WHOI-78-67

Woods Hole Oceanographic Institution

VOLUME I

NOTES ON THE 1979 SUMMER STUDY PROGRAM ON DYNAMO MODELS OF GEOMAGNETISM IN GEOPHYSICAL FLUID DYNAMICS AT THE WOODS HOLE OCEANOGRAPHIC INSTITUTION

by

Willem V. R. Malkus, Director
and
Mary Thayer, Editor

November 1978

TECHNICAL REPORT

Prepared for the Office of Naval Research under Contract N00014-78-G-0072.

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WOODS HOLE, MASSACHUSETTS 02543

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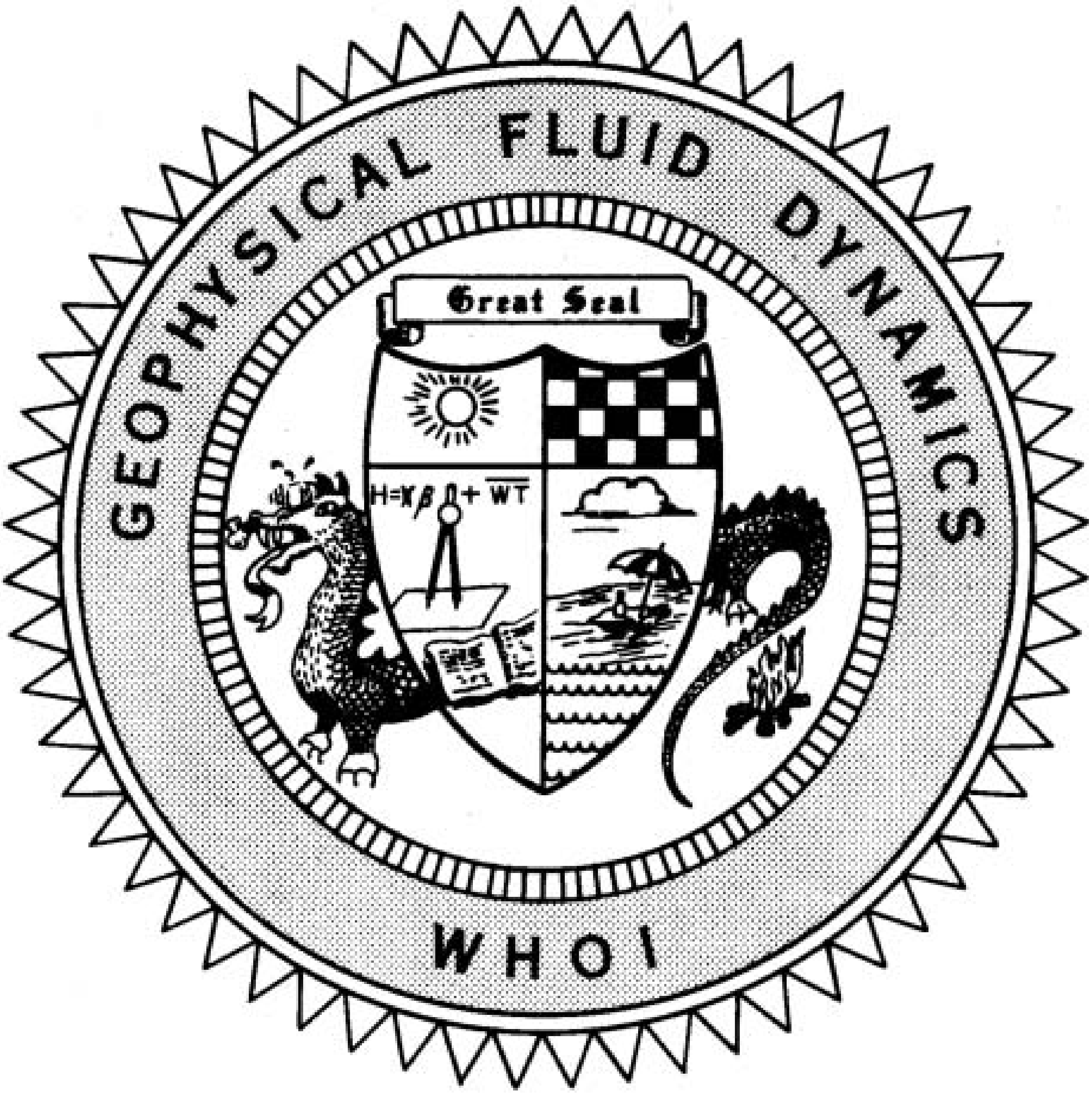

COURSE LECTURES

SEMINARS

MINI-SYMPOSIUM ABSTRACTS

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EDITOR'S PREFACE

VOLUME I

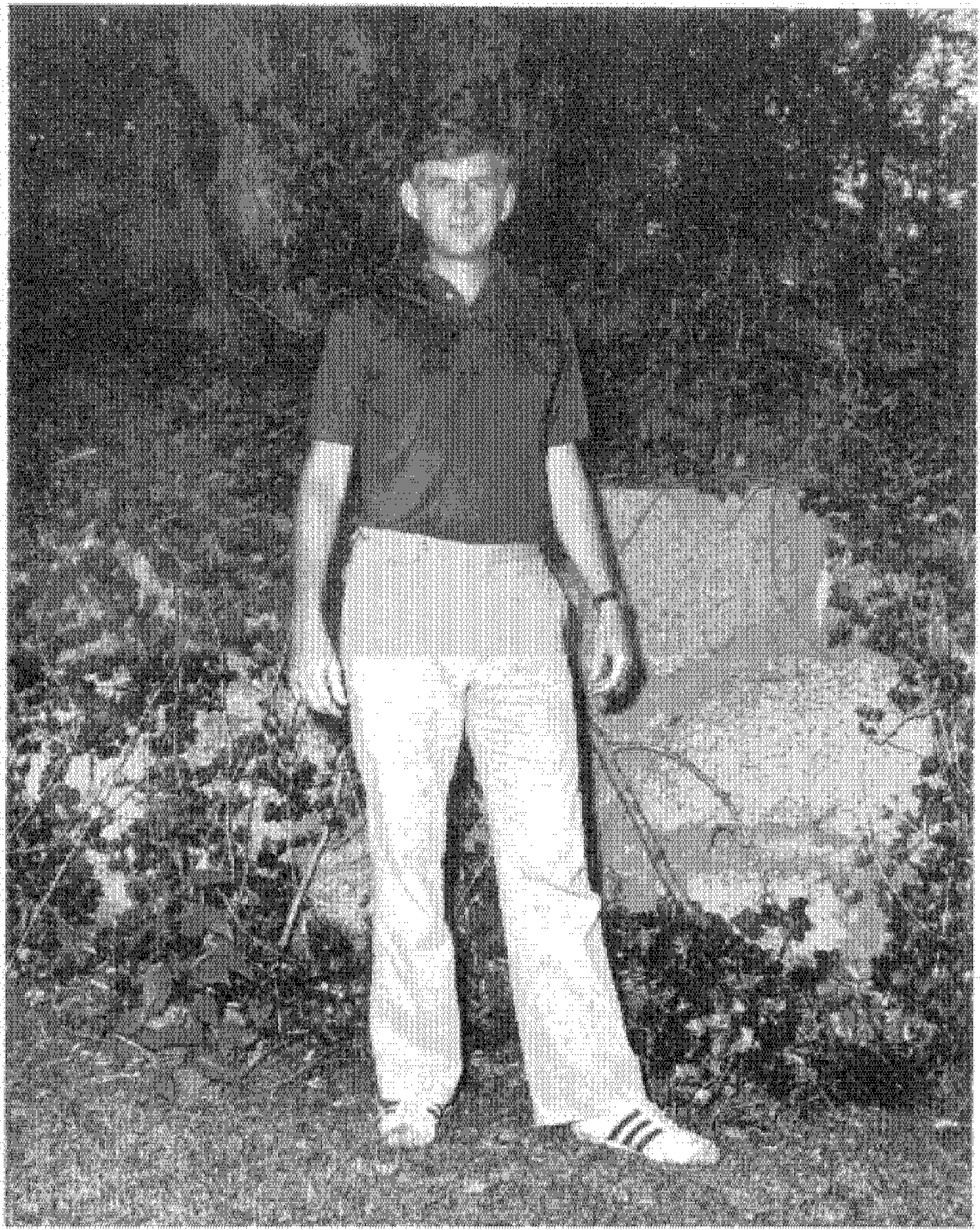
This was the twentieth Geophysical Fluid Dynamics program at Woods Hole. Stephen Childress of the Courant Institute was our principal lecturer, Dynamo theory, with all its interdisciplinary facets was our central theme. Geomagnetism and the solar magnetic cycle were brought closer to comprehension, yet none claimed a detailed predictive theory was near at hand. Perhaps J. Keller's lecture, entitled "Smooth equations for rough problems", best characterized the nature of these studies. Even then, the smooth equations are quite nonlinear, with Finite-amplitude magnetic solutions yet to be explored. Lectures intertwined with those of Childress exposed us to topics beside and outside his emphasis on a convective geodynamo.

The fellows of the summer program were responsible for the notes of the principal lectures and checking their content with Childress. Extended abstracts of addresses by program staff members and the ten participants in the July mini-symposium on magnetohydrodynamics were prepared by the speakers. The eleven lectures of the Fellows are recorded in the second of this two-volume report.

Mary C. Thayer has gathered and typed all the abstracts, lecture reports and fellowship papers -- for a twentieth year! Fellows and staff salute her skill and patience with an often recalcitrant crew.

We thank particularly Dr. Ralph Cooper, and through him the Office of Naval Research and the National Aeronautics and Space Administration for continuing support and encouragement.

Willem V. R. Malkus



Dr. Stephen Childress, Principal Lecturer.

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COURSE LECTURES ON GEOMAGNETIC DYNAMO THEORY

by

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Stephen Childress
Courant Institute of Mathematical Sciences
New York University

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COURSE LECTURES

by

Stephen Childress
New York University
Courant Institute of Mathematical Sciences

Lecture #1. INTRODUCTION TO GEOMAGNETIC DYNAMO THEORY

1.1 Historical Introduction

The problem of explaining the origin of the magnetic fields of the earth and sun is a difficult one, and only in recent years has substantial progress been made towards its solution. The following is a list of some of the decisive contributions made by scientists and mathematicians who have tackled the problem.

1919 Larmor asked how a rotating body, such as the sun, could become a magnet. One of the suggestions he put forward was that the magnetic field was maintained by the motion of the electrically conducting fluid of which the sun is composed.

1934 Cowling found that a steady axi-symmetric magnetic field could not be maintained by fluid with an axi-symmetric velocity field (with the same axis of symmetry). This was a 'great step backward' for dynamo theory, since it is very natural to look for axi-symmetric fields when dealing with a rotating body.

1946 Elsasser studied non-axi-symmetric magnetic and velocity fields.

1954 Bullard and Gellman pointed out the importance of differential rotation for generating a "toroidal" field from a "poloidal" one, and the importance of non-axi-symmetric motion's for distorting the toroidal field to produce "poloidal" field. (We shall define these terms presently.) However, the fluid motions they considered were not capable of indefinitely sustaining a magnetic field.

1955 Parker provided a physical argument to explain how irregular upwellings of fluid could produce a mean magnetic field when their inductive effect was averaged over space and time. This was a major break-through in dynamo theory.

1964 Braginskii considered nearly axi-symmetric systems with very high fluid conductivity; using a formal asymptotic procedure.

1966 Steenbeck, Krause and Radler considered turbulent dynamos, the length scale for the turbulent component being much shorter than that for the mean component. Since this paper. much work has been done on turbulent dynamos.

1.2 The Earth's Magnetic Field

The earth can be regarded as a sphere of radius 6.4×10^6m, of which a shell of inner radius 1.4×10^6m and outer radius 3.5×10^6m is composed of electrically conducting fluid, mainly molten iron. Inside the shell is a solid core, and outside the shell is the mantle, which can often be regarded as a solid insulator, although sometimes its visco-elastic deformations or small conductivity need to be taken into account. The basic facts of the earth's magnetic field which any theory must explain are:

(A) The field is permanent; that is, it has been in existence for the whole of the earth's history, thought to be about 10^9 years.

(B) There are large-scale changes of structure, namely reversals of polarity, on a time scale of order 10^5 years.

(C) There are small-scale variations on a time scale of the order of a hundred years.

Table 1 shows estimates of the main physical parameters of the earth relevant to the subject-matter of these lectures. Since the decay time, 10^5 years, for the earth's field in the absence of any fluid motions in the core, is much less than the age of the earth, 10^9 years, it is clear that we must look to fluid motions for the explanation of the persistence of the field.

The reversals of polarity have been statistically analysed by Cox (1968) who claimed that the probability that the time between successive reversals lies between t and $t + dt$ is

$$\frac{1}{t_o} \exp\left(-\frac{t}{t_o}\right) dt$$

where $t_o = 2 \times 10^5$ years. However, it is questionable to fit a particular type of probability distribution in the absence of a theory of the underlying mechanism. Further, as more data becomes available of the history of the earth's field, it may become necessary to put in more reversals, so that what is now believed to be a period of one particular polarity may subsequently need to be split into smaller periods of different polarity. Thus statistical formulae may need to be revised.

The most interesting feature of the small-scale variations is the westward drift of the non-dipole field. After performing a harmonic analysis of the earth's field it is easy to remove the dipole component, and a contour map can then be drawn showing, for example, lines of constant vertical component

Table 1. Physical Parameters of the Geodynamo

Symbol	Meaning	Units	Value
L	core radius	m	3.5×10^6
σ	electrical conductivity	$m^{-3}k^{-1}sq^2$ = mho/m	3×10^5
η	magnetic diffusivity	m^2s^{-1}	3
τ	diffusion time	s	$4 \times 10^{12} = 10^5$ yr
ρ	fluid density	km^{-3}	10^4
ν	kinematic viscosity	m^2s^{-1}	10^{-6}(?)
Ω	rotation viscisity	s^{-1}	7.4×10^{-5}
T	core temperature	^oK	4000
c_p	specific heat	$m^2s^{-2}\ {}^oK^{-1}$	670
λ	thermal conductivity	$mks^{-3}\ {}^oK^{-1}$	60
α	coef.of volume expansion	$^oK^{-1}$	5×10^{-6}
κ	thermal diffusivity	m^2s^{-1}	10^{-5}
g	accel. of gravity	ms^{-2}	5
β	mean temp. gradient	$^oK\ m^{-1}$	2×10^{-3}(?)
B	magnetic field	$ks^{-1}q^{-1}$	10^{-2} = 100 gauss
V_A	Alfvén speed	ms^{-1}	10^{-1}
v	speed	ms^{-1}	10^{-4}
Q	core heating rate	m^2ks^{-3}	10^{12}-10^{13}

1 gauss = $10^{-4}\ kq^{-1}s^{-1}$

1 joule = 1 m^2ks^{-2} = .239 calorie

1 volt = 1 $m^2ks^{-2}q^{-1}$

1 ohm = 1 $m^2K\ s^{-1}q^{-2}$ = 1 mho^{-1}

$\mu = 4\pi \times 10^{-7}\ mkq^{-2}$

of the non-dipole field. The main features of such a map drift westward at a typical rate of 0.2^o of longitude per year, although some features move faster than others. It is not known whether this is caused by wave motion in the core, or bulk motion, or both; and it is difficult to do an experiment to find out! It should be noted that the slight conductivity of the mantle places a lower limit on the time scale of magnetic effects observable at the earth's surface. For if η_m is the magnetic diffusivity of the mantle, and L its thickness, then any magnetic field at the bottom of the mantle varying on a time scale less than L^2/η_m will be greatly reduced in magnitude by diffusion. Using an estimate of η_m, this time scale is about 10 yrs.

1.3 The Basic Equations

The physical quantities needed in the analysis are:

$\underline{B}(x,t)$ = magnetic field,

$\underline{j}(x,t)$ = current density,

$\underline{E}(x,t)$ = electric field,

$\underline{u}(x,t)$ = fluid velocity

σ = electrical conductivity,

μ = magnetic permeability,

$\eta = (\sigma\mu)^{-1}$ = magnetic diffusivity.

The equations satisfied by these quantities are the 'pre-Maxwell' equations and Ohm's Law:

$$\operatorname{div} \underline{B} = 0 \qquad (1.1)$$

$$\operatorname{curl} \underline{B} = \mu \underline{j} \qquad (1.2)$$

$$\operatorname{curl} \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (1.3)$$

$$\underline{j} = \sigma(\underline{E} + \underline{u} \wedge \underline{B}) \qquad (1.4)$$

The pre-Maxwell equations (which neglect the displacement current term in (1.2) hold on the assumption that the time taken by light to traverse the region of interest is small compared with the time scale of the events being described. On taking the curl of (1.4) and eliminating $\underline{j}$ and $\underline{E}$ using (1.2) and (1.3) we obtain

$$\frac{\partial \underline{B}}{\partial t} = \operatorname{curl}(\underline{u} \wedge \underline{B}) + \eta \nabla^2 \underline{B}. \qquad (1.5)$$

This is the fundamental equation for $\underline{B}$. Equation (1.5), and the Navier-Stokes equation (with a forcing term of $\underline{j} \wedge \underline{B}$ per unit volume), form the foundation of theories of the evolution of magnetic fields in fluids.

To simplify the problem, we often regard $\underline{u}$ as being given, and use (1.5) to determine the evolution of $\underline{B}$; the equation is then **linear.** This is called the <u>kinematic</u> approach. The resulting problem is still difficult, however, and further simplifying assumptions need to be made. The most obvious way to proceed is to see when one or other of the terms on the right-hand side of (1.5) is negligible compared with the other. To this end, let

$\mathcal{U}$ = typical fluid velocity,

L = typical length scale for variation of $\underline{u}$ and $\underline{B}$

Then
$$\frac{|\mathrm{curl}\,(\underline{u}\wedge\underline{B})|}{|\eta\nabla^2\underline{B}|} \sim \frac{\mathcal{U}L}{\eta}$$

The quantity $\mathcal{U}L/\eta$ is called the magnetic Reynolds number, and is denoted by R_m. (In the earth's core, R_m is typically about 100). So from (1.5) we obtain

$$\frac{\partial\underline{B}}{\partial t} \simeq \mathrm{curl}\,(\underline{u}\wedge\underline{B}) \quad (R_m \gg 1) \tag{1.6}$$

$$\frac{\partial\underline{B}}{\partial t} \simeq \eta\nabla^2\underline{B} \qquad (R_m \ll 1) \tag{1.7}$$

Equation (1.6) is exact when $\eta = 0$ (perfect conductivity, i.e. $\sigma = \infty$), and (1.7) is exact when $\underline{u} = 0$. Equation (1.7) is just the diffusion equation, and when $R_m \ll 1$ often gives a good approximation everywhere to the true solution. When $R_m \gg 1$, however, (1.6) fails to be a good approximation in boundary layers, where diffusion is important, and in these regions a closer approximation to the full Eq. (1.5) must be used.

1.4 Exact Solution for a Perfect Conductor

The equation for $\underline{B}$ in a perfect conductor is
$$\frac{\partial\underline{B}}{\partial t} = \mathrm{curl}\,(\underline{u}\wedge\underline{B}). \tag{1.8}$$

Taking $\underline{u}$ as given, this can be solved exactly using Lagrangian coordinates. Let the position at time t of a fluid particle initially at $\underline{a}$ be $\underline{x}(\underline{a},t)$, so that $\underline{x}(\underline{a},o) = \underline{a}$, and let

$$D = \det\left(\frac{\partial x_i}{\partial a_j}\right),$$

$$\rho(\underline{x},t) = \text{fluid density.}$$

For any function $f(\underline{x},t)$ let the function $\hat{f}(\underline{a},t)$ be defined by the equation
$$\hat{f}(\underline{a},t) = f\,(\underline{x}(\underline{a},t),t),$$
and let $\frac{D}{Dt}$ denote differentiation with respect to time, keeping $\underline{a}$ fixed, so that
$$\frac{Df}{Dt} = \frac{\partial\hat{f}}{\partial t} = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\underline{\nabla}\right)f$$

Then conservation of mass implies that

$$D = \frac{\hat{\rho}(\underline{a},0)}{\rho(\underline{x},t)}$$

and

$$\frac{D\rho}{Dt} = -\rho \operatorname{div} \underline{u}.$$

Hence

$$\frac{D}{Dt}(D) = -\frac{\hat{\rho}(\underline{a},0)}{\rho^2}\frac{D\rho}{Dt} = (\operatorname{div} \underline{u}) D \tag{1.9}$$

(This can also be established directly from the expression for the determinant D.) Equation (1.9) enables us to show that the solution of (1.8) is

$$B_i(\underline{x},t) = \frac{1}{D}\frac{\partial x_i}{\partial a_j}\hat{B}_j(\underline{a},0). \tag{1.10}$$

For (1.9) and (1.10) imply that

$$\frac{D}{Dt}(DB_i) = \frac{\partial u_i}{\partial a_j}\hat{B}_j(\underline{a},0).$$

and

$$\frac{D}{Dt}(DB_i) = (\operatorname{div} \underline{u}) DB_i + D\frac{DB_i}{Dt}$$

Hence

$$\begin{aligned}\frac{DB_i}{Dt} &= \frac{1}{D}\frac{\partial u_i}{\partial a_j}\hat{B}_j(\underline{a},0) - (\operatorname{div}\underline{u}) B_i \\ &= \frac{\partial u_i}{\partial a_j}\frac{\partial a_j}{\partial x_k} B_k - (\operatorname{div}\underline{u}) B_i \; [\text{by } (1.10)] \\ &= \frac{\partial u_i}{\partial x_k} B_k - (\operatorname{div}\underline{u}) B_i \;,\end{aligned}$$

which is the same equation as (1.8). This proves that (1.10) solves (1.8). The meaning of (1.10) can be shown by writing it in the form

$$\frac{B_i(\underline{x},t)}{\rho(\underline{x},t)} = \frac{\partial x_i}{\partial a_j}\frac{\hat{B}_j(\underline{a},0)}{\hat{\rho}(\underline{a},0)},$$

and comparing it with

$$dx_i = \frac{\partial x_i}{\partial a_j} da_j$$

it follows that $\underline{B}/\rho$ is transformed like a material element, and that the field acts as if it is 'frozen' in the fluid.

1.5 Dynamos

The fundamental question of kinematic dynamo theory is this: given a volume V of electrically conducting fluid, which velocity fields $\underline{u}$ are such that when $\underline{B}$ evolves according to (1.5) it does not ultimately decay to zero? This question will be considered in some detail in Lecture 2. For the time being, define a dynamo as a system eomprising V, $\underline{B}$, $\underline{u}$, η such that the magnetic energy

$$\int_{\text{all space}} \tfrac{1}{2}\mu^{-1}\underline{B}^2 dV$$

is bounded away from zero for all time. (Of course other measures of magnetic field strength might be used.) If V is not the whole of space then the conditions outside V must be specified. It is common to take V as a sphere and the volume outside V as an insulator; this is a good model for the earth's dynamo. Roughly speaking, for a system to act as a dynamo, the convection term curl $(\underline{u}\wedge\underline{B})$ in (1.5) must be such as to compensate for the dissipation term $\eta\nabla^2\underline{B}$.

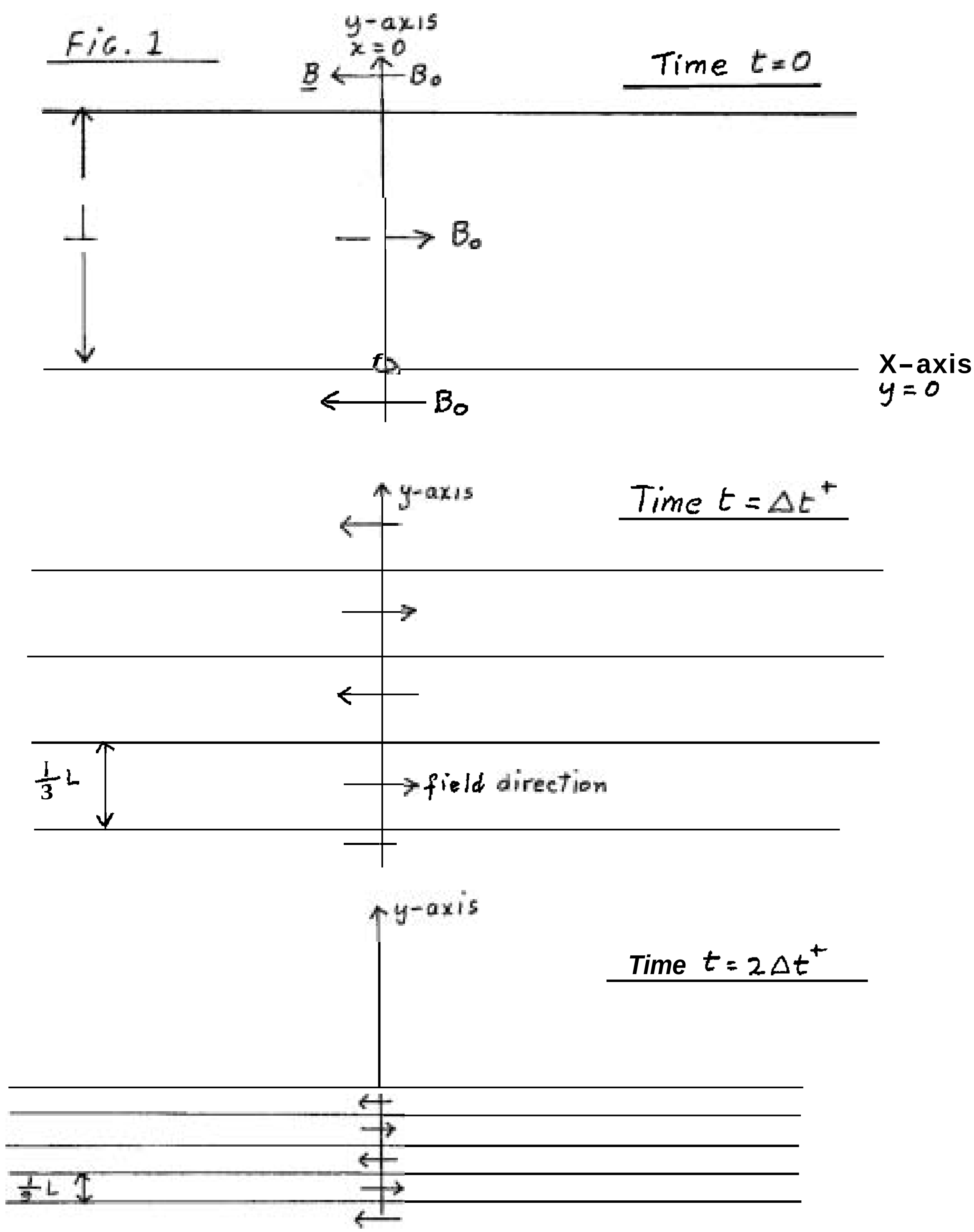
1.6 A Non-dynamo

Consider the following means of trying to make a dynamo. At t = 0, a magnetic field is present in a stationary conductor. We wait a time Δt, during which time $\underline{u} = 0$ and diffusion operates accoeding to (1.6). At time $t = \Delta t$ we instantaneously stretch the conductor and fold it over so that it occupies the same region of space as before. During the folding, diffusion has no time to act and so the field satisfied (1.8) (note that $|\underline{u}| = \infty$ during the stretching and folding, so that $R_m = \infty$). Assume that the conductor is imcompressible; then the field can be intensified by stretching in a direction parallel to $\underline{B}$, because of the term $\partial x_i/\partial a_j$ in the solution (1.10) of (1.8). We now repeat the process by waiting from $t = \Delta t$ to $2\Delta t$ and then folding again; this is continued indefinitely. Thus we alternate between field intensification and decay. By choosing suitable deformation, the net result might be thought to be a relentless increase in $\underline{B}$ (i.e. $|\underline{B}| \to \infty$ as $t \to \infty$), thus giving a dynamo. However, we do not always obtain a dynamo, and an example of this is now given.

