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Classifications of dense languages

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Abstract Let X be a finite alphabet containing more than one letter. A dense language over X is a language containing a disjunctive language. A language L is an n-dense language if for any distinct n words $w_1, w_2, \ldots, w_n \in X^+$, there exist two words $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots uw_nv \in L$. In this paper we classify dense languages into strict n-dense languages and study some of their algebraic properties. We show that for each $n \geq 0$, the n-dense language exists. For an n-dense language L, $n \neq 1$, the language $L \cap Q$ is a dense language, where Q is the set of all primitive words over X. Moreover, for a given $n \geq 1$, the language L is such that $L \cap Q \in D_n(X)$, then $L \in D_m(X)$ for some $m, n \leq m \leq 2n+1$. Characterizations on 0-dense languages and n-dense languages are obtained. It is true that for any dense language L, there exist $w_1 \neq w_2 \in X^+$ such that $uw_1v, uw_2v \in L$ for some $u, v \in X^*$. We show that every n-dense language, $n \geq 0$, can be split into disjoint union of infinitely many n-dense languages.

Keywords Primitive words · Dense languages · n-dense languages · Strict n-dense languages

1 Introduction and definitions

Let X be a finite alphabet with more than one letter and let X^* be the free monoid generated by X. Every element of X^* is a *word* and let $X^+ = X^* \backslash \{1\}$, where 1 is

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the empty word. Every subset of X^* is a *language*. The cardinality of a language A is denoted by $|A|$.

A word $f \in X^+$ is a *primitive word* if f is not a power of any other word. It is known that every word $x \in X^+$ is a power of a primitive word and the expression is unique. Let Q be the set of all primitive words over X. For $u = f^i, f \in Q, i \geq 1$, let $\sqrt{u} = f$, and call f the *primitive root* of u. For a language $L \subseteq X^+$, let $\lambda(L) = \{\sqrt{u} \mid u \in L\}$ [13]. Clearly, $\lambda(L) \subseteq Q$ for every language $L \subseteq X^+$. We define the length of $w \in X^*$, denoted by $lg(w)$, to be the number of letters in w. For any finite language $A \subseteq X^*$, we let $Lg(A) = \max\{lg(x) \mid x \in A\}$.

For a word $u \in X^+$, if $u = vw$ for some $v, w \in X^*$, then v is called a *prefix* of u, denoted by $v \leq_p u$, and w is called a *suffix* of u, denoted by $w \leq_s u$. Similarly, by $v <_p u$, we mean that $v \leq_p u$ but $v \neq u$, and call v a *proper prefix* of u. By $w <_s u$, we mean that $w \leq_s u$ but $w \neq u$, and call w a *proper suffix* of u. For two given words v and u, by $v \leq_d u$ we mean that the v is both a prefix and suffix of the word u. A nonempty language $L \subset X^+$ is a *code* if $x_1x_2\cdots x_n = y_1y_2\cdots y_m$, $x_i, y_j \in L$ imply that $m = n$ and $x_i = y_i$, $i = 1, 2, \ldots, n$. A code L is a *prefix code* (*suffix code*) if $L \cap LX^+ = \emptyset$ ($L \cap X^+L = \emptyset$). A code L is called a *bifix code* if L is both a prefix code and a suffix code.

A language L over X is called a *disjunctive language* if for every $x, y \in X^*$, $x \neq y$, there exist $u, v \in X^*$ such that $uxv \in L$ and $uyv \notin L$ or vice versa. A language L is said to be a *discrete language* if any distinct words $x, y \in L$, $lg(x) \neq lg(y)$. A language $L \subseteq X^+$ is called a *dense language* if for every $w \in X^+$, $X^*wX^* \cap L \neq \emptyset$. In this paper we investigate several properties about dense languages. It is known that a discrete dense language must be disjunctive [5]. In fact, every dense language contains a disjunctive language [5]. Some other theories related to dense languages refer to references [2,7–9,11,12].

Now we provide some new concepts about dense languages. Recall that a language L which satisfies the condition $X^*wX^* \cap L \neq \emptyset$ for all $w \in X^+$ is a dense language. That is for a dense langauge L, it is true that to each given single word $w \in X^+$, there exist $u, v \in X^*$ such that $uxv \in L$. In this paper, we like to generalize the above classical concept of dense languages and make some classifications of new dense languages. We now give the formal definition of our new family of classified dense languages.

Let $L \subseteq X^*$ and let $n \geq 1$. We call a language L an *n-dense language* if for any distinct n words $w_1, w_2, \ldots, w_n \in X^+$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$. We note that with this definition, 1-*dense language* is exactly the classical dense language. It is immediate that every n-dense language is an $(n-1)$-dense language for all $n \geq 2$.

If L is a language such that for any $k \geq 1$ and for any $w_1, w_2, \ldots, w_k \in X^+$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_kv \in L$, then we call such a language L a *0-dense language*. Thus a 0-dense language is an n-dense language for all $n \geq 1$. We will show that for any $n \geq 2$, there exists an n-dense language which is not an $(n+1)$-dense language. We will call such a particular language a *strict n-dense language*. For $X = \{a, b\}$, the so call Balanced language $H = \{w \in X^* \mid w_a = w_b\}$ is an example of 1-dense language but not a 2-dense language, where w_a stands for the number of the letter a occurring in the word w.

We define the new families of languages related to the n-dense property. Let $D(X)$ be the family of all dense language over X.

$D_0(X) = \{L \in D(X) \mid L \text{ is a 0-dense language}\}$;

$D_1(X) = \{L \in D(X) \mid L \text{ is a strict 1-dense language}\}$.

For $n \geq 2$,
$D_n(X) = \{L \in D(X) \mid L \text{ is a strict } n\text{-dense language}\}$.
We have the following disjoint decomposition:

$$D(X) = D_0(X) \cup D_1(X) \cup D_2(X) \cup \cdots .$$

It is know that for any $u \in X^+$, $a, b \in X$, $a \neq b$, one of ua, ub must be primitive [4]. In Sect. 2, we first provide a stronger version about this known result. We also show that $D_n(X) \neq \emptyset$ for all $n \geq 0$ and we investigate some elementary properties of each family $D_n(X)$. For any nonempty set $L \subseteq X^*$, we define $L_{n,r} = \{w \in L \mid lg(w) \equiv r \pmod{n}\}$ where $0 \leq r < n$. Some results are presented in the final part of Sect. 2 about the properties of $L_{n,r}$ when the given language L is dense.

We study the relations between a dense language L and its sublanguage $L \cap Q$ in Sect. 3. Let $L \subseteq X^*$ be a language and let $L \cap Q \in D_m(X)$ for some $m \geq 1$. Then we can find the range of the index n such that $L \in D_n(X)$. Otherwise, if $L \in D_n(X)$ for some $n \geq 2$, then $L \cap Q$ is dense and we also can find the range of the index m such that $L \cap Q \in D_m(X)$. Let $L \in D_n(X)$ and $L \cap Q \in D_m(X)$. Then we provide some relations between the positive integers n and m. Furthermore, some of these relations are even optimal relations.

In the final section, the Sect. 4, we want to discuss how to split a given n-dense language into a disjoint union of infinitely many n-dense languages. It is known that any dense language which split into finitely many languages, then one of these languages must be dense. However, the disjunctive languages also have similar properties [5]. And in this section, we show that in some particular divisions, we can split a dense language into a disjoint union of two (three, finitely many or even infinitely many) dense languages. Furthermore, since the disjunctive language and the strict n-dense language both are dense languages. we also show that the disjunctive language and the strict n-dense language both have similar separated properties.

2 Elementary properties of *n*-dense languages

It is easy to see that the languages X^* and Q both are 0-dense languages. It is true that 0-dense languages exist. A 0-dense language is by definition an n-dense language for every $n \geq 1$, one has that n-dense languages exist.

Remarks

(1) Every n-dense language is an m-dense language for all $0 < m < n$.
(2) If a language L is not n-dense, then L is not an m-dense language for all $m > n > 0$.
(3) A language which contains an n-dense language is also an n-dense language.

Lemma 2.1 [3] *If $uv = vu, u \neq 1, v \neq 1$, then u and v are powers of a common word.*

Proposition 2.2 *Let u, v be two distinct words such that $lg(u) = lg(v)$. Then for any $w \in X^+$, $lg(w) \geq lg(u)$, one of wu, wv must be primitive.*

Proof Suppose to the contrary that $wu = f^i$ and $wv = g^j$ for some $f, g \in Q$ and $i, j \geq 2$. For $lg(w) \geq lg(u) = lg(v)$, one has that $f \leq_p w$ and $g \leq_p w$. This implies that there exist $s, t \geq 1$, $f_1, g_1 \in X^*$, $f_2, g_2 \in X^+$, $f_1 f_2 = f$, $g_1 g_2 = g$ such that

$$w = f^s f_1 = g^t g_1, \quad u = f_2 f^{i-s-1}, \quad v = g_2 g^{j-t-1}.$$

Since $lg(u) = lg(v)$, we have $lg(wu) = lg(wv)$. If $f = g$, then $i = j$. This implies that $u = v$, a contradiction. Hence $f \neq g$. Without loss of generality, assume that $f <_p g$ and $i > j \geq 2$, then there exist $s_1 \geq 1, f_3, f_4 \in X^+$, $f_3f_4 = f$ such that $g = f^{s_1}f_3$ and $f_4f^{s-s_1-1}f_1 = g^{t-1}g_1$.

If $s - s_1 - 1 \geq 1$, then $f_4f_3 <_p g = f^{s_1}f_3$. One has that $f_4f_3 = f_3f_4$ and by Lemma 2.1, we have $f = f_3f_4 \notin Q$, a contradiction. Thus $s - s_1 - 1 = 0$. Hence we have $g = f^{s-1}f_1$ and $f_4f_1 = g^{t-1}g_1$. If $f_3 \leq_p f_1$, then $f_4f_3 \leq_p f_4f_1 = g^{t-1}g_1$, contradicts to the fact that $g = f^{s-1}f_1$. This implies that $f_1 <_p f_3$ and then follows that $lg(g^{t-1}g_1) = lg(f_4f_1) < lg(f_3f_4) = lg(f) < lg(g)$. One has that

$$t-1=0, \quad g_1 = f_4f_1 \in X^+, \quad w = gg_1,\ v = g_2g^{j-2}, \quad f_4 \leq_p g_1 <_p f <_p g.$$

Since $lg(w) \geq lg(v)$, we have $j = 2$ or $j = 3$.

From above two paragraphs, we organize the following formulas:

$$wu = f^i \text{ and } wv = g^j, i > j \geq 2,\ j = 2 \text{ or } j = 3, \tag{I}$$

$$f = f_1f_2 = f_3f_4,\ g = g_1g_2,\ f_1 \in X^*,\ f_2, f_3, f_4, g_1, g_2 \in X^+, \tag{II}$$

$$w = f^sf_1 = gg_1,\ u = f_2f^{i-s-1},\ v = g_2g^{j-2}, \tag{III}$$

$$g = f^{s-1}f_3,\ f_4f_1 = g_1, \tag{IV}$$

$$f_4 \leq_p g_1 <_p f <_p g, \tag{V}$$

$$f_1 <_p f_3. \tag{VI}$$

Now we discuss the following two cases, $j = 2$ and $j = 3$.

(1) $j = 2$. Then by formula (III), we have $w = f^sf_1 = gg_1,\ u = f_2f^{i-s-1},\ v = g_2$. Since $w = g_1g_2g_1 = f_1(f_2f^{s-1})f_1$, $g_1vg_1 = f_1(f_2f^{s-1})f_1$. For $lg(g_1) > lg(f_1)$, $lg(u) = lg(v)$ and $u = f_2f^{i-s-1}$, one has that $s-1 \geq i-s-1$, that is $2s - i \geq 0$. Hence we have $f_4f_1 \cdot v \cdot f_4f_1 = f_1 \cdot u \cdot (f^{2s-i})f_1$. This implies that $2s - i = 1$ and $f_4f_1 \cdot v \cdot f_4f_1 = f_1 \cdot u \cdot ff_1$. It follows that $lg(f_3) = lg(f_4)$ since $f = f_3f_4$. Thus by formula (V), $f_3 = f_4$ and $f = f_4^2 \notin Q$, a contradiction.

