $z = 0.8$. Parameters as in Fig. 1.

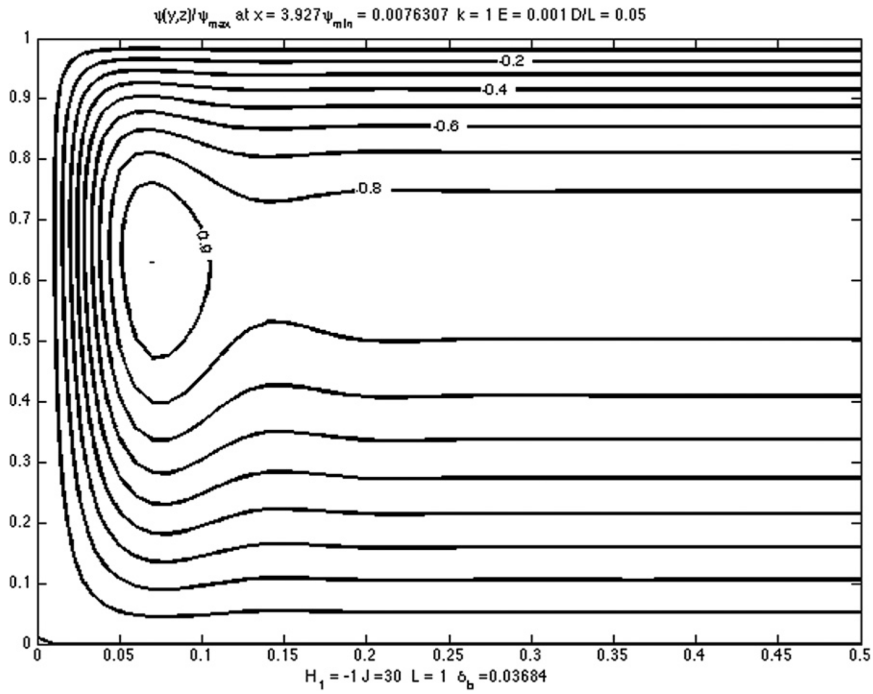

FIG. 4. The circulation streamfunction of the ageostrophic velocity in the *y–z* plane. All parameters are as in Fig. 1. The streamfunction contours are scaled with its numerical maximum = 0.007 63.

nonlinear advection is included. In particular, the cooling has been chosen so that the geostrophic velocity in the cross-channel direction, forced by the cooling, drives an amplified response in a narrow sidewall boundary layer that is, for all intents and purposes, the same as the boundary layer introduced by Stewartson (1957). The principal difference in the treatment here is that for the layer to bring the *geostrophic* cross-channel velocity to rest at the boundary requires an amplitude for the boundary layer correction that is much larger than in the classical theory, where the interior flow is two-dimensional and the cross-channel velocity in the interior is weak and ageostrophic.

The analysis confirms the interpretation of Spall (2008) that the narrow zone to which strong vertical motion is limited in the numerical model that he investigated is essentially the same as the linear Stewartson layer. This is somewhat intriguing since the numerical model is strongly nonlinear, but it does seem to imply that the basic process determining the region of sinking by cooling is robustly governed by the balance between vorticity generated by vortex stretching and the dissipation of that vorticity in narrow regions near the boundary. In particular, the net downwelling at the boundary is set by the magnitude of the geostrophic flow in the interior, driven to the boundary by the along-channel pressure gradient produced by the cooling. The local boundary enhancement of the along-channel velocity is also well described by this simple analytical model and is, again, a fundamental feature of the Stewartson $E_H^{1/3}$ layer when that layer is forced by a geostrophic interior impinging flow.