Let the whole of space be filled by an incompressible substance of magnetic diffusivity %, and at time $t = 0$ let the magnetic field $B(x,t)$ be

$$\underline{B}(x,y,z,0) = \begin{cases} (B_0, 0, 0) & (2mL \le y < (2m+1)L,\ m = 0, \pm 1, \pm 2, \dots) \\ (-B_0, 0, 0) & ((2m+1)L \le y < 2mL,\ m = 0, \pm 1, \pm 2, \dots) \end{cases}$$

as shown in Figure 1. At time Δt each band of width L is stretched to three times its original length in the x-direction and folded over on itself to produce the configuration shown in the second part of Fig.1. To achieve this, we can conceive of each band being chopped into sections of length $L_1 \gg L$ and then folded into W-shapes. The end regions of these sections, where the field is not as shown in the diagram, will have a negligible effect. At time $t = 2\Delta t$, a similar procedure is



The stretching and folding procedure. (See Section 1.6.)

used to give the configuration shown in the third part of **Fig.1.**

The stretching intensifies the field by a factor of 3 (this is a consequence of (1.10), and so we have

$$\underline{B}(x,y,z,n\Delta t^{+}) = 3\underline{B}(x,3y,z,n\Delta t^{-}),$$

where $n = 1, 2, 3, \ldots$ It is clear that $\underline{B}$ is always in the x-direction and depends only on y and t; so write

$$\underline{B}(\underline{x},t) = (B(y,t), 0, 0)$$

We wish to find $B(y,t)$ explicitly. Hence the following problem must be solved:

$$\frac{\partial B}{\partial t} = \eta_0 \frac{\partial^2 B}{\partial y^2} \qquad \begin{pmatrix} t > 0 \\ t \neq \Delta t, 2\Delta t, \ldots \\ -\infty < y < \infty \end{pmatrix}$$

$$B(y,n\Delta t^{+}) = 3B(3y,n\Delta t^{-}) \quad (n = 1,2,\cdots)$$

$$B(y,0) = \begin{cases} B_0 & (2mL \le y < (2m+1)L) \\ -B_0 & ((2m+1)L \le y < 2mL) \end{cases}$$

The simplest way to solve this is to change the space variable by a factor of 3 at times $t = \Delta t, 2\Delta t, \ldots$ So define a function $f(y,t)$ satisfying:

$$\frac{\partial f}{\partial t} = \eta(t)\frac{\partial^2 f}{\partial y^2} \qquad \begin{pmatrix} t > 0 \\ -\infty < y < \infty \end{pmatrix}$$

$$f(y,0) = \begin{cases} 1 & (2mL \le y < (2m+1)L \\ -1 & ((2m+1)L \le y < 2mL) \end{cases}$$

(The fact that $\partial^2 f/\partial y^2$ does not exist at $t = 0$ does not matter; it will exist for all $t > 0$.) The function $\eta(t)$ is defined by

$$\eta(t) = \begin{cases} \eta_0 & (0 \le t < \Delta t) \\ 9\eta_0 & (\Delta t \le t < 2\Delta t) \\ 9^2\eta_0 & (2\Delta t \le t \le 3\Delta t) \\ etc. & \end{cases}$$

Then $B(y,t)$ is given by

$$B(y,t) = 3^{N(t)} B_0 f(3^{N(t)} y, t)$$

where $N(t)$ is defined as the integer n satisfying

$$n\Delta t \le t < (n+1)\Delta t.$$

It only remains to find $f(y,t)$. By putting

$$f(y,t)=\sum_{n=0}^{\infty} A_n(t)\sin(2n+1)\pi\frac{y}{L},$$

It is easily seen that

$$f(y,t)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\exp\left\{-\frac{(2n+1)^2\pi^2}{L^2}\int^{t}\eta(\tau)d\tau\right\}\sin(2n+1)\pi\frac{y}{L},$$

where

$$\eta(t)=9^{N(t)}\eta_0.$$

We can now show that $B(y,t)\longrightarrow 0$ as $t\longrightarrow\infty$, uniformly in y. For

$$\left|f(y,t)\right|\le\exp\left\{-\frac{\pi^2}{L^2}\int_0^t\eta(\tau)d\tau\right\}\left|\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)\frac{\pi y}{L}\right|$$

$$=\exp\left\{\frac{\pi^2}{L^2}\int_0^t\eta(\tau)d\tau\right\}$$

Now when $t>2\Delta t$, we have

$$\int_0^t\eta(\tau)d\tau>\int_{\{N(t)-1\}\Delta t}^{N(t)\Delta t}\eta(\tau)d\tau=9^{N(t)-1}\eta_0\Delta t,$$

and so

$$\left|f(y,t)\right|<\exp\left\{-\frac{\pi^2}{9L^2}\cdot{}^{N(t)}\eta_0\Delta t\right\}$$

Hence

$$\left|B(y,t)\right|<3^{N(t)}\exp\left\{-\frac{\pi^2}{9L^2}9^{N(t)}\eta_0\Delta t\right\}$$

and so the magnetic energy density tends to 0 as $t\longrightarrow\infty$. Thus the stretching procedure described above is not capable of sustaining a magnetic field. The physical explanation of this is that the stretching increases the field by a factor $3^{N(t)}$, but the diffusion decreases it at least by a factor of

$$\exp\left\{-\frac{\pi}{9L^2}9^{N(t)}\eta_0\Delta t\right\};$$

clearly the latter term is dominant except possibly during an initial period, This is in fact a feature of all two-dimensional systems where there are no z-components of any of the fields, and everything is independent of

An alternative procedure is to adjust the times between foldings so that the energy does not decay. But it then turns out that the time intervals required become shorter and shorter so rapidly that the sum tends to a limit as the number of them tends to infinity. So this method does not work either.

It is of interest to contrast this situation with what might be achieved by three-dimentional deformations. Alfven has suggested the process shown in

Fig.2, which, at each step, doubles the field intensity at the expense of a small "x-point" where diffusion can be expected to be important. Nevertheless, diffusion penalty here would appear to be quite small. Unfortunately there is no simple way to compute the process and establish that a dynamo effect occurs when $\eta \neq 0$.

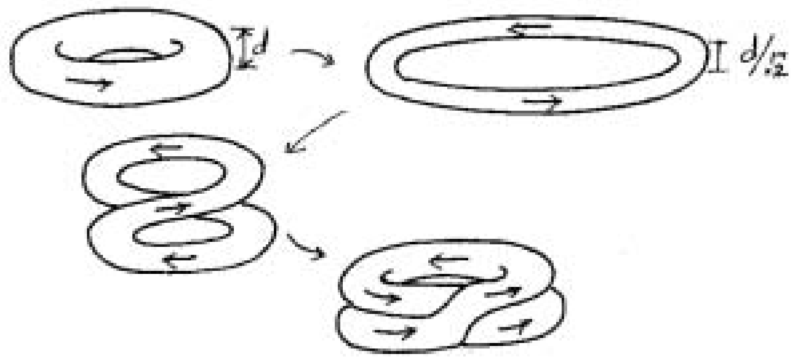


Fig.2

Notes submitted by
John Chapman and
Frank Condi.

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Lecture #2. SOME NECESSARY CONDITIONS FOR DYNAMO ACTION

(2.1) The Rate of Change of Magnetic Energy

In this lecture we shall consider a finite volume V of electrically conducting fluid, of constant, uniform magnetic diffusivity η, surrounded by a motionless insulator occupying the rest of space, $\hat{V}$. Then as shown in Lecture #1, the equations for the magnetic field are

$$\frac{\partial \underline{B}}{\partial t} = \text{curl}(\underline{u} \wedge \underline{B}) + \eta \nabla^2 \underline{B}, \nabla \cdot \underline{B} = 0 \quad (\text{in } V) \tag{2.1}$$

$$\text{curl } \underline{B} = 0, \nabla \cdot \underline{B} = 0 \quad (\text{in } \hat{V}), \tag{2.2}$$

where $\underline{u}$ is the velocity of the fluid. The boundary conditions on S, the surface of V, are

$$[\underline{n} \cdot \underline{B}] = 0, [\underline{n} \times \underline{B}] = 0 \tag{2.3}$$

$$[\underline{n} \wedge \underline{E}] = 0 \tag{2.4}$$

where [ ] denotes the jump across S. The first equation (2.3) follows from div $\underline{B}$ = 0, and the second follows from curl $\underline{B} = \mu \cdot \underline{j}$, since there are no surface currents and hence $\underline{j}$ is everywhere finite. Equation (2.4) follows from (1.3) and boundedness of $\underline{B}$. Equations (2.3) and (2.4) say merely that $\underline{B}$ and tangential $\underline{E}$ are continuous across S. We also assume that there are no 'sources at infinity', so that

$$|\underline{B}| = O(r^{-3}) \text{ as } r \longrightarrow \infty \tag{2.5}$$

where $r = |\underline{x}|$. In the kinematic theory, we take $\underline{u}$ as being given, and so (2.1) = (2.5), together with the initial value of $\underline{B}$ everywhere, provide a complete specification of the problem of determining the evolution of $\underline{B}$.

The magnetic energy is

$$E_m(t) = \int_{V+\hat{V}} \tfrac{1}{2} \mu^{-1} \underline{B}^2 \, dV,$$

and we now derive two expressions for dE_m/dt, using the equations above plus the equation (1.3),

$$\text{curl } \underline{E} = -\partial \underline{B}/\partial t.$$

We also use the fact that

$$|\underline{E}| = O(r^{-2}) \text{ as } r \longrightarrow \infty, \tag{2.6}$$

(since the charge is confined to the finite region V), and we use Ohm's Law (1.4) in the form

$$\eta \text{ curl } \underline{B} = \underline{E} + \underline{u} \wedge \underline{B}. \tag{2.7}$$

Thus
$$\mu \frac{dE_m}{dt} = \int_{V+\hat{V}} \underline{B} \cdot \frac{\partial \underline{B}}{\partial t}\, dV$$
$$= -\int_{V+\hat{V}} \underline{B} \cdot \mathrm{curl}\, \underline{E}\, dV$$
$$= -\int_{V+\hat{V}} \left\{ \mathrm{div}\left(\underline{E} \wedge \underline{B}\right) + \underline{E}\, \mathrm{curl}\, \underline{B} \right\} dV$$

The first term vanishes (using the divergence theorem and (2.5), (2.6) and the second term is zero in $\hat{V}$. So by (2.7) we obtain

$$\frac{dE_m}{dt} = \frac{1}{\mu}\int_V \underline{u} \cdot \left(\underline{B} \cdot \mathrm{curl}\, \underline{B}\right) dV - \frac{\eta}{\mu}\int_V (\mathrm{curl}\, \underline{B})^2\, dV \qquad (2.8)$$

This is our first expression for dE_m/dt. Using the fact that

$$\int_V \underline{u} \cdot \left(\underline{B} \wedge \mathrm{curl}\,(\underline{B})\right) dV = \int_V \underline{u} \cdot \left\{ \mathrm{grad}\left(\tfrac{1}{2}\, \underline{B}^2\right) - \underline{B} \cdot \nabla\, \underline{B} \right\} dV$$
$$= \int_V \left\{ u_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} B_k B_k\right) - u_j\, B_k \frac{\partial B_j}{\partial x_k} \right\} dV$$

we can obtain a second expression provided that we assume more, namely that $\mathrm{div}\, \underline{u} = 0$ (incompressible flow) and $\underline{u}$ on S. Then the first term above is

$$\int_V \frac{\partial}{\partial x_j}\left(u_j \tfrac{1}{2} B_k B_k\right) dV = 0,$$

by the divergence theorem, and the second term is

$$-\int_V u_j \frac{\partial}{\partial x_k}\left(B_k B_j\right) dV = -\int_V \left\{ \frac{\partial}{\partial x_k}\left(u_j B_k B_j\right) - \frac{\partial u_j}{\partial x_k} B_k B_j \right\} dV$$

The first term here is zero (by the divergence theorem again), and we are finally left with

$$\int_V \underline{u} \cdot \left(\underline{B} \wedge \mathrm{curl}\, \underline{B}\right) dV = \int_V B_i B_j e_{ij}\, dV,$$

where

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial x_j}{\partial u_i} \right).$$

Hence from (2.8) we obtain a second expression,

$$\frac{dE_m}{dt} = \frac{1}{\mu}\int_V B_i B_j e_{ij}\, dV - \frac{\eta}{\mu}\int_V (\mathrm{curl}\, B)^2\, dV. \qquad (2.9)$$

The basic difference between (2.8) and (2.9) is that $\underline{u}$ enters (2.9) only through its spatial derivatives, while it enters (2.8) only through its undifferentiated values.

In both (2.8) and (2.9) the first term on the right-hand side represents rate of creation of magnetic energy by the agency providing the fluid motion, and the second term represents rate of destruction of magnetic energy by ohmic decay.

(2.2) Two Necessary Conditions for Dynamo Action

We say that a velocity field $\underline{u}$ in the volume V defined above capable of dynamo action if for some initial magnetic field there exists a constant E_m^0, strictly greater than zero, such that

$$E_m(t) > E_m^0 \quad \text{for all} \quad t \geq 0.$$

Hence if it is possible to find a function f such that

$$\frac{dE_m}{dt}(t) \leq f$$

then a necessary condition for dynamo action is that f is not always negative (when $\underline{B}$ is not everywhere zero). It is shown below that Eqs. (2.8) and (2.9) provide us with two such f's. We need to use the length $\mathcal{L}$ defined by

$$\frac{1}{\mathcal{L}^2} = \min_{\underline{B} \in \mathbb{B}} \left\{ \frac{\int_V (\text{curl } \underline{B})^2 dV}{\int_{V+\hat{V}} \underline{B}^2 dV} \right\}$$

where $\mathbb{B}$ is the class of admissible functions over which the minimum is found, taken to be the set of all solenoidal fields continuously differentiable in V, irrotational in $\hat{V}$, continuous across S, and $O(r^{-3})$ at infinity. (It can be shown that if V is sphere of radius L, then $\mathcal{L} = L/\pi$.) Let u_m be the maximum value of $|\underline{u}|$, and let λ_m be the maximum eigenvalue of the tensor (e_{ij}). In each case the maximum is taken over all time and all $\underline{x}$ in V. It can easily be shown that if $\underline{u}$ is not everywhere and always zero, then $\lambda_m > 0$. Using the Cauchy-Schwarz inequality and the definition of $\mathcal{L}$ we obtain

$$\begin{aligned}
\left| \int_V \underline{u} \cdot (\underline{B} \wedge \text{curl } \underline{B}) dV \right| &\leq u_m \int_V |\underline{B}| \, |\text{curl } \underline{B}| \, dV \\
&\leq u_m \left[\left\{ \int |\underline{B}|^2 dV \right\} \left\{ \int_V |\text{curl } \underline{B}|^2 dV \right\} \right]^{1/2} \\
&\leq u_m \mathcal{L} \int |\text{curl } \underline{B}|^2 dV,
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_V B_i B_j e_{ij} dV \right| &\leq \lambda_m \int_V |\underline{B}|^2 dV \leq \lambda_m \int_{V+\hat{V}} |\underline{B}|^2 dV \\
&\leq \lambda_m \mathcal{L}^2 \int_V |\text{curl } \underline{B}|^2 dV
\end{aligned}$$

Hence (2.8) and (2.9) give the inequalities

$$\left.\begin{aligned}
\frac{dE_m}{dt} &\leq \frac{\eta}{\mu} \left(-1 + \frac{u_m \mathcal{L}}{\eta}\right) \int_V |\text{curl } \underline{B}|^2 dV \\
\frac{dE_m}{dt} &\leq \frac{\eta}{\mu} \left(-1 + \frac{\lambda_m \mathcal{L}^2}{\eta}\right) \int_V |\text{curl } \underline{B}|^2 dV
\end{aligned}\right\} \qquad (2.10)$$

Therefore two very simple necessary fonditions for dynamo action are

$$\frac{u_m \mathcal{L}}{\eta} > 1 , \qquad \frac{\lambda_m \mathcal{L}^2}{\eta} > 1$$

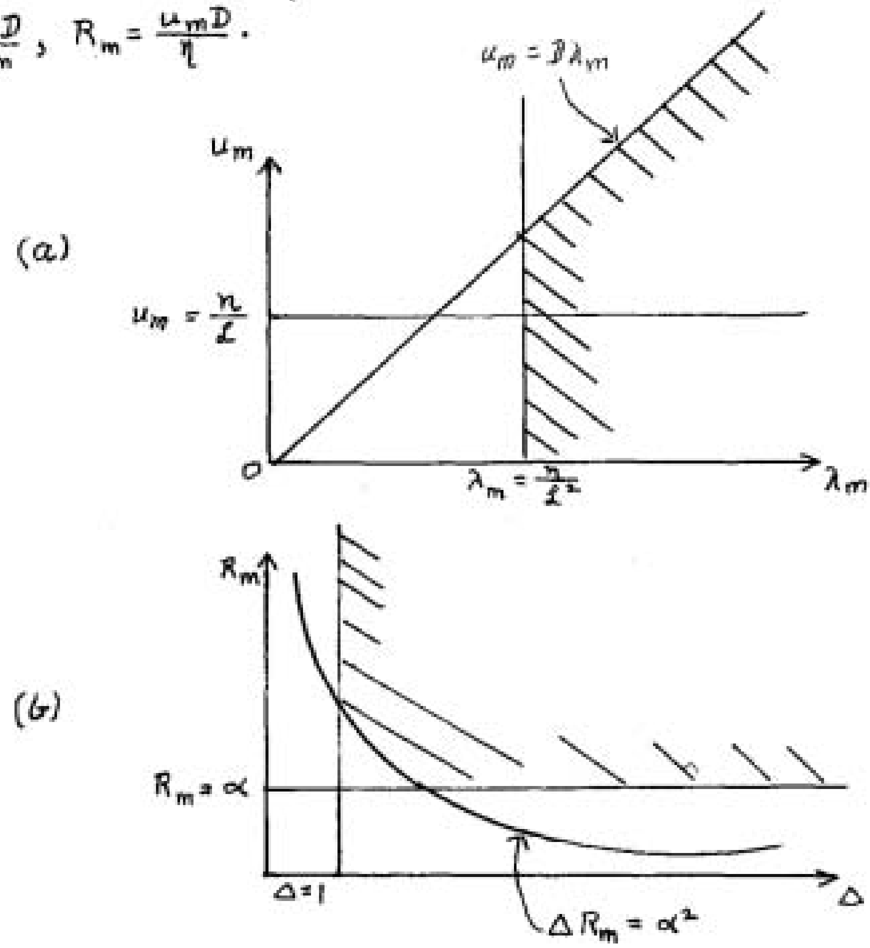
Note that $u_m \mathcal{L}/\eta$ is the magnetic Reynolds number based on the length scale $\mathcal{L}$. If V is convex the maximum distance apart of any two points in V is denoted by D, then by the vector mean-value theorem

$$u_m \text{ a } D\lambda_m .$$

If we regard $\mathcal{L}$, D , and η as given, then (2.11) and (2.12) imply that any velocity field permitting dynamo action must be such that u_m and λ_m lie within the shaded region shown in Fig. 1(a). We can express this result in terms of the dimensionless variables defined by

$$\alpha = \frac{D}{L} , \quad \Delta = \frac{\lambda_m D}{u_m} , \quad R_m = \frac{u_m D}{\eta} .$$

Fig. 1



A necessary condition for dynamo action is that the parameters shown lie in the shaded areas. (See section 2.2 for the definitions of these parameters.)

Then (2.1 1) and (2.12) become

$$\Delta R_m > \alpha^2 , \quad \Delta \geq 1 , \quad R_m > \alpha ,$$

and so regarding α as given, we see that Δ and R_m must lie in the shaded region shown in Fig.1(b). It should be emphasized that these conditions are not sufficient for dynamo action.

It is very plausible that an inequality such as the second part of (2.11) should be a necessary condition, since it was shown in Lecture #1 that shear can greatly intensify $\underline{B}$ by stretching the field lines. The first part of (2.11) is not so obvious, however, because this inequality depends only on the magnitude of $\underline{u}$, not its derivatives. Note however that we have not had to select a special coordinate system, so u_m could have been chosen relative to any convenient coordinate frame. In particular, in a spherical domain we deduce that core motions must differ from solid-body rotation by an amount u_m consistent with these inequalities.

It is very probable that these estimates have counterparts in electrical circuit theory. The inequality involving λ_m was first derived by Backus (1958) while that involving u was noted by Childress (1969). There is considerable room for improved estimates of dynamo action. Recemt;u Proctor (1978) has observed that Backus' estimate can be improved by 20% if the integral of $|\underline{B}|^2$ over V is retained in the estimate of $\mathcal{L}$ instead of extending the integral to V + $\hat{V}$). Such refinements complicate the variational problem which must be solved to complete the estimate, but presumably move us closer to realistic'estimates for the dynamo process.

It is also possible to sharpen necessary conditions by elaborating the structure of admissible magnetic fields. We consider next a condition of this kind.

(2.3) A Third Necessary Condition for Dynamo Action

Busse (1975) has obtained another necessary condition for dynamo action by splitting $\underline{B}$ up into its toroidal and poloidal parts:

$$\underline{B} = \mathrm{curl}\,(T\underline{x}) + \mathrm{curl}\ \mathrm{curl}\ (P\underline{x})$$

This decomposition is always possible for a solenoidal field; P and T are scalar functions of position, to which an arbitrary function of r can be added without altering $\underline{B}$. Busse showed that

$$\frac{d}{dt}\int_V \frac{1}{2}\left(\underline{B}\cdot\underline{x}\right)^2 dV \le \left\{-\eta + \max_V\,(\underline{u}\cdot\underline{x})\left(\frac{E_m}{2E_p}\right)^{1/2}\right\}\int_{V+\hat{V}}\left\{\mathrm{grad}\left(\underline{B}\cdot\underline{x}\right)\right\}^2 dV,$$

where E_m is again the total magnetic energy, E_p is magnetic energy in the poloidal part of $\underline{B}$, and V is a sphere. It is assumed that div $\underline{u} = 0$ and $\underline{u} = 0$ on S, the surface of the sphere. Thus we obtain a third necessary condition for dynamo action:

$$\max_V\,(\underline{u}\cdot\underline{x}) > \left(\frac{2E_p}{E_m}\right)^{1/2}\eta$$

This condition is of rather a different type from those derived in (2.2), since E_p/E_m depends on $\underline{B}$. As an example of its use, we can deduce, from the fact that the magnetic field in the earth has a poloidal component, that there are radial fluid motions there. This is relevant when considering convection.

(2.4) A One-dimensional Analog

It would be of interest to solve Eq.(2.1) for $\underline{B}$, given some particular class of velocity fields $\underline{u}$, and then relate the necessary conditions derived to the occasions when dynamo action actually occurs. Unfortunately this is difficult to do, because the equation is so hard to solve; indeed, this is why it is worthwhile to derive the necessary conditions in the first place. So we look for a simpler equation, which we hope retains the important features of (2.1) and which we can solve exactly. The corresponding necessary conditions for dynamo action can be derived, and then compared with the exact solutions of the simple equation.

As such an equation take u and B to be complex-valued functions of x and t satisfying

$$\frac{\partial B}{\partial t} = \frac{1}{R}\frac{\partial^2 B}{\partial x^2} + i\frac{\partial}{\partial x}(uB^*) \quad (-\infty < x < \infty, t \geq 0) \qquad (2.14)$$

(it is something of an act of faith that the solutions of this equation behave, in some sense, like the solutions of Eq.(2.1) !). Note that (2.14) has been made nondimensional; assume that this has been done by measuring x in units of L, and u in units of the maximum velocity, u_m ; then t is measured in units of L/u_m. The quantity R in (2.14) is then the magnetic Reynolds number $u_m L/\eta$.

The first step is to derive an inequality corresponding to the first part of (2.10). To do this multiply (2.14) by B^* and integrate with respect to x from a to b, say, to obtain

$$\int_a^b \frac{\partial B}{\partial t} B^* dx = \frac{1}{R}\left\{\left[\frac{\partial B}{\partial x}B^*\right]_a^b - \int_a^b \frac{\partial B}{\partial x}\frac{\partial B^*}{\partial x}dx\right.$$
$$\left. + \left\{\left[uB^{*2}\right]_a^b - \int_a^b uB^*\frac{\partial B^*}{\partial x}dx\right\}\right. \qquad (2.15)$$

Assume that a and b can be chosen so that the integrated parts vanish; this will be possible if, for example, u and B are space-periodic. Taking the complex conjugate of (2.15) and adding to (2.15) gives

$$\frac{d}{dt}\int_a^b |B|^2 dx = -i\int_a^b \left(uB^*\frac{\partial B^*}{\partial x} + u^* B\frac{\partial B}{\partial x}\right)dx - \frac{2}{R}\int_a^b \left|\frac{\partial B}{\partial x}\right|^2 dx \qquad (2.16)$$

Define $\mathcal{L}$ by the equation

$$\frac{L^2}{\mathcal{L}^2} = \min_{B \in \mathbb{B}} \left\{ \frac{\int_a^b \left|\frac{\partial B}{\partial x}\right|^2 dx}{\int_a^b |B|^2 dx} \right\} \tag{2.17}$$

where $\mathbb{B}$ is some suitable class of functions. Now $u \leq 1$, because u is measured in units of u_m, the maximum value of $|u|$, and so using the Cauchy-Schwarz inequality and the definition of $\mathcal{L}$, (2.16) gives,

$$\begin{aligned} \frac{d}{dt}\int_a^b |B|^2 dx &\leq 2\left\{\int_a^b |B|^2 dx \int_a^b \left|\frac{\partial R}{\partial x}\right|^2 dx\right\}^{1/2} - \frac{2}{R}\int_a^b \left|\frac{\partial B}{\partial x}\right|^2 dx \\ &\leq 2\left(\frac{\mathcal{L}}{L} - \frac{1}{R}\right)\int_a^b \left|\frac{\partial B}{\partial x}\right|^2 dx \end{aligned} \tag{2.18}$$

This is the analog of the first part of (2.10). It implies that a necessary condition for dynamo action is $R > L/\mathcal{L}$, that is

$$\frac{u_m \mathcal{L}}{\eta} > 1. \tag{2.19}$$

We now solve (2.14) exactly for the particular velocity field

$$u(x,t) = e^{ix + i\omega t}, \tag{2.20}$$

where ω is a constant. Trying the solution

$$B(x,t) = e^{inx+\sigma t} + A e^{i\bar{n}x + \bar{\sigma}t}, \tag{2.21}$$

where n, σ, $\bar{n}$, and A are complex constants, as yet unknown, we obtain

$$i\frac{\partial}{\partial x}(uB^*) = (n^*-1)e^{-i(n^*-1)x + (\sigma^* + i\omega)t}$$

$$+ A^*(\bar{n}^*-1)e^{-i(\bar{n}^*-1)x + (\bar{\sigma}^* + i\omega)t}$$

and

$$\frac{\partial B}{\partial t} - \frac{1}{R}\frac{\partial^2 B}{\partial x^2} = \left(\sigma + \frac{n^2}{R}\right)e^{inx+\sigma t}$$

$$+ A\left(\bar{\sigma} + \frac{\bar{n}^2}{R}\right)e^{i\bar{n}x + \bar{\sigma}t}$$

Therefore (2.21) solves (2.14) if

$$\begin{aligned} \bar{n} &= -(n^* - 1), \\ \bar{\sigma} &= \sigma^* + i\omega, \\ A\left(\bar{\sigma} + \frac{\bar{n}^2}{R}\right) &= n^* - 1 \\ A(\bar{n} - 1) &= \sigma^* + \frac{n^{*2}}{R} \end{aligned}$$

If it is assumed that B is bounded when $t = 0$, then n must be real; so assume this. Hence a solution of (2.14) is

$$B(x,t) = e^{inx+\sigma t} - \left(\frac{\sigma^*}{n} + \frac{n}{R}\right) e^{-i(n-1)x + (\sigma^* + i\omega)t} \tag{2.22}$$

where

$$n(1-n) = \left(\sigma^* + \frac{n^2}{R}\right)\left\{\sigma^* + i\omega + \frac{(n-1)}{R}\right\} \tag{2.23}$$

We can regard (2.22) as a family of solutions labelled by the single parameter n, since for given n Eq.(2.23) determines just two available values of σ

We are interested in whether the solution (2.22) decreases to zero or grows without limit as $t \longrightarrow \infty$. This is determined solely by the sign of the real part of σ; for a given n, the system acts as a dynamo if and only if the larger real part of the two possible values of σ is positive.

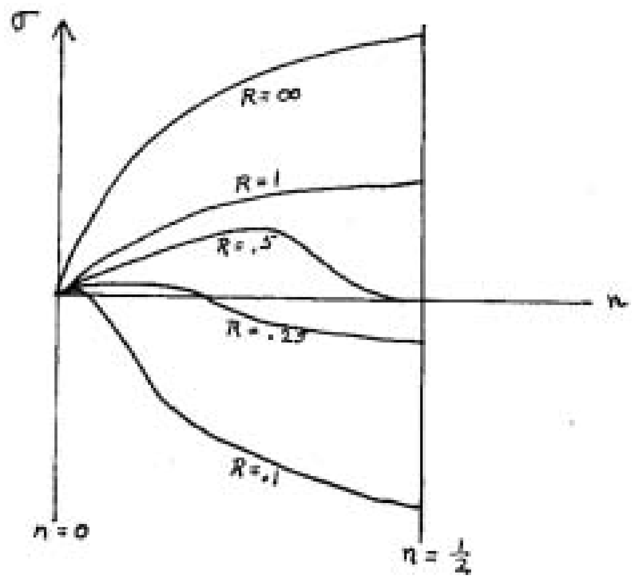
We now deal with the case $\omega = 0$ in more detail. Equation (2.23) when gives

$$\sigma = \frac{1}{2}\left[\frac{N-1}{R} + \sqrt{\left\{\frac{1}{R^2} + 2N\left(1 - \frac{1}{R^2}\right)\right\}}\right] \tag{2.24}$$

where $N = 2n(1-n)$.