(2) $j = 3$. Then by formula (III), we have $w = f^sf_1 = gg_1,\ u = f_2f^{i-s-1},\ v = g_2g$. Since $lg(w) \geq lg(u)$, we have $s \geq i - s - 1$, that is $i \leq 2s + 1$. For $v = g_2g = g_2f^{s-1}f_3,\ u = f_2f^{i-s-1},\ lg(v) = lg(u)$, one has that $i - s - 1 \geq s - 1$, that is $i \geq 2s$. This implies that $i = 2s$ or $i = 2s + 1$.

(2-1) $i = 2s + 1$. Then $w = f^sf_1 = gg_1,\ u = f_2f^s,\ v = g_2g$. This implies that $f_2 \cdot f^sf_1 \cdot g_2 = f_2 \cdot gg_1 \cdot g_2$ and then follows that $uf_1g_2 = f_2g_1v$. For $lg(g_1) = lg(f_1) + lg(f_4)$, one has that $lg(g_2) = lg(f_2) + lg(f_4)$. Hence $lg(g) = lg(g_1g_2) = lg(f) + 2lg(f_4)$. Since $lg(u) \leq lg(w)$, we have $lg(f_2) \leq lg(f_1)$. Thus by formula (VI), $lg(f_2) \leq lg(f_1) < lg(f_3)$. By formula (II), we can see that $lg(f_4) < lg(f_2) \leq lg(f_1) < lg(f_3)$. This implies that $lg(f_4) < \frac{1}{2}lg(f)$ and $lg(g) = lg(f) + 2lg(f_4) < 2lg(f)$. For $g = f^{s-1}f_3$, we have $s = 2$ and $i = 5$. Thus by formula (I), $wu = f^5,\ wv = g^3$. One has that $5lg(f) = 3lg(g) = 3lg(f) + 6lg(f_4)$, that is $lg(f) = 3lg(f_4)$. Since $lg(f) = 3lg(f_4)$ and $f = f_3f_4$, we have $lg(f_3) = 2lg(f_4)$. This implies that $f_4 <_p f_3$ from formula (V). For $f_1 <_p f_3$ (formula (VI)) and $lg(f_4) < lg(f_1)$, one has that $f_4 <_p f_1$ and then follows that $f_4^2 <_p f_4f_1 = g_1 <_p f = f_3f_4$. That is $f_3 = f_4^2$ and $f \notin Q$, a contradiction.

(2-2) $i = 2s$. Then $w = f^s f_1 = gg_1$, $u = f_2 f^{s-1}$, $v = g_2 g$. For $f_1 \cdot u \cdot f_1 g_2 = g_1 v$ and $lg(u) = lg(v)$, one has that $lg(g_2) = lg(f_4) - lg(f_1)$. Hence we have $lg(g) = 2lg(f_4)$ and $lg(u) = lg(v) = lg(g_2 g) = 3lg(f_4) - lg(f_1) < 3lg(f)$. For $u = f_2 f^{s-1}$, then $s - 1 \leq 2$. Clearly, $s = 1$ is impossible (see formula (IV)). This implies that $s = 2$ or $s = 3$.

(2-2-1) $s = 2$. Then $wu = f^4$ and $wv = g^3$. This implies that $4lg(f_3)+4lg(f_4) = 3lg(g) = 6lg(f_4)$. That is $2lg(f_3) = lg(f_4)$. Thus by formula (V), we have $f_4 <_p f = f_3 f_4$. It is immediate that $f_4 = f_3^2$ and $f \notin Q$, a contradiction.

(2-2-2) $s = 3$. Then $wu = f^6$ and $wv = g^3$. This implies that $lg(g) = 2lg(f)$, contradicts to the fact that $lg(g) = 2lg(f_4)$.

Finally, by above cases, one of wu, wv must be primitive and we are done.

In general, Proposition 2.2 may not hold when $lg(w) < lg(u)$. The following is an example:

Example Let $X = \{a, b\}$ and let $u = abaab$, $v = baaba$, $w = a$. Then $lg(u) = lg(v)$. But $wu = (aab)^2$, $wv = (aba)^2 \notin Q$.

From the above proposition, the following known result is immediate.

Corollary 2.3 [4] *Let $u \in X^+$ and let $a, b \in X$, $a \neq b$. Then one of ua, ub must be primitive.*

Recall that the alphabet X has at least two letters, the following proposition is immediate.

Proposition 2.4 *Let u, v be two distinct words such that $lg(u) = lg(v) \leq 2$. Then for any $w \in X^+$, one of wu, wv must be primitive.*

Lemma 2.5 [10] *Let $uv = f^i$, $u, v \in X^+$, $f \in Q$, $i \geq 1$. Then $vu = g^i$ for some $g \in Q$.*

For a given language $L \subseteq X^*$ and $i \geq 1$, we define $L^{(i)}$ to be the language $L^{(i)} = \{w^i \mid w \in L\}$ (see [9]).

Proposition 2.6 *Let $L \subseteq X^*$ and $i \geq 2$. The language $L^{(i)}$ is not n-dense for any $n \geq 2$.*

Proof Since every n-dense language, $n > 2$, is a 2-dense language, we only need to show that $L^{(i)}$ is not a 2-dense language. Now suppose to the contrary that $L^{(i)}$ is a 2-dense language. Then with the two distinct letters a and b, there exist $u, v \in X^*$ such that $uav, ubv \in L^{(i)}$. This implies that uav, ubv are both not primitive words. Thus by Lemma 2.5, both vua, vub are also not primitive words. This then contradicts to Corollary 2.3. Thus the language $L^{(i)}$ is not a 2-dense language.

Remarks The language $Q^{(i)}$, $i \geq 2$, is not an n-dense langauge for all $n \geq 2$.

From Proposition 2.6, the following two corollaries are immediate.

Corollary 2.7 *Let $L \subseteq X^+ \setminus Q$. Then L is not an n-dense langauge for all $n \geq 2$.*

Corollary 2.8 *For $n \geq 2$ or $n = 0$, an n-dense language L contains a primitive word and hence contains infinitely many primitive words.*

Proposition 2.9 *Every discrete language over X can never be an n-dense language for all $n \geq 2$.*

Proof Immediate from the definition of n-dense language for all $n \geq 2$.

Lemma 2.10 [3] *Let $f, g \in Q, f \neq g$. Then $f^m g^n \in Q$ for all $m \geq 2, n \geq 2$.*

Lemma 2.11 [6] *Let $f, g \in Q, f \neq g$ and $n \geq 1$. If $fg^n \notin Q$, then $fg^{n+k} \in Q$ for all $k \geq 2$.*

Lemma 2.12 *Let $L \subseteq X^*$ be a 2-dense language. Then $L \cap Q$ is a dense language.*

Proof Let $x \in X^+$. We want to show that $(L \cap Q) \cap X^*xX^* \neq \emptyset$. Let $a, b \in X,\ a \neq b$ and assume that $b \leq_s x$. In this case $a \not\leq_s x$. Consider the two words ax and ax^3. Since L is 2-dense, there exist $u, v \in X^*$ such that $uaxv, uax^3v \in L$. Now consider the words $(vua)x$ and $(vua)x^3$. Since $a \not\leq_s x$, we have $\sqrt{vua} \neq \sqrt{x}$. If $vua \notin Q$, then by Lemma 2.10, $(vua)x^3 \in Q$. By Lemma 2.5, we have $uax^3v \in Q$. This implies that $uax^3v \in L \cap Q$. Now if on the other hand $vua \in Q$, then by Lemma 2.11, one has that $(vua)x \in Q$ or $(vua)x^3 \in Q$. Thus by Lemma 2.5 again, we have $uaxv \in Q$ or $uax^3v \in Q$. This implies that $uaxv \in L \cap Q$ or $uax^3v \in L \cap Q$. In either case, the condition $(L \cap Q) \cap X^*xX^* \neq \emptyset$ holds, that is $L \cap Q$ is a dense language and we are done.

The following is in fact a stronger version of Corollary 2.8.

Proposition 2.13 *Let $L \subseteq X^*$ be an n-dense language for some $n \geq 2$ or $n = 0$. Then $L \cap Q$ is a dense language.*

Proof Since every n-dense language, $n \geq 2$ or $n = 0$ is a 2-dense language the result follows immediately from Lemma 2.12.

In general, an 1-dense language, the classical dense language, may not have the property of Proposition 2.13. The following is an example:

Example Let $L = X^* \setminus Q$. Then L is an 1-dense language but $L \cap Q = \emptyset$ and it is not dense.

From Proposition 2.13, the following corollary is immediate.

Corollary 2.14 *Let $L \subseteq (X^+ \setminus Q) \cup I$, where I is a non-dense subset of Q. Then L is not an n-dense language for all $n \geq 2$.*

Next, we study some properties of the so called strict n-dense languages. We need to assure the existence of each strict n-dense language first.

Recall that $D(X)$ is the family of all dense languages over X and other notations like $D_n(X),\ n \geq 0$ are defined in Section 1. We also recall that for any nonempty language $L \subseteq X^*$, $L_{n,r}$ is the set $L_{n,r} = \{w \in L \mid lg(w) \equiv r \pmod{n}\}$, where $0 \leq r < n$.

Proposition 2.15 *For any $n \geq 0$, $D_n(X) \neq \emptyset$.*

Proof Clearly, $D_0(X) \neq \emptyset$, since $Q \in D_0(X)$. Consider the language $L = Q \setminus Q_{n+1,0}$, where $n \geq 1$. We want to show that $L \in D_n(X)$. Since $Q \setminus Q_{n+1,0} = Q_{n+1,1} \cup Q_{n+1,2} \cup \cdots \cup Q_{n+1,n}$, we see that $lg(x) \not\equiv 0 \pmod{n+1}$ for all $x \in Q \setminus Q_{n+1,0}$. Our aim is to show that L is an n-dense language but not an $(n+1)$-dense language. Let $a, b \in X,\ a \neq b$. First we show that L is not $(n+1)$-dense by considering the

$n+1$ words, $a, a^2, \ldots, a^{n+1}$. If L is $(n+1)$-dense, then there exist $u, v \in X^*$ such that $uav, ua^2v, \ldots, ua^{n+1}v \in Q \setminus Q_{n+1,0}$. That is $lg(ua^iv) \not\equiv 0 \pmod{n+1}$ for all $1 \le i \le n+1$. But this is impossible. Hence the language $L = Q \setminus Q_{n+1,0}$ is not $(n+1)$-dense. Next we show that the language L is indeed an n-dense language. For this, let $w_1, w_2, \ldots, w_n \in X^+$ be any n words and let $m = \max\{lg(w_1), lg(w_2), \ldots, lg(w_n)\}$. Then it is immediate that $w_1ab^k, w_2ab^k, \ldots, w_nab^k \in Q$ for any $k \ge m$ since every factor would need to be in b^*. Since the set $\{w_1ab^k, w_2ab^k, \ldots, w_nab^k\}$ has only n words, it is true that for some $h \ge m$, $w_1ab^h, w_2ab^h, \ldots, w_nab^h \in Q$ and $lg(w_iab^h) \not\equiv 0 \pmod{n+1}$ for all $1 \le i \le n$. This implies that $w_1ab^h, w_2ab^h, \ldots, w_nab^h \in Q \setminus Q_{n+1,0}$ and $Q \setminus Q_{n+1,0}$ is an n-dense language. It is clear now that $L = Q \setminus Q_{n+1,0} \in D_n(X)$ and we are done.