If the stratification were increased enough to reverse the inequality (4.4) so that $\sigma_H S \gg E_H^{2/3}(D/L)^{2/3}$ (so that the deformation radius exceeds the width of the Stewartson layer), then the Stewartson layer would split (Barcilon and Pedlosky 1967) into a hydrostatic layer of thickness $(\sigma_H S)^{1/2}$ and a very narrow buoyancy layer whose thickness is $(E_H D/L)^{1/2}/(\sigma_H S)^{1/4}$; however, with insulating sidewalls this second layer is essentially absent. The vertical velocity is much reduced in the stratified hydrostatic layer, and the constraint on the interior velocity eliminates the forcing of the boundary layer by the geostrophic cross-channel flow. Thus, as might be expected, the strong downwelling seen in the model described here depends essentially on the weakness of the stratification, which in the numerical model of Spall is self-generated as the cooling-forced mixed layer.

Acknowledgments. I am grateful to M. Spall for sharing with me his unpublished numerical calculations, which suggested the analysis of the present paper and for many helpful conversations concerning this work. This research was supported in part by NSF Grant OCE 0451086.

APPENDIX A

The Boundary Layer Correction Functions

From (4.3) and (4.6) we find that

$$\bar{u} = \sum_{j=1} \gamma_j \left(A_{1j} e^{-\gamma_j \eta} + (1/2 + i\sqrt{3}/2) A_{2j} e^{-\gamma_j (1/2 + i\sqrt{3}/2)\eta} + (1/2 - i\sqrt{3}/2) A_{3j} e^{-\gamma_j (1/2 - i\sqrt{3}/2)\eta}\right) \cos(j\pi z),$$

$$\bar{v}_g = \sum_{j=1} \left(\frac{\partial A_{1j}}{\partial x} e^{-\gamma_j \eta} + \frac{\partial A_{2j}}{\partial x} e^{-\gamma_j (1/2 + i\sqrt{3}/2)\eta} + \frac{\partial A_{3j}}{\partial x} e^{-\gamma_j (1/2 - i\sqrt{3}/2)\eta}\right) \cos(j\pi z),$$

$$\bar{v}_a = -\frac{1}{2} \sum_{j=1} \gamma_j^3 \left(A_{1j} e^{-\gamma_j \eta} - A_{2j} e^{-\gamma_j (1/2 + i\sqrt{3}/2)\eta} - A_{3j} e^{-\gamma_j (1/2 - i\sqrt{3}/2)\eta}\right) \cos(j\pi z),$$

$$\bar{w} = \frac{1}{2} \sum_{j=1} \gamma_j^4 \left(-A_{1j} e^{-\gamma_j \eta} + (1/2 + i\sqrt{3}/2) A_{2j} e^{-\gamma_j (1/2 + i\sqrt{3}/2)\eta} + (1/2 - i\sqrt{3}/2) A_{3j} e^{-\gamma_j (1/2 - i\sqrt{3}/2)\eta}\right) \frac{\sin(j\pi z)}{j\pi}. \quad \text{(A.1a–d)}$$

The lateral temperature gradient correction is

$$\bar{T}_\eta = \sum_{j=1} \gamma_j{}^3 \left(A_{1j} e^{-\gamma_j \eta} - \frac{1}{2} A_{2j} e^{-\gamma_j (1/2 + i\sqrt{3}/2)\eta} - \frac{1}{2} A_{3j} e^{-\gamma_j (1/2 - i\sqrt{3}/2)\eta}\right) \frac{\sin(j\pi z)}{j\pi}. \quad \text{(A.1e)}$$

The real part of the above expression is understood.

APPENDIX B

The Coefficients for the Interior Barotropic Flow

Applying (5.4) to (5.5) yields

$$B = A,$$

$$F = E,$$

$$E = F - \frac{A}{2k^2} \left(\frac{1 + 1/\cosh k + k \tanh k}{\coth k + 1/\sinh k}\right),$$

and

$$A = 2kH_1 \tanh K / K^3 \Bigg/ \left(1 - \frac{1 + 1/\cosh k + \tanh k}{\coth k + 1/\sinh k}\right). \quad \text{(B.1a–d)}$$

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