From (2.22) it can be seen that the appropriate length scale $\mathcal{L}$ for the solution B is the larger of $1/n$ and $1/(n-1)$ (times the unit L). For simplicity, consider $0 \le n \le \frac{1}{2}$; then we have $n = L/\mathcal{L}$. Figure 2 shows a graph of σ as function of n for $0 \le n \le \frac{1}{2}$, for different values of R.

Fig.2



Graoh of 6 against n for $0 \le n \le \frac{1}{2}$ when $\omega = 0$.
(See section 2.4 for the explanation of these quantities.)

Putting $\sigma = 0$ in (2.24) gives $R^2/n^2 = 1/n - 1$; so the necessary and sufficient condition for dynamo action is

This can be written as

$$\frac{R}{n^2} > \frac{1}{n} - 1.$$

$$\left(\frac{u_m \mathcal{L}}{\eta}\right)^2 > \frac{1}{n} - 1.$$

from the definitions of R and $\mathcal{L}$.

Equation (2.27) is what we are looking for. Our aim is to compare it with the crudely derived necessary condition given in (2.19). Since we are only considering $0 \leq$ k-~$< 1/2$, we see that they are consistent. Further, (2.19) is the best possible condition of its type, because a condition of the form $u_m \mathcal{L}/\eta > 1 + \epsilon$ where $\epsilon > 0$, is violated by taking n close enough to 1/2. In this sense, (2.19) is a 'good' result about the solutions of the simple Equation (2.1)

Notes submitted by
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Lecture #3 CONSTRUCTIONS BASED ON SMOOTHING

3.1 Introduction

In this lecture we shall discuss an asymptotic method for treating the kinematic dynamo problem. In general there are at least three methods which have been used.

(i) Filtering method

(a) Temporal filtering is based on the fact that the fluctuating magnetic field decays much faster than the mean one. We are interested primarilly in the mean field. The velocity field is turned on and off periodically. While the velocity is on, the magnetic field is created. While it is off, the magnetic field decays. Hence the turn-off time should be long enough to allow the unwanted fluctuating field to decay. For example, Tverskoy (1965) applied this method to prove dynamo action by a toroidal eddy in a solid conductor.

(b) Geometric filtering is based on the similar idea that certain harmonic decay spatially faster than the others. Hence the harmonics will affect each other selectively according to the distance between sources (see for example Herzenberg's two-sphere dynamo, 1957).

(ii) Symmetry breaking

Cowling's theorem (1933) does not allow steady dynamos when both fields are axially symmetric. This has been studied by Braginsky (1964, 1965). In this method the variables are divided into axially summetric and small asymmetric parts.

(iii) Smoothing

This method is the main topic of this lecture. Here the variables are assumed to consist of a spatially rapidly varying part (small length scales $\approx \ell$) and of a spatially slowly varying part (large length scales $\approx \mathcal{L}$). An averaging process (e.g. over an intermediate scale) is essential to separate the mean field from the fluctuating field. This idea has been initiated by Parker (1955) and been explored since then by Steenbeck & Krause (1966), Childress (1967), G. O. Roberts (1970), Moffatt (1970), and many others.

3.2 First-order smoothing

We start with the induction equation:

$$\frac{\partial \underset{\sim}{B}}{\partial t} - \eta \nabla^2 \underset{\sim}{B} - \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) = 0$$

where $\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1$ and $\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{B}_1$, $\underset{\sim}{u}_0$ and $\underset{\sim}{B}_0$ are respectively the smooth (large scale) parts of $\underset{\sim}{u}$ and $\underset{\sim}{B}$, while $\underset{\sim}{u}_1$ and $\underset{\sim}{B}_1$ are their fluctuating (small scale) parts. We define an averaging operator $\langle \cdot \rangle$. As we have noted, this might be a spatial or temporal average, but it might also have other meanings, e.g. ensemble averaging.

Then by definition

$$\langle \underset{\sim}{u}_0 \rangle = \underset{\sim}{u}_0 \quad ; \quad \langle \underset{\sim}{u}_1 \rangle = 0$$

$$\langle \underset{\sim}{B}_0 \rangle = \underset{\sim}{B}_0 \quad ; \quad \langle \underset{\sim}{B}_1 \rangle = 0$$

Also to simplify the notation let us introduce the following operators

$$L \equiv \frac{\partial}{\partial t} - \eta \nabla^2 - \nabla \times (\underset{\sim}{u} \times (\cdot))$$

$$L_0 \equiv \frac{\partial}{\partial t} - \eta \nabla^2 - \nabla \times (\underset{\sim}{u} \times (\cdot))$$

$$L_1 \equiv - \nabla \times (\underset{\sim}{u}_1 \times (\cdot))$$

With these above defined operators the induction e.g. (3.1) can be written as:

$$L B = 0 \tag{3.2}$$

By applying the averaging procedure to (3.2) we get:

$$L_0 \underset{\sim}{B}_0 + \langle L_1 \underset{\sim}{B}_1 \rangle = 0 \tag{3.3}$$

where the terms $\langle L_1 \underset{\sim}{B}_0 \rangle$ and $\langle L_0 \underset{\sim}{B}_1 \rangle$ are assumed to vanish due to the "smoothness" of L_0 and $\underset{\sim}{B}_0$.

Subtracting (3.3) from (3.2) we get:

$$L_1 \underset{\sim}{B}_0 + L_0 \underset{\sim}{B}_1 = \langle L_1 \underset{\sim}{B}_1 \rangle - L_1 B_1 \tag{3.4}$$

In the <u>first-order smoothing</u> the right-hand side of (3.4) is neglected. Hence in this case we have (formally)

$$\underset{\sim}{B}_1 = - L_0^{-1} L_1 \underset{\sim}{B}_0 \tag{3.5}$$

Then (3.3) gives

$$L_0 \underset{\sim}{B}_0 = \langle L_1 L_0^{-1} L_1 \rangle \underset{\sim}{B}_0 \tag{3.6}$$

which is the equation determining $\underset{\sim}{B}_0$. Given the solution $\underset{\sim}{B}_0$ of (3.6) the approximate (1st order) solution for the complete magnetic field is then

$$\underset{\sim}{B} \simeq \underset{\sim}{B}_0 - L_0^{-1} \underset{\sim}{B}_0 \tag{3.7}$$

if (3.6) is simpler than (3.2) we have gained something, and this seems likely since now the coefficients are smooth functions. Compared with $L_0 \underset{\sim}{B}_0 = 0$, (3.6) has a new term on the right-hand side, which is crucial for the dynamo action in those cases where $\underset{\sim}{u}_0$ fails by itself to give dynamo action.

To have a rough idea of the physical conditions under which this first order smoothing is valid we assume $\underset{\sim}{u}_0 \equiv 0$. Then (3.4) gives

$$\underbrace{\frac{\partial \underset{\sim}{B}}{\partial t}}_{O(\omega B_1)} - \underbrace{\eta \nabla^2 \underset{\sim}{B}_1}_{O(\eta \frac{B_1}{\ell^2})} - \underbrace{\nabla \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_0)}_{O\,\frac{UB}{\ell}} = \underbrace{\nabla \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_1) - \langle \nabla \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_1) \rangle}_{O\left(\frac{UB}{\ell}\right)}, \tag{3.8}$$

where U, ℓ, ω^{-1} are the characteristic scales of $\underset{\sim}{u}_1$. For the right-hand side to be negligible compared to at least one of the first two terms on the left-hand side either $U/\ell\omega$ or $U\ell/\eta$ has to be small compared to 1. The first condition does not involve the resistivity η and that means that one must be careful when using it. The second condition turns out to be sufficient when $U/\ell\omega$ is arbitrary and $O(1)$. In what follows we use the magnetic Reynolds numbers R and $\mathcal{R}$ based on ℓ and $\mathcal{L}$ respectively.

Also, the mean equation (3.3) gives:

$$\underbrace{\frac{\partial}{\partial t}\underset{\sim}{B}_0}_{O\left(\frac{B_0}{T}\right)} - \underbrace{\eta\nabla^2\underset{\sim}{B}_0}_{O\left(\frac{\eta B_0}{\mathcal{L}^2}\right)} = \underbrace{\langle\nabla\times(\underset{\sim}{u}_1\times\underset{\sim}{B}_1)\rangle}_{O\left(\frac{UB_1}{\mathcal{L}}\right)} \tag{3.9}$$

where $\mathcal{L}$, T are the characteristic scales of $\underset{\sim}{B}$. In (3.8) the fact that the last two terms on the left are comparable gives:

$$B_1 \simeq \frac{U\ell}{\eta} B_0 \equiv R B_0$$

so that the right-hand side of (3.9) is of order $\frac{U^2\ell B_0}{\eta\mathcal{L}}$ which is comparable to the terms on the left if $\mathcal{R} \equiv \frac{U\mathcal{L}}{\eta} \gg 1$ since $R = \frac{U\ell}{\eta} \ll 1$. Thus the two scales must be widely separated. This is equivalent to saying that $\frac{\ell}{\mathcal{L}} = O(R^2)$. The different scales can be pictured in Fig.1. Also from (3.9) we have

$$\left(\frac{\partial}{\partial t}\right)^{-1} = T = O\left(\frac{\mathcal{L}^2}{\eta}\right).$$

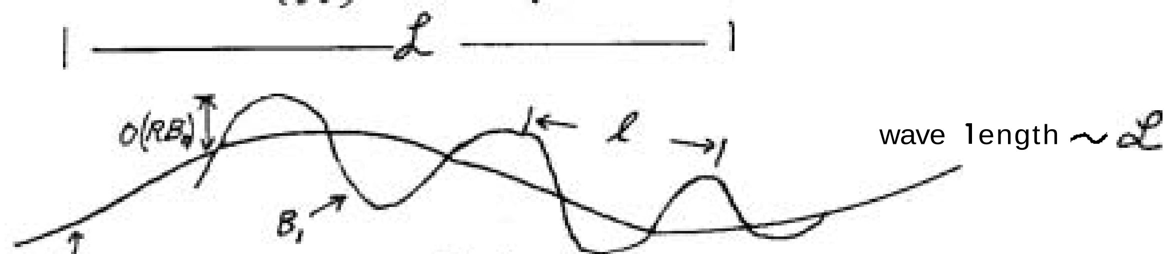


Fig.1

When $\underset{\sim}{u}_0 \neq 0$ the situation remains the same provided its magnetic Reynolds is not too large, although the determination of L_0^{-1} is not easy then.

3.3 The α-effect

By the α-effect (terminology of Steenbeck & Krause, 1966) we mean the case in which the term $\langle\underset{\sim}{u}_1 \times \underset{\sim}{B}_1\rangle$, in the mean field equation can be written in the form:

$$\langle\underset{\sim}{u}_1 \times \underset{\sim}{B}_1\rangle = \alpha\,\underset{\sim}{B}_0 \tag{3.10}$$

where α is a constant. In the general case we have $\langle\underset{\sim}{u}_1 \times \underset{\sim}{B}_1\rangle = \underset{\approx}{\alpha}\cdot\underset{\sim}{B}_0$ where $\underset{\approx}{\alpha}$ is a pseudo-tensor.

We consider the periodic velocity field in Cartesian coordinates

$$\underset{\sim}{u}_1 = U(0, \cos kx\, , \sin kx) \tag{3.11}$$

To the first-order smoothing Eq. (3.4) gives

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \eta \nabla^2 \underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_0)$$

or

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \eta \nabla^2 \underset{\sim}{B}_1 = \underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{u}_1 - \underset{\sim}{u}_1 \cdot \nabla \underset{\sim}{B}_0 \tag{3.12}$$

In (3.12) we can neglect the term $\underset{\sim}{u}_1 \cdot \nabla \underset{\sim}{B}_0$ compared to $\underset{\sim}{B}_0 \cdot \nabla \underset{\sim}{u}_1$ since $\ell \ll \mathcal{L}$. Hence from (3.11)

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} - \eta \nabla^2 \underset{\sim}{B}_1 \simeq B_{ox} U k\, (0, -\sin kx, \cos kx). \tag{3.13}$$

After a time t such that

$$\frac{\ell^2}{\eta} \ll t \ll T = \frac{\mathcal{L}^2}{\eta}$$

the effect of initial transients disappears and (3.13) gives

$$\underset{\sim}{B}_1 \simeq B_{ox} \frac{U}{k\eta} (0, -\sin kx, \cos kx)$$

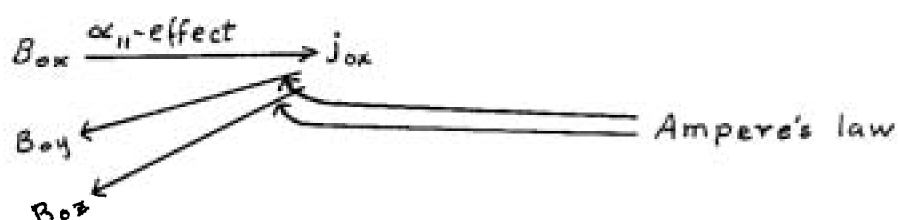
we then have

$$\langle \underset{\sim}{u}_1 \times \;, \quad \simeq \left(B_{ox} \frac{U^2}{k\eta}, 0, 0 \right) \equiv \underset{\approx}{\alpha} \cdot \underset{\approx}{B}_0$$

where

$$\underset{\approx}{\alpha} = \frac{U^2}{k\eta} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If $\underset{\sim}{B}_0$ is a uniform field in the x-direction, then the α-effect here produces a mean induced current $\langle \underset{\sim}{u}_1 \times \underset{\sim}{B}_1 \rangle = \underset{\sim}{j}_0$ in the same direction as $\underset{\sim}{B}_0$.



But in this case there is no feedback upon B_{ox} so that B_{ox} would decay away and no dynamo action is possible. Note here the important effect of phase shift. The flow deforms the field in just the right way to give a maximum interaction between the perturbed field and the flow (see Fig.2).

The helicity of $\underset{\sim}{u}_1$ (terminology introduced by Moffatt, 1970) is defined as $H = \langle \underset{\sim}{u}_1 \cdot \nabla \times \underset{\sim}{u}_1 \rangle$ and is a measure of the knottedness of vortex lines; in this case we obtain $H = -U^2$

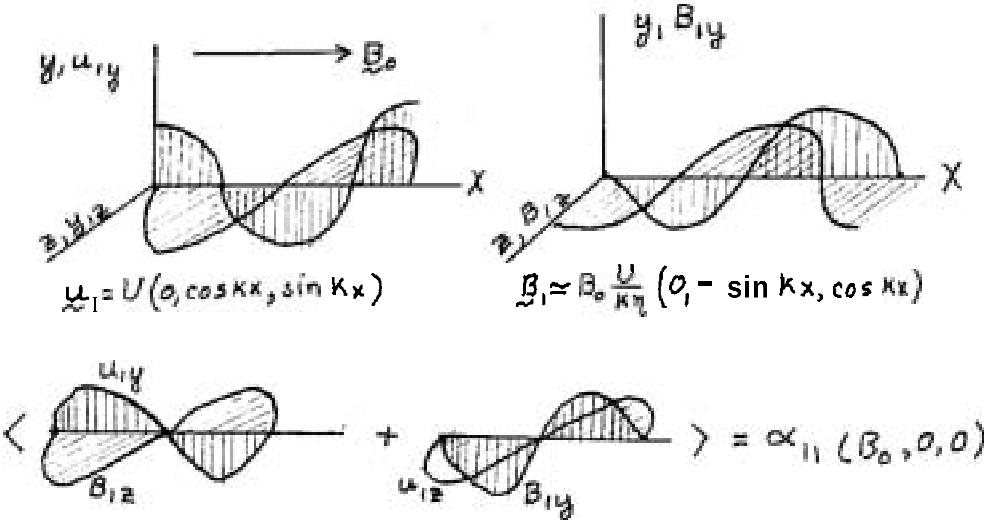


Fig.2

The motion that is next simplest is

$$\underset{\sim}{u}_1 = U(\sin ky, \cos kx, \sin kx + \cos ky) \tag{3.14}$$

Using (3.12) this gives

$$\underset{\sim}{B}_1 \simeq \frac{U}{k\eta}\left(B_{oy}\cos ky - B_{ox}\sin kx,\ B_{ox}\cos kx - B_{oy}\sin ky\right)$$

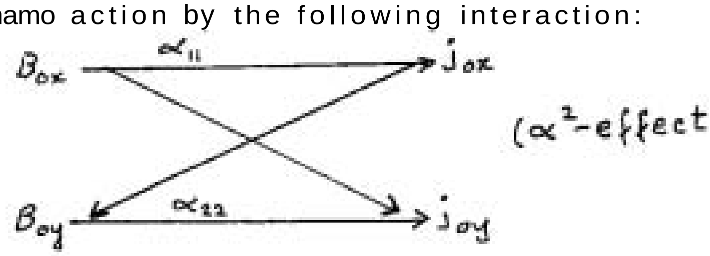
and

$$\langle \underset{\sim}{u}_1 \times \underset{\sim}{B}_1 \rangle \simeq \frac{U^2}{k\eta}\left(B_{ox}, B_{oy}, 0\right) \equiv \alpha \cdot \underset{\sim}{B}_o$$

or

$$\underset{\approx}{\alpha} = \frac{U^2}{k\eta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This model can cause dynamo action by the following interaction:



The cross arrows in the above diagram are ude to Ampere's law. This kind of interaction is called the α^2-effect.

To obtain

$$\underset{\approx}{\alpha} = \alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can take

$$\underset{\sim}{u}_1 = (\sin ky + \cos kz,\ \sin kz + \cos kx,\ \sin kx + \cos ky) \tag{3.15}$$

where H is the helicity.

In all of these examples we have

$$\nabla \times \underset{\sim}{u}_1 = -k\underset{\sim}{u}_1 .$$

This kind of field is called Beltrami (velocity is parallel to vorticity). In general orbits of such a flow are known to be topologically complicated.

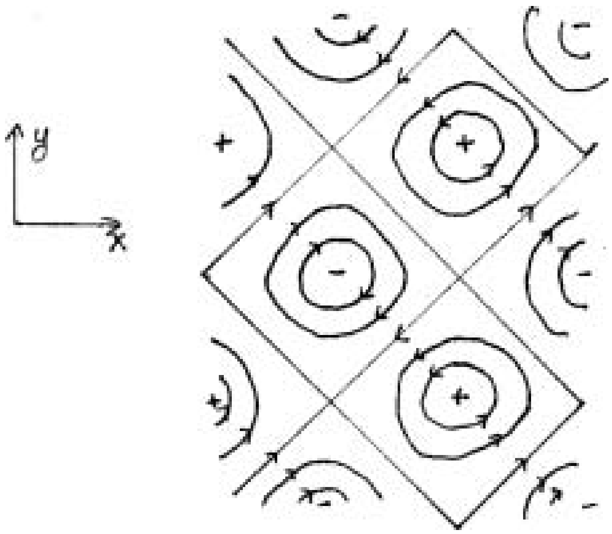


Fig.3. Stream lines for x and y components of $\underset{\sim}{u}_1$ given by (3.14).

+: upwelling

-: downwelling.

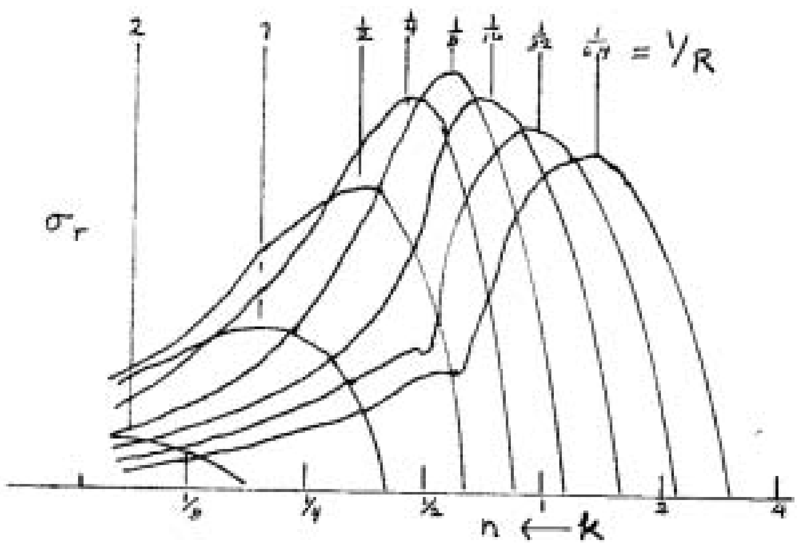


Fig.4. Numerical computation (G.O.Roberts, 1972) of the growth rate σ_r as function of k a d $'$ for $\underset{\sim}{u}_1$ given by (3.14).

Figure 3 represents the stream line for x and y components of the velocity field $\underset{\sim}{u}_1$ given by (3.14). The instabilities take the form of large scale circularly polarized stationary waves with magnetic field components perpendicular to the axis of the "eddies". Figure 4 shows G.O.Roberts (1972) computation of the growth rate σ_r as a function of k and R for the same velocity field. The growth rate increases to a maximum and then decreases as R^{-1} decreases. Figure 5 shows the lattice of straight particle paths (correcting stagnation points) for the three-dimensional motion (3.15).

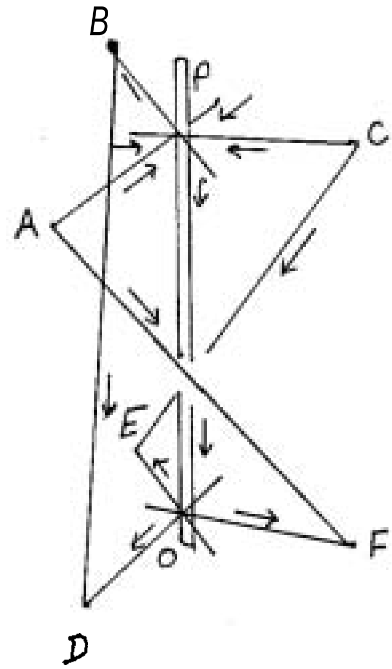


Fig.5. An element of the orbit structure of the motion (3.15). All labeled points are stagnation points, equivalent under a rotation and translation. All lines are particle trajectories in the indicated direction. Lines such *as* OP involve a divergence (in the plane ABC) and a convergence (toward the plane DEF). Note the helicity evident in the structure.

Note that Fig.4 bears a resemblance to Fog.2 of Lecture 2. It can easily be seen that 1st order smoothing is actually <u>exact</u> for the one-dimensional analog.

3.4 Periodic dynamos

In all three examples in Sec. 3.3 the velocity fields $\underset{\sim}{u}$ are steady and periodic in space. In general when $\underset{\sim}{u}$ is periodic in space and time we expect the magnetic field to incorporate the same periodicity in addition to large-scale components (G.O.Roberts 1970, 1972; Childress 1967, 1970).

We now consider, for such periodic fields, some interesting points related to "infinite-order" smoothing of the induction equations. Consider the modal form

$$\underset{\sim}{B}_o = \underset{\sim}{\Gamma}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}+\sigma t} \qquad (3.16)$$

With $\underset{\sim}{u}_1$ given to be periodic we can write the mean field Eq.(3.3) as:

$$(\sigma + n^2\eta)\underset{\sim}{r} = i\underset{\sim}{n} \times [\underset{\approx}{A}(\underset{\sim}{n}, \sigma, R, \underset{\sim}{u}_1) \cdot \underset{\sim}{\Gamma}]$$

where $n = |\underset{\sim}{n}|$. (We show how $\underset{\approx}{A}$ is derived in the next lecture.) The matrix $\underset{\approx}{A}$ can be written in the form of a cumulant expansion

$$\underset{\approx}{A} = \sum_{j=2}^{\infty} \underset{\approx}{A}_j$$

in terms of cumulants of the velocity field $\underset{\sim}{u}_1$. The $\underset{\approx}{A}_j$ have some nice symmetry properties and in particular $\underset{\approx}{A}_j(0, 0, R, \underset{\sim}{u}_1)$ is real and symmetric or anti-symmetric depending on whether j is even or odd. Also note that

$$\underset{\approx}{A}_2(0, 0, R, \underset{\sim}{u}_1) = \underset{\approx}{\alpha}.$$

It is interesting to note that in order to have dynamo action it is sufficient that $\det(\underset{\approx}{A}_2(0, 0, R, \underset{\sim}{u}_1))$ be different from zero, since in this case two eigenvalues of this matrix are of the same sign (α-effect). This is true for almost all periodic motions (in a precise sense based on the representation of the admissible $\vec{u}$, as a Hilbert space, and the non-dynamos as confined to a "lower dimensional" hypersurface in that space). Also, $\underset{\approx}{A}_2(0, 0, R, \underset{\sim}{u}_1)$ is analytic in R and thus if $\underset{\approx}{A}_2(0, 0, R, \underset{\sim}{u}_1)$ is such that $\det(\underset{\approx}{A}_2(0, 0, 0, \underset{\sim}{u}_1)) \neq 0$ then $\det(\underset{\approx}{A}_2(0, 0, R, \underset{\sim}{u}_1)) \neq 0$ for almost all R.

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Lecture #4 MEAN-FIELD ELECTRODYNAMICS

4.1 Smoothing to all orders

Applied to the equations of magnetohydrodynamics, the smoothing method has come to be known as mean-field electrodynamics. As a theory it encompasses not only the derivation of the mean-field equation but also the solving of the mean-field equation under conditions pertinent to the dynamo problem. We examine these two aspects of the theory in this lecture, beginning with the general form of the mean-field equation.

As in the last lecture we let

$$L_o = \frac{\partial}{\partial t} - \eta\nabla^2 - \nabla\times(\vec{u}_o\times(\cdot))$$

$$L_1 = -\nabla\times(\vec{u}_1\times(\cdot)$$

Using these, the dynamo problem can be written

$$(L_o + L_1)B = 0$$

We now define P as an averaging or smoothing operator.

$$P(\cdot) \equiv \langle(\cdot)\rangle$$

We require only that P be a projection, i.e. that it satisfy

$$P^2 = P$$

It is important to note that when $\underline{u}$ is small scale $L_o^{-1}L_1$ may not be a "small" operator on $\underline{B}$ even though $L_o^{-1}(L_1 - PL_1)$ is "small". (This has already been used at the level of first-order smoothing.).

Since

$$(L_o + L_1)\underline{B} = 0$$

we have

$$((L_o + L_1 - P(L_o + L_1))\underline{B} = 0$$

or

$$(L_o - PL_o)\underline{B} = -(L_1 - PL_1)\underline{B}$$

We assume that P and L_o (or L_o^{-1}) commute, that is

$$PL_o = L_oP, \quad PL_o^{-1} = L_o^{-1}P,$$

and that L_1 produces only "rough" field when applied to smooth:

$$PL_1 P = 0.$$

Commutation implies

$$(L_o - PL_o)\underline{B} = L_o(\underline{B} - P\underline{B}).$$

Letting $P\underline{B} = \underline{f}$ equal the mean field, then

$$L_o(\underline{B} - \underline{f}) = (L_o - PL_o)\underline{B} = -(L_1 - PL_1)\underline{B}.$$

By inverting L_o, this can be rewritten as

$$\underline{B} - \underline{f} = -L_o^{-1}(L_1 - PL_1)\underline{B} = S\underline{B}$$

where the operator S is

$$S = -L_o^{-1}(L_1 - PL_1).$$

Hence

$$(I - S)\underline{B} = \underline{f}.$$

If S is a small operator as claimed (the formal smallness of S is implied by the smallness of $\mathcal{R} = \frac{U\ell}{\lambda}$ or $\frac{U}{\ell\omega}$ from the last lecture), we should be able to write $\underline{B} = (I-S)^{-1}\underline{f}$.

The mean field equation now becomes

$$PL_o\underline{B} + PL_1\underline{B} = L_o\underline{f} + PL_1\underline{B}$$

which implies

$$L_o\underline{f} + PL_1(I - S)^{-1}\underline{f} = 0. \qquad (4.1)$$

This is an exact equation for the mean field.