Let us see a few cases.

$n = 1$. $L = Q\setminus Q_{n+1,0} = Q\setminus Q_{2,0} = Q_{2,1} = Q_{\text{odd}} \in D_1(X)$.
$n = 2$. $L = Q\setminus Q_{n+1,0} = Q\setminus Q_{3,0} = Q_{3,1} \cup Q_{3,2} \in D_2(X)$.
$n = 3$. $L = Q\setminus Q_{n+1,0} = Q\setminus Q_{4,0} = Q_{4,1} \cup Q_{4,2} \cup Q_{4,3} \in D_3(X)$.
$n = 4$. $L = Q\setminus Q_{n+1,0} = Q\setminus Q_{5,0} = Q_{5,1} \cup Q_{5,2} \cup Q_{5,3} \cup Q_{5,4} \in D_4(X)$.

By using a similar proof as in Proposition 2.15 we can show the following proposition.

Proposition 2.16 *Let L be a language and $Q \subseteq L$. Then $L \setminus L_{n+1,0} \in D_n(X)$ for all $n \ge 1$.*

Proposition 2.17 *Let $L \in D(X)$. Then the following statements are true:*

(1) *Let $n \ge 1$ and let $L \in D_n(X)$. For any $A \subseteq L$, if $L\setminus A$ is dense, then $L\setminus A \in D_m(X)$ for some $1 \le m \le n$.*
(2) *Let $L \in D_0(X)$. If $A \subseteq L$ and $L\setminus A$ is dense, then $L\setminus A \in D_m(X)$ for some $m \ge 0$.*
(3) *Let $n \ge 1$ and $L \in D_n(X)$. If $L \subseteq L'$, then $L' \in D_m(X)$ for some $m \ge n$ or $m = 0$.*
(4) *Let $L \in D_0(X)$. If $L \subseteq L'$, then $L' \in D_0(X)$.*

Proof Follows directly from the definition of strict n-dense language.

Proposition 2.18 *Let $L \in D_n(X)$ and $A \in D_m(X)$, where $n > m \ge 1$. Then the language $L \setminus A$ is a dense language.*

Proof Since $A \in D_m(X)$, A is not an $(m+1)$-dense language. This implies that there exist $z_1, z_2, \ldots, z_{m+1} \in X^+$ such that for any $u, v \in X^*$, $uz_iv \notin A$ for some $1 \le i \le m+1$. For $L \in D_n(X)$, L is n-dense. One has that L is $(m+1)$-dense, since $n \ge m+1$. Thus for any $w \in X^+$, there exist $u', v' \in X^*$ such that $u'wz_1v', u'wz_2v', \ldots, u'wz_{m+1}v' \in L$. This implies that there exists $1 \le j \le m+1$ such that $u'wz_jv' \notin A$, that is $u'wz_jv' \in L\setminus A$. This shows that the $L \setminus A$ is a dense language and we are done.

Since a 0-dense language is n-dense for all $n \ge 1$, the following corollary is immediate.

Corollary 2.19 *Let $L \in D_0(X)$ and $A \in D_m(X)$ for some $m \ge 1$. Then $L \setminus A$ is a dense language.*

The following lemma is a known result and we provide a simple proof.

Lemma 2.20 *Let $L \subseteq X^*$ be a dense language and $L = A \cup B$, $A \cap B = \emptyset$. Then A or B is a dense language.*

Proof If A is not dense, then $A \cap X^*zX^* = \emptyset$ for some $z \in X^+$. Let $w \in X^+$. Since L is dense, there exist $u, v \in X^*$ such that $uzwv \in L = A \cup B$. For $A \cap X^*zX^* = \emptyset$, one has that $uzwv \notin A$, that is, $uz \cdot w \cdot v \in B$. This implies that B is dense and the proof is complete.

Proposition 2.21 *For a fixed $n \geq 0$ and $L \in D_n(X)$, it is true that for any non-dense language $A \subset L$, $L \setminus A \in D_n(X)$.*

Proof

1° $n \geq 1$. Since L is dense and $A \subset L$, A non-dense, by Lemma 2.20, we have that $L \setminus A$ is dense. Thus by Proposition 2.17(1), $L \setminus A \in D_m(X)$, for some m, $1 \leq m \leq n$. Since $L = (L \setminus A) \cup A$, $L \in D_n(X)$, $L \setminus A \in D_m(X)$ (in here see $L \setminus A$ as A in Proposition 2.18), and A is not dense, by Proposition 2.18, we have $n = m$ and $L \setminus A \in D_n(X)$.

2° $n = 0$. Since L is 0-dense by assumption, L is dense. Now from the fact that $A \subset L$, A non-dense, by Lemma 2.20 again, $L \setminus A$ is dense. Thus there exists an $m \geq 0$ such that $L \setminus A \in D_m(X)$. By similar argument as in 1°, we can show that $m = 0$ and $L \setminus A \in D_0(X)$.

Proposition 2.21 may not be true when the language $A \subset L$ is a dense language. The following is an example:

Example Let $L = X^*$ and let $A = (X^*)_{2,0}$. Then $L \in D_0(X)$ and A is dense. But $L \setminus A = (X^*)_{2,1} \notin D_0(X)$. In fact, $L \setminus A \in D_1(X)$.

Proposition 2.22 *For any $n \geq 0$, $|D_n(X)| = \infty$.*

Proof It follows directly from the above proposition.

The following corollary is immediate.

Corollary 2.23 *Let L be an n-dense language for some $n \geq 0$ and let A be a non-dense language. Then $L \setminus A$ is an n-dense language.*

Proposition 2.24 *Let $n \geq 2$ or $n = 0$. Then for any language $L \subset X^*$, the language $L_{k,r}$ can never be n-dense, for all $k \geq 2$, $0 \leq r < k$.*

Proof Clearly, this proposition holds true when $L_{k,r} = \emptyset$. Let $L_{k,r} \neq \emptyset$ and $w \in L_{k,r}$. Then $lg(w) \equiv r \pmod{k}$. Consider the two words, a, a^2, where $a \in X$. It is easy to see that for any $u, v \in X^*$, we have $uav \notin L_{k,r}$ or $ua^2v \notin L_{k,r}$. One has that $L_{k,r}$ is not 2-dense. Hence $L_{k,r}$ is not n-dense for all $n \geq 2$. Furthermore, $L_{k,r}$ is also not 0-dense.

From the above proposition, we can make a remark here that the particular languages $(X^+)_{k,r}$ and $Q_{k,r}$ both are strict 1-dense languages, where $k \geq 2$, $0 \leq r < k$.

Lemma 2.25 *Let $L \in D_n(X)$ for some $n \geq 1$. Then $L_{n,r} \in D_1(X)$ for all $0 \leq r < n$.*

Proof If $n = 1$, then $r = 0$ and $L_{1,0} = L \in D_1(X)$. Let $n \geq 2$. From the results in Proposition 2.24, we only need to show that $L_{n,r}$ is a dense language. Now if $L_{n,r}$ is not dense, then by Proposition 2.21, we have $L \setminus L_{n,r} \in D_n(X)$. Since $L \setminus L_{n,r} = L_{n,0} \cup L_{n,1} \cup \cdots \cup L_{n,r-1} \cup L_{n,r+1} \cup \cdots \cup L_{n,n-1}$, one has that for any $u, v \in X^*$, $\{uav, ua^2v, \ldots, ua^nv\} \not\subseteq L \setminus L_{n,r}$ where $a \in X$. This contradicts to the fact that $L \setminus L_{n,r} \in D_n(X)$. Hence we have $L_{n,r}$ is dense and by Proposition 2.24, $L_{n,r} \in D_1(X)$.

Since a 0-dense language is n-dense for all $n \geq 1$ and an n-dense language is m-dense for all $1 \leq m \leq n$, by Lemma 2.25 the following proposition is immediate.

Proposition 2.26 *Let $L \subseteq X^*$ be a dense language. Then the following statements are true:*

(1) *If $L \in D_n(X)$ for some $n \geq 1$, then $L_{m,r} \in D_1(X)$ for all $1 \leq m \leq n$, $0 \leq r < m$.*
(2) *If $L \in D_0(X)$, then $L_{n,r} \in D_1(X)$ for any $n \geq 1$, $0 \leq r < n$.*

In general, if $L \in D_n(X)$ for some $n \geq 1$, then $L_{k,r}$ may not be a strict 1-dense language when $k > n$. The following is an example:

Example Let $L = Q \setminus Q_{n+1,0}$. Then by Proposition 2.16, we have $L \in D_n(X)$. But $L_{n+1,0} = \emptyset$ is not a strict 1-dense language.

Proposition 2.27 *Let $L \subseteq X^*$ be an n-dense language for some $n \geq 0$ and let $A_1, A_2, \ldots, A_h$, $B_1, B_2, \ldots, B_k$ be nonempty subsets of X^* for some $h, k \geq 1$. Then $A_1A_2 \cdots A_h \cdot L \cdot B_1B_2 \cdots B_k$ is an n-dense language.*

1° $n \geq 1$. In this case L is n-dense. Let $w_1, w_2, \ldots, w_n \in X^+$. Since L is n-dense, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$. Let $x_i \in A_i$, $y_j \in B_j$, where $1 \leq i \leq h$, $1 \leq j \leq k$. Then $x_1x_2 \cdots x_h \cdot uw_1v \cdot y_1y_2 \cdots y_k$, $x_1x_2 \cdots x_h \cdot uw_2v \cdot y_1y_2 \cdots y_k$, $\ldots$, $x_1x_2 \cdots x_h \cdot uw_nv \cdot y_1y_2 \cdots y_k \in A_1A_2 \cdots A_h \cdot L \cdot B_1B_2 \cdots B_k$. This implies that $A_1A_2 \cdots A_h \cdot L \cdot B_1B_2 \cdots B_k$ is an n-dense language.
2° $n = 0$. In this case, L is n-dense for all $n \geq 1$. Thus by case 1°, we have $A_1A_2 \cdots A_h \cdot L \cdot B_1B_2 \cdots B_k$ is n-dense for all $n \geq 1$. That is $A_1A_2 \cdots A_h \cdot L \cdot B_1B_2 \cdots B_k \in D_0(X)$.

The following corollary a direct consequence of Proposition 2.27.

Corollary 2.28 *For any n-dense language $L \subseteq X^*$, $n \geq 0$, the language L^m is an n-dense for all $m \geq 2$.*

In general, L^m may not be strict n-dense when L is a strict n-dense language, see the following example:

Example Let $X = \{a, b\}$ and $L = \{a, b\} \cup (X^*)_{2,0}$. Then $L \in D_1(X)$ and $L^2 = X^* \notin D_1(X)$.

Recall that a strict 1-dense language L is 1-dense but not 2-dense. That is there exist $w_1, w_2 \in X^+$ such that $\{uw_1v,\ uw_2v\} \nsubseteq L$ for all $u, v \in X^*$. Next we investigate a particular language with similar property of strict 1-dense language.

Proposition 2.29 *A language L which has the property that for any $w_1 \neq w_2 \in X^+$, there are no two words $u, v \in X^*$ such that both uw_1v, uw_2v are in L, then L is finite and $|L| \leq 2n + 1$ where $|X| = n \geq 2$. Furthermore, L is not dense.*

Proof Let $|X| = n$. If $|L| \geq 2n+2$, then there exist distinct words $w_1, w_2, \ldots, w_{n+1} \in L$ with $lg(w_i) \geq 2$ for all $1 \leq i \leq n + 1$. Since $|X| = n$ and $|\{w_1, w_2, \ldots, w_{n+1}\}| = n + 1$, there exist $a \in X$, $1 \leq i < j \leq n + 1$ such that $a <_s w_i$ and $a <_s w_j$, say $w_i = w_i'a$, $w_j = w_j'a$ for some $w_i', w_j' \in X^+$. This implies that $w_i' \neq w_j'$ and $w_i'a$, $w_j'a \in L$, a contradiction. Hence $|L| \leq 2n + 1$ is finite. It is immediate that L is not dense.