We can easily rederive first order smoothing from this equation as follows:

We have approximately: $(I - S)^{-1} \approx I + S$

Substituting into the mean field equation, we get

$$L_o\underline{f} + PL_1(I + S)\underline{f} = 0$$

or

$$0 = L_o\underline{f} + PL_1(-L_o^{-1}(L_1 - PL_1))\underline{f}$$

$$= L_o\underline{f} - PL_1 L_o^{-1} L_1 \underline{f} \quad (\text{since } PL_1L_o^{-1}PL_1 = PL_1PL_o^{-1}L_1 = 0.$$

Of course, one may also use the previous method in which

$$\underline{B}_o = P\underline{B},\ \underline{B} - P\underline{B} = \underline{B}_1,\ L_o\underline{B}_1 = -L_1\underline{B}_o$$

We then again have

$$\underline{B}_1 \simeq -L_0^{-1} L_1 \underline{B}_0 , \; L_0 \underline{B}_0 + \langle L_1 \underline{B}_1 \rangle = L_0 \underline{B}_0 - \langle L_1 L_0^{-1} L_1 \rangle \underline{B}_0 = 0.$$

This simpler sequence of steps is all that need be considered if one stops at first order.

The full mean-field equation has, however, a much richer structure. It can be thought of, at least when $R = U\ell/\eta \sim U/\ell_\omega$ is small, as containing a double expansion, both in R and in the ratio $\ell/\mathcal{L}$ of spatial scales. (A "slow" time derivative in the mean field equation is regarded as order $\eta/\mathcal{L}^2$ and counts in the ordering as equivalent to a double space derivative.) A symbolic representation of the expansion goes as follows: First we write (upon expanding $(I - S)^{-1}$ in (4.11)

$$L_0 \underline{f} + \nabla \times \sum_{j=2}^{\infty} L_j \underline{f} = 0 \quad L_j(\cdot) = -\mathcal{P}(\underline{u}_1 \times S^{j-1}(\cdot))$$

The operators L_j have dimensions of a speed U and an order of magnitude (after division by U) of $O(R^{j-1})$. Moreover, it can be shown that L_j involves only the j^{th} order cumulant of $\underline{u}_1$.

Second, each L_j is, when expressed as a sum of a series of differential operators, formally of infinite order. We can then write

$$L_j = \sum_{K=0}^{\infty} L_{jK}$$

where

$$U^{-1} L_{jK} = O(R^{j-1} (\ell/\mathcal{L})^K).$$

In customary terminology L_{20} represents the α-effect and L_{21} the "/-effect" (when $\underline{u}_0$ vanishes). The last effect thus accounts for large-scale gradients of the mean field insofar as these affect mean induction. Because of the curl in (4.2) we see that the β-effect involves second derivatives of the mean field, and hence the capacity to modify the effective diffusion.

4.2 Examples

1) As we saw in the previous lecture it is easy to compute for certain simple motions. For a progressive wave of the form

$$\vec{u} = U(0, \sin\xi, \sin(\xi - \varphi)), \quad \xi = kx + \omega t,$$

we have, as the only non-zero entry in the pseudo-tensor

$$\alpha_{11} = \frac{-U^2 \eta k^3 \sin\phi}{\omega^2 + \eta^2 k^4}$$

Note that if $\omega \neq 0$ the effect vanishes in the limit $\eta \to 0$. (We consider the matter of small and zero resistivity below in an appendix).

2) Let $\underline{u} = \underline{u}_1$ be periodic, solenoidal, and representable in the form

$$\underline{u} = U \sum_{\underline{k} \in K,\, \omega \in \Omega} \underline{\mu}(k, \omega)\, e^{i(\underline{k}\cdot\underline{x} + \omega t)}$$

where K and Ω are suitable sets. As we noted previously, the mean field equation, if

$$\underline{f} = \underline{\Gamma}\, e^{i(\underline{n}\cdot\underline{x} + \sigma t)}$$

has the form

$$(\sigma + \eta n^2)\underline{\Gamma} = i\, \underline{n} \times \underline{\underline{A}} \cdot \underline{\Gamma}$$

The matrix $\underline{\underline{A}}$ has the cumulant expansion $\underline{\underline{A}} = \sum_{j=2} \underline{\underline{A}}_j$, and if we write

$$\underline{\underline{A}}_2 \cdot \underline{f} = \alpha_{ij} f_j + \beta_{ijk} f_j n_k + O(n^2)$$

we obtain the α and β effects with

$$\alpha_{ij} = i U^2 \sum_{K,\Omega} \frac{\epsilon_{ilm} \mu_l^* \mu_m k_j}{i\omega + \eta k^2}$$

$$\beta_{ij} = -\mathrm{Re}\left[\sum_{K,\Omega}\left(\frac{2\epsilon_{ilm}\mu_l^* \mu_m k_j k_k}{(i\omega + \eta k^2)^2} - \frac{\epsilon_{ijl}\mu_l^* \mu_k}{i\omega + \eta k^2}\right)\right]$$

3) For stochastic $\underline{u}$ with energy spectrum tensor $\Phi_{ij}(\underline{k}, \omega)$, expressions for α and β can be obtained from those just given by the replacement

$$\sum_{K,\Omega} \mu_i^* \mu_j (\cdot) \longrightarrow \iint \Phi_{ij}\ (\cdot)\, d\underline{k}\, d\omega$$

The expression for β is equivalent to that obtained by applying first-order smoothing to a mean field with constant gradient, up to a distant surface integral in wave number vector space; the latter will vanish for most physically realizable flows.

4) An important generalization of the method allows $\underline{u}_1$ to have, in addition to its basic small-scale features, a slow variation of structure. The computation of α in the first-order theory treats such motions as if the slow variation of parameters were not there, so there is no special difficulty at that level. A formal study of this and other generalizations has been carried out by Roberts and Soward (1975).

To take one example that will be important later let

$$\underline{u} = \left(-\frac{a}{k}\sin kx\cos az,\ \sin kx\cos az,\ \cos kx\sin az\right)$$

If $a/k \ll 1$ this can also be written

$$\underline{u} = (0, 1/2, 1/2)\sin(kx+az) + (0, 1/2, -1/2)\sin(kx-az) + O(a/k).$$

Thus $\underline{u}$ is approximately the sum of two nearby modes in $\underline{k}$-space, each having components in phase. The full spatial average of helicity is, moreover, zero. However, the two modes in combination produce a non-trivial α-effect, given by

$$\alpha_{11} = \frac{1}{2}\frac{U^2}{k\eta}\sin 2az + O(a/k).$$

4.3 Boundary-value problems

We now examine several examples of boundary value problems arising in mean field electrodynamics.

Consider the boundary value problem for the kineamtic α^2-dynamo in a slab. This model relies solely on the α-effect; we take $\underline{u}_0 = 0$ so that there is no contribution from the $\nabla\times(\underline{u}_0\times\underline{B}_0)$ term.

The model is as follows:

$\underline{B} = (B_1(z,t), B_2(z,t), 0)$, a two-dimensional field,

and the α matrix is taken to be

$$\alpha = \alpha_0\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

The dynamo equations become

$$\frac{\partial B_1}{\partial t} - \eta\frac{\partial^2 B_1}{\partial z^2} = -\alpha_0\frac{\partial B_2}{\partial z}$$

$$\frac{\partial B_2}{\partial t} - \eta\frac{\partial^2 B_2}{\partial z^2} = \alpha_0\frac{\partial B_1}{\partial z}$$

Setting $B = B_1 + iB_2$, we have a complex equation

$$\frac{\partial B}{\partial t} - \eta\frac{\partial^2 B}{\partial z^2} = i\alpha_0\frac{\partial B}{\partial z}$$

(which compares closely with the model problem of lecture 2). We assume motion to occur in a slab $0 \le z \le L$, suppose both magnetic components to vanish elsewhere, and therefore set $B = 0$ at $z = 0, L$. With $B = e^{\sigma t + i\lambda z}$ and α_0 an eigenparameter, we have the eisenfunctions

$$B = e^{\sigma t} e^{-\frac{\alpha_0}{2\eta}(1+K)iz}\left[e^{\frac{i\alpha_0 K z}{\eta}} - 1\right]$$

where $K = 1 - \frac{4\sigma\eta}{\alpha_0^2}$

and $\frac{\alpha K L}{\eta} = 2n\pi$, n an integer, provided that $\frac{L^2\sigma}{\eta} = \frac{R^2}{4} - \pi^2 n^2$, $R = \frac{\alpha_0 L}{n}$

For $n = 1$, $R > 2\pi$ gives dynamo action. There are no oscillatory modes. Induction regenerates the field by the "α^2" interaction:

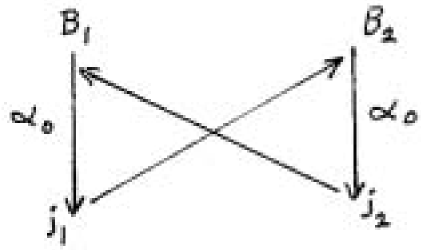

Fig.1

Consider now the same problem in a spherical core. A number of these cases have been studied by Steenbeck and Krause, and re-examined by Roberts (1972). The .case of constant $\alpha = \alpha_0 I$ can be solved explicitly and the minimum critical value of $\mathcal{R}$ (based on sphere radius) for steady dynamo action was found to be 4.49 corresponding to a vacuum dipole field. For the cases where is odd with respect to the equatorial plane the critical g for dipole and quadrupole field x, while not identical, are so close as to make them for all practical purposes indistinguishable. For example, with

$$\alpha = 24\sqrt{3}\,\frac{r^2}{L^2}(L - r)^2 U \cos\theta \sin^2\theta$$

the dipole eigenvalue is 10.09 and the quadrupole eigenvalue 10.45. An explanation of this coincidence has been given by Proctor (1977), using a comparison problem which exactly admits these degeneracies and appears to be close to the realiz'ed structures.

We now consider the migratory dynamo model of Parker (1955). Consider an infinite domain with mean field of form

$$\vec{B} = -\frac{\partial A}{\partial z}\hat{i}_x + B\hat{i}_y + \frac{\partial A}{\partial x}\hat{i}_z$$

where A and B are independent of y. A is a stream function for the poloidal field while B is the torroidal field, we take

$$\vec{u}_0 = (0, \omega(z), 0) \quad \text{with } \frac{d\omega}{dz} = \text{constant}.$$

The equations are

$$\frac{\partial B}{\partial t} - \eta\frac{\partial^2 B}{\partial x^2} = \frac{d\omega}{dz}\frac{\partial A}{\partial x} - \alpha\nabla^2 A$$

$$\frac{\partial A}{\partial t} - \eta\frac{\partial^2 A}{\partial x^2} = \alpha B$$

Looking for modes proportional to $e^{ikx+\sigma t}$ we find

$$(\sigma + \eta k^2) = i\eta\alpha\gamma + k^2\alpha^2 \quad \text{where } \gamma = \frac{d\omega}{dz} = \text{constant}.$$

If $\sigma = \sigma_r + i\sigma_i$, for neutral stability ($\sigma_r = 0$) and $\alpha << 1$, $2\sigma_i \eta k^2 = k\alpha\gamma$

or $\sigma_i = \dfrac{\alpha\gamma}{2\eta k}$ so waves propagate in the direction of negative x when $\alpha\gamma < 0$.

Applied to the surface of a fluid sphere, these waves suggest how the poloidal field components migrate across latitude lines under the combined influence of microscale motions and large-scale subsurface shear. This is particularly interesting because of the migration of sunspots to the solar equator.

These migratory waves illustrate what is known as the "α-ω" effect. When $\alpha << 1$ and $\alpha\gamma = O(1)$ the $\alpha^2 k^2$ term in the dispersion relation may be neglected, as may the α term in the equation for $\frac{\partial A}{\partial x}$. Simultaneously $A << B$, and this is the most reasonable parameter range for the geodynamo. In this limit, the "α-ω" effect looks like

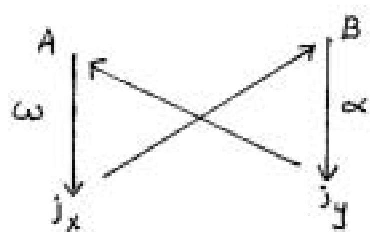

Fig.2

$\alpha\gamma > 0$

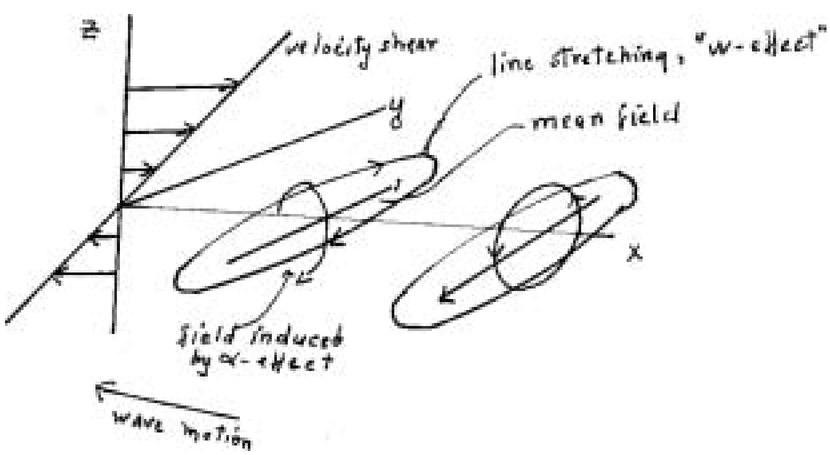

Fig.3

A number of $\alpha\omega$ dynamos have been tried in a spherical core. For a recent assessment see Roberts (1972); see also, Deinzer et.al. (1974), and Roberts and Stix (1971). If one allows for some meridional flow in addition to the differential rotation responsible for the ω-effect, the fields have the forms

$$\underset{\sim}{u}_0 = \nabla\times t(r,\theta)\,\underline{r} + \nabla\times\nabla\times p(r,\theta)\,\underline{r}, \quad \omega = (r\sin\theta)^{-1}\frac{\partial t}{\partial\theta}$$

$$\underline{B} = \nabla\times T(r,\theta)\,\underline{r} + \nabla\times\nabla\times P(r,\theta)\,\underline{r}$$

Since we are dealing with axially-symmetric fields there are two principle parities (components parallel or perpendicular to the equatorial plane)

dipole symmetry: $B_{||}$ odd, $B_{\perp}$ even

quadrupole symmetry: $B_{||}$ even, $B_{\perp}$ odd.

Roberts (1972) examines a model of Steenbeck and Krause,

$$\alpha = \alpha_0 \cos\theta,$$

$$\omega = \gamma_0 r, \; P = 0.$$

As in the α^2 cases the dipole and quanrupole critical $\mathcal{R}$ are quite close, there being a slight preference for the quadrupole mode when $\alpha_0\gamma_0 > 0$ and for the dipole when $\alpha_0\gamma_0 < 0$. In both cases the modes are oscillatory. Similar results are obtained with other choices of α and ω of the same basic parity, although for some configurations the preference for one of the symmetries becomes more pronounced.

It is found that the oscillatory solutions exhibit a lateral migration of poloidal field structures, toward the equator if $\alpha_0\gamma_0 < 0$, thus suggesting that the physical mechanism isolated by Parker plans an important role in the oscillation.

For a model of Braginsky incorporating both components of $\underset{\sim}{u}_0$ we have

$$t = -r^a(1-r^2)^2 P_1(\cos\theta)$$

$$p = 10\, m\, r^6(1-r)^2 P_2(\cos\theta)$$

$$\alpha = {}_{5}\; r^2(1-r)^2\left[P_1(\cos\theta) - P_3(\cos\theta)\right]$$

Roberts finds that, in the range $.52 \leq m \leq -.012$ the most easily excited mode is a steady dipole when $\alpha_0\gamma_0 > 0$, the smallest critical $\mathcal{R}$ occurring when $m = -.3$. If the sign of $\alpha_0\gamma_0$ is reversed, the quadrupole mode replaces dipole, critical $\mathcal{R}$'s and $|m|$'s are again close, but the sign of m is changed. This surprising symmetry property has recently been discussed by Proctor, (1978) and explained in terms of the proximity of solutions to those of a comparison problem where the property holds exactly. If we were to seek the model most relevant to the earth, we would have to pick this one, with $\alpha_0\gamma_0$ positive in the northern hemisphere (to select the dipole mode). In the range of parameters studied the most easily excited fields were steady. In this reapect reversal phenomena are not predicted and indeed the oscillatory kinematic dynamos are probably misleading as models for reversals.

Notes submitted by
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Appendix

The α-effect in the limit

Stephen Childress

We record here some observations regarding the difficult and controversial matter of applying smoothing to perfect or near-perfect conductors. (A discussion of the problem may be found in Moffatt's book.)

Formally, asymptotic smoothing was seen to be valid if $U/\ell\omega \ll 1$ and this condition is independent of η. This leads to the hope that rapid movements, or a stochastic field with short correlation time, can be made to achieve dynamo action in first-order smoothing even for vanishing resistivity.

One immediate point, if one really wants actual convergence of the mean-field equation, is whether such convergence can ever be achieved for vanishing η. One's suspicions in this regard are confirmed by estimates on an appropriate norm of the operator S considered above. It is not sufficient that $U/\ell\omega$ be small, but is sufficient that $U^2/\omega\eta$ be small, the condition again involving η. In fact the condition $U/\ell\omega \ll 1$ is "only" asymptotic, with $(I - S)^{-1}$ having a divergent majorant series of the form

$$\sum_{j=0}^{\infty} (U/\ell\omega)^j j!$$

when $\eta = 0$. Of course most frequently this is a minor point, but in at least one class of dynamos, rapidly periodic in time and periodic in space, first-order smoothing induces a secondary flow and a mean-field equation with an effective magnetic Reynolds number $U^2/\omega\eta$. As this last parameter tends to infinity the α-effect is found to vanish (as in the steady flows considered below).

On the other hand, Alfvèn's twisting of a torus to amplify the field (lecture #1) is a tempting mechanism and may work for sufficiently small resistivity. By its very nature, however, this process cannot be accessible to modeling by asymptotic first-order smoothing, since mean and perturbational fields are comparable during the twisting process. One possibility then, is that there are difficulties to smoothing a perfectly conducting dynamo. In addition, it is not immediately clear what unnatural zero resistivity phenomena might be introduced by ensemble averaging over admissible motions, if the latter contain singularities capable of severing and reconnecting lines of force.

Actually little is known about the limit of small η even in the case of induction by steady spatially-periodic flows. G.O.Roberts' numerical results (see lecture #3) included some values of α out to $R = 64$ and these can be compared with calculations based on boundary-layer theory. The latter makes strong use of symmetry and evaluates α once flux is concentrated near the boundaries of cells. It is found that $\alpha = \text{const} \times R^{-1/2}$ as $R \to \infty$ where the constant obtained produces rough agreement with Roberts' values. It may be conjectures that this ordering persists for any steady motion independent of one coordinate.

For three-dimensional steady spatially-periodic motions one expects concentration of flux into tubes and possibly also sheets. The former can be shown (using the asymptotics described elsewhere by Proctor) to produce an α-effect nominally $O(1/R)$, but at the present time there are no worked-out examples to support this estimate.

Lecture #5

ALMOST SYMMETRIC DYNAMOS

5.1 Graginskii's (1964) solution to the kinematic dynamo problem

Consider the dimensionless equation

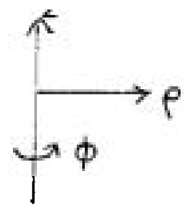
$$\frac{1}{R}\left(\frac{\partial}{\partial t}-\nabla^2\right)\underline{B} = \underline{\nabla}\wedge\underline{u}\wedge\underline{B}$$

with $\nabla\cdot\underline{B}=0$ and $R=\frac{uL}{\eta}$ in the limit $R\to\infty$. The time scale of the field has been taken as the diffusive time scale t = time

It is.known from Cowling's theorem that axisymmetric fields cannot maintain a dynamo. Braginskii's idea was that fields which were close to axisymmetric might be able to maintain a dynamo for sufficiently large R. He assumed

$$\underline{B} = B(z,\rho)\,\underline{i}_\varphi + o(1)$$

$$\underline{u} = W(z,\rho)\,\underline{i}_\varphi + o(1) \text{ as } R\to\infty$$



His analysis then investigates whether the small additional, o(1) components of $\underline{u}$ can act with W to create a kinematic dynamo such that B can be maintained or amplified. Braginskii sought a symmetric "mean-field" equation,. An averaging operator P is defined to be averaging over φ , so if

$$\underline{V} = V_z\underline{i}_z + V_\rho\underline{i}_\rho + \varphi\cdot\underline{\varphi}$$

then

$$P\underline{V} = \left(\frac{1}{2\pi}\int_0^{2\pi} V_z\,d\varphi\right)\underline{i}_z + \left(\frac{1}{2\pi}\int_0^{2\pi} V_\rho\,d\varphi\right)\underline{i}_\rho + \left(\frac{1}{2\pi}\int_0^{2\pi} V_\varphi\,d\varphi\right)\underline{i}_\varphi$$

A velocity field is postulated of the form exemplified by

$$\underline{u} = W(z,\rho)\,\underline{i}_\varphi + R^{-1}\underline{\nabla}\wedge\left(\bar{\Psi}(z,\rho)\,\underline{i}_\varphi\right) + R^{-1/2}\,\underline{u}^{(1)}(z,\rho,\phi)$$

with $P\underline{u}^{(1)} = 0.$

The results following from this choice can then be viewed as a special asymptotic version of 1st order smoothing.

It is found that a self-consistent expansion of $\underline{B}$ takes the form

$$\underline{B} = B(z,\rho)\,\underline{i}_\varphi + R^{-1}\underline{\nabla}\wedge\left(\bar{A}(z,\rho)\,\underline{i}_\varphi\right) + \sum_{j=1}^{3} R^{-j/2}\,\underline{B}^{(j)}(z,\rho,\varphi) + o(R^{-3/2}).$$

$$P\underline{B}^{(j)} = 0 \qquad j = 1,2,3.$$

The mean field equations obtained from these fields can be written in a form almost identical to those obtained for an axisymmetric $\alpha\omega$ dynamo from 1st order smoothing. Given A, $\underline{u}$ "effective" variables are defined

$$\tilde{A}_{eff} = \tilde{A} + \gamma B$$
$$\tilde{\Psi}_{eff} = \tilde{\Psi} + \gamma W, \quad \underline{\tilde{V}}_{eff} = \underline{\nabla} \wedge \tilde{\Psi}_{eff} \underline{i}_{\varphi}$$

where

$$\gamma = \frac{1}{2} \frac{P(u_z \hat{u}_\rho - u_\rho \hat{u}_z)\rho}{W^2}$$

and

$$\left(\frac{\partial}{\partial\varphi} \hat{u}_z\right) i_z + \left(\frac{\partial}{\partial\varphi} \hat{u}_\rho\right) i_\rho + \left(\frac{\partial}{\partial\varphi} \hat{u}_\varphi\right) \underline{i}_\varphi = \underline{u}.$$

Then, in the limit $R \to \infty$, the equations are

$$\frac{\partial}{\partial t} A_{eff} + \rho^{-1} \underline{\nabla}(\rho \tilde{A}_{eff}) = \left(\nabla^2 - \frac{1}{\rho^2}\right) \tilde{A}_{eff} + \tilde{\alpha} B$$
$$\frac{\partial B}{\partial t} + \rho \underline{\tilde{V}}_{eff} \cdot \nabla\left(\frac{B}{\rho}\right) = \left(\nabla^2 - \frac{1}{\rho^2}\right) B + \left(\nabla \frac{W}{\rho} \wedge \nabla \rho \tilde{A}_{eff}\right)_\phi$$

where $\tilde{\alpha}(z, \rho)$ is quadratic in $\underline{u}^{(1)} R^{1/2}$

These equations are precisely the same as those of an axisymmetric dynamo, with the exception that the equation for B would also contain the term

$$\left(\underline{\nabla} \wedge \alpha (\underline{\nabla} \wedge A \underline{i}_\varphi)\right)$$

on the right-hand side. The absence of this term in Braginskii's formalism means that he can only obtain $\alpha\omega$ dynamos (and that α^2 dynamos are not obtainable, except possibly through higher-order calculations).

Since it can be shown that the same boundary conditions apply to the effective variables in the boundary value problem, we have two independent asymptotic theories which produce the same "smooth" mathematical problem. However, Braginskii's method has the advantage that there is no assumption about length scales associated with $\underline{u}^{(1)}$. Instead $\underline{u}^{(1)}$ is chosen to be slightly asymmetric and the singular limit $R \to \infty$ is used to make possible the expansion of the non-axisymmetric component of $\underline{B}$.

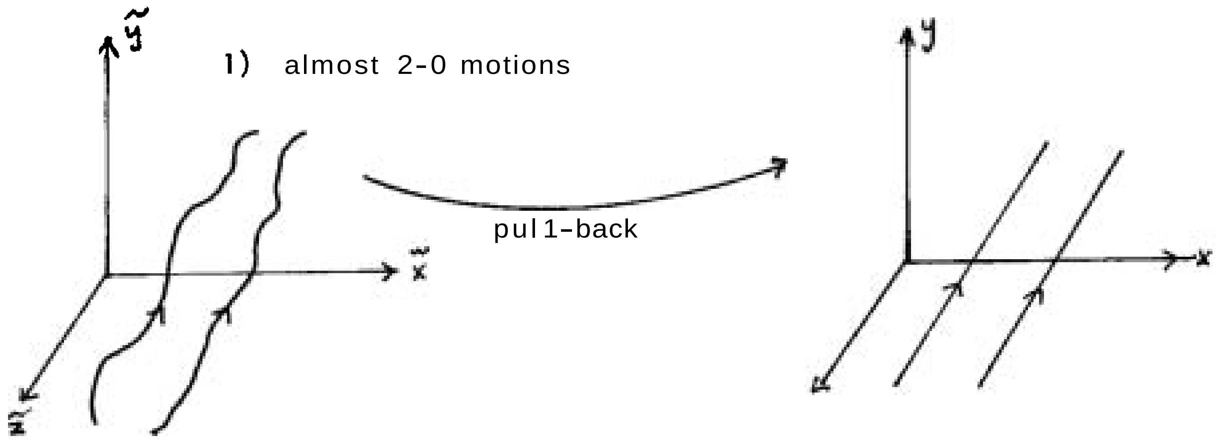
The emergence of the effective variables is startling and led Soward to reinterpret Braginskii's work by considering it to be an instance of "diffusive modification" of the kinematics of an essentially perfect conductor. This point of view is useful here because the perfect fluid kinematics are close to a simple form, because the fields are almost axially symmetric.

5.2 Soward's pull-back method

The idea of this method is to explain the form of Braginskii's equations by using a transformation property of the perfect conductor.

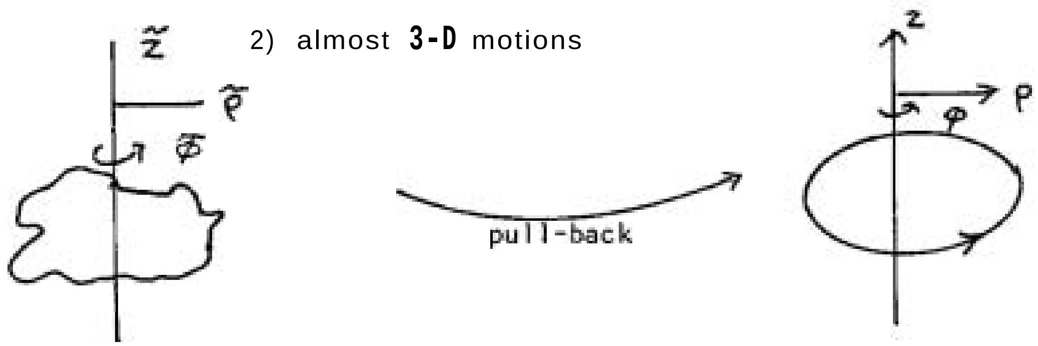
We are interested in motions close to simple ones, e.g.

1) almost 2-0 motions



The pull-back maps R to R and straightens out the "wiggles".

2) almost 3-D motions



In order to erase the "wiggles" at a given instant we can imagine a smooth volume-preserving map of space into itself, which we write as

$$\mathcal{F}: \; \underline{x} = (\tilde{\underline{x}}, t)$$

We want to see the effect of such a transformation on the equations for a moving perfect conductor. Let fields $\tilde{\underline{B}}(\tilde{\underline{x}}, t), \tilde{\underline{E}}(\tilde{\underline{x}}, t), \tilde{\underline{u}}(\tilde{\underline{x}}, t)$ be given as solutions of

$$\tilde{\underline{E}} + \tilde{\underline{u}} \wedge \tilde{\underline{B}} = 0$$
$$\underline{\nabla} \cdot \tilde{\underline{B}} = 0$$
$$\underline{\nabla} \wedge \tilde{\underline{E}} = -\frac{\partial \tilde{\underline{B}}}{\partial t}$$

Then $\frac{\partial \tilde{B}}{\partial t} = \underline{\nabla} \wedge (\tilde{\underline{u}} \wedge \tilde{\underline{B}})$.

The Lagrangian form of this equation is

$$\frac{D}{Dt}\left\{\tilde{B}_i(\tilde{x},t)\frac{\partial a_j}{\partial \tilde{x}_i}\right\} = 0$$

This equation can easily be transformed using the transformation $\mathcal{J}$ and we find

$$\frac{D}{Dt}\left\{B_k(x,t)\frac{\partial a_j}{\partial x_k}\right\} = 0$$

where

$$B_k(x,t) = B_i(\tilde{x},t)\frac{\partial x_k}{\partial \tilde{x}_i}$$

Hence

$$\frac{\partial B}{\partial t} = \nabla \wedge (u \wedge B)$$

where

$$u_k(x,t) = \left(\frac{\partial x_k}{\partial t}\right)_a = \frac{\partial x_k}{\partial t} + \tilde{u}_i\frac{\partial x_k}{\partial \tilde{x}_i}$$

We can now find $E(x,t)$ since $E = -u \wedge B$. Using the fact that $\mathcal{J}$ is volume preserving, so

$$\epsilon_{ijk}\frac{\partial x_i}{\partial \tilde{x}_p}\frac{\partial x_j}{\partial \tilde{x}_q}\frac{\partial x_k}{\partial \tilde{x}_r} = \epsilon_{pqr}$$

Then $$E_i = E_j\frac{\partial \tilde{x}_j}{\partial x_i} - \epsilon_{ikl}\frac{\partial x_k}{\partial t}\tilde{B}_j\frac{\partial x_l}{\partial \tilde{x}_j}.$$

So we see the equations are invariant under the transformation of $\tilde{x}, \tilde{B}, \tilde{E}, \tilde{u}$ into x, B, u.

However, if we have chosen $\mathcal{J}$ as a pull-back to symmetric flow, we can average along the direction of symmetry (z for 2-D, ϕ for 3-D). So

$$P\left[\frac{\partial B}{\partial t} + u\cdot\nabla B - B\cdot\nabla u\right] = 0$$
$$= \frac{\partial}{\partial t}PB + u\cdot\nabla PB - PB\cdot\nabla u$$

since the equation is linear in B and since $Pu = 0$ as u is symmetric.

We now claim we may identify u with $\tilde{u}_{eff}$ and PB with $\tilde{B}_{eff}$ of Braginskii's dynamo. This is particularly easy to argue if $\tilde{u}$ is a steady motion and all stream-lines are unlinked closed curves, since it is plausible that in this case that the necessary mapping, $\mathcal{J}$, can be found. In general the method hinges on the existence of a smooth $\mathcal{J}$, and it is not obvious when it exists. All $\tilde{u}$ for which the method works can, of course, be obtained by smooth transformations, $\mathcal{J}^{-1}$ of symmetric flows u. Since Braginskii's equations are asymptotic for $R \to \infty$, the pull-back need only have certain asymptotic properties (e.g. near closure of orbits).

Having examined how the perfect conductor equations transform, we now wish to see how the exact equations transform. One finds,

$$\frac{\partial \underline{B}}{\partial t} - \eta \nabla^2 \underline{B} = \underline{\nabla} \wedge \underline{u} \wedge \underline{B} + \underline{\nabla} \wedge \underline{\mathcal{E}}$$

where

$$\mathcal{E}_i = \alpha_{ij} B_j + \beta_{ijk} \left(\frac{\partial B_j}{\partial \tilde{x}_k} \right)$$

$$\alpha_{ij} = \eta \epsilon_{ikl} \frac{\partial x_k}{\partial \tilde{x}_m} \frac{\partial}{\partial x_j} \left(\frac{\partial x_l}{\partial \tilde{x}_m} \right)$$

$$\beta_{ijk} = \eta \epsilon_{ikl} \frac{\partial x_l}{\partial \tilde{x}_m} \frac{\partial x_j}{\partial \tilde{x}_m}$$

It can be shown for choices of $\mathfrak{F}$ appropriate to Braginskii's special choice of velocity field ($O(R^{-1/2})$ axisymmetric part etc.) that the β_{ijk} term is negligible and that the equations obtained already, by Braginskii's method,
3 can be rederived.

Notes submitted by
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Lecture #6 CONVECTIVE DYNAMOS: General Principles

We turn now to the full magnetohydrodynamic dynamo problem, and in particular the convection driven model, which may be diagrammed as follows:

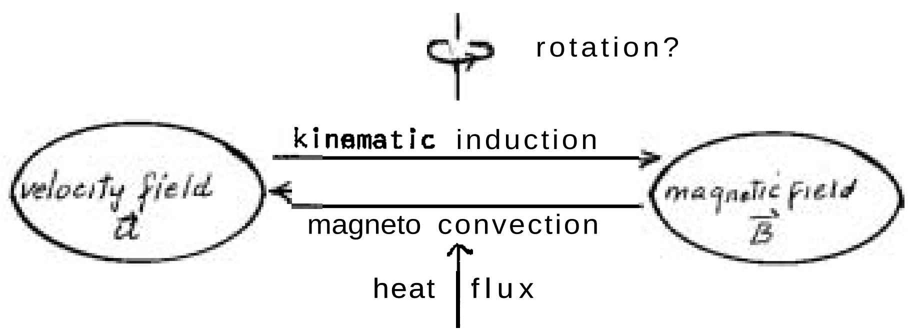

Fig.1

The most relevant problem - the convective MHD dynamo in a rotating spherical annulus - is quite formidable, and has not been in any sense "solved", On the other hand, one might argue for the existence of solutions as follows:

(1.) Show that nonmagnetic ($\vec{B} = 0$) solutions are unstable to the addition of field, i.e. infinitesimal magnetic fields are amplified. (In this case, in the weak-field theory, the kinematic induction uncouples from the dynamics.)

(2.) Show that the magnetic field energy is necessarily bounded.

Given that the dynamo is being driven by applying energy at a fixed rate, the latter point is very plausible. Today we shall see how it goes for a convective dynamo, driven by heat sources.

We use the term "convective" to mean there is some scalar field (e.g. temperature), affecting the fluid density, which can itself be advected and diffused (one nonthermal process, suggested by Braginskii, involves the floating up of light elements released during the growth by accretion of the inner core). There is no real concensus concerning the energy source for the geodynamo, but we feel that convection, (thermal or nonthermal), is as plausible as any of the other proposed mechanisms (e.g. core turbulence driven by precession, or baroclinic instability analogous to that in the atmosphere). Moreover it rests on a well-understood process, so that various models can be formulated rather easily.

A profound (and controversial) criticism of the thermal convection model has been put forward by Higgins and Kennedy (1971, 1973). They propose that the core is, for the most part, stably stratified (their second paper says that a convecting region may exist within 500 km of the inner (solid) core). Their argument is that the adiabatic gradient is shallower than the melting point gradient. Assuming that the mantle/core and core/inner core boundaries are melting-point transitions, they argue that the temperature profile must follow the melting point curve: greater temperatures would lead to melting of some of the solid core, lower temperature would tend to solidify some of the liquid core.

To compute the adiabatic gradient we have $\nabla p = -\rho g \frac{\underline{r}}{L}$ for $L \equiv$ radius of core;

$$\frac{c_p}{T}\, dT = \frac{\alpha}{\rho}\, dp, \text{ so } \left(\frac{\partial T}{\partial p}\right)_{ad} = \frac{\alpha T}{\rho C_p} \text{ and } \left(\frac{dT}{dr}\right)_{ad} = -\frac{\alpha T g r}{C_p L}.$$

The melting-point gradient was obtained by extrapolation of shock-tube data for melting temperature vs pressure.

For convection, the temperature profile must increase (with depth) faster than the adiabatic increase.

$$-\frac{\partial T}{\partial r} > -\left(\frac{\partial T}{\partial r}\right)_{ad} = \frac{\alpha g T r}{C_p L}$$

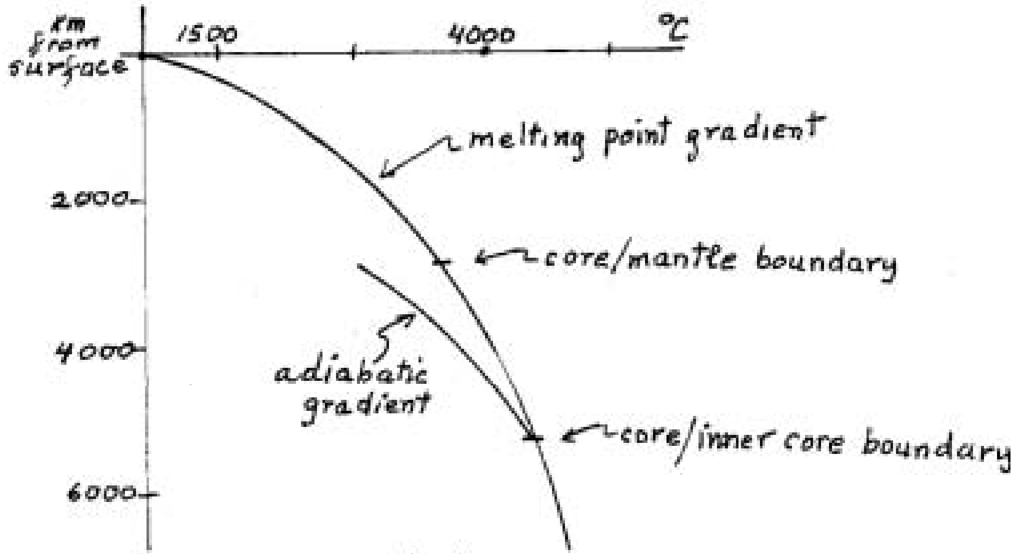

Fig.2

With α, T, C_p as in Table 1 (Lecture 1), $g\frac{r}{L} \doteq 5 m/sec^2$, we get

$$-\left(\frac{\partial T}{\partial r}\right)_{ad} \doteq 2 \times 10^{-4} \,°K/m.$$

Assuming all heat sources are in the core (quite unlikely), and that the radial gradient at the core/mantle boundary is given by the last figure, we get a surface heat flux of approximately 6×10^{-4} cal/m^{2}s. If we assume the Kennedy-Higgins melting-point curve, we obtain 4×10^{-4} cal/m^{2}s. Both values are considerably below the observed flux of 1.5×10^{-2} cal/m^{2}s.

Another basic criticism of a thermal model of convection has been given by Braginskii (1964), which lead him to his proposal of a geochemical mechanism. Treating the thermal dynamo as a heat engine with optimal efficiency $\frac{\Delta T}{T} \doteq \frac{500}{5000} = \frac{1}{10}$, actual efficiency perhaps $\frac{1}{100}$, the work done in sustaining the field (most of which appears as Joule heating) is about $\frac{1}{100}$ of the net core heating (presumably by radioactive decay). Assuming 1/5 of the 10^{15} joule/sec heat flux at the surface is created in the core, Joule heating should amount to 2×10^{10} joules/sec. Suggested values for Joule heating have ranged from 4×10^{9} to 4×10^{12} joules/sec; Braginskii favored the latter, higher figure, based on a kinematic dynamo model of his.

Both of the above objections (popularly known as "core paradoxes") are themselves subject to criticism; in what follows, we treat convection explicitly as thermal.

Our equations now become (in a rotating frame)

$$\rho\frac{d\vec{u}}{dt} + 2\rho\vec{\Omega}\wedge\vec{u} + \rho\dot{\vec{\Omega}}\wedge\vec{r} + \rho\vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r}) + \nabla p + \vec{B}\wedge\vec{J} + \nabla\cdot\vec{\vec{\pi}} = \rho\vec{F}, \qquad (6.1)$$

where $\vec{F}$ includes all gravitational forces and π_{ij} is the viscous stress tensor;

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0;$$

$$\rho C_p \frac{dT}{dt} = \alpha T \frac{dp}{dt} + \nabla\cdot\mathcal{F} + Q_{source} + Q_{visc} + Q_{joule} , \qquad (6.2)$$

where $\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$, $Q_{visc} = \pi_{ij}\frac{\partial u_i}{\partial x_j}$, $Q_{joule} = \frac{1}{\sigma}J^2$, and $\mathcal{F} = \lambda\nabla T$,

the thermal diffusivity.

We assume $\vec{u} = 0$ on S, the boundary of the sphere which is our domain, and that $T = T_S$ is constant on S. Setting $\vec{F} = -\nabla\Psi$ and $\Psi + \frac{1}{2}|\vec{\Omega}\wedge\vec{r}|^2$, taking the dot product with the momentum equation gives

$$\frac{\partial}{\partial t}\rho\left(\frac{u^2}{2} + \Psi\right) + \nabla\cdot\left(\rho\vec{u}\left(\frac{u^2}{2} + \Psi + \frac{p}{\rho}\right)\right) =$$
$$= p\nabla\cdot\vec{u} - \vec{u}\cdot(\vec{B}\wedge\vec{J}) - \vec{u}\cdot\nabla\pi - \rho\vec{u}\cdot(\dot{\vec{\Omega}}\wedge\vec{r}) + \rho\frac{\partial\Psi}{\partial t} . \qquad (6.3)$$

Integrating over the core,

$$\frac{\partial}{\partial t}(E_k + E_p) = W_c + W_p - \mathcal{Q}_{visc} - \int \vec{u}\cdot(\vec{B}\wedge\vec{J})\,d\mathrm{vol} , \qquad (6.4)$$

where E_k, E_p are kinetic and potential energy given by $\frac{1}{2}\int\rho u^2\,d\mathrm{vol}$, $\int\rho\Psi\,d\mathrm{vol}$ respectively, the compressional work $W_c = \int p\nabla\cdot\vec{u}\,d\mathrm{vol}$ and the precessional (and tidal) work

$$W_p = \int\rho\left(\frac{\partial\Psi}{\partial t} - \vec{u}\cdot(\dot{\vec{\Omega}}\wedge\vec{r})\right)d\mathrm{vol}.$$

Recall, from the kinematic dynamo problem,

$$\frac{\partial E_m}{\partial t} = \int\vec{u}\left(\vec{B}\wedge\vec{J}\right)d\mathrm{vol} - \mathcal{Q}_{Joule}, \qquad (6.5)$$

where E_m is the magnetic energy. Putting this together with the previous equation gives

$$\frac{\partial}{\partial t}\left(E_m + E_k + E_p\right) = W_c + W_p - \mathcal{Q}_{visc} - \mathcal{Q}_{Joule} \qquad (6.6)$$

Introduce entropy, s and internal energy density e, we rewrite the temperature equation as

$$\rho T\frac{ds}{dt} = \rho\frac{De}{Dt} - \frac{p}{\rho}\frac{D\rho}{Dt} = \frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e\vec{u}) + p\nabla\cdot\vec{u}$$
$$= \nabla\cdot\vec{\mathcal{F}} + Q_{Source} + Q_{visc} + Q_{Joule} \qquad (6.7)$$

Again we integrate, and get

$$\frac{\partial E_i}{\partial t} = \frac{\partial}{\partial t}\int\rho e\,d\mathrm{vol} = -W_c + \int_S\vec{\mathcal{F}}\cdot\vec{n}\,ds + \mathcal{Q}_{source} + \mathcal{Q}_{visc} + \mathcal{Q}_{Joule} \qquad (6.8)$$

so if $E = E_m + E_K + E_p + E_i$,

$$\frac{\partial E}{\partial t} = W_p + \mathcal{Q}_{source} + \int_s \vec{\mathcal{F}} \cdot \vec{n}\, ds. \qquad (6.9)$$

Denote time average by an **overbar,** and assume the system is near equilibrium. Then

$$W_c + W_p = \mathcal{Q}_{visc} + \mathcal{Q}_{Joule} \qquad (6.10)$$

$$\text{and} \quad -\overline{\int_s \vec{\mathcal{F}} \cdot \vec{n}\, ds} = \overline{W}_p + \overline{\mathcal{Q}}_{source}$$

We can consequently sketch the system as a heat engine driving a kinematic dynamo.

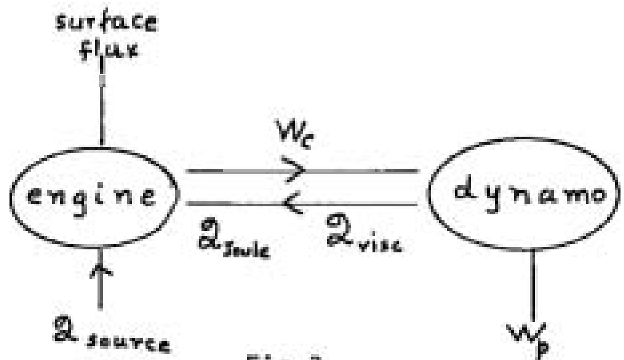

Fig.3

Of course, there is but a single earth, and so the two must be superimposed, as they consist of the same fluid particles.

We want to show that $\mathcal{Q}_{Joule}$ + , is bounded in terms of $\mathcal{Q}_{source}$. **The** idea is that parcels of fluid deep in the core absorb heat $\mathcal{Q}_i$ at temperature Ti, and give **it** up at the surface at temperature T_S; entropy increases, so $\sum \frac{\mathcal{Q}_i}{T_i} \le \frac{\mathcal{Q}_s}{T_s}$, However, $\sum \frac{\mathcal{Q}_i}{T_i} = \sum \frac{(\mathcal{Q}_s + \mathcal{Q}_v + \mathcal{Q}_J)_i}{T_i} \ge \sum \frac{(\mathcal{Q}_s + \mathcal{Q}_v + \mathcal{Q}_J)_i}{T_{max}}$,

so $\mathcal{Q}_{source} + \mathcal{Q}_{visc} + \mathcal{Q}_{Joule} - \frac{T_{max}}{T_s}\mathcal{Q}_s$ and $\mathcal{Q}_{visc}$ $\mathcal{Q}_{Joule} \le \left(\frac{T_{max}}{T_s} - 1\right)\mathcal{Q}_s$. This still involves the unknown ,T however, For further discussion, see Malkus (1973), Hewitt ~~et al~~ (1975), and **Backus** (1975).

THE BOUSSINESQ APPROXIMATION

We assume at the start that $\vec{\Omega}$ = constant.

We then treat the density ρ as a constant ρ_0, EXCEPT in the body force **term** $\rho F = -\rho \nabla \Psi = -\rho_0 (1 - \alpha g T) \frac{E}{L}$, where L = radius of earth and $\alpha = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_p$; he effect of density fluctuations on the gravitational potential is also ignored. **We** also neglect, in the temperature equation, the terms $\mathcal{Q}_{visc}$ $\mathcal{Q}_{Joule}$ and $\alpha T \frac{D\rho}{Dt}$. The equations (6.1) - (6.2) then become

$$\rho_0 \frac{d\vec{u}}{dt} + 2\rho_0 \Omega \wedge \vec{u} + \nabla p + \vec{B} \wedge \vec{J} + \nabla \cdot \vec{\pi} \qquad (6.1')$$
$$= \rho_0 \alpha T g \frac{\vec{r}}{|\vec{r}|},$$
$$\nabla \cdot \vec{u} = 0,$$

and
$$\rho_0 C_p \frac{dT}{dt} - \lambda \nabla^2 T = Q_s \qquad (6.2')$$

These assumptions require that the thickness of the convecting region, L, be much less than the "temperature scale height", $L_T = \frac{C_p}{\alpha g}$; for the earth, one obtains $L = 2 \times 10^6$ m, $L_T = 2 \times 10^7$ m, so this is not too bad. This condition arises from the supposition that buoyancy and magnetic (Lorentz) terms are comparable, the Q_{Joule} much smaller.

A remark: in the Boussinesq approximation, if all time-averaged quantities are independent of initial conditions, then the mean total helicity must vanish. One sets the initial conditions alternately to $(\vec{u}_1, p_1, T_1, \vec{B}_1) = (\vec{u}_0(\vec{r}), p_0(\vec{r}),$ $(\vec{r}),$, for some given functions $\vec{u}_0$, p_0, T_0 and $\vec{B}_0$, and to $(\vec{u}_2, p_2, T_2,$ $\vec{B}_2) = (\vec{u}_0(-r), p_0(r), T_0(-\vec{r}), \vec{B}_0(-\vec{r})$.

Then $H_1 = \overline{\int \vec{u}_1 \cdot (\nabla \wedge \vec{u}_1)\, d\,vol} = -\overline{\int \vec{u}_2 \cdot (\nabla \wedge \vec{u}_2)\, d\,vol} = -H_2$;

since we assumed $H_1 = H_2$, we conclude that they must be zero.

We now consider a bound on magnetic energy in the Boussinesq limit. One can show (see the appendix) that $\overline{\mathcal{Q}}_{visc} + \overline{\mathcal{Q}}_{Joule}$

$$\frac{3\lambda\beta T_s}{L^2}\left(\frac{4}{3}\pi L^3 - \frac{\overline{\mathcal{Q}}_s}{(Q_s)_{max}}\right) + \frac{4\pi\beta}{L^2}\int_0^L r \int_0^r (r')^2 Q(r')\, dr'\, dr, \qquad (6.11)$$

where $\beta = \frac{L}{L_T} \ll 1$ and $Q(r) = \langle \overline{Q}_s \rangle_r$, $\langle(\cdot)\rangle_r$ denoting the mean over a spherical shell of radius r, and $Q_s = Q_{source}$. The first term vanishes if Q_s is constant, so

$$\overline{\mathcal{Q}}_{visc} + \overline{\mathcal{Q}}_{Joule} \leq 4\pi\beta L\, \overline{\mathcal{Q}}_{source}.$$

We now use (cf. Lecture 2) $\overline{\mathcal{Q}}_{Joule} \geq \frac{\pi^2}{\mu^2\sigma L^2}\int B^2 d\,vol$ and set $\int \overline{B^2} d\,vol = B_0^2 \cdot vol$, define the Rayleigh number as $R_a = \frac{Q_s \alpha g L^5}{\rho_0 C_p \nu K^2}$, the Hartman number as $M = \frac{B_0^2 L^2}{\mu \eta \rho_0 \nu}$ in which case one obtains $\frac{R_a}{M^2} \geq 5\pi^2 \doteq 50$.

(Our standard values (Lecture #1) give $R_a/M^2 \doteq 190$ for fields of 100 G.) Thus in the Boussinesq case, the field must be bounded ("most of the small-scale stuff is wiped out, so only about 1/10 of the α-effect remains"). This bound was obtained (using a somewhat different argument) by Hewitt et al. (1975).

Notes submitted by
David C.W.Hart

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Appendix (Stephen Childress)

To obtain (6.11), first multiply (6.2') by $T-T_s$, integrate over V and average over time. After use of Gauss' theorem and the constancy of T_s we obtain

$$\lambda \int_V \overline{(\Delta T)^2 dvol} = \int_V \overline{Q_s (T-T_s) dvol}$$

and therefore

$$T_s \mathcal{Q}_{source} \leq \int_V Q_s T\, dvol.^*$$

We will use this relation presently. Now define

$$\overline{\langle Q_s \rangle} = Q(r),$$

where

$$\langle (\cdot) \rangle (r) = \frac{1}{4\pi r_0^2} \int_{r=r_0} (\cdot)\, d\,area$$

is the spherical mean at radius r_0. The spherical mean of (6.2') gives, after integrating once with respect to v,

$$\rho_0 C_p r^2 \overline{\langle u_r T \rangle} - \lambda r^2 \overline{\left\langle \frac{\partial T}{\partial r} \right\rangle} = \int_0^r \rho^2 Q(\rho)\, d\rho.$$

But in the present case

$$\overline{W_0} = 4\pi \frac{\rho_0 \alpha}{L} g \int_0^L r^2 \overline{\langle u_r T \rangle}\, dr = \mathcal{Q}_{Joule} + \mathcal{Q}_{viscous}$$

and so, now using (''),

$$4\pi\int_0^L r^3\frac{\partial}{\partial r}\langle\bar{T}\rangle\,dr = 4\pi L^3 T_s - 3\int T\,d\mathrm{vol}$$

$$\le 4\pi L^3 T_s - \frac{3}{(Q_s)_{max}}\int Q_s T\,d\mathrm{vol}$$

$$\le 4\pi L^3 T_s - \frac{3T_s}{(Q_s)_{max}}\,\overline{\mathcal{Q}_{source}}$$

where the maximum is over space and time.

Combining these results, we have (recall $\beta = L/L_1 = L\alpha g/c_p$)

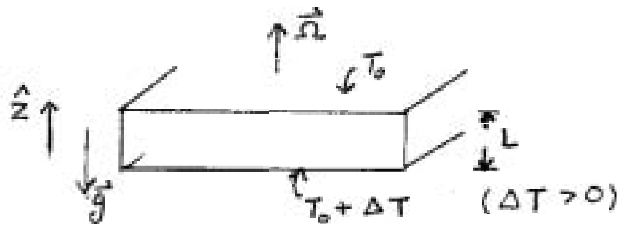
$$\mathcal{Q}_{viscous} + \mathcal{Q}_{Joule} \le \frac{3\lambda\beta T_s}{L^2}\left[\frac{4}{3}\pi L^3 - \frac{\overline{\mathcal{Q}}_{source}}{(Q_s)_{max}}\right] + \frac{4\pi\beta}{L^2}\int_0^L r\int_0^r \rho^3 Q(\rho)\,d\rho$$

which is (6.11).

Lecture #7 ROTATING MAGNETO CONVECTION

Now we consider the effect of putting in heat to support a magnetic field and a convective velocity field. There are two aspects of the complete dynamical problem that we will consider. 1) The kinematic problem of generating a magnetic field from the free convective motion of the fluid and 2) the effect of an imposed uniform magnetic field on the fluid motion.

First we examine the Bénard convection problem, i.e. a rotating layer of fluid heated from below, in a uniform gravitational field, (c.f. S.Chandrasekhar "Hydrodynamic and Hydromagnetic Stability" Chap.3). Our interest from the point of view of dynamo theory is the structure of the realized modes of convection. We make the Boussinesq assumption that density changes are important only in the buoyancy term and inertial effects are negligible. Also we linearize the equations of motion which eliminates the advective nonlinearity of the substantial derivative.



∧ denotes unit vector

In the temperature we subtract out the initially imposed linear gradient:

$$T = T_0 + \Delta T - \frac{z}{L}\Delta T + \theta$$

The momentum equation is:

$$\frac{\partial \vec{U}}{\partial t} + 2\vec{\Omega}\times\vec{U} + \nabla P - \nu\nabla^2\vec{U} = g\alpha\theta\hat{z}\ ;\ \nabla\cdot\vec{U} = 0$$

and temperature equation:

$$\frac{\partial\theta}{\partial t} - W\frac{\Delta T}{L} - K\nabla^2\theta = 0$$

We apply "slippery" boundary conditionr: $\theta = W = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial t} = 0$ at $z = 0, L$ for isothermal bounding surfaces.

Keller: In elasticity theory these boundary conditions are termed a "greased surface".

Malkus: This is achieved experimentally at $R_a \sim 640$ with silicone oil bounded by mercury below and helium above both of which are good conductors.

We will obtain asymptotic results which in any case are relatively insensitive to choice of boundary conditions.

To put the equations in more tractable form apply $\hat{z}\cdot\nabla\times$ and $\hat{z}\cdot\nabla\times(\nabla\times$ to the momentum equation and let $\xi = (\nabla\times\vec{U})_z$, the vertical component of vorticity, obtaining (extemporaneously)

$$\frac{\partial\xi}{\partial t} - 2\Omega\frac{\partial w}{\partial z} - \nu\nabla^2\xi = 0\ \left[\nabla\times(\vec{\Omega}\times\vec{u}) = -\Omega\frac{\partial w}{\partial z}\right]$$

$$-\nabla^2\frac{\partial w}{\partial t} + \nu\nabla^4 W - 2\Omega\frac{\partial\xi}{\partial z} = -g\alpha\nabla_1^2\theta$$

Spiegel: "That deserves a round of applause . . . even the signs are right".

Childress: "Is that right?"

Spiegel: "How should I know?"