The following corollary is immediate.

Corollary 2.30 *Let L be an infinite language. Then there exist $w_1 \neq w_2 \in X^+$ and $u, v \in X^*$ such that $uw_1v, uw_2v \in L$.*

3 Relations between a strict n-dense language L and its subset $L \cap Q$

In this section, we want to investigate relations between L and $L \cap Q$, where $L \in D_n(X)$, $n \geq 0$. Recall that an n-dense language L for some $n \geq 0$, $n \neq 1$ it is true that $L \cap Q$ is dense. The following proposition is immediate.

Proposition 3.1 *Let $L \subseteq X^*$ be a dense language. If $L \cap Q$ is not dense, then $L \in D_1(X)$.*

We remark here that $Q^{(i)} \in D_1(X)$ for all $i \geq 2$. The following is a characterization of n-dense languages.

Proposition 3.2 *Let $L \subseteq X^*$ and $n \geq 1$. Then the following statements are equivalent:*

(1) *L is n-dense.*
(2) *For any $w_1, w_2, \ldots, w_n \in X^*$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$.*
(3) *Let $A \subseteq X^*$ be an n-dense language and for any $w_1, w_2, \ldots, w_n \in A$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$.*
(4) *Let $x, y \in X^*$. Then for any $w_1, w_2, \ldots, w_n \in X^+$, $x \leq_p w_i$, $y \leq_s w_i$, $1 \leq i \leq n$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$.*
(5) *For any $w_1, w_2, \ldots, w_n \in Q$ and $a \leq_d w_i$, $1 \leq i \leq n$, $a \in X$, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in L$.*

Proof The implications that (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate.

(5) $\Rightarrow$ (2) Let $w_1, w_2, \ldots, w_n \in X^*$ and $a, b \in X$, $a \neq b$. Then there exists $k \geq 1$ such that $a^kbw_1ba^k$, $a^kbw_2ba^k$, $\ldots$, $a^kbw_nba^k \in Q$. Thus by condition (5), there exist $u, v \in X^*$ such that $(ua^kb)w_1(ba^kv), (ua^kb)w_2(ba^kv), \ldots, (ua^kb)w_n(ba^kv) \in L$. This implies that L is n-dense and condition (2) holds.

(3) $\Rightarrow$ (2) Let $w_1, w_2, \ldots, w_n \in X^*$. Then for condition (1) is equivalent to condition (2) and A is n-dense, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv \in A$. Thus by assumption, there exist $u', v' \in X^*$ such that $u'uw_1vv', u'uw_2vv', \ldots, u'uw_nvv' \in L$. Hence the implication (3) $\Rightarrow$ (2) holds true.

We now turn to discuss density property between L and $L \cap Q$ when $L \cap Q$ is dense. Before we start our work, we need the following lemmas first.

The proof of following lemma is immediate.

Lemma 3.3 *Let $a \neq b \in X$ and let $w_1, w_2 \in X^+$, $w_1 \neq w_2$, $a \leq_d w_1$, $a \leq_d w_2$. Then for any $i_1, i_2 \geq lg(w_1w_2)$, the language $\{w_1b^{i_1}, w_2b^{i_2}\}$ is a prefix code.*

Lemma 3.4 [1] *Let x, $w \in X^+$, $i > 0$. Then the following statements are true:*

(1) *If $x <_p w^ix$, then $x = w^jw_1$ for some $w_1 \leq_p w$, $j \geq 0$.*
(2) *If $x <_s xw^i$, then $x = w_2w^j$ for some $w_2 \leq_s w$, $j \geq 0$.*

Lemma 3.5 *Let $a \neq b \in X$, $n \geq 2$, $w_1, w_2, \ldots, w_n \in X^+$ with $w_i \neq w_j$ for all $i \neq j$ and let $k = lg(w_1w_2 \cdots w_n)$, $a \leq_d w_i$, $1 \leq i \leq n$. Then for any $u \in X^*$, $i_1, i_2, \ldots, i_n \geq k$, the language $\{uw_1b^{i_1}, uw_2b^{i_2}, \ldots, uw_nb^{i_n}\}$ has at most one non-primitive word, that is, $|\{uw_1b^{i_1}, uw_2b^{i_2}, \ldots, uw_nb^{i_n}\} \setminus Q| \leq 1$.*

Proof Suppose to the contrary that the language $\{uw_1b^{i_1}, uw_2b^{i_2}, \ldots, uw_nb^{i_n}\}$ has more than one non-primitive word. Without loss of generality, we may assume that $uw_1b^{i_1} = f^s$ and $uw_2b^{i_2} = g^t$ for some $f, g \in Q$, $s, t \geq 2$. For $a \leq_d w_1$ and $i_1 \geq k = lg(w_1w_2\cdots w_n)$, one has that $w_1b^{i_1} \leq_s f$ and there exist $u_1 \in X^+$, $u_2 \in X^*$, $u_1 = f^{s-1}$ such that $u = u_1u_2 = f^{s-1}u_2$ and $u_2w_1b^{i_1} = f$. This implies that $g^t = u_1u_2w_2b^{i_2} = (u_2w_1b^{i_1})^{s-1}u_2w_2b^{i_2}$. If $u_2 = 1$, then $g^t = (w_1b^{i_1})^{s-1}(w_2b^{i_2})$. Since $a \leq_d w_1$, $a \leq_d w_2$ and i_1, i_2 both are sufficient large, it is easy to see that $g = (w_1b^{i_1})^{l_1} = (w_1b^{i_1})^{l_2}w_2b^{i_2}$ for some $l_1 > l_2 \geq 0$, that is, $w_2b^{i_2} = (w_1b^{i_1})^{l_1-l_2}$, which by Lemma 3.3 is a contradiction. Hence we have $u_2 \in X^+$. Now for $g^t = (u_2w_1b^{i_1})^{s-1}u_2w_2b^{i_2}$, one has that $w_2b^{i_2} \leq_s g$ and we proceed the proof by discussing the following cases:

(1) Let $g = (u_2w_1b^{i_1})^r u_2w_2b^{i_2}$ for some $r \geq 0$. Then we have

$$(u_2w_1b^{i_1})^r u_2w_2b^{i_2} <_p (u_2w_1b^{i_1})^{s-1}u_2w_2b^{i_2}.$$

This implies that $r < s-1$ and $u_2w_2b^{i_2} <_p (u_2w_1b^{i_1})^{s-1-r}u_2w_2b^{i_2}$, that is, $w_2b^{i_2} \leq_p w_1b^{i_1}$ or $w_1b^{i_1} \leq_p w_2b^{i_2}$, contradicting to the result in Lemma 3.3.

(2) Let $g = (b^{i_1}u_2w_1)^r b^{i_1}u_2w_2b^{i_2}$ for some $r \geq 0$. Then $g^{t-1} = (u_2w_1b^{i_1})^{s-2-r}u_2w_1$ and $a <_s g$. This implies that $a <_s b^{i_2}$, a contradiction.

(3) Let $g = (w_1b^{i_1}u_2)^r w_2b^{i_2}$ for some $r \geq 0$.

(3-1) If $r \geq 1$, then $u_2w_1b^{i_1} = w_1b^{i_1}u_2$. Thus by Lemma 2.1, $f = u_2w_1b^{i_1} \notin Q$, a contradiction.

(3-2) If $r = 0$, then $g = w_2b^{i_2}$ and $g^{t-1} = (u_2w_1b^{i_1})^{s-1}u_2 = (u_2w_1b^{i_1})^{s-2}u_2w_1 \cdot (b^{i_1}u_2)$. This implies that $a \leq_p g$ and $a \leq_p u_2$. For $b^{i_1}u_2 <_s g^{t-1} = (w_2b^{i_2})^{t-1}$ and $a \leq_p u_2$, one has that $u_2 = (w_2b^{i_2})^l = g^l$ for some $l \geq 1$. Thus $w_2b^{i_2} = g \leq_p g^{t-1-l} = w_1b^{i_1}(u_2w_1b^{i_1})^{s-2}u_2$, contradicts to the fact that $\{w_1b^{i_1}, w_2b^{i_2}\}$ is a prefix code.

(4) Let $g = b^{i_4}(u_2w_1b^{i_1})^r u_2w_2b^{i_2}$ for some $r \geq 0$, $i_3, i_4 \geq 1$, $i_3 + i_4 = i_1$. Then we have $u_2 <_p b^{i_4}u_2$. Thus by Lemma 3.4, it follows that $u_2 \in b^+$. Since $b^{i_4}(u_2w_1b^{i_1})^r u_2w_2b^{i_2} <_p (u_2w_1b^{i_1})^{s-1}u_2w_2b^{i_2}$, we have $b^{i_4}u_2 <_p u_2w_1b^{i_1}$ and by $a \leq_p w_1$, $u_2 \in b^+$, one has that $a = b$, a contradiction. Similarly, the case $g = w_{12}(b^{i_1}u_2w_1)^r b^{i_1}u_2w_2b^{i_2}$ for some $r \geq 0$, $w_{11}, w_{12} \in X^+$, $w_{11}w_{12} = w_1$ is also a contradiction.

(5) Let $g = u_{22}(w_1b^{i_1}u_2)^r w_2b^{i_2}$ for some $r \geq 0$, u_{11}, $u_{12} \in X^+$, $u_{21}u_{22} = u_2$. Then $g^{t-1} = (u_2w_1b^{i_1})^{s-1-r}u_{21}$, where $s-1-r \geq 1$. If $r \geq 1$, then $u_{21}u_{22}w_1b^{i_1} = u_{22}w_1b^{i_1}u_{21}$. Thus by Lemma 2.1, $f \notin Q$, a contradiction. Hence we have $r = 0$, that is $g = u_{22}w_2b^{i_2}$ and $g^{t-1} = (u_2w_1b^{i_1})^{s-1}u_{21}$. For $(u_{22}w_2b^{i_2})^{t-1} = (u_2w_1b^{i_1})^{s-1}u_{21}$, it is immediate that $u_{21} \notin (u_{22}w_2b^{i_2})^+$. Now we discuss the following subcases:

(5-1) If $(u_{22}w_2b^{i_2})^h <_s u_{21} \leq_s b^{i_2}(u_{22}w_2b^{i_2})^h$ for some $h \geq 0$, then $u_{21} = b^{l_1}(u_{22}w_2b^{i_2})^h$ where $l_1 \geq 1$. For $u_{22} <_p g <_p (u_2w_1b^{i_1})^{s-1}u_{21}$, we have $u_{22} <_p b^{l_1}u_{22}$ and then it follows that $u_{22} = b^{l_2}$ for some $l_2 \geq 1$. This implies that $b^{l_1+l_2}w_1b^{i_1} \leq_p (b^{l_2}w_2b^{i_2})^{t-1}$ when $h = 0$ and $b^{l_1+l_2}w_2b^{i_2} \leq_p (b^{l_2}w_2b^{i_2})^{t-1}$ when $h \geq 1$. In either case, we have $a = b$, a contradiction.

(5-2) If $b^{i_2}(u_{22}w_2b^{i_2})^h <_s u_{21} <_s w_2b^{i_2}(u_{22}w_2b^{i_2})^h$ for some $h \geq 0$, then there exist $w_{21}, w_{22} \in X^+$, $w_{21}w_{22} = w_2$ such that $u_{21} = w_{22}b^{i_2}(u_{22}w_2b^{i_2})^h$ and $(u_{22}w_2b^{i_2})^{t-2-h}u_{22}w_{21} = (u_2w_1b^{i_1})^{s-1}$. This implies that $w_{21} <_s b^{i_1}$. It contradicts to the fact that $a \leq_p w_{21}$.

(5-3) If $w_2b^{i_2}(u_{22}w_2b^{i_2})^h \leq_s u_{21} <_s u_{22}w_2b^{i_2}(u_{22}w_2b^{i_2})^h$ for some $h \geq 0$, then there exist $u' \in X^+$, $u'' \in X^*$, $u'u'' = u_{22}$ such that $u_{21} = u''w_2b^{i_2}(u_{22}w_2b^{i_2})^h$. For $(u_{22}w_2b^{i_2})^{t-1} = (u_2w_1b^{i_1})^{s-1}u_{21} = u_{21}u_{22}w_1b^{i_1}(u_2w_1b^{i_1})^{s-2}u_{21}$, one has that $u'u''w_2b^{i_2} = u''w_2b^{i_2}u'$. Thus by Lemma 2.1 again, $g = u'u''w_2b^{i_2} \notin Q$, a contradiction.

Finally, by accomplishing the above discussions, our proof is completed.

Proposition 3.6 *Let $L \subseteq X^*$ be n-dense, $n \geq 1$. Then for any distinct words $w_1, w_2, \ldots, w_n \in X^+$, there exist $u, v \in X^*$ such that $\{uw_1v, uw_2v, \ldots, uw_nv\} \subseteq L$ and $|\{uw_1v, uw_2v, \ldots, uw_nv\} \setminus Q| \leq 1$.*

Proof Let $w_1, w_2, \ldots, w_n \in X^+$ and let $a \in X$. By Proposition 3.2, we may assume that $a \leq_d w_i$ for all $1 \leq i \leq n$. Since L is n-dense, there exist $u, v \in X^*$ such that $uw_1b^kv, uw_2b^kv, \ldots, uw_nb^kv \in L$ where $k = lg(w_1w_2 \cdots w_n)$, $b \in X$, $b \neq a$. By Lemma 3.5, we have $|\{vuw_1b^k, vuw_2b^k, \ldots, vuw_nb^k\} \setminus Q| \leq 1$. Hence by Lemma 2.5, $|\{uw_1b^kv, uw_2b^kv, \ldots, uw_nb^kv\} \setminus Q| \leq 1$ and we are done.

Proposition 3.7 *Let $L \subseteq X^+$ be such that $L \cap Q \in D_n(X)$ for some $n \geq 1$. Then $L \in D_m(X)$ for some m, $n \leq m \leq 2n + 1$.*

Proof Since $L \cap Q$ is not $(n+1)$-dense by the given condition, there exist $w_1, w_2, \ldots, w_{n+1} \in X^+$ such that for any $u', v' \in X^*$, $u'w_iv' \notin L \cap Q$ for some i, $1 \leq i \leq n+1$. If L is $(2n+2)$-dense, then by Proposition 3.6, there exist $u, v \in X^*$ such that $uw_1v, uw_2v, \ldots, uw_nv$, $uw_1av, uw_2av, \ldots, uw_nav \in L$ and $|\{uw_1v, uw_2v, \ldots, uw_nv, uw_1av, uw_2av, \ldots, uw_nav\} \setminus Q| \leq 1$. This implies that $uw_1v, uw_2v, \ldots, uw_nv \in L \cap Q$ or $uw_1av, uw_2av, \ldots, uw_nav \in L \cap Q$. In either case, we will have a contradiction. It follows that L is not a $(2n+2)$-dense language, that is L is not 0-dense.

Since $L \cap Q$ is n-dense, by Proposition 2.17, we have $L \in D_m(X)$ for some $m \geq n$ or $m = 0$. This implies that the case $m \geq n$ holds since L is not 0-dense. For L is not $(2n+2)$-dense, we have $m \leq 2n+1$. That is $L \in D_m(X)$ for some $m, n \leq m \leq 2n+1$.

We now give the following characterization of 0-dense languages.

Proposition 3.8 *Let $L \subseteq X^+$ be a language. Then the following statements are equivalent:*

(1) $L \in D_0(X)$.
(2) $L \cap Q \in D_0(X)$.
(3) $L \setminus L_{n+1,r} \in D_n(X)$ *for all* $n \geq 1$, $0 \leq r < n+1$.

Proof (2) $\Rightarrow$ (1) Trivial.

(1) $\Rightarrow$ (2) Since L is 0-dense, by Proposition 2.13, we have the language $L \cap Q$ is dense. This implies that $L \cap Q \in D_n(X)$ for some $n \geq 0$. If $n \geq 1$, then by Proposition 3.7, $L \in D_m(X)$ for some $m, n \leq m \leq 2n+1$. One has that L is not $(2n+2)$-dense. It follows that L is not 0-dense, a contradiction. Hence $n = 0$ must be true and $L \cap Q \in D_0(X)$.

(3) $\Rightarrow$ (1) For any $n \geq 1$, $0 \leq r < n$. Since $L \setminus L_{n+1,r} \subseteq L$ and $L \setminus L_{n+1,r}$ is n-dense, one has that L is n-dense, that is $L \in D_0(X)$.

(1) $\Rightarrow$ (3) Let $n \geq 1$, $0 \leq r < n$. Then for L is 0-dense, we have L is $n \cdot (n+1)$-dense. This implies that for any $w_1, w_2, \ldots, w_n \in X^+$, there exist $u, v \in X^*$, $a \in X$ such that

$$\begin{gathered} uw_1av, uw_2av, \ldots, uw_nav \in L, \\ uw_1a^2v, uw_2a^2v, \ldots, uw_na^2v \in L, \\ uw_1a^3v, uw_2a^3v, \ldots, uw_na^3v \in L, \\ \vdots \\ uw_1a^{n+1}v, uw_2a^{n+1}v, \ldots, uw_na^{n+1}v \in L. \end{gathered}$$

One has that there exists $1 \leq k \leq n+1$ such that $lg(uw_ia^kv) \not\equiv r \pmod{n+1}$ for all $1 \leq i \leq n$. This implies that $uw_1a^kv, uw_2a^kv, \ldots, uw_na^kv \in L \setminus L_{n+1,r}$. It is clear now that $L \setminus L_{n+1,r}$ is an n-dense language. That $L \setminus L_{n+1,r}$ not $(n+1)$-dense is easy to see and $L \setminus L_{n+1,r} \in D_n(X)$, we are done.

We remark here that the equivalent relation (1) $\Leftrightarrow$ (3) in Proposition 3.8 is a stronger version of Proposition 2.16. Now by using a similar method for the proof of Proposition 3.8, we can show that following proposition is true.

Proposition 3.9 *Let $L \subseteq X^*$ be n-dense for some $n \geq 1$ and let $m \geq 1$, $m(m+1) \leq n$. Then $L \setminus L_{m+1,r} \in D_m(X)$ for all $0 \leq r < m+1$.*

Before proving the next proposition, we need to define the following known notation. For any real number r, the greatest number $[r]$ is the largest integer that is less than or equal to r. (For instance, $[4] = 4$, $[3.2] = 3$, $[-2.8] = -3$.)

Proposition 3.10 *Let $L \in D_n(X)$ for some $n \geq 2$. Then $L \cap Q \in D_m(X)$ for some m, $[\frac{n}{2}] \leq m \leq n$.*

Proof Since L is n-dense with $n \geq 2$, by Proposition 2.13, $L \cap Q$ is dense, This implies that $L \cap Q \in D_m(X)$ for some $m \geq 0$. Clearly, $m \neq 0$ and $m \leq n$. Let $[\frac{n}{2}] = k$ for some $k \geq 1$. Then $2k \leq n \leq 2k+1$. If $m \leq k-1$, then by Proposition 3.7, $L \in D_t(X)$ for some t, $t \leq 2(k-1)+1 = 2k-1 < n$, a contradiction. Hence $m \geq k = [\frac{n}{2}]$ and we are done.

In general, Proposition 3.10 may not hold true when $n = 1$. The following is an example:

Example 1. Let $L = Q^{(2)} \in D_1(X)$. Then $L \cap Q = \emptyset$ which is not dense.

Proposition 3.7 states that the condition $L \cap Q \in D_n(X)$ implies that $L \in D_m(X)$, $n \leq m \leq 2n+1$. In fact the condition $n \leq m \leq 2n+1$ in Proposition 3.7 is optimal. In the final part of this section we will provide two examples to show this fact.

Example Let $L = Q \setminus Q_{n+1,0}$. Then by Proposition 2.16, we have $L \in D_n(X)$ and $L \cap Q = L \in D_n(X)$.

Example We construct an example of language L such that $L \cap Q \in D_n(X)$ and $L \in D_{2n+1}(X)$. To this end let $L = X^* \setminus Q_{n+1,0}$. It is easy to see that $L \cap Q = Q \setminus Q_{n+1,0}$ and $L \cap Q \in D_n(X)$. Now we want to show that $L \in D_{2n+1}(X)$. If L is $(2n+2)$-dense, then there exist $u, v \in X^*$ such that $uav, ubv, uaav, ubav, \ldots, uaa^nv, uba^nv \in L$, where $a \neq b \in X$. This implies that there exists $0 \leq i \leq n$ such that $lg(uba^iv) \equiv 0 \pmod{n+1}$. Clearly, we have $uaa^iv \in Q$ or $uba^iv \in Q$. This implies that $uaa^iv \in Q_{n+1,0}$ or $uba^iv \in Q_{n+1,0}$, that is $uaa^iv \notin L$ or $uba^iv \notin L$, a contradiction. Hence L is not a $(2n+2)$-dense.

Next, we show that L is $(2n+1)$-dense. Let $w_1, w_2, \ldots, w_{2n+1} \in X^+$. Since $X^+ = (X^+)_{n+1,0} \cup (X^+)_{n+1,1} \cup \cdots \cup (X^+)_{n+1,n}$, we have $w_1, w_2, \ldots, w_{2n+1} \in (X^+)_{n+1,0} \cup (X^+)_{n+1,1} \cup \cdots \cup (X^+)_{n+1,n}$. Let $A = \{w_1, w_2, \ldots, w_{2n+1}\}$. Then we discuss the following cases:

1° If there exists $0 \le r < n$ such that $A \cap (X^+)_{n+1,r} = \emptyset$, then $lg(w_i) \not\equiv r \pmod{n+1}$ for all $1 \le i \le 2n+1$. Let $s = n+1-r$. Then $lg(w_i a^s) \not\equiv 0 \pmod{n+1}$ for all $1 \le i \le 2n+1$. This implies that $w_1 a^s, w_2 a^s, \ldots, w_{2n+1} a^s \in L$.

2° If $A \cap (X^+)_{n+1,i} \neq \emptyset$ for all $0 \le i \le n$, then for $|A| = 2n+1$, there exists $0 \le p \le n$ such that $|A \cap (X^+)_{n+1,p}| = 1$, say $\{w_j\} = A \cap (X^+)_{n+1,p}$ where $1 \le j \le 2n+1$. This implies that $lg(w_j) \equiv p \pmod{n+1}$ and $lg(w_i) \not\equiv p \pmod{n+1}$ for all $i \neq j$. Let $q = n+1-p$. Then $lg(w_j a^q) \equiv 0 \pmod{n+1}$ and $lg(w_i a^q) \not\equiv 0 \pmod{n+1}$ for all $i \neq j$. Hence we have $lg(w_j a^q (w_j a^q)) \equiv 0\ (mod\ \ n+1)$ and $lg(w_i a^q (w_j a^q)) \not\equiv 0 \pmod{n+1}$ for all $i \neq j$. One has that $w_j a^q (w_j a^q) \in L$ since $w_j a^q (w_j a^q) \notin Q$ and $w_i a^q (w_j a^q) \in L$ for all $i \neq j$ since $lg(w_i a^q (w_j a^q)) \not\equiv 0 \pmod{n+1}$, respectively. That is, $w_1 a^q (w_j a^q), w_2 a^q (w_j a^q), \ldots, w_j a^q (w_j a^q), \ldots, w_{2n+1} a^q (w_j a^q) \in L$.