ξ may be eliminated between the two equations, multiplying the first by $2\Omega\frac{\partial}{\partial z}$, the second by $\left(\frac{\partial}{\partial t} - \nabla^2\right)$ and adding. This result and temperature equation are nondimensionalized with the following scales:

length $\sim$ L

Rayleigh number $R_a = \dfrac{L^3\Delta T g\alpha}{\nu K}$

time $\sim \frac{L^2}{\nu}$ (typical vertical diffusion time)

Taylor number $T_a = \dfrac{4\Omega^2 L^4}{\nu^2}$ (measures effects of rotation relative to viscosity)

Also note $T_a \equiv \lambda^2(U\cdot K_1) \equiv \frac{1}{E^2}$ (E = Ekman number), and finally $\nabla_1^2 = \nabla^2 - \frac{d^2}{\partial z^2}$. Note the Taylor number for the Earth using $L \sim 3\times10^6$ $\Omega = 7.4\times10^{-5}$ $\nu \sim 10^{-6}$ (MKS) is $\sim 1.8\times10^{30}$!!

in final form:

$$\nabla^2\left(\frac{\partial}{\partial t} - \nabla^2\right)^2 W + T_a \frac{\partial^2 W}{\partial z^2} = R_a \nabla_\perp^2 \left(\frac{\partial}{\partial t} - \nabla^2\right)\theta$$

$$\left(P_r \frac{\partial}{\partial t} - \nabla^2\right)\theta = W$$

for which there is a class of solutions with

$$W = \sin(n\pi z)\cos(\vec{k}\cdot\vec{x})\,e^{\sigma t} \quad n = 1, 2, \cdots \quad (\vec{k}, \vec{x} \text{ in plane } \perp \hat{z})$$

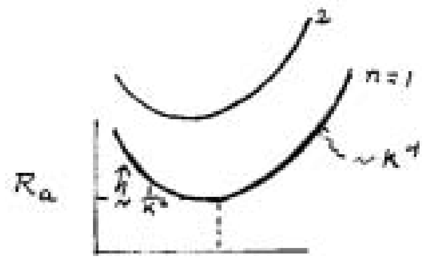
where the wave vector $(k_x,\ k\ \ n)$ satisfies the relation

$$R_a = \frac{(k^2+\pi^2 n^2)^3 + \pi^2 n^2 T_a}{k^2} \quad \text{for neutral stability}$$

Proctor: This is for $P_r = 1$?

Childress: Assume $r \geq .6-.7$. This insures exchange of stability at large Ta.

We are concerned with the onset of instability for σ crossing through zero, in the limit Ta $\rightarrow \infty$. Qualitatively the relation above may be sketched as



Clearly $(R_a)_{min}$ is achieved for n = 1, but for what value of k? (as $T_a \rightarrow \infty$). If we assume that $k \sim 1$, then $R_a \sim T_a$. But if $k \gg 1$, then $R_a \approx k^4 + \frac{\pi^2 T_a}{k^2}$. Taking $\frac{dR_a}{dk} = 0$ for a minimum yields

$$k_c = \left(\frac{\pi^2}{2} T_a\right)^{1/6} \Rightarrow R_a = 3\left(\frac{\pi^2}{2} T_a\right)^{2/3}$$

(which is smaller than the estimate from the $k \sim 1$ analysis. From the point of view of dynamo theory this is nice, for a fast enough spin the convection modes have bery large horizontal wave number, i.e. small spatial extent ℓ. Thus

$$\frac{\ell}{L} \sim \left(\frac{2}{\pi^2 T_a}\right)^{1/6} \ll 1 \quad \text{as } T_a \rightarrow \infty.$$

The appearance of this new scale in the problem may be thought of as arising from strong Coriolis forces which cause a large orthogonal deflection in the trajectory of a particle initially moving horizontally, leading to thin vertical convection cells.

Malkus: Veronis found as $T_a \rightarrow \infty$ and k_c gets larger the actual particle trajectory in a roll is the same as for $T_a \sim 1$, just tilted. The balance is feostrophic.

Stern: Think of a top, the rotation stabilizes the motion even though it is top-heavy. The only thing that makes it fall is friction as in this case, small cells are formed in which frictional forces offset centrifugal forces.

Malkus: Yes, but the viscosity destabilizes the flow.

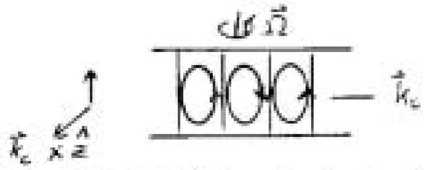
Spiegel: The point is, why is k_c large? Viscosity kills large wave numbers.

Malkus: Suppose you increase the heat flux, what happens to ΔT_c? How does it depend on ν ? ΔT_c goes to infinity as viscosity goes to zero! The k^4 is a viscous term.

Spiegel: The inviscid problem has a critical wave number such that all smaller wave numbers are stable and which tends to infinity for infinite rotation rate.

Childress: Perhaps the term to focus on is $-\frac{\pi^2 T_a}{K^2}$, where we see a balance of buoyancy in the numerator with vertical forcing in the denominator, i.e. not a geostrophic balance to lowest order.

Now we consider the velocity field of a roll.



$$\vec{u}_k = \sin(\pi z)\cos(\vec{K}_c\cdot\vec{x})\hat{z} + \frac{\sqrt{2}}{K_c}(\hat{z}\times\vec{K}_c)\cos(\pi z)\sin(\vec{K}_c\cdot\vec{x}_c) - \frac{\pi\vec{K}_c}{K_c^2}\cos(\pi z)\sin(\vec{K}_c\cdot\vec{x})$$

Note that the first two components, vertical and perpendicular to the plane of the page, are $O(1)$ quantities while the last component which makes u_k divergence free is $O(T_a^{-1/6})$. We expect from examples already studied that the $\frac{\pi}{2}$ phase lag between the first two components should lead to regeneration. The pressure of a small scale ℓ justified first order smoothing and we find the dyad

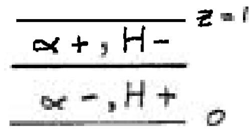
$$\underset{\sim}{\alpha} = -\frac{1}{\sqrt{2}}\sin(2\pi z)\frac{\vec{K}_c\cdot\vec{K}_c}{K_c^2} \sim O(T_a^{-1/6})$$

Proctor: This is for fixed η .

We can add any number of rolls with the same k_c but at different angles, e.g. a hexagonal or square arrangement leading to a matrix of the form:

$$\underset{\sim}{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with a positive definite upper block. A mean current in the plane suggests an α^2 dynamo is possible in principle, having a periodic repeated cell structure. Although the net helicity $\int(\vec{u}\cdot\nabla\times\vec{u})\,d(vol)$ is zero, there is a "polarization of helicity by rotation" with respect to the midplane:



We can produce a dynamo with large scale magnetic field by slow (horizontal) modulation of $\underset{\sim}{\alpha}$.

Stern: Are there modes which aren't dynamos?

Childress: The only case which would not be a dynamo would be a degenerate matrix, e.g. one roll alone ($\underset{\sim}{\alpha}$ diagonal with one nonvanishing element).

Pedlosky: Why isn't this realistic, that is, does $\vec{B}$ have to develop?

Childress: this is a highly degenerate plane problem, in a sphere one can't arbitrarily combine rolls, for example you might have an α-effect, but no α^2.

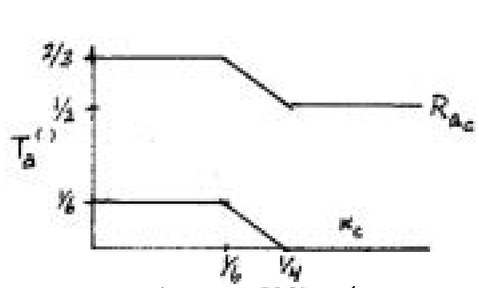
Proctor: According to Soward with two or three rolls you can have a field with R_a smaller than the critical value, here.

Childress: Actually the problem is even worse as there is golbal subcritical instability as opposed to just local.

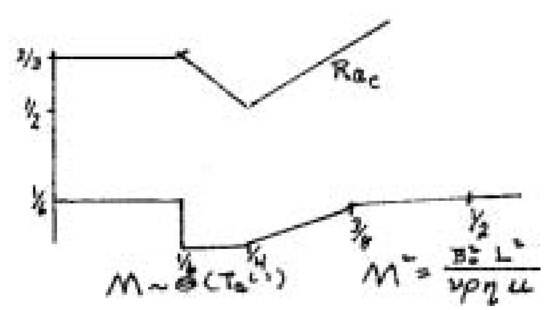
Proctor: A smaller value of Ra_c for two rolls with a field suggests the only possible mode is a dynamo.

Malkus: Roberts following Chandrasekhar suggests that "gyroscopic constraints are stabilizing" which is widely accepted. It is interesting here that the two constraints oppose each other, relaxing the conditions for instabiliry.

Now we add a uniform imposed magnetic field following Eltayeb and Roberts, and examine the asymptotic dependence of Ra_c and k_c on T_a.



$M - \Theta(T_a^{1/2})$ (Hartman number)
Horizontal field



Vertical field

The minimum R_{a_c} in both cases is for $M^2 \sim T_a^{1/2}$ in the intermediate magnetic field region where magnetic and rotational constraints come closest to cancelling. In the weak field regime, not surprisingly, rotation dominates and the exponents are those found earlier, while for strong fields, horizontal and vertical (cf. J.Pedlosky on "Inviscid Stabilization"), results differ markedly.

Spiegel: There is a dip for the vertical field?

Childress: Remember, the vertical field responds to both horizontal motions.

Note that the ratio $\frac{M^2}{T_a^{1/2}}$ is independent of viscosity and putting in the numbers for the earth, the ratio is ~ 16.

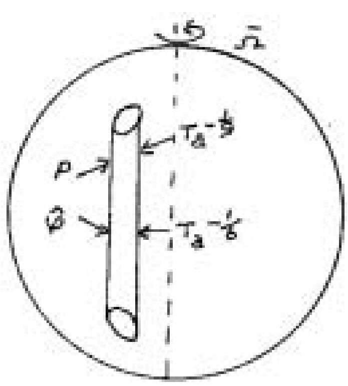
Proctor: $R_a \sim O(T_a^{1/2})$ puts one out in the high field regime where the instability is a diffusive one.

Knobloch: For no field, $R - T_a^{2/3}$, and with a strong (horizontal) field $R_a - T_a^{1/2}$?

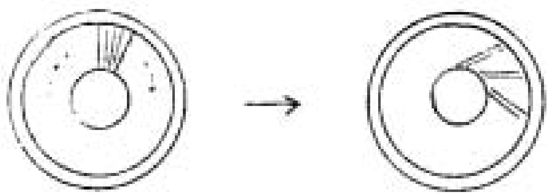
Childress: Yes, and notice that the change in exponent of $T_a^{1/6}$ now means $\ell - L \sim \mathcal{L}$ for the earth which would invalidate the mean field equation and α-effect.

Malkus: In a spherical geometry, though, the constraints might lead to an optimal k such that $\ell \sim \frac{1}{10}$ Radius.

For a sphere the problem is complicated considerably. Busse and Roberts used linear stability theory to obtain asymptotic results for $T_a \rightarrow \infty$. The convective mode for a spherically symmetric gravitational field consists of slender rolls oriented along the rotation axis which propagate eastward, this being one of the consequences of a loss of a geostrophic balance.



There is an expanded radial structure and a rapid azimuthal variation; thus one neglects r derivatives relative to φ derivati'ves obtaining an equation in φ and z which reduces to an eigenvalue problem in z with the assumption of a mode $e^{im\varphi}$. The loss of radial dependence leads to an infinitely degenerate class of modes. Soward did a multiple scale analysis supposing $R_a \gtrsim R_{a_c}$ and found a radial dependence of σ and thus of the wave apeed. If one does an initial value problem, eventually viscosity kills the mode.



Spiegel: Perhaps the time involved is too great in the context of other approximations.

Soward employed nonlinear stability theory and was able to resolve the structure on a long time scale for

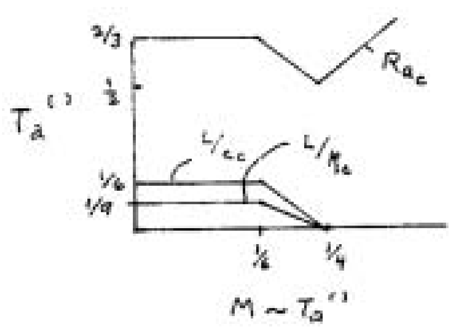
$$\frac{R_a - R_{a_c}}{R_{a_c}} \sim \theta(T_a^{1/9})$$

There is a complicated radial structure on slow space $(T_a^{-5/36})$ and long time scale resulting in a Stewart-Robertson type evolution equation, Heat is transferred radially by a soliton-1ike structure (rather than the laminar Plow of a conven-. tional cell) of dimension T_a . In Busse's experimental apparatus one would expect $T_a^{-1/18} \sim 1$, thus his observation of convection cells propagating eastward is not in conflict with Soward's asymptotic results. (Also, the gravitational field is cylindrical not radial.)

Eltayeb and Kumar considered the effect of a magnetic field

$$\vec{B} = B_0 \frac{(x^2+y^2)^{1/2}}{L} \hat{\varphi}$$

in a sphere, obtaining the following results:

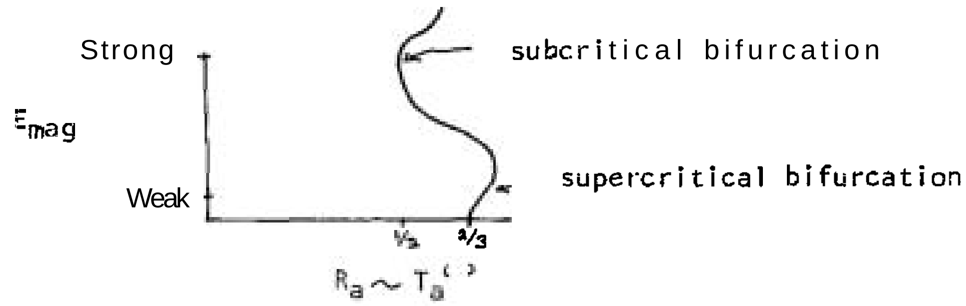


For $M \sim T_a^{1/6} - T_a^{1/4}$ the drift direction switches so that very strong fields $(M^2 >> T_a^{1/2})$ (with essentially no dissipation) are associated with westward drift. From numerical results the velocity is found to be

$$L\omega_c = 13.33 \frac{M^2}{T_a^{1/2}} \frac{\eta}{L} \sim 10^{-5} \frac{M^2}{T_a^{1/2}} \text{ m/sec}$$

which with $M^2/T_a^{1/2} \sim 16$ for the earth is comparable to the observed westward drift. (This is the phase velocity, the group velocity is always westward even for small B. Also these results are for $\eta \sim \chi \sim \nu$).

For a dynamo one would like to find a simple problem which contains the essential physics without having to consider a sphere. Soward explored the weak field limit of a Benard dynamo, and found a local regime of stable dynamo operation. His results coupled with those of Etayeb & Roberts and Eltayeb & Kumar suggest the following sketch for general rotating bodies with convective dynamos.



A Bénard dynamo in the strong field regime is rendered extraordinarily complicated by these aspects of magnetoconvection. Thus one might consider, rather than the full spherical model, either a planar limit or, as Eusse has done, an annular model with top and bottom surfaces inclined at a small angle to allow Rossby-like waves.

Malkus: Also one might examine a cylinder with spherical caps.

Any model of an α^2 dynamo is subject to the criticism that one would expect an $\alpha\omega$ dynamo due to large scale motion in the strong field regime since convection will distort the initial radial symmetry of the temperature field leading to a "thermal wind" to provide large scale aximuthal motion. in any case either approximate models or approximate analysis through a truncated modal expansion seems crucial to obtaining a tractable problem. In the next lecture we shall examine the case of a weak field planar Benard dynamo, where some of these ideas can be examined in what is probably their simplest setting.

Notes submitted by
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and Hisashi Haukuda

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Lecture #8 THE BÉNARD DYNAMO

In this lecture we consider a successful hydrodynamical dynamo, studied by Soward (1974) in the weak field regime. Physically we have in mind a rotating Bénard dynamo in the $T_a \to \infty$ limit. This implies, as we have seen, a small scale convective motion (from linear stability considerations). We will put in a small field and see if it is amplified. As the structure has local helicity we anticipate it will excite magnetic fields. Finally we use a modified finite amplitude theory which incorporates first order magnetoconvective back reaction on the velocity field, and enables us to investigate dynamical equilibration. By weak field we mean $\mathcal{M} = \frac{B_0 L}{\sqrt{\mu \nu \rho \eta}} \sim O(1)$.

Malkus: How is it that equipartition doesn't enter into the problem?

Childress: This requires $M \sim O(T_a{}^{1/6})$ in which range there may be more than one stable state.

Pedlosky: $M \sim O(1)$ means all the terms in the dynamic equation are comparable and in particular the buoyancy roughly balances the magnetic energy.

We fix $\frac{R_a - R_{a_c}}{R_{a_c}} = O(T_a{}^{-1/4})$ in which case the convection process rapidly achieves an equilibrium convective heat flux (essentially total kinetic energy) equal to that realized without a field, while the dynamo process changes on a very much longer time scale. Effectively this decouples the energy of the convective flow from the magnetic energy (cf. discussion on this point re: validity of first order smoothing). The magnetic field in the weak field limit merely redistributes the energy among the available modes; thus the question of mode degeneracy is important. In the Bénard problem all rolls of fixes $|k_c|$ are equivalent and direction is then a continuous parameter at our disposal.

Malkus: These are assertions not duductions, one wishes to establish that this is what emerges.

For our choice of R_a there is a band of allowed $|k|$ of width $\sim T_a^{-1/12}$ but for convenience we shall restrict ourselves to a band of $O(T_a^{-1/6})$ about $|k_c|$, to avoid multiple scale analysis in the horizontal.

Malkus: Is this essential? Why are you being so restrictive considering supercritical equilibration when in fact a subcritical disturbance may be important? It seems a kinematic approach, essentially you are saying a square lattice of rolls works.

Childress: No, you do not know that one mode isn't selected dynamically.

Malkus: There is even a convection experiment with triangles! With an appropriate bottom surface you could probably even get a convection pattern in the form of the GFD dragon!

Pedlosky: You impose no planform restrictions other than no variation on small scales.

Spiegel: You can always put walls in the problem and select a single k.

Scales: $M \sim o(1)$, $k_c \sim o(T_a^{1/6})$ $R_{a_c} \sim O(T_a^{2/3})$

Let u_0 = velocity scale,

expansion parameter = $\varepsilon = T_a^{-1/6}$

Recall, $\alpha = T_a^{-1/6} \sim O\left(\frac{u_0 L \varepsilon}{\eta}\right)$ so then, balancing large scale diffusion against the $\nabla \times (\alpha \vec{B})$ term in the mean field equation, we have

$$\frac{-\eta B_0}{L^2} \sim \frac{\alpha B_0}{L} \implies u_0 = \frac{\eta}{L}\,\varepsilon^{-1/2} = O(T_a^{-1/12})$$

For the unit of time we choose the magnetic diffusion time: .

Finally write the temperature as

$$T = T_0 + (1 - z/L)\Delta T + \varepsilon \Delta T \theta$$

Keller: It is interesting that u_0 is determined independently of the magnetic field even though it is the back reaction of the field that should limit the velocity.

Childress: That's basically due to the linearity of the kinematic dynamo problem.

Pedlosky: **It** seems that your choice of $\frac{R_a - R_{a_c}}{R_{a_c}}$ imposes a **develop**ment time for an unstable mode. A dissipative time for scales of $T_a^{-1/6}$ leads to **a** Landau-type evolution equation. The other time unit, L^2/η, is much longer so that there is a rapid hydrodynamic equilibration which fixes in quasi-static equilibration and then one can imagine turning down the magnetic diffusivity to produce a slow drift.

Stern: This approach presumes that one can solve the stability problem for a finite amplitude velocity field without a magnetic field.

Malkus: We have for example Busse's $O(\varepsilon^2)$ expansion around the band or Pedlosky's technique.

We set $R_a = \varepsilon^{-4}\tilde{R}_a$ so that $\tilde{R}_a \sim O(1)$ (since $R_a \sim T_a^{2/3}$) and we presume **all** diffusivities are about equal i.e. $\nu \sim \kappa \sim \eta$ (although for the Earth $\frac{\kappa}{\eta} \sim 10^{-5}$ for which singular perturbation there is a "sharp" temperature field which does not diffuse while momentum and magnetic fields do).

Pedlosky: Strong differences in diffusivity hence a lag between fields introduces fluctuations in finite amplitude states which are otherwise stable.

Introduce horizontal scaling: $\frac{x}{\varepsilon} = \tilde{x}$, $\frac{y}{\varepsilon} = \tilde{y}$ (and by extension $\tilde{\nabla}_h$) and we have

<u>Continuity</u>: $\tilde{\nabla}_h \cdot \vec{u} + \varepsilon \frac{\partial w}{\partial z} = 0$

<u>Temperature</u>: $\varepsilon^{3/2}\frac{\partial \theta}{\partial t} + (\vec{u}\cdot\tilde{\nabla}_h)\theta + \varepsilon w \frac{\partial \theta}{\partial z} - \varepsilon^{-1/2}\frac{\kappa}{\eta}\tilde{\nabla}_h^2\theta - \varepsilon^{3/2}\frac{\kappa}{\eta}\frac{\partial^2\theta}{\partial z^2} - w = 0$

<u>Magnetic Field</u>: $\varepsilon^2\frac{\partial \vec{B}}{\partial t} - \tilde{\nabla}_h^2\vec{B} - \varepsilon^2\frac{\partial^2\vec{B}}{\partial z^2} = \varepsilon^{1/2}\tilde{\nabla}_h \times (\vec{u}\times\vec{B}) + \varepsilon^{3/2}\frac{\partial}{\partial z}(\hat{z}\times(\vec{u}\times\vec{B}))$

<u>Momentum</u>: $\frac{\eta}{\nu}\left[\varepsilon^3\frac{\partial \vec{u}}{\partial t} + \varepsilon^{3/2}(\vec{u}\cdot\tilde{\nabla}_h)\vec{u} + \varepsilon^{5/2} w\frac{\partial \vec{u}}{\partial z}\right] + \varepsilon\hat{z}\frac{\partial P}{\partial z} + \tilde{\nabla}_h P + \hat{z}\times\vec{u}$

$$= \varepsilon^2 M^2 \vec{B}\times(\tilde{\nabla}_h\times\vec{B}) + \varepsilon^3 M^2\left(\vec{B}\times\left(\hat{z}\times\frac{\partial\vec{B}}{\partial z}\right)\right) - \varepsilon\tilde{\nabla}^2\vec{u} - \varepsilon^3\frac{\partial^2\vec{u}}{\partial z^2} = \frac{\kappa}{\eta}\tilde{R}_a\,\varepsilon\,\theta'\,\hat{z}$$

where in the last equation we further define $\theta = \Theta(z,t) + \varepsilon^{1/2}\theta'(\tilde{x},\tilde{y},z,t,\varepsilon)$ (where Θ = mean temperature)

$$\varepsilon^{3/2}\frac{\partial\Theta}{\partial t} + \varepsilon^2\frac{\partial\theta'}{\partial t} + \varepsilon^{1/2}(\vec{u}\cdot\tilde{\nabla}_h)\theta' + \varepsilon w\left[\frac{\partial\Theta}{\partial z} + \varepsilon^{1/2}\frac{\partial\theta'}{\partial z}\right] - \frac{\kappa}{\eta}\tilde{\nabla}_h^2\theta' - \varepsilon^{3/2}\frac{\kappa}{\eta}\left[\frac{\partial^2\Theta}{\partial z^2} + \varepsilon^{1/2}\frac{\partial^2\theta'}{\partial z^2}\right] - w = 0$$

Notice in the momentum equation that the Coriolis force is a dominant term.

Finally we introduce expansions in the other fields:

$\vec{u} = \vec{u}_h + w\hat{z}$ where

$$\vec{u}_h = \vec{u}_h^{(0)} + \varepsilon^{1/2}\vec{u}_h^{(1)} + \ldots$$

$$w = w^{(0)} + \varepsilon^{1/2} w^{(1)} + \ldots$$

and

$$\vec{B} = \vec{B}^{(0)}(z,t) + \varepsilon^{1/2} B^{(1)}(\tilde{x},\tilde{y},z,t) + \ldots$$

$$p = p^{(0)} + \ldots$$

$$\theta' = \theta^{(0)} + \ldots$$

where fluctuations first enter at order $\varepsilon^{1/2}$ (eventually but produce other effects, e.g. a mean flow in the velocity field, through nonlinear coupling).

Pedlosky: What about a boundary layer on the horizontal surface?

Childress: The boundary conditions are stress free, isothermal, so there is no boundary layer. But even if you inserted one with no slip conditions it doesn't affect results, to the order we shall work, in the high T_a limit.

Zeroth order $\quad \tilde{\nabla}_h \cdot \vec{u}_h^{(0)} = 0 \quad$ Continuity

$$\tilde{\nabla}_h p^{(0)} + \hat{z} \times \vec{u}_h^{(0)} = 0 \quad \text{Momentum}$$

which is readily solved with $\vec{u}_h^{(0)} = \hat{z} \times \tilde{\nabla}_h p^{(0)} \equiv -\nabla \times (\hat{z}\, p^{(0)})$ (div(curl) = 0 in continuity equation)

Note that $\frac{\partial p}{\partial z} - O(1)$ so the velocity field is a baroclinic geostrophic balance of pressure field and Coriolis forces.

In $O(\varepsilon^{1/2})$ we find the same set of equations for $\vec{u}_h^{(1)}$ and $p^{(1)}$ so that a suitable choice of normalization allows us to set $\vec{u}_h^{(1)} = p^{(1)} = 0$

$$O(\varepsilon):\ \tilde{\nabla} p^{(2)} + \hat{z} \times \vec{u}_h^{(2)} = \tilde{\nabla}^2 u_h^{(0)}$$
$$= -\frac{\partial w^{(0)}}{\partial z}$$

Oliver; Where is the fluctuating temperature field?

Childress: That enters only in the vertical component of the equations.

The first equation is solved by $\vec{u}_h^{(2)} = \tilde{\nabla}\tilde{\nabla}^2 p^{(0)} + \hat{z} \times \tilde{\nabla} p^{(2)}$ then the second equation yields $\tilde{\nabla}^4 p^{(0)} = -\frac{\partial w^{(0)}}{\partial z}$. The z component at order ε yields

$$\frac{\partial p^{(0)}}{\partial z} = \frac{\kappa}{\eta}\, \tilde{R}_a^{(0)} \theta^{(0)} + \tilde{\nabla}^2 w^{(0)} \text{ where } \tilde{R}_a = R_a^{(0)} + \varepsilon\, \tilde{R}_a^{(2)} + \ldots$$

Pedlosky: Taking the curl of the $\vec{u}_h^{(2)}$ equation we see that the diffusion of the vertical component of vorticity is balanced by fortex tube stretching, the pressure gradient is in geostrophic equilibrium, it is not just hydrostatic; as in Stewartson's work, involving an $\epsilon^{1/3}$ expansion with buoyancy, internal stretching is much greater than Ekman boundary layer stretching.

The $O(\epsilon^0)$ temperature equation is $\frac{\kappa}{\eta}\tilde{\nabla}^2\theta^{(0)} + W^{(0)} = 0$. Applying ∇^2 to this, ∇^4 to the $O(\epsilon)$ z component of the momentum equation, and eliminating $\tilde{\nabla}^4 p^{(\epsilon)}$ yields

$$\left(\tilde{\nabla}^6 - \tilde{R}_a^{(0)}\tilde{\nabla}^2 + \frac{\partial^2}{\partial z^2}\right)W^{(0)} = 0$$

Let $W^{(0)} = \sum_{|\vec{k}|=k_c} \hat{W}\,(e^{i\vec{k}\cdot\tilde{\vec{x}}_h}\sin(\pi z)$ then $\tilde{R}_a^{(0)} = \frac{k^6+\pi^2}{k^2}$ which recovers the asymptotic result from the last lecture. Closure at any order, n, requires the $n+2$ order equation.

Also the heat flux is known from $\Sigma|\hat{W}|^2$ by the adjointness (or solubility) condition applied to $\tilde{R}^{(2)}$. In this weak field model $(M \sim O(1))$ the B field enters only in the $O(\epsilon^3)$momentum equations and thus does not affect the heat flux. A different scaling appropriate to the intermediate field regime would bring in B at second order (albeit for this model it seems to be no stable solutions in such a range) while in the strong field regime magnetoconvection would greatly modify the heat flux and B would be present in lowest order and it would be inappropriate to expand about the zero field Bénard problem.

Stern: When does the linear stability problem for $\vec{B}$ enter?

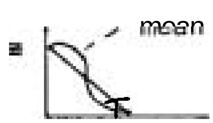
Childress: The $O(\epsilon^3)$ equations give the modal amplitudes.

Pedlosky: Note the field corrections are of order (amplitude)^2.