By 1° and 2°, L is a $(2n+1)$-dense language. For L is not a $(2n+2)$-dense language, $L \in D_{2n+1}(X)$ and we are done.

4 Decomposition of a dense languages into disjoint union of infinitely many dense languages

In this section, let the alphabet be $X = \{a, b\}$ and we want to investigate the decompositions of general dense languages first, that decomposition of n-dense languages will be dealt at the end of this section.

Our aim will be that every n-dense language can be split into m parts for any $m \ge 2$ such that all parts are all n-dense languages. Furthermore, we will show that every n-dense language can be decomposed into a disjoint union of infinitely many n-dense languages.

Before we start our works, we need to consider a known total order $\le$ defined on X^* and it is called the *length-lexicographic order*. Our total order is defined as follows: For two words of different lengths u and v, $u \le v$ if $lg(u) < lg(v)$. For the two words with same length u and v, our order is the *lexicographic order*. The order so defined can be found in [9]. Thus X^* with the order can be demonstrated as

$$X^* = \{1, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, \ldots\}$$

and

$$1 < a < b < aa < ab < ba < bb < aaa < aab < aba < abb \cdots.$$

In this section, for convenience, we always assume that $X^* = \{x_1, x_2, x_3, x_4, x_5, \ldots\}$, where

$$x_1 = 1, \quad x_2 = a, \quad x_3 = b, \quad x_4 = aa, \quad x_5 = ab, \ldots.$$

Let us recall the following known result from [5].

Lemma 4.1 [5] *Let $S \subseteq X^*$. Then the following are equivalent:*

(1) *S contains a disjunctive language.*

(2) $S \cap X^*wX^* \neq \emptyset$ *for all* $w \in X^*$.
(3) $|S \cap X^*wX^*| = \infty$ *for all* $w \in X^*$.

Proposition 4.2 *Let* $L \subseteq X^*$ *be a dense language. Then there exist* $L_1, L_2 \subseteq L$, $L_1 \cup L_2 = L$, $L_1 \cap L_2 = \emptyset$ *such that* L_1 *and* L_2 *are both dense languages.*

Proof Let $X^* = \{x_i \mid i \geq 1\}$. For L is dense, by Lemma 4.1, $|L \cap X^*x_iX^*| = \infty$ for all $x_i \in X^*$. From this one has that for each x_i there exist $u'_i, u''_i, v'_i, v''_i \in X^*$ such that $u'_ix_iv'_i \neq u''_ix_iv''_i \in L$. Without loss of generality, we may assume that

$$lg(u'_1x_1v'_1) < lg(u''_1x_1v''_1) < lg(u'_2x_2v'_2) < lg(u''_2x_2v''_2) < \cdots < lg(u'_kx_kv'_k) \\ < lg(u''_kx_kv''_k) < \cdots .$$

Let $L_1 = \{u''_ix_iv''_i \mid i \geq 1\}$ and $L_2 = L \setminus L_1$. Clearly, both L_1 and L_2 are dense languages, since $\{u'_ix_iv'_1 \mid i \geq 1\} \subseteq L_2$. It is easy to see that $L_1 \cap L_2 = \emptyset$ and our proof is completed.

The following proposition is immediate.

Proposition 4.3 *Let* $L \subseteq X^*$ *be a dense language and let* $k \in N$. *Then there exist* $L_1, L_2, \ldots, L_k \subseteq L$, $L_1 \cup L_2 \cup \cdots \cup L_k = L$, $L_i \cap L_j = \emptyset$, $i \neq j$ *such that* $L_1, L_2, \ldots, L_k$ *are all dense languages.*

Proposition 4.4 *Every dense language can be split into disjoint union of infinitely many dense languages.*

Proof Let $X^* = \{x_i \mid i \geq 1\}$. For L is dense, by Lemma 4.1, $|L \cap X^*x_iX^*| = \infty$ for all $x_i \in X^*$. One has that there exist $u_{i1}, u_{i2}, \ldots, u_{ii}$, $v_{i1}, v_{i2}, \ldots, v_{ii} \in X^*$ such that $u_{i1}x_iv_{i1}, u_{i2}x_iv_{i2}, \ldots, u_{ii}x_iv_{ii} \in L$ for all $i \geq 1$. Since $|L \cap X^*x_iX^*| = \infty$ for all $x_i \in X^*$, without loss of generality, we may assume that

$$lg(u_{11}x_1v_{11}) < lg(u_{21}x_2v_{21}) < lg(u_{22}x_2v_{22}) < lg(u_{31}x_3v_{31}) < lg(u_{32}x_3v_{32}) \\ < lg(u_{33}x_3v_{33}) < \cdots .$$

Let $L_i = \{\, u_{ji}x_jv_{ji} \mid j \geq i \,\}$ and $L_1 = L \setminus (\cup_{i \geq 2}L_i)$. It is immediate that $L = \cup_{i \geq 1}L_i$ and $L_i \cap L_j = \emptyset$ for all $i \neq j$. Clearly, L_1 is dense, since $\{u_{i1}x_iv_{i1} \mid i \geq 1\} \subseteq L_1$. Let $n \geq 2$ and $w \in X^*$. Since $X^* = \{x_i \mid i \geq 1\}$, we may assume that $w = x_m$ for some $m \geq 1$. If $m \geq n$, then we have $u_{mn}x_mv_{mn} \in L_n$, that is $L_n \cap X^*wX^* \neq \emptyset$. If $m < n$, then we can consider the words wa^n, where $a \in X$. One has that there exists $t \geq n$ such that $wa^n = x_t$. For $t \geq n$, we have $u_{tn}x_tv_{tn} \in L_n$. This implies that $u_{tn}wa^nv_{tn} \in L_n$, i.e. $L_n \cap X^*wX^* \neq \emptyset$. Since the number n is chosen arbitrarily, we have L_n is dense for all $n \geq 1$. The proof is completed.

It is known that every dense language contains a disjunctive language, for example, see Lemma 4.1. And in the following we discuss decompositions of disjunctive languages.

Lemma 4.5 *Let* $L \subseteq X^*$ *be a disjunctive language. Then for any* $x, y \in X^*$, $x \neq y$, *there exist infinitely many ordered pairs* $\{(u_i, v_i) \mid u_i, v_i \in X^+, i \geq 1\}$ *such that one of the following statements is true:*

(1) $u_i x v_i \in L$ *and* $u_i y v_i \notin L$ *for all* $i \geq 1$.
(2) $u_i x v_i \notin L$ *and* $u_i y v_i \in L$ *for all* $i \geq 1$.

Proof Let $x \neq y \in X^*$. Then for L is disjunctive, there exist $u_1, v_1 \in X^*$ such that $u_1 x v_1 \in L$ and $u_1 y v_1 \notin L$ (or vice versa). Since $u_1 x v_1 \neq u_1 y v_1$, there exist $u_2, v_2 \in X^+$, $u_1 <_s u_2$, $v_1 <_p v_2$ such that $u_2 x v_2 \in L$ and $u_2 y v_2 \notin L$ (or vice versa). Continuing this process, there exist infinitely many words $u_j, v_j \in X^*$, $u_j <_s u_{j+1}$, $v_j <_p v_{j+1}$ for all $j \geq 1$ such that $u_j x v_j \in L$ and $u_j y v_j \notin L$ (or vice versa). It is immediate that either condition (1) or condition (2) holds.

Proposition 4.6 *Let* $L \subseteq X^*$ *be a disjunctive langauge. Then there exist* $L_1, L_2 \subseteq L$, $L_1 \cup L_2 = L$, $L_1 \cap L_2 = \emptyset$ *such that* L_1 *and* L_2 *are both disjunctive languages.*

Proof Recall that $X^* = \{x_1, x_2, \ldots, x_n, \ldots\}$. Then $X^* \times X^* = \{(x_i, x_j) \mid i, j \geq 1\}$. Let the subset $I = \{(x_i, x_j) \mid j > i \geq 1\} \subseteq X^* \times X^*$. That is $I = \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_2, x_4), (x_3, x_4), \ldots\}$. Since $x_1 \neq x_2$, by Lemma 4.5, there exist infinitely many ordered pairs $\{(u_i(1,2), v_i(1,2)) \mid u_i(1,2), u_i(1,2) \in X^+, \ i \geq 1\}$ such that either $u_i(1,2) x_1 v_i(1,2) \in L$ and $u_i(1,2) x_2 v_i(1,2) \notin L$ for all $i \geq 1$ (or vice versa). This implies that there exist $u_\alpha(1,2), u_\beta(1,2), v_\alpha(1,2), v_\beta(1,2) \in X^+$ with $lg(u_\alpha(1,2) x_2 v_\alpha(1,2)) < lg(u_\beta(1,2) x_1 v_\beta(1,2))$ such that

$$u_\alpha(1,2) x_1 v_\alpha(1,2), u_\beta(1,2) x_1 v_\beta(1,2) \in L$$

and

$$u_\alpha(1,2) x_2 v_\alpha(1,2), u_\beta(1,2) x_2 v_\beta(1,2) \notin L. \text{ (or vice versa)}$$

For $x_1 \neq x_3$, by a similar procedure, there exist $u_\alpha(1,3), u_\beta(1,3), v_\alpha(1,3), v_\beta(1,3) \in X^+$ with $lg(u_\beta(1,2) x_2 v_\beta(1,2)) < lg(u_\alpha(1,3) x_1 v_\alpha(1,3))$ and $lg(u_\alpha(1,3) x_3 v_\alpha(1,3)) < lg(u_\beta(1,3) x_1 v_\beta(1,3))$ such that

$$u_\alpha(1,3) x_1 v_\alpha(1,3), u_\beta(1,3) x_1 v_\beta(1,3) \in L$$

and

$$u_\alpha(1,3) x_3 v_\alpha(1,3), u_\beta(1,3) x_3 v_\beta(1,3) \notin L. \text{ (or vice versa)}$$

Continuing this process, for any $x_n \neq x_m$, $n < m$, there exist $u_\alpha(n,m), u_\beta(n,m), v_\alpha(n,m), v_\beta(n,m) \in X^+$ with $lg(u_\alpha(n,m) x_m v_\alpha(n,m)) < lg(u_\beta(n,m) x_n v_\beta(n,m))$ and $k < lg(u_\alpha(n,m) x_n v_\alpha(n,m))$ where $k = lg(u_\beta(n-1,m) x_m v_\beta(n-1,m))$ when $n \geq 2$ or $k = lg(u_\beta(m-2,m-1) x_{m-1} v_\beta(m-2,m-1))$ when $n = 1$ such that

$$u_\alpha(n,m) x_n v_\alpha(n,m), u_\beta(n,m) x_n v_\beta(n,m) \in L$$

and

$$u_\alpha(n,m) x_m v_\alpha(n,m), u_\beta(n,m) x_m v_\beta(n,m) \notin L. \text{(or vice versa)}$$

Let $w_\alpha(n,m) = u_\alpha(n,m) x_n v_\alpha(n,m)$ if $u_\alpha(n,m) x_n v_\alpha(n,m) \in L$ or $w_\alpha(n,m) = u_\alpha(n,m) x_m v_\alpha(n,m)$ if $u_\alpha(n,m) x_m v_\alpha(n,m) \in L$, respectively. Similarly, we also let $w_\beta(n,m) = u_\beta(n,m) x_n v_\beta(n,m)$ if $u_\beta(n,m) x_n v_\beta(n,m) \in L$ or $w_\beta(n,m) = u_\beta(n,m) x_m v_\beta(n,m)$ if $u_\beta(n,m) x_m v_\beta(n,m) \in L$, respectively. Then we have $\{w_\alpha(n,m) \mid m > n \geq 1\} \cap \{w_\beta(n,m) \mid m > n \geq 1\} = \emptyset$ and $\{w_\alpha(n,m) \mid m > n \geq 1\} \cup \{w_\beta(n,m) \mid m > n \geq 1\} \subseteq L$.