If we consider terms of $O(\epsilon^{3/2})$ in the temperature equation and average over the horizontal we obtain

$$\frac{\partial\Theta}{\partial t} + \frac{\partial}{\partial z}\langle W^{(0)}\theta'^{(0)}\rangle - \frac{\kappa}{\eta}\frac{\partial^2\Theta}{\partial z^2} = 0$$

where the second term represents the convective heat flux, that is, there is a balance of kinetic energy and the mean field. This is how via the α-effect large scale motions are driven in a sphere. Convective motion leads to a new temperature profile which stabilizes quickly.



mean temperature with convective heat flux

We will write the mean field equation with this convective mode for small scale motion in the horizontal (compared with a length scale of $O(1)$ in the vertical). Smoothing is done by horizontal averaging in both the mean field and induction equations. Recall our expansion for the magnetic field

$$\vec{B} = \vec{B}^{(0)} + \epsilon^{?} \vec{B}^{(1)} + \ldots$$

where we write the mean field $\vec{B}^{(0)}$ as $\langle \vec{B} \rangle = (B_1(z,t), B_2(z,t), c$ i.e. no horizontal mean on the fluctuating part of the field. From $O(\epsilon^{1/2})$ terms in the magnetic field equation we have

$$-\tilde{\nabla}^2 \vec{B}^{(1)} = \vec{B}^{(0)} \cdot \bar{\nabla} \vec{U}^{(0)}$$

which expresses the interaction of the small scale velocity field with the mean magnetic field. Then, taking terms of $O(\epsilon^2)$ one obtains

$$\frac{\partial \vec{B}^{(0)}}{\partial t} - \frac{\partial^2 \vec{B}^{(0)}}{\partial z^2} - \frac{\partial}{\partial z}\left[\hat{z} \times (\vec{u}^{(0)} \times \vec{B}^{(1)})\right] = 0$$

which becomes

$$\frac{\partial \vec{B}^{(0)}}{\partial t} - \frac{\partial^2 \vec{B}^{(0)}}{\partial z^2} - \frac{\partial}{\partial z}\left[\hat{z} \times (\vec{u}^{(0)} \times \left[\tilde{\nabla}^2\right]^{-1} (\vec{B}^{(0)} \cdot \tilde{\nabla}) \vec{u}^{(0)})\right] = 0$$

Averaging in the horizontal (indicated above by brackets) yields, ultimately,

$$\frac{\partial B_i}{\partial t} - \frac{\partial^2 B_i}{\partial z^2} + 2\pi\lambda \frac{\partial}{\partial z}\left(\sin(2\pi z) m_{ij} B_j\right) = 0$$

where

$$m_{ij} = \begin{pmatrix} -\alpha_{21} & -\alpha_{22} \\ \alpha_{11} & \alpha_{12} \end{pmatrix}, \quad \alpha_{ij} = \sum_{\vec{k}} \frac{k_i k_j}{k^2} e(\vec{k}, t)$$

and e is essentially the energy in the mode $\vec{k}$. We can normalize by requiring $\alpha_{11} + \alpha_{22} = 2$. (For the isotropic case $\alpha_{11} = \alpha_{22} = 1$.) Note the factor of $\sin(2\pi z)$ which corresponds to the polarization of helicity by rotation. Specifying some initial distribution of amplitudes one finds there is a certain minimum value of λ required for a solution, which corresponds to a critical kinematic energy for the convective field.

The self-consistency for the problem enters in third order by an evolution equation for the amplitudes, given the magnetic field. Thus one can imagine iterating the problem numerically be giving $e(\vec{k}, c)$, finding $\vec{B}^{(0)}(z, \Delta t)$ which determines $e(\vec{k}, \Delta t)$. . . The evolution equation is of the form

$$\frac{de(\vec{k}, t)}{dt} = \left[\Gamma(t) - \sum_{\vec{k}'} \gamma\, \vec{k}, \vec{k}', t; B_i(z,t)\, e(\vec{k}', t)\right] e(\vec{k}, t)$$

with the requirement that since the kinetic energy (heat flux) is fixed, ( ) on both sides is identically zero. To simplify the problem one can imagine a system with discrete wave vectors and certain selection rules.

Keller: If the initial value (amplitude) of a mode is zero, it stays zero.

Stern: But only at first order, there are certainly mode-mode interactions in higher order.

Pedlosky: The total energy is determined hydrodynamically, so the rolls could arrange themselves to make $\vec{B} = 0$ couldn't they?

Childress: It doesn't happen, apparently, if two or more rolls are excited.

Pedlosky: Shouldn't the system try to break constraints?

Childress: Yes, but it never quite makes it.

Keller: It's like the oscillatory case with no dissipation.

Motivated by the spirit of the kinematic problem we will set

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Pedlosky: In considering all rolls with all possible distributions, how do you choose α ?

Childress: There is only one scalar, $\frac{\alpha_{11}}{\alpha_{22}}$ to within a rotation of coordinates. We now look for steady solutions.

The field equations are

$$2\pi\lambda \frac{\partial}{\partial z}\left(\sin(2\pi z)(-B_2)\right) - \frac{\partial^2 B_1}{\partial z^2} = 0$$

$$2\pi\lambda \frac{\partial}{\partial z}\left(\sin(2\pi z)\, B_1\right) - \frac{\partial^2 B_2}{\partial z^2} = 0$$

with boundary conditions $\frac{\partial B_i}{\partial t} = 0 \quad z = 0, 1$, i.e. perfectly conducting walls, carrying no tangential current. To eliminate uniform fields we require

$$\int_0^1 B_i = 0 \, .$$

If we define $B_1 + iB_2 = \phi$ the equations become

$$2\pi\lambda i \left(\phi \sin(2\pi z)\right)' - \phi'' = 0$$

integrating once $\phi(2\pi\lambda i \sin(2\pi z)) - \phi' = 0$ thus

$$\phi = e^{-i\lambda\cos(2\pi z)}$$

and $\int_0^1 e^{-j\lambda\cos(2\pi z)}\, dz = J_0(\lambda) = 0$ is the associated eigenvalue problem with a minimum of

$$\lambda_c \doteq 2.404.$$

Soward considered nonstationary fields and found a smaller λ_c.

For a more general mode distribution the steady result is

$$\delta \lambda_c \sim 2.404 \text{ where } \delta^2 (\det \alpha)^{1/2}$$

Proctor: Don't bounding techniques for the unsteady case give a (an absolute lower limit) value of $\pi/2$?

Childress: The smallest exhibited is λ x 1.

Keller: It doesn't seem you should beat the steady result with bounding techniques which are insensitive to unsteady $\vec{v}$.

If one considers a wave-like field of the form

$$B_i = e^{i\omega t} \sum A_i^{(m)} \cos (2m+1) \pi z$$

with α imposed the result is $\lambda_c \delta = 1.597$ and $\frac{\omega_c}{\pi^2} = -1.394$.

Alternatively we can compute the α matrix for a single, slowly rotating, roll (rotational frequency ω) with the result that $\lambda_c \delta = 1.08$ and $\frac{\omega}{\pi^2} = -.92$.

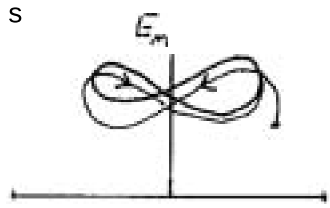
Spiegel: A rotating roll is not an exact solution of the convection problem.

Pedlosky: But the rotational rate is on a very long time scale (compared to hydrodynamic equilibration time).

Keller: This approach gives rise to an ordinary differential equation with periodic coefficients soluble with Floquet theory.

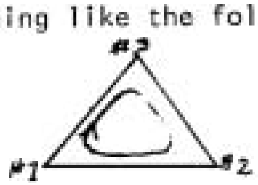
Now we turn on feedback and look at the results of some numerical experiments.

1) For two orthogonal modes there is a limit cycle behavior for energy distribution in the field and rolls



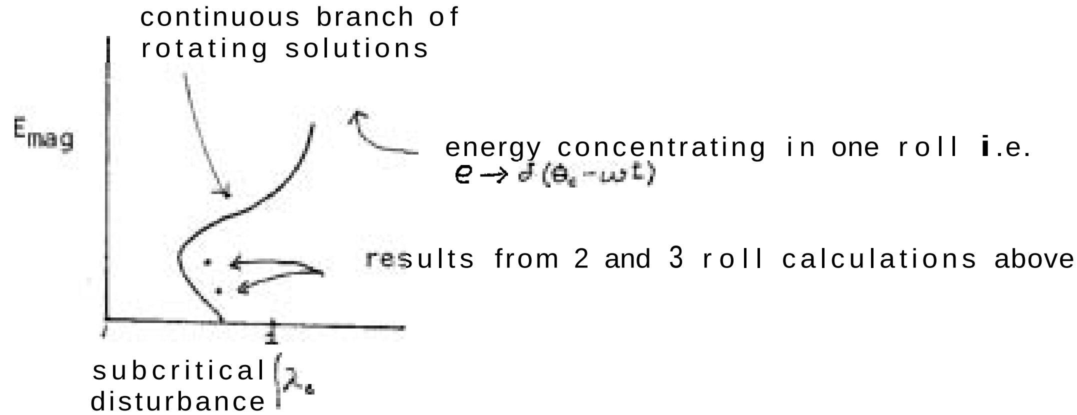
kinetic energy all in roll #2   kinetic energy all in roll #1

2) With three rolls oriented at 120° intervals the distribution of kinetic energy in the modes is something like the following

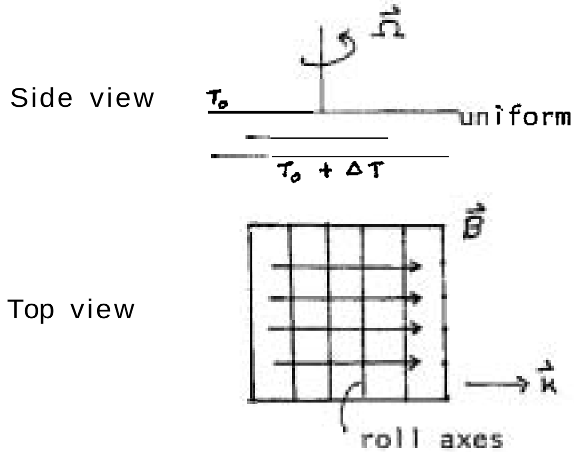


which can be thought of as the system mimicking one roll rotating.

3) Following Soward one considers a continuous angular distribution of energy. $e(\theta)\ 0 \leq \theta \leq 2\pi$ (corresponding to $N \to \infty$) for a class of motion with $e(\theta)$ corresponding to a fixed structure rigidly rotating at fixed angular velocity, i.e. the distribution has the form $e(\theta - \omega t)$. The following results are obtained:

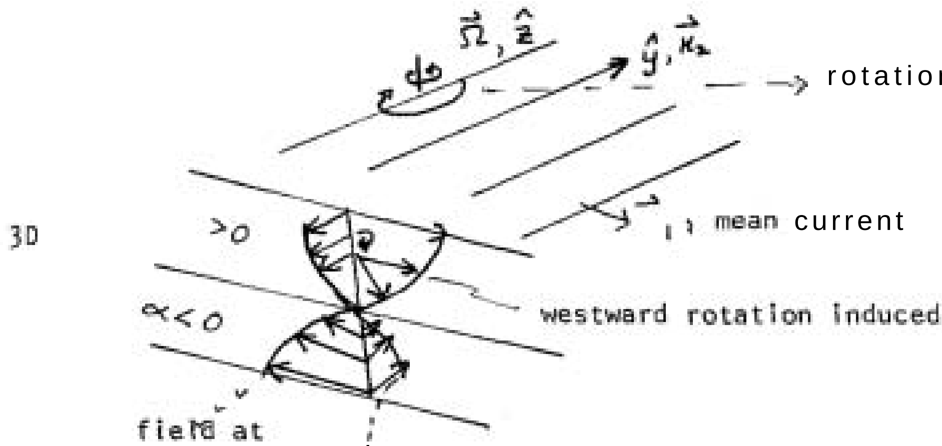


The dynamical picture for the weak field limit is as sketched below:



Malkus: This orientation of roll axes is the same as in the case of a weak shear field, for a strong field the roll axes line up with the field.

This suggests a physical mechanism to explain the rotating solutions. Roughly, roll axes stay orthogonal to some representative vertical average of the magnetic field. By the α-effect, mean current is generated perpendicular to roll axes, which acts to feed the orthogonal component of the field. This changes the actually direction of the field, and so rotation occurs. When one considers the polarization of helicity, the boundary conditions on the field, and the signs of the α-effect, one gets the following sketch (B_i is taken a proportional to $\cos \pi z$):



The rotation of rolls to the "west" is observed in the numerical solutions.

Keller: How about a different means for limiting B besides back reaction? It seems B has to be too large to limit the motion.

Proctor: No, it is otherwise a kinematical problem.

Malkus: Yes, there are certainly other mechanisms though.

Spiegel: Yes, saturation for example where $\eta = \eta(B^2)$.

Notes submitted by
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Lecture #9 COMMENTS ON THE SMOOTHING METHOD

Yesterday afternoon Joe Keller presented a general method of applying smoothing methods to "rough" problems. It might be useful to consider what the smoothing method applied to the dynamo problem is giving in its entirety.

In Keller's formalism the operator which gives the total field from the mean field is $[I + (M^{-1}(1-P)V)]^{-1}$, and in Lecture #3 we called it $[I-S]^{-1}$. Each operator is a differential operator. If we invert and substitute into the Mean Field Equation the result is

$$\left(\frac{\partial}{\partial t} - \eta\nabla^2\right)f = \nabla\times\left[S^2 + S^3 + S^4\right]f$$

where f is the smoothed field. The magnitude of the S^2 term is

$\frac{U^2 \ell}{(\ell^2\omega+\eta)}\left[1+\frac{\ell}{L}-\frac{\ell^2}{L^2}+\ldots\right]$ where the first term represents the α-effect and the second term is β. If we assume that $\ell^2\omega$ is smaller than or comparable to η and balance the diffusion with the α-effect we find that $\frac{\ell^2 L U^2}{\eta^2}\sim 1$ or, if $R=\frac{\ell u}{\eta}$ then, $\frac{L}{\ell}R^2\sim 1$, $R\sim\sqrt{\epsilon}$.

The β-effect is then always smaller than the 4-effect by a factor of ℓ/L. (This need not be true if ω is ordered differently, however), even if the α term were vanishingly small and we attempted to balance diffusion with the β term we discover that the ratio of the two terms is

$$\frac{\ell}{L}\,\frac{U^2\ell}{\eta L}\Big/\frac{\eta}{L^2}=R^2$$

so the β-effect is $O(R^2)$ compared to diffusion.

If we suppose that the second order statistics disappear (or at least that the α-effect vanishes), then the diffusion term must balance with the S^3 term. The ratio of the largest terms is $R^3\frac{L}{\ell}\sim 1$ so $R\sim\epsilon^{1/3}$ and in general, if (j - 1) order statistics vanish then $R\sim\epsilon^{1/j}$.

The Bénard Dynamo

In lecture #8 we learned that the kinetic energy of the flow in the rotating Bénard Dynamo tends to concentrate in a very narrow spectrum in wave number space and that the roll axes that are most easily excited tend to be aligned perpendicularly to the main field. In order to maintain this alignment the rolls rotate slowly with the main field. Compared to the spherical dynamo this model is ery easy to analyze but we should also consider the possible defects of the rotating Bénard dynamo.

One problem may arise from the degeneracy which allows multiple-roll solutions. If walls could be imposed which would not allow rotation of the rolls while still allowing the existance of several nearby wave numbers then an α-effect would still be present. In this case however the α matrix would be very anisotropic and as a result the components of the magnetic field will be of very different magnitudes.

It is also very likely that the Bénard Dynamo is unstable for larger fields than we considered. In the weak field analysis the total heat flux was fixed. Physically we would imagine that the addition of $\vec{B}$ field would further decrease the eigenvalues and the dynamo would take off.

If the Hartmann number is in the intermediate field range, $M \sim \frac{1}{\sqrt{E}}$, then the magnetic field terms enter into the perturbation equations at the same level that the heat flux is determined. The steady state solution with two orthogonal rolls was found to be unstable to the intermediate field, and numerical calculations by Yves Fautrell indicate that for certain conditions, Bénard dynamos in the strong field regime are also unstable.

Convective Dynamo Action in a Sphere

This topic will be difficult to study directly since no working models of a spherical convective dynamo have been fully explores. We will therefore limit ourselves to a few general properties imposed by the sphericity.

We start with the simplest system and work up in complexity by incorporating more of the physics. The simplest model is simply geostrophic motion.

Geostrophic Motion: Here we have only the balance of the pressure gradient with the Coriolis acceleration.

$$\nabla P + 2\vec{\Omega} \times \mathbf{u} \qquad \nabla \cdot \vec{u} = 0$$

and we require the normal component of the velocity to vanish on the sphere, S. Taking the curl we have

$$\frac{\partial \vec{u}}{\partial z} = 0$$

The only constant vector which can be tangent to both intercepts of S with a vertical line is parallel to $\vec{1}_\varphi$. Hence, by incompressibility, the velocity is only a function of ρ and we write it as

$$\vec{u} = \rho\,\omega(\rho)\,\vec{1}_\varphi, \quad \rho = (x^2 + y^2)^{1/2}$$

Geostrophic balance alone is not enough since it neglects convection and the coupling of the flow to the magnetic field, but we are interested in how motion similar to this might be set up by nonlinear processes in a convective system. Note that no geostrophic motion is possible if $\vec{u} = 0$ on S.

Taylor's Constraint: We expect the magnetic field to play some role in the dynamic balance in the core. Therefore we add a magnetic field and "everything else".

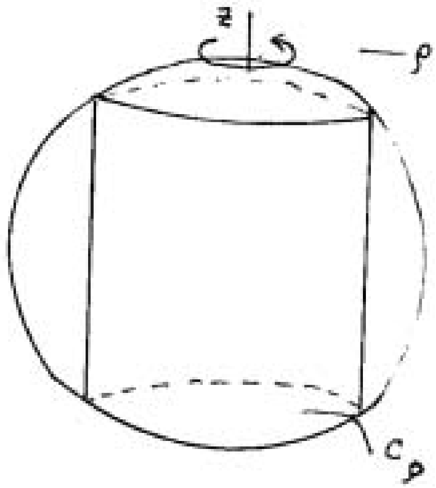
$$\nabla P + 2\vec{\Omega} \times \vec{u} + \frac{1}{\mu\rho_0}\vec{B} \times (\vec{\nabla} \times \vec{B}) = \vec{\mathcal{F}}^{ee}$$

In cylindrical component form this can be written

$$\mathcal{F}_z^{ee} = \frac{\partial P}{\partial z} + (\mu\rho_0)^{-1}\left[B_\rho\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right) - B_\varphi\left(\frac{1}{\rho}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right)\right]$$

$$\mathcal{F}_\varphi^{ee} = \frac{\partial p}{\partial \rho} + (\mu\rho_0)^{-1}\left[B_\varphi\left(\frac{1}{\rho}\frac{\partial \rho B_\varphi}{\partial\rho} - \frac{1}{\rho}\frac{\partial B_\rho}{\partial\varphi}\right) - B_z\left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial\rho}\right)\right] - 2\Omega u_\varphi$$

$$\mathcal{F}_\rho^{ee} = \frac{1}{\rho}\frac{\partial p}{\partial\varphi} + (\mu\rho_0)^{-1}\left[B_z\left(\frac{1}{\rho}\frac{\partial B_z}{\partial\varphi} - \frac{\partial B_\varphi}{\partial z}\right) - B_\rho\left(\frac{1}{\rho}\frac{\partial\rho B_\varphi}{\partial\rho} - \frac{1}{\rho}\frac{\partial B}{\partial\varphi}\right)\right] 2\Omega u_\rho,$$



Consider the φ-component. **If** we integrate this equation over the cylindrical surface inbedded in the sphere, the result will be the torque on the cylinder. We are assuming that all of the physics is included so we **may** set $\vec{u} = 0$ on S, and so there is no flux through the spherical caps. **Only** two of the terms contribute to the integral of the φ =equation.

$$I(\rho) = \int_{C_\rho} \mathcal{F}_\varphi^{ee}\, dS = -\frac{1}{\mu\rho_0}\int_{C_\rho}\left(B_z\frac{\partial B_\varphi}{\partial z} + \frac{\partial B_\rho}{\rho}\frac{\partial\rho B_\varphi}{\partial\rho}\right) dS$$

This is Taylor's Constraint. This condition must be satisfied by $\vec{B}$ at every instant. We can write Taylor's Constraint in a slightly different form **if** $\vec{B}$ is axisymmetric and

$$\vec{B} = B\,\vec{1}_\varphi + \nabla\times A\,\vec{1}_\varphi \quad , \quad B = 0 \text{ on } S.$$

We substitute the axisymmetric form for $\vec{B}$ into the equation

$$-\mu\rho_0 I(\rho) = \int_{-\sqrt{L^2-\rho^2}}^{\sqrt{L^2+\rho^2}}\left(\frac{\partial\rho A}{\partial\rho}\frac{\partial B}{\partial z} - \frac{\partial A}{\partial z}\frac{\partial\rho B}{\partial\rho}\right) dz$$

$$= -2\pi\frac{d}{d\rho}\,\rho^2\int_{-\sqrt{L^2-\rho^2}}^{\sqrt{L^2+\rho^2}}\frac{\partial A}{\partial z}\,B\,dz$$

Thus, **if** the φ-component of force is negligible, **the** integral **must** vanish .learly

$$\int_{-\sqrt{L^2-\rho^2}}^{\sqrt{L^2-\rho^2}}\frac{\partial A}{\partial z}\,B\,dz = \frac{C}{\rho^2} = 0$$

Note that the parity of A and B for dipole symmetry (i.e. A even, B odd) should make this a non-trivial constraint. What are the implications on the dynamo problem?

A Non-existance Theorem. Consider a dynamo with the following four properties:

(i) the dynamo is essentially axisymmetric,

(ii) $\vec{\mathfrak{F}}^{ee} = 0$ or at least negligible,

(iii) B >> A (e.g. Braginskii's Dynamo where $A = O\left(\frac{1}{R} B\right)$

(iv) there is no α-effect in the mean field equation for B.

Then the dynamo runs down in that B decays to 0.

Proof: Integrate the z-component equation for B, keeping only the dominant terms.

$$P = -\frac{1}{2\mu\rho_0} B_\varphi^2 + \pi(\rho, t)$$

imilarly, from the ρ-component equation we get

$$\frac{\partial P}{\partial \rho} + \frac{1}{\mu\rho_0} B_\varphi^2 \frac{1}{\rho} \frac{\partial(\rho B_\varphi)}{\partial \rho} - 2\Omega u_\varphi = 0$$

Combine these two equations

$$2\Omega u_\varphi = \frac{1}{\mu\rho_0} \frac{B_\varphi^2}{\rho} + \frac{\partial \pi}{\partial \rho}$$

If we define $u_\varphi = \rho\omega$ then $\omega = \frac{1}{2\Omega\mu\rho_0}\left(\zeta(\rho) + \frac{B^2}{\rho^2}\right)$.

Now B satisfies Braginskii's Equation based on smoothing,

$$\frac{\partial B}{\partial t} + \rho \vec{V}_p \cdot \nabla_p \rho^{-1} B = \eta\left(\nabla^2 - \frac{1}{\rho^2}\right) B + \left[\nabla\omega \times \nabla\rho A\right]_\varphi$$

Substitute for ω from the above expression, multiply by B/ρ^2 and integrate over the spherical core

$$\frac{d}{dt}\int_V \frac{B}{\rho^2}\, dV + \int_V \frac{B}{\rho} \vec{V} \cdot \nabla\left(\rho^{-1} B\right) dV = \int_V \frac{B\eta}{\rho^2}\left(\nabla^2 - \frac{1}{\rho^2}\right) B\, dV + \int_V \frac{B}{\rho^2}\left[\nabla\omega \times \nabla\rho A\right]_\varphi dV$$

The second term vanishes because $\vec{V}$ is divergence Free and B vanishes on S. The third term can be rewritten as

$$\int \frac{B\eta}{\rho^2}\left(\nabla^2 - \frac{1}{\rho^2}\right) B\, dV = \eta \int \frac{B}{\rho^2}\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\right) B\, d\rho\, dz$$

$$= \eta \int \nabla\left(\frac{B}{\rho}\right)^2 dV.$$

The last term can be separated into two parts.

$$\int \frac{B}{\rho^2}\left[\nabla\omega \times \nabla\rho A\right]_\varphi dV = \frac{1}{2\Omega\mu\rho_0}\int \frac{B}{\rho^2}\left(\nabla\zeta \times \nabla\rho A\right)_\varphi dV + \frac{1}{2\Omega\mu\rho_0}\int \frac{B}{\rho^2}\left(\nabla\frac{B}{\rho^2} \times \nabla\rho A\right)_\varphi dV$$

But $\int \frac{B}{\rho}\left(\nabla\zeta \times \nabla\rho A\right)_\varphi dV = 0$ by Taylors constraint and the final term can be further expanded.

$$\int \frac{B}{\rho^2}\left(\nabla \frac{B^2}{\rho^2} \times \nabla \rho A\right)_\varphi dV = \int\left[\int \frac{B}{\rho}\left[\frac{\partial}{\partial z}(B^2/\rho^2)\frac{\partial \rho A}{\partial \rho} - \frac{\partial (B^2/\rho^2)}{\partial \rho}\frac{\partial \rho A}{\partial z}\right] d\rho\right] dz$$

$$= \iint \frac{2}{3}\left[\frac{\partial}{\partial z}(B^3/\rho^3)\frac{\partial \rho A}{\partial \rho} - \frac{\partial}{\partial \rho}(B^3/\rho^3)\frac{\partial \rho A}{\partial z}\right] d\rho\, dz$$

$$= \iint \frac{2}{3}\left[\frac{\partial}{\partial z}\left(\frac{B^3}{\rho^3}\frac{\partial \rho A}{\partial \rho}\right) - \frac{\partial}{\partial \rho}\left(\frac{B^3}{\rho^3}\frac{\partial \rho A}{\partial z}\right)\right] d\rho\, dz$$

$$= 0$$

since $B = 0$ on S.

Thus, the energy equation for B reduces to

$$\frac{1}{2}\frac{d}{dt}\int\left(\frac{B}{\rho}\right)^2 dV = -\eta\int\left(\nabla \frac{B}{\rho}\right)^2 dV.$$

B must decay exponentially, if the four assumptions we made initially are to hold. It is of interest to note that the addition of Taylor's Constraint was sufficient to destroy the α-effect in Braginskii's kinematic model.

Breaking the theorem. In order to have dynamo action in a sphere one or more of the conditions must be relaxed. Braginskii (1975) hypothesized that $\vec{\mathcal{F}}^{ae}$, though small, is not negligible. He proposes that $A_z << A_\rho$ so that the poloidal field is almost vertical. The core-mantle coupling is important due to eddy currents induced in the mantle. α is prescribed and there is buoyancy in the Braginskii model. The object is to discover the function $\zeta(\rho,t)$ given all of the assumptions.

Another possible wav to break the theorem is to include two α-effects or put in a radial dependence of $\mathcal{F}^{ee}$. Malkus and Proctor proposed a model where the α^2-effect is important. If we take a plausible α (one that is odd in z), the B field can be prevented from dying away. If we write the force equation as

$$\vec{\mathcal{F}}^{ee} = -\varepsilon\frac{d\vec{u}}{dt} + E\nabla^2\vec{u}$$

we can regard this as a predictive equation. By evolving the fields numerically, the role Taylor condition can be investigated. The result seems to be numerical equilibration to the Taylor condition.

The Convective Dynamo. Nondimensionalizing the equations whould give a good indication of the problems we face in attempting to solve the problem of a complete convective dynamo.

The variables in the problem can be scaled as follows:

Temp $\sim \frac{Q_s L^2}{\rho_0 C_p \eta}$ Magnetic Field $\sim (2\Omega \mu \rho_0 \eta)^{1/2}$

Unit of speed $\sim \frac{\eta}{l}$

The resulting nondimensional equations are

$$S \frac{d\vec{u}}{dt} + \vec{1}_z \times \vec{u} + \nabla P + \vec{B} \times \nabla \times \vec{B} = -\tilde{R}_a \vec{1}_r T + E \nabla^2 \vec{u}$$

$$\frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T - \frac{K}{\eta} \nabla^2 T = 1 \qquad \text{(uniform heating)}$$

$$\frac{\partial B}{\partial t} - \nabla^2 \vec{B} = \nabla \times \vec{u} \times \vec{B}$$

The values of the coefficients for the earth core can be estimated as,

$$S = \frac{\eta}{2\Omega L^2} \sim 10^{-9} \qquad \tilde{R}_a = \frac{\alpha g Q_s L_s}{2\rho C_p \eta^2 \Omega} \sim 100$$

$$E = \text{Ekman No.} \sim 10^{-14} \qquad \frac{K}{\eta} \sim 10^{-6}$$

Calculations by Proctor indicate that we can probably neglect terms multiplied by S and E. The effect of large $\tilde{R}_a$ still remains to be solved. If there were some region in a sphere where convection were allowed to occur, it might be possible to make the convective system thin while increasing the R_a. This would force the existance of one small length scale to be important in the problem so a smoothing method could be used. Then, in a thin geometry we could see if we recovered results obtained previously.