Let $L_1 = \{w_\alpha(n,m) \mid m > n \geq 1\}$ and $L_2 = L \setminus L_1$. Then $\{w_\beta(n,m) \mid m > n \geq 1\} \subseteq L_2$. Next we show that both L_1 and L_2 are disjunctive languages. Let $x, y \in X^*$, $x \neq y$, say $x = x_n,\ y = x_m,\ n < m$. Then $w_\alpha(n,m) \in L_1$. This implies that

$$u_\alpha(n,m)x_n v_\alpha(n,m) \in L \text{ and } u_\alpha(n,m)x_m v_\alpha(n,m) \notin L \text{ when } w_\alpha(n,m) \\ = u_\alpha(n,m)x_n v_\alpha(n,m)$$

or

$$u_\alpha(n,m)x_m v_\alpha(n,m) \in L \text{ and } u_\alpha(n,m)x_n v_\alpha(n,m) \notin L \text{ when } w_\alpha(n,m) \\ = u_\alpha(n,m)x_m v_\alpha(n,m).$$

That is L_1 is a disjunctive language. Similarly, L_2 is also a disjunctive language.

The following proposition is immediate.

Proposition 4.7 *Let $L \subseteq X^*$ be a disjunctive language. Then for any $k \geq 1$, there exist $L_1, L_2, \ldots, L_k \subseteq L$, $L_1 \cup L_2 \cup \cdots \cup L_k = L$, $L_i \cap L_j = \emptyset$, $i \neq j$ such that $L_1, L_2, \ldots, L_k$ are all disjunctive languages.*

Lemma 4.8 [5] *Let L be a discrete language for which $X^*wX^* \cap L \neq \emptyset$ for all $w \in X^+$. Then L is disjunctive.*

Proposition 4.9 *Every disjunctive language can be split into disjoint union of infinitely many disjunctive languages.*

Proof Let $L \subseteq X^*$ be a disjunctive language. By using the same definition of L_1 and L_2 in the proof of Proposition 4.6, we can see that L_1 is discrete and both L_1, L_2 are disjunctive languages. For L_1 is dense, by Proposition 4.4, L_1 can be split into a disjoint union of infinitely many dense languages. Since a subset of a discrete language is also discrete, by Lemma 4.8, we have L_1 can be split into a disjoint union of infinitely many disjunctive languages. This implies that $L = L_1 \cup L_2$ can be split into a disjoint union of infinitely many disjunctive languages.

Next, we investigate the decomposition of n-dense languages. The following lemma is immediate.

Lemma 4.10 *Let $L \subseteq X^*$ be an n-dense language for some $n \geq 1$. Then for any $w_1, w_2, \ldots, w_n \in X^*$, there are infinitely many pairs of words $u_i, v_i \in X^+$, $i \in N$ such that $u_i w_1 v_i, u_i w_2 v_i, \ldots, u_i w_n v_i \in L$.*

Proposition 4.11 *Let $L \subseteq X^*$ be an n-dense language for some $n \geq 1$. Then there exist $L_1, L_2 \subseteq L$, $L_1 \cup L_2 = L$, $L_1 \cap L_2 = \emptyset$ such that L_1 and L_2 are both n-dense languages.*

Proof Clearly, the case $n = 1$ holds from Proposition 4.2. Let $n = 2$. Then for $X^* \times X^* = \{(x_i, x_j) \mid i, j \geq 1\}$. We can assume that the subset $I = \{(x_i, x_j) \mid j > i \geq 1\} \subseteq X^* \times X^*$. That is $I = \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_1, x_4), (x_2, x_4), (x_3, x_4), \ldots\}$. Since $x_1 \neq x_2$, by Lemma 4.10, there exist infinitely many ordered pairs $\{(u_i(1,2), v_i(1,2)) \mid u_i(1,2), u_i(1,2) \in X^+,\ i \geq 1\}$ such that $u_i(1,2)x_1 v_i(1,2)$ and $u_i(1,2)x_2 v_i(1,2) \in L$ for all $i \geq 1$. Since the language $\{u_i(1,2)x_1 v_i(1,2), u_i(1,2)x_2 v_i(1,2) \mid i \geq 1\}$ has infinitely many words, there exist $\alpha, \beta \geq 1$ such that

$$u_\alpha(1,2)x_1 v_\alpha(1,2), u_\alpha(1,2)x_2 v_\alpha(1,2), u_\beta(1,2)x_1 v_\beta(1,2), u_\beta(1,2)x_2 v_\beta(1,2) \in L$$

with $lg(u_\alpha(1,2)x_2 v_\alpha(1,2)) < lg(u_\beta(1,2)x_1 v_\beta(1,2))$.

For $x_1 \neq x_3$, by a similar procedure, there exist $u_\alpha(1,3), u_\beta(1,3), v_\alpha(1,3), v_\beta(1,3) \in X^+$ with $lg(u_\beta(1,2)x_2v_\beta(1,2)) < lg(u_\alpha(1,3)x_1v_\alpha(1,3))$ and $lg(u_\alpha(1,3)x_3v_\alpha(1,3)) < lg(u_\beta(1,3)x_1v_\beta(1,3))$ such that

$$u_\alpha(1,3)x_1v_\alpha(1,3), u_\alpha(1,3)x_3v_\alpha(1,3), u_\beta(1,3)x_1v_\beta(1,3), u_\beta(1,3)x_3v_\beta(1,3) \in L.$$

Continuing this process, for any $x_n \neq x_m$, $n < m$, there exist $u_\alpha(n,m), u_\beta(n,m), v_\alpha(n,m), v_\beta(n,m) \in X^+$ with $lg(u_\alpha(n,m)x_mv_\alpha(n,m)) < lg(u_\beta(n,m)x_nv_\beta(n,m))$ and $k < lg(u_\alpha(n,m)x_nv_\alpha(n,m))$ where $k = lg(u_\beta(n-1,m)x_mv_\beta(n-1,m))$ when $n \geq 2$ or $k = lg(u_\beta(m-2,m-1)x_{m-1}v_\beta(m-2,m-1))$ when $n = 1$ such that

$$u_\alpha(n,m)x_nv_\alpha(n,m),\ u_\alpha(n,m)x_mv_\alpha(n,m),\ u_\beta(n,m)x_nv_\beta(n,m),$$
$$u_\beta(n,m)x_mv_\beta(n,m) \in L.$$

Let $L_1 = \{u_\alpha(n,m)x_nv_\alpha(n,m),\ u_\alpha(n,m)x_mv_\alpha(n,m) \mid m > n \geq 1\}$ and $L_2 = L \setminus L_1$. Then $\{u_\beta(n,m)x_nv_\beta(n,m),\ u_\beta(n,m)x_mv_\beta(n,m) \mid m > n \geq 1\} \subseteq L_2$. Next we show that both L_1 and L_2 are 2-dense languages. Let $x, y \in X^*$, $x \neq y$, say $x = x_n$, $y = x_m$, $n < m$. Then by the definition of L_1 and L_2, $u_\alpha(n,m)x_nv_\alpha(n,m)$, $u_\alpha(n,m)x_mv_\alpha(n,m) \in L_1$ and $u_\beta(n,m)x_nv_\beta(n,m), u_\beta(n,m)x_mv_\beta(n,m) \in L_2$, respectively. This implies that L_1 and L_2 both are 2-dense and we complete the case $n = 2$.

Finally, by using a similar method, the cases $n \geq 3$ also hold true.

From Proposition 4.11, the following proposition is immediate.

Proposition 4.12 *Let $L \subseteq X^*$ be an n-dense language for some $n \geq 1$ and let $k \in N$. Then there exist $L_1, L_2, \ldots, L_k \subseteq L$, $L_1 \cup L_2 \cup \cdots \cup L_k = L$, $L_i \cap L_j = \emptyset$, $i \neq j$ such that $L_1, L_2, \ldots, L_k$ are all n-dense languages.*

Proposition 4.13 *Let $n \geq 1$. Then every n-dense language can be split into disjoint union of infinitely many n-dense languages.*

Proof From Proposition 4.4, clearly case $n = 1$ holds true. Let L be a 2-dense language and let $I = \{(x_i, x_j) \mid j > i \geq 1\} = \{(x_1,x_2), (x_1,x_3), (x_2,x_3), (x_1,x_4), (x_2,x_4), (x_3,x_4), \ldots\}$, where $X^* = \{x_1, x_2, \ldots, x_n, \ldots\}$. For convenience, we may define the set $I = \{w_1, w_2, w_3, \ldots\}$, where $w_1 = (x_1,x_2), w_2 = (x_1,x_3), w_3 = (x_2,x_3), w_4 = (x_1,x_4), \ldots$.

Since $w_1 = (x_1,x_2)$, by Lemma 4.10, there exist $u_1(1,2), v_1(1,2) \in X^+$ such that $u_1(1,2)x_1v_1(1,2)$, $u_1(1,2)x_2v_1(1,2) \in L$. For $w_2 = (x_1,x_3)$, by Lemma 4.10 again, one has that there exist $u_1(1,3)$, $v_1(1,3)$, $u_2(1,3), v_2(1,3) \in X^+$ such that

$$u_1(1,3)x_1v_1(1,3), u_1(1,3)x_3v_1(1,3), u_2(1,3)x_1v_2(1,3), u_2(1,3)x_3v_2(1,3) \in L,$$

where $lg(u_1(1,2)x_2v_1(1,2)) < lg(u_1(1,3)x_1v_1(1,3))$ and $lg(u_1(1,3)x_3v_1(1,3)) < lg(u_2(1,3)x_1v_2(1,3))$.

Continuing this process, for any $x_n \neq x_m$, $n < m$, say $(x_n, x_m) = w_h$, $h \geq 3$ and $w_{h-1} = (x_p, x_q)$, there exist $u_i(n,m), v_i(n,m) \in X^+, 1 \leq i \leq h$ such that

$$u_i(n,m)x_nv_i(n,m), u_i(n,m)x_mv_i(n,m) \in L$$

with

$$lg(u_{h-1}(p,q)x_qv_{h-1}(p,q)) < lg(u_1(n,m)x_nv_1(n,m))$$

and

$$lg(u_j(n,m)x_m v_j(n,m)) < lg(u_{j+1}(n,m)x_n v_{j+1}(n,m),$$

where $j = 1, 2, \dots, h-1$.

Let $L_k = \{u_k(n,m)x_n v_k(n,m),\ u_k(n,m)x_m v_k(n,m) \mid (x_n, x_m) = w_t \in I,\ t \geq k\}$ for all $k \geq 2$ and let $L_1 = L \setminus \cup_{k \geq 2} L_k$. Then $\{u_1(n,m)x_n v_1(n,m),\ u_1(n,m) x_m v_1(n,m) \mid (x_n, x_m) \in I\} \subseteq L_1$ and $L_i \cap L_j = \emptyset$ for all $i \neq j$. Next we want to show that each L_i is 2-dense for all $i \geq 1$. Clearly, it is immediate that L_1 is 2-dense since $\{u_1(n,m)x_n v_1(n,m),\ u_1(n,m)x_m v_1(n,m) \mid (x_n, x_m) \in I\} \subseteq L_1$. Let j be a fixed number. For any $x \neq y \in X^+$, say $x = x_p,\ y = x_q,\ n < m$ and $(x_p, x_q) = w_k \in I$. Now we discuss the following cases:

(1) If $k \geq j$, then for $L_j = \{u_j(n,m)x_n v_j(n,m),\ u_j(n,m)x_m v_j(n,m) \mid (x_n, x_m) = w_t \in I,\ t \geq j\}$, one has that $u_j(p,q)x_p v_j(p,q),\ u_j(p,q)x_q v_j(p,q) \in L_j$. That is $u_j(p,q)xv_j(p,q),\ u_j(p,q)yv_j(p,q) \in L_j$.