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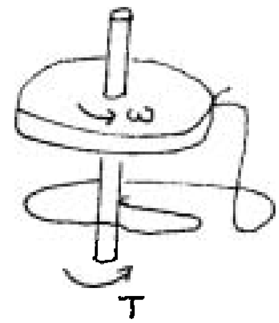
In order to approximately study the convective dynamo there are several useful approaches quite different from formal asymptotics but still very useful for understanding the dynamics of the system at an almost structural level. There are many examples of simple models that exhibit behavior very similar to the record of magnetic reversals of the earth's field. We will examine several of these models.

The simple disc dynamo consists of a disc driven by a constant torque and spinning in a magnetic field. This model has the equivalent of an ω-effect without an α-effect (although at this level we cannot attach much of a distinction between them). If the rotation rate is large enough the field will increase.



The idea in looking at this simple laboratory model of a dynamo is to formulate ordinary differential equations which capture the essential elements of the system. The two equations describing the currents and the torque are

$$L\dot{I} + RI = M\omega I$$

and

$$C\dot{\omega} = T - MI^2$$

where C is the inertia and M is the mutual inductance of the disc. If these are then nondimensionalized the result is

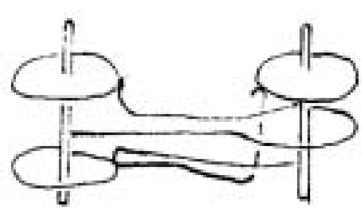
$$\dot{x} + \mu x = xy$$
$$\dot{y} = 1 - x^2$$

These can be solved in the form

$$(y - \mu)\,dy = \left(\frac{1 - x^2}{x}\right) dx.$$

If we look then at the phase plane we see that oscillitory solutions exist but the dynamo cannot reverse.

The coupled-disc dynamo of Rikitake is slightly more complex than the simple disc model.



We can think of the current through one loop as representing the poloidal field while the other represents the toroidal field. Insofar as the kinematic dynamo is concerned we would call this a model of an α^2-dynamo (or ω^2-dynamo). Unlike simple disc, this coupled system exhibits aperiodic reversals (see e.g. Cook and Roberts, 1970).

Approximate models can be considered from several points of view. The simple disc dynamo is an example of an analog for dynamo. They are similar in some aspects of their behavior but that is all. The coupled discs dynamo is an example of a heuristic model. An attempt has been made to put some physics into the model: in this case the coupling of the poloidal and toroidal fields.

The simple disc with a shunt carried the heuristic process one step further. Malkus and Howard proposed that we should reason not only from how the laboratory dynamo works but also on the basis of how the theoretical models we have been looking at would work. Consider a mode1 with a poloidal field, A, and a toroidal field, B. In an $\alpha\omega$-kinematic dynamo we could write the following equations to describe the time dependence of A and B.

$$\dot{A} + \mu A = \alpha B$$
$$\dot{B} + \eta B = \omega A$$

μ and η are the effective diffusivities for the two fields. If we set $\alpha = \omega$, and $\omega = \omega_2$, we would be back to the two-disc dynamo model. Instead of this, we take α to be a constant as before, but allow ω to vary according to the equation,

$$C\dot{\omega} = T - \nu\omega - AB$$

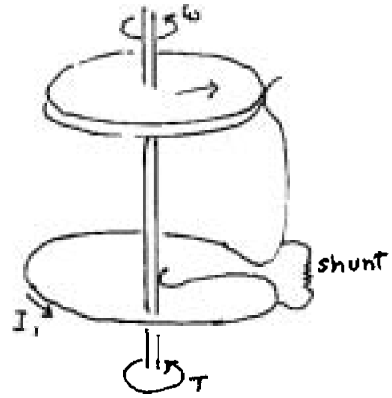
T is the external torque on the system, $\nu\omega$ describes the viscous torque and AB is the reaction of the field back on the system. It is reminiscent of the $\hat{\varphi}$-component of the momentum equation in cylindrical coordinates,

$$\frac{\partial u_\varphi}{\partial t} + \frac{1}{\mu\rho}(\vec{B} \times \nabla \times \vec{B})_\varphi - \nu L_\varphi = F_\varphi^{ee}$$

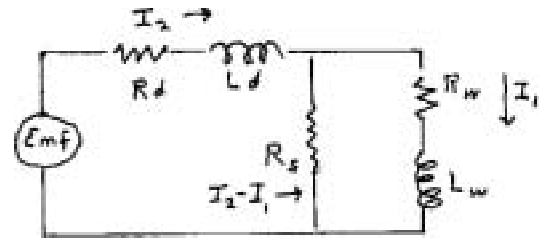
In order to realize this system mechanically, it is only necessary to add a shunt- to the simple disc dynamo.

The equations for this dynamo model are:

$$M\omega I_1 = L_d \dot{I}_2 + R_d I_2 + R_s (I_2 - I_1)$$
$$(I_2 - I_1) R_s = L_w \dot{I}_1 + I_1 R_w$$
$$C\dot{\omega} = T - M I_1 I_2 - \nu\omega$$



The Shunted Dynamo



Circuit Diagram

Kay Robbins transformed the variables in the following manner:

$$[t, \omega, I_1, I_2] \qquad [\tau t, \delta(x - \tfrac{R_s}{M}), \beta y, \alpha z]$$

where $\tau \equiv \dfrac{L_d}{R_d + R_s}$; $\delta \equiv \dfrac{(R_w + R_s)(R_d + R_s)}{R_s M}$

$$\alpha^2 \equiv \frac{R_w + R_s}{M R_s \nu \delta} \qquad ; \beta = \frac{R_s \alpha}{R_w + R_s}$$

Then the equations become

$$dx = R - yz - \nu x$$

$$\frac{dz}{dt} = xy - z$$

$$\frac{dy}{dt} = \sigma(x - y)$$

where $\sigma \equiv \dfrac{(R_w + R_s)\tau}{L}$,

$$R \equiv \frac{1}{\delta}\left[\frac{T}{\nu} + \frac{R_s}{M}\right]$$

Three steady-state solutions to these equations exist. The easiest to consider is the case of convection without fields so that $y = z = 0$ and $x = R/\nu$. To examine the stability of this case we linearize the equations around the point $(R/\nu, 0, 0)$ by putting $x \equiv R/\nu$; $y, z \propto e^{\lambda t}$. The equations reduce to

$$\begin{vmatrix} \lambda + \sigma & -\sigma \\ -R/\nu & \lambda + 1 \end{vmatrix} = 0$$

So $\lambda = -\frac{1}{2}(\sigma + 1) + \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(R/\nu - 1)}$ for the unstable root. If $R/\nu < 1$ all solutions approach the zero field solution. If $R/\nu > 1$ the eigenvalue is positive and the field is unstable to the addition of magnetic field in the weak limit. The other critical points have nonzero magnetic field. If we repeat the above procedure and linearize around the points $(1, \pm\sqrt{R-\nu}, \pm\sqrt{R-\nu})$ the eigenvalue problem reduces to a solution of the equation

$$\lambda^3 + (\nu + 1 + \sigma)\lambda^2 + (\nu\sigma + R)\lambda + 2(R-\nu)\sigma = 0$$

As the constant term, $2\sigma(R-\nu)$ is positive, the necessary and sufficient condition for λ to have a positive real part is that the cubic equation can be written in the following form:

$$\{\lambda - (\lambda_r + i\lambda i)\}\{\lambda - (\lambda_r - i\lambda i)\} \qquad (\lambda + \lambda_r') = 0 \qquad \lambda_r' > 0$$

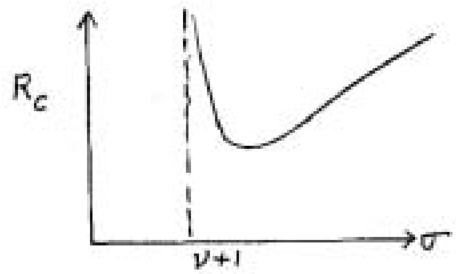
For the critical point $\lambda_r \equiv 0$ then

$$\lambda^3 + \lambda_r'\lambda^2 + \lambda_i^2\lambda + \lambda_i^2\lambda_r' = 0$$

If we compare terms with the eigenvalue equation we find that at the critical point $2\sigma(R-\nu) = (\sigma + \nu + 1)(\sigma\nu + R)$.

Therefore $R_c = \dfrac{\sigma\nu(\sigma+\nu+3)}{\sigma - \nu - 1}$

As R increases, the first positive real value of R occurs for complex The dynamo is unstable to growing oscillations for $R > R_c$.



The question of behavior below R_c is not answered by the linear theory, but solutions of the dynamo equations can be unstable well below the point of linear stability. It turns out that there is a critical value of R such that the fields characterized by $R < R_{sc}$ are completely stable. R_{sc} has been determined numerically for the system of equations we have considered but it has not proved possible to calculate R_{sc} analytically.

<u>Convective models</u>

We keep the same two basic equations for the poloidal and toroidal fields,

$$\dot{A} + \mu A = \alpha B$$
$$\dot{B} + \eta B = \omega A$$

but now we attach a slightly different meaning to the forcing equation. In a convective model ω should represent the response to thermal driving in the presence of a magnetic field, i.e. a modified thermal wind. If we let Θ denote the mean temperature perturbation then for ω we write (neglecting "inertia")

$$\omega = k_1 \Theta - k_2 q (A, B).$$

In other words, the distortion of the mean temperature field gives rise instantaneously to ω.

We still need equations for α and Θ so we should consider the effects of small scale quantities. Let θ be the small scale temperature and let u be the small scale velocity. Ideally θ should be correlated to u by an equation of the form:

$$\dot{\theta} + \chi' \theta = K u (1 - k \Theta).$$

The term on the right represents the effect of the advection on the small scale of the distortion of the mean temperature profile. We assume, instead, that θ is related by the simpler equation,

$$\theta = k_3 u (1 - k \Theta),$$

so that if $\langle u\theta \rangle$ is the convective heat flux then

$$\langle u\theta \rangle = k_3 \langle u^2 \rangle (1 - k\Theta \rangle$$

It is the convective heat flux which distorts the mean temperature profile, so

$$\dot{\Theta} + \chi \Theta = k_3 \langle u^2 \rangle (1 - k \Theta).$$

Only α^2 remains to be determined. In first-order field models α can be written

$$\alpha = \langle u^2 \rangle \lambda (A, B).$$

Letting λ be a function of A and B allows for effects absent in two scale smoothing. But take λ to be a constant here. We still need an equation for α or $\langle u^2 \rangle$. Multiply the momentum equation by u and average, then, if the inertia is small $(\varepsilon << 1)$ the equation is

$$\varepsilon \frac{d}{dt} \langle u^2 \rangle = \left[R_a (1 - k \Theta) - Q(A, B) - R(A, B) \alpha \right] \langle u^2 \rangle$$

where $Q(A,B)$ is the back-reaction of the mean field and $R(A,B)$ of the small scale magnetic field. After rapid equilibration, with $R(A,B)$ taken as 1.

$$\alpha = R_a (1 - k \Theta) - Q (A, B)$$

The equations are then

$$\dot{A} + \mu A = \left[R_a (1 - k \Theta) - Q \right] B$$

$$\dot{B} + \eta B = (k_1 \Theta - k_2 q) A$$

$$\dot{\Theta} + \chi \Theta = k_3 \left[R_a (1 - k \Theta) - Q \right] (1 - k \Theta)$$

Notice that if ω is identified with the mean temperature perturbation and some nonlinear effects are ignored, one recovers the model of Malkus and Howard. Equilibria and stability are highly dependent on the values of the parameters. In some models AB is put in for Q. We would at least like to know if Q is positive definite and perhaps we can get it from the Physics.

Kennet's "ABCDE" Model looks at the full equations but considers only certain modes. It includes both convection and the Lorenz force. The model assumes only one temperature, velocity and mean temperature perturbation modes.

$$\vec{u} = \left(C\frac{\partial f}{\partial x}(x,y)\cos m\pi z,\ C\frac{\partial f}{\partial y}\cos m\pi z,\ f\sin m\pi z\right)\ u(t)$$

$$\theta = f\sin m\pi z\ \ \theta(t)$$

$$\Theta = \sin 2m\pi z\ \ \Theta(t)$$

$$\vec{B} = \left(d\frac{\partial}{\partial x}g(x,y)\cos n\pi z,\ d\frac{\partial g}{\partial y}\cos n\pi z,\ g\sin n\pi z\right) A(t) + \left(\frac{\partial h(x,y)}{\partial y}, -\frac{\partial h}{\partial x}, 0\right)$$

Then, if we disregard positive constant multipliers the equations have the form:

$$\dot{u} = \theta - u - AB$$
$$\dot{\theta} = -\theta + u(1 - k\Theta)$$
$$\dot{\Theta} = -\Theta + u\theta$$
$$\dot{A} = -A + uB$$
$$\dot{B} = -B + uA$$

The equations are very similar to Soward's model. The magnetic field has two components. The convective system without the magnetic back reaction can be compared with the shunted disc model.

We end with one possible idea for further research. It may be possible to apply variational principles to obtain dynamos. We would look at the system which maximizes the growth of magnetic energy given a particular state. If we assume that all the energy flows into the mode which causes *the* greatest maximum growth, an α-effect would be generated.

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SEMINARS
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ABSTRACTS OF SEMINARS

ON FLUID MOTION AT THE SURFACE OF THE CORE

Edward R. Benton

Two different methods of obtaining unique horizontal fluid motions adjacent to the core-mantle boundary using geomagnetic measurements at earth's surface are considered. The mantle is taken as a spherical annulus of inner (outer) radius $r = b = 3485$ KM, ($r = a = 6371.2$ KM). The core fluid is assumed to be sufficiently highly conducting that the frozen flux assumption holds to leading order, and to the same order the mantle is an insulator. However, core conductivity is not supposed sufficiently large that a current sheet exists at $r = b$ and the magnetohydrodynamic boundary layer there is largely ignored (on the basis that, because the core magnetic Prandtl number is small, then so also should be the jump in $\hat{r} \times \vec{B}$ across the layer, at least if it is of Ekman-Hartmann type).

For the first method, continuity of $\vec{B}$ across $r = b$, together with the insulating nature of the mantle assures that one can find $\vec{B}$ at the top of the core. Backus (1968) considered what velocity information could be obtained primarily from the radial frozen flux induction equation. His result, that null-flux curves (on which $B_r = 0$) move with the fluid, only determines the core motion orthogonal to those contours, as he emphasized. Here we add further ingredients from other components of $\partial \vec{B}/\partial t = \nabla \times (\vec{v} \times \vec{B})$. The following results are obtained:

(1) At 18 points on $r = b$ as the field presently stands B_r and B_ϕ simultaneously vanish and these points are fluid tracers if the angular speed of westward drift about earth's axis is locally uniform in latitude, i.e. if $\partial(w/\sin\theta)/\partial\theta \doteq 0$, where $w = \vec{v}\cdot\hat{\phi}$, and θ is colatitude. This can subsequently be checked.

(2) At six points on $r = b$, B_r and B_θ simultaneously vanish and these points are fluid tracers if $\partial v/\partial\phi \doteq 0$, where $v = \vec{v}\cdot\hat{\theta}$. This can subsequently be checked.

(3) At 17 points on $r = b$, B_θ and B_ϕ simultaneously vanish and these points are fluid tracers if the horizontal motion, at the edge of the boundary

layer is, at most, a linear function of depth, i.e. if $\partial(V/r)/\partial r \doteq 0$, $\partial(W/r)/\partial r \doteq 0$. There is no direct way to verify or refute this assumption.

In the second method, we attempt to construct a systematic perturbation procedure for the magnetohydrodynamics at the core-mantle boundary which can lead to unique velocities, not just at isolated points or curves on r = b, but rather nearly everywhere on r = b. Additional physical assumptions are, of course, needed and what is put forward (still very much in the formative stages) is not what one would claim to be a picture of what is actually occurring, but rather only that it is one of many models that could be achieved. Its virtue is its solvability.

Scaled equations of interest are

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \cdot \vec{E} = q$$
$$\nabla \times \vec{B} = \vec{j}, \qquad \nabla \times \vec{E} = -\partial \vec{B}/\partial t, \qquad (1)$$
$$\vec{j} = \sigma(\vec{E} + \vec{V} \times \vec{B})$$

where the scale factors for ($\vec{B}$; $\vec{V}$; $\vec{j}$; $\vec{E}$; q; and σ) are, respectively, $(B_0; U; \mu_0^{-1} B_0 L^{-1}; UB_0; \epsilon_0 U B_0 L^{-1}; \sigma_0)$. Here, we think of $B_0 \sim 10^{-3}$ tesla (10 gauss), U = 4.10^{-4}m/sec (westward drift speed), L = 300 KM (a bit less than the smallest scales resolved at r = b by spherical harmonic field models truncated at N = 12). Time is scaled with advection, LU^{-1} ($\doteq$ 24 years with the above values). The two small dimensionless parameters of the problem are then (in terms of dimensionsal parameters)

$$\eta_m \equiv \mu_0 \sigma_m(b) UL, \qquad \eta_c = (\mu_0 \sigma_c UL)^{-1}$$

where $\sigma_m(b) = \sigma_0$ is the conductivity at the base of the mantle, say 200 mho/m and σ_c the uniform core conductivity of $3 \cdot 10^5$ mho/m. For these values, $\eta_m \doteq 0,030$, $\eta_c \doteq 0.022$.

The idea is to solve the system (1) throughout the mantle and sown to the edge of the boundary layer subject to $|B| = O(r^{-3})$, $|\vec{E}| = O(r^{-2})$ as $r \to \infty$, with $\vec{B}$ known at r = a at two epochs separated by a few decades (or $\vec{B}$ and $\dot{\vec{B}}$ known at a single epoch) and with the following continuity conditions across the core-mantle boundary:

$$\langle \vec{B} \rangle = \langle \vec{j} \rangle = \langle \hat{r} \times \vec{E} \rangle = \langle \hat{r} \cdot \vec{V} \rangle = 0, \langle \sigma \rangle \neq 0, \langle \hat{r} \cdot \vec{r} \rangle \neq 0, \langle \hat{r} \times \vec{V} \rangle \neq 0.$$

The velocity in the mantle is zero and the object is to find the slip velocity just inside the core-mantle boundary layer.

The quantities $\vec{B}$, $\vec{V}$, $\vec{j}$, $\vec{E}$ are assumed to have expansions of the form

$( )_0 + \eta_m ( )_1 + \eta_c ( )_2 + \cdots$ However, (and this is a central assumption that needs verification or refutation) volume charge in the mantle is assumed absent at zeroth order. The argument is that if σ_m were truly zero, no charge from the core could teak into it and we assume no one put charge in the mantle to begin with; so q_m ought to be proportional to σ_m as $\sigma_m \longrightarrow 0$. We now have $q = \eta_m q_{M_1} + O(\eta_m^2, \eta_m \eta_c)$, $q_c = q_{c_0} + \eta_m q_{c_1} + \eta_c q_{c_2} + , \cdots$

The first three perturbation problems are then:

MANTLE	CORE
$\nabla \cdot \vec{B}_{M_0} = 0$	$\nabla \cdot \vec{B}_{c_0} = 0$
$\nabla \times B_{M_0} = \vec{j}_{M_0}$	$\nabla \times \vec{B}_{c_0} = \vec{j}_{c_0}$
$\nabla \cdot \vec{E}_{M_0} = 0$	$\nabla \cdot \vec{E}_{c_0} = q_{c_0}$
$\nabla \times \vec{E}_{M_0} = -\partial \vec{B}_{M_0}/\partial t$	$\nabla \times \vec{E}_{c_0} = -\partial \vec{B}_{c_0}/\partial t$
$\vec{j}_{M_0} = 0$	$\vec{E}_{c_0} = -\vec{V}_{c_0} \times \vec{B}_{c_0}$
$\nabla \cdot \vec{B}_{M_1} = 0$	$\nabla \cdot \vec{B}_{c_1} = 0$
$\nabla \times \vec{B}_{M_1} = \vec{j}_{M_1}$	$\nabla \times \vec{B}_{c_1} = \vec{j}_{c_1}$
$\nabla \cdot \vec{E}_{M_1} = q_{M_1}$	$\nabla \cdot \vec{E}_{c_1} = q_{c_1}$
$\nabla \times \vec{E}_{M_1} = -\partial \vec{B}_{M_1}/\partial t$	$\nabla \times \vec{E}_{c_1} = -\partial$
$\vec{j}_{M_1} = \sigma_M \vec{E}_{M_0}$	$\vec{E}_{c_1} = -\vec{V}_{c_0} \times \vec{B}_{c_1} - \vec{V}_{c_1} \times \vec{B}_{c_0}$
$\nabla \cdot \vec{B}_{M_2} = 0$	$\nabla \cdot \vec{B}_{c_2} = 0$
$\nabla \times \vec{B}_{M_2} = \vec{j}_{M_2}$	$\nabla \times \vec{B}_{c_2} = \vec{j}_{c_2}$
$\nabla \cdot E_{M_2} = 0$	$\nabla \cdot \vec{E}_{c_2} = q_{c_2}$
$\nabla \times E_{M_2} = -\partial \vec{B}_{M_2}/\partial t$	$\nabla \times \vec{E}_{c_2} = -\partial \vec{B}_{c_2}/\partial t$
$\vec{j}_{M_2} = 0$	$\vec{j}_{c_0} = \vec{E}_{c_2} - \vec{V}_{c_2} \times \vec{B}_{c_0} - \vec{V}_{c_0} \times \vec{B}_{c_2}$

The zeroth order magnetic problem in the mantle is solved **by** the usual geomagnetic field model in terms of a (truncated) spherical harmonic expansion for the scalar potential. For the zeroth order electric problem in the mantle, note that from the next order problem, since $\vec{j}_{M_1} = \sigma_M \vec{E}_{M_0}$, in order that $\nabla \cdot \vec{j}_{M_1} = 0$ it is necessary that $\hat{r} \cdot \vec{E}_{M_0} = 0$, assuming spherically symmetric mantle conductivity. Then, $\vec{E}_{M_0}$ is obtained from the magnetic vector potential:

$$\vec{B}_{M_0} = \nabla \times \vec{A}_{M_0} = \nabla \times [\nabla \times (S_{M_0} \hat{r})] \quad \text{and}$$

$$\vec{E}_{M_0} = -\frac{\partial \vec{A}_{M_0}}{\partial t} .$$

No potential part of $\vec{E}_{M_0}$ is needed because it would have to be a solution of Laplace's equation without radial variation. The form of S_M is

$$S_{M_0} = a^2 \sum_{n=1}^{N} \sum_{m=0}^{n} \frac{\dot{a}}{n} \left(\frac{a}{r}\right)^n \left[\dot{g}_n^m \cos m\phi + \dot{h}_n^m \sin m\phi \right] P_n^m(\theta)$$

and this series converges rapidly.

With $\vec{B}_{M_0}$, $\vec{E}_{M_0}$ known and the continuity boundary conditions, we turn to the zeroth order problem in the core, but evaluated at the core surface where $\vec{V} \cdot \hat{r} = 0$ for all θ, ϕ . Ohm's law gives (with subscripts 0 temporarily suppressed and the c on $\vec{V}$ dropped since the core is the only place where there is motion):

$$E_{c_r} = W B_{c_\theta} - V B_{c_\phi} ,$$

$$E_{c_\theta} = -W B_{c_r} , \quad E_{c_\phi} = V B_{c_r}$$

Equation (3) gives the desired horizontal fluid motion at $r = b$ in terms of the horizontal electric field and the vertical magnetic field there; but these latter quantities are all continuous across $r = b$, so we can use the values at the base of the mantle. Only at null flux curves where $B_r = 0$ is the motion undetermined (even there, field models should be adjusted or constrained to make null curves of E_θ, E_ϕ coincide with those of B_r).

Once V, W have been found, (2) gives E_{c_r} which measures the surface charge needed to bring the ground state core radial electric field to the value zero in the mantle. From $\vec{E}_{M_0}$, previously found, we also immediately obtain the leading (horizontal) system of currents in the mantle from the last equation in the mantle problem at order η_M (provided a model of mantle conductivity is supplied).

It is interesting that, from (3), the <u>direction</u> of core fluid motion, depending as it does on the ratio of V to W, is independent of the relatively poorly convergent series for B_r. Thus, the streamline pattern is obtainable, on this theory, purely from the secular variation of the magnetic vector potential.

TRANSITION OF DECAYING TURBULENCE TO DECAYING INTERNAL GRAVITY WAVES

Thomas Dickey

Turbulence properties of neutral and stratified fluids have been studied experimentally. Temporal decaying turbulence was created by towing a grid through an initially quiescent fluid. Streak photographs of neutrally buoyant particles and a photodigitizing system were used for velocity measurements. Conductivity measurements were employed in the stratified experiments in order to ascertain both stationary and dynamic values of density.

Results of the decaying turbulence experiment (neutral) indicated that the initial period decay law, $q^2 \propto t^{-1}$, applies through $W_g t/M \lesssim 800$ for a relatively high mesh Reynolds number, 48,260. Previous measurements at comparable Reynolds numbers have been limited to the range $W_g t/M \lesssim 400$. Anistropy was small, $\overline{w^2}/\overline{u^2} \sim 1.1$, throughout the experiment. The dissipation rate was found to decay inversely with the square of time. The Taylor microscale was evaluated and the turbulence Reynolds number was found to be $Re_\lambda + 90.7$ for the experiment. Two point turbulence velocity correlation measurements were utilized in evaluating the macro or integral length scale. A method of determining dissipation from these correlations was presented.

The effect of stratification upon a turbulent flow created by a vertically towed grid was determined for the first time. Conductivity probe measurements of density variations indicated a turbulence-dominated regime through approximately $W_g t/M = 275$, after which internal gravity waves were predominant. The transition period features properties of both internal gravity waves and turbulence. The decay rate of turbulence was virtually identical to that of the neutral case through $W_g t/M - 275$. However, after this time the decay rate was much lower. Integral length scales were computed as before with greater values (by $\sim$ 20%) being determined for the stratified case. A model for this experiment was developed so that a general set of parameters could be used in predicting the initiation time of the internal gravity waves. The results of the decaying turbulence experiments are relevant to modeling dissipation in geophysical systems.

THE APPLICATION TO THE EARTH OF A NEW METHOD FOR DETERMINING THE RADIUS OF THE ELECTRICALLY-CONDUCTING FLUID CORE OF A PLANET FROM EXTERNAL MAGNETIC OBSERVATIONS

Raymond Hide

The proposed new method (Hide, R., 1978, Nature 271: 640) when applied to the Earth gives a core radius differing by less than 2% from the "seismological" value. This finding strongly implies that effects due to ohmic decay, though crucial in the dynamo process by which the magnetic field is produced, can be treated as small perturbations in theories of the geomagnetic secular variation. It also sets limits on the electrical conductivity of the lower mantle and the viscosity of the core. The new method could be exploited in the investigation of the internal structure of other magnetic planets': it will be particularly important to use the method to determine the size of the electrically-conducting fluid core of Jupiter.

NONLINEAR OSCILLATIONS

Louis N. Howard

Three lectures about nonlinear oscillations and techniques for studying them beginning with a description of various examples such as pendulums and clocks, electronic oscillators and the van der Pol equation, and models of oscillations in chemical or ecological systems. This was followed by a discussion of small amplitude weakly nonlinear oscillations which can be regarded as arising from a change in stability of a stationary solution (critical point) as some parameter is varied. When this change in stability occurs because a single conjugate pair of complex eigenvalues crosses the imaginary axis ("overstability"), and crosses at a nonzero rate with respect to the parameter variation, the 'Hopf bifurcation theorem' asserts the existence - somewhere in the neighborhood of the critical point in the phase space and the crossing point in parameter space ('bifurcation point') - of a one-parameter family of periodic solutions might not occur all for the same value of the parameter (the bifurcation value); the latter indeed happens for an exactly linear system, as well as some nonlinear ones. With some additional hypotheses about the nonlinear terms, amounting to the statement that in an appropriate sense some quadratic and/or cubic terms are genuinely present, one can be sure that a periodic solution will occur for each value of the parameter sufficiently close, on one side or the other (but not both), to the bifurcation value. Furthermore, if the bifurcation point is one at which a stable critical point loses its