(2) If $k < j$, then there exist $h \geq 1,\ a \in X$ such that $a^h x = x_{p_1},\ a^h y = x_{q_1}$ and $(x_{p_1}, x_{q_1}) = w_{k_1}$, where $p_1 < q_1$ and $k_1 > j$. Thus by case (1), we have $u_j(p_1,q_1)x_{p_1} v_j(p_1,q_1),\ u_j(p_1,q_1)x_{q_1} v_j(p_1,q_1) \in L_j$. That is $u_j(p_1,q_1)a^h xv_j(p_1,q_1),\ u_j(p_1,q_1)a^h yv_j(p_1,q_1) \in L_j$.

By cases (1) and (2), it follows that L_j is a 2-dense language. Since the positive integer j is chosen arbitrarily, we have L_i is 2-dense for all $i \geq 1$. That is the language $L = \cup_{i \geq 1} L_i$ can be split into disjoint union of infinitely many 2-dense languages. This completes the case $n = 2$.

Finally, let $n \geq 3$ and we may consider the set I_n be the set of all n-tuples over $X^* = \{x_1, x_2, \dots, x_n, \dots\}$. Thus by using a similar method, the cases $n \geq 3$ also hold true and we are done.

Proposition 4.14 *Let $L \in D_n(X)$ for some $n \geq 1$ and let $k \in N$. Then there exist $L_1, L_2, \dots, L_k \subseteq L$, $L_1 \cup L_2 \cup \dots \cup L_k = L$, $L_i \cap L_j = \emptyset$, $i \neq j$ such that $L_1, L_2, \dots, L_k$ are all strict n-dense languages.*

Proof Since L is n-dense, by Proposition 4.12, there exist $L_1, L_2, \dots, L_k \subseteq L$, $L_1 \cup L_2 \cup \dots \cup L_k = L$, $L_i \cap L_j = \emptyset, i \neq j$ such that $L_1, L_2, \dots, L_k$ are all n-dense languages. For $L \in D_n(X)$, we have L is not $(n+1)$-dense. This implies that $L_1, L_2, \dots, L_k$ are all not $(n+1)$-dense. That is $L_1, L_2, \dots, L_k$ are all strict n-dense languages and we are done.

Proposition 4.15 *Let $n \geq 1$. Then every strict n-dense language can be split into disjoint union of infinitely many strict n-dense languages.*

Proof Immediate.

In the next proposition, we want to show that every 0-dense language can be split into disjoint union of infinitely many 0-dense languages. Before we start our work, we need to define the following notations:

Let $X^* = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$. Then the following sets I_n are defined as

$$
\begin{aligned}
I_1 &= \{x_1, x_2, x_3, x_4, x_5, x_6, \ldots\} \\
&= \{w_{11}, w_{12}, w_{13}, \ldots\}, \quad \text{where } w_{11} = x_1,\ w_{12} = x_2,\ w_{13} = x_3, \ldots \\
I_2 &= \{(x_1,x_2),(x_1,x_3),(x_2,x_3),(x_1,x_4),(x_2,x_4),(x_3,x_4),\ldots\} \\
&= \{w_{21}, w_{22}, w_{23}, \ldots\}, \quad \text{where } w_{21} = (x_1,x_2),\ w_{22} = (x_1,x_3),\ w_{23} = (x_2,x_3), \ldots \\
I_3 &= \{(x_1,x_2,x_3),(x_1,x_2,x_4),(x_1,x_3,x_4),(x_2,x_3,x_4),(x_1,x_2,x_5),(x_1,x_3,x_5),\ldots\} \\
&= \{w_{31}, w_{32}, w_{33}, \ldots\}, \quad \text{where } w_{31} = (x_1,x_2,x_3),\ w_{32} = (x_1,x_2,x_4), \\
&\quad\ w_{33} = (x_1,x_3,x_4), \ldots \\
I_4 &= \{(x_1,x_2,x_3,x_4),(x_1,x_2,x_3,x_5),(x_1,x_2,x_4,x_5),(x_1,x_3,x_4,x_5),(x_2,x_3,x_4,x_5), \\
&\quad\ (x_1,x_2,x_3,x_6),\ldots\} \\
&= \{w_{41}, w_{42}, w_{43}, \ldots\}, \quad \text{where } w_{41} = (x_1,x_2,x_3,x_4),\ w_{42} = (x_1,x_2,x_3,x_5), \\
&\quad\ w_{43} = (x_1,x_2,x_4,x_5), \ldots \\
&\qquad\qquad\qquad \vdots \\
I_n &= \{(x_1,x_2,\ldots,x_{n-1},x_n),(x_1,x_2,\ldots,x_{n-1},x_{n+1}),\ldots\} \\
&= \{w_{n1}, w_{n2}, \ldots\}, \quad \text{where } w_{n1} = (x_1,x_2,\ldots,x_{n-1},x_n), \\
&\quad\ w_{n2} = (x_1,x_2,\ldots,x_{n-1},x_{n+1}), \ldots \\
&\qquad\qquad\qquad \vdots
\end{aligned}
$$

From above definitions of I_n, we have the following table:

$$
\begin{array}{llllll}
I_1 : w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & \cdots \\
I_2 : w_{21} & w_{22} & w_{23} & w_{24} & w_{25} & \cdots \\
I_3 : w_{31} & w_{32} & w_{33} & w_{34} & w_{35} & \cdots \\
I_4 : w_{41} & w_{42} & w_{43} & w_{44} & w_{45} & \cdots \\
I_5 : w_{51} & w_{52} & w_{53} & w_{54} & w_{55} & \cdots \\
 & & \vdots & & &
\end{array}
$$

Now we consider the sequence w_{11}, w_{12}, w_{21}, w_{13}, w_{22}, w_{31}, w_{14}, w_{23}, w_{32}, $w_{41}, \ldots$ and define the set $S = \{w_{11}, w_{12}, w_{21}, w_{13}, w_{22}, w_{31}, w_{14}, w_{23}, w_{32}, w_{41}, \ldots\} = \{s_1, s_2, s_3, s_4, s_5, \ldots\}$, where $s_1 = w_{11}$, $s_2 = w_{12}$, $s_3 = w_{21}, \ldots$.

Proposition 4.16 *Every* 0*-dense language can be split into disjoint union of infinitely many 0-dense languages.*

Proof Let $L \in D_0(X)$. Then L is n-dense for all $n \geq 1$.

For $s_1 = w_{11} = x_1$ and L is 1-dense, there exist $u_{11}, v_{11} \in X^*$ such that $u_{11}x_1v_{11} \in L$. Let $S_{11} = \{u_{11}x_1v_{11}\}$.

For $s_2 = w_{12} = x_2$ and L is 1-dense, then by Lemma 4.10, there exist $u_{21}, u_{22}, v_{21}, v_{22} \in X^*$ such that $u_{21}x_2v_{21}, u_{22}x_2v_{22} \in L$. with $lg(u_{11}x_1v_{11}) < lg(u_{21}x_2v_{21}) < lg(u_{22}x_2v_{22})$. Let $S_{21} = \{u_{21}x_2v_{21}\}$, $S_{22} = \{u_{22}x_2v_{22}\}$.

For $s_3 = w_{21} = (x_1, x_2)$ and L is 2-dense, then by Lemma 4.10, there exist $u_{31}, u_{32}, u_{33}, v_{31}, v_{32}, v_{33} \in X^*$ such that

$$\begin{aligned} &u_{31}x_1v_{31}, u_{31}x_2v_{31} \in L \quad \text{with } lg(u_{22}x_2v_{22}) < lg(u_{31}x_1v_{31});\\ &u_{32}x_1v_{32}, u_{32}x_2v_{32} \in L \quad \text{with } lg(u_{31}x_2v_{31}) < lg(u_{32}x_1v_{32});\\ &u_{33}x_1v_{33}, u_{33}x_2v_{33} \in L \quad \text{with } lg(u_{32}x_2v_{32}) < lg(u_{33}x_1v_{33}). \end{aligned}$$

Let $S_{31}=\{u_{31}x_1v_{31}, u_{31}x_2v_{31}\}$, $S_{32}=\{u_{32}x_1v_{32}, u_{32}x_2v_{32}\}$, $S_{33}=\{u_{33}x_1v_{33}, u_{33}x_2v_{33}\}$.

$$\vdots$$

Continuing this process, we have the following table:

$$\begin{array}{lllll} S_{11} & & & & \\ S_{21} & S_{22} & & & \\ S_{31} & S_{32} & S_{33} & & \\ S_{41} & S_{42} & S_{43} & S_{44} & \\ & & \vdots & & \\ S_{n1} & S_{n2} & S_{n3} & S_{n4} & \cdots S_{nn} \\ & & \vdots & & \end{array}$$

and $s_{ij} \subseteq L$ for all $i \geq j \geq 1$.

For any $m \geq 2$, let $L_m = \cup_{n \geq m} S_{nm}$. We also define the language $L_1 = L \setminus \cup_{m \geq 2} L_m$. Then $\cup_{n \geq 1} S_{n1} \subseteq L_1$. Since S_{nm} are all finite languages for all $n, m \in N$, $n \geq m$ and

$$Lg(S_{11}) < Lg(S_{21}) < Lg(S_{22}) < Lg(S_{31}) < Lg(S_{32}) < Lg(S_{33}) < \cdots,$$

one has that $L_i \cap L_j = \emptyset$ for all $i \neq j$. Now we want to show that L_m is 0-dense for all $m \geq 1$. Let $n \geq 1$ be given and let $w_1, w_2, \ldots, w_n \in X^+$. Without loss of generality, we may assume that $w_1 < w_2 < w_3 < \cdots < w_n$, where the order $<$ is defined in the beginning of this section. Then there exist $a \in X$, $h \geq 1$ such that $a^h w_1 = x_{n1}$, $a^h w_2 = x_{n2}$, $\ldots$, $a^h w_n = x_{nn}$ and $(x_{n1}, x_{n2}, \ldots, x_{nn}) = s_k \in I_n$ for some $k \geq m$. This implies that $S_{km} \subseteq L_m$ and then follows $S_{km} = \{u_{km}x_{n1}v_{km},\ u_{km}x_{n2}v_{km},\ \ldots,\ u_{km}x_{nn}v_{km}\} \subseteq L_m$. One has that $(u_{km}a^h)w_1v_{km}$, $(u_{km}a^h)w_2v_{km}$, $\ldots$, $(u_{km}a^h)w_nv_{km} \in L_m$, that is L_m is an n-dense language. Since the positive integer n is chosen arbitrarily, L_m is 0-dense for all $m \geq 1$ and we are done.

Proposition 4.17 *Let $L \in D_0(X)$ and $k \in N$. Then there exist $L_1, L_2, \ldots, L_k \subseteq L$, $L_1 \cup L_2 \cup \cdots \cup L_k = L$, $L_i \cap L_j = \emptyset$, $i \neq j$ such that $L_1, L_2, \ldots, L_k$ are all 0-dense languages.*

Proof Clearly, $k = 1$ is immediate. Let $k \geq 2$ and let $L_m \subseteq L$, $m \geq 1$ by using the same definition in the proof of Proposition 4.16. Then we have L_m are all 0-dense for all $m \geq 1$. Since a language which contains a 0-dense language is also a 0-dense language, the language $\cup_{n \geq k} L_n$ is a 0-dense language. Hence $L = L_1 \cup L_2 \cup \cdots \cup L_{k-1} \cup (\cup_{n \geq k} L_n)$ and we are done.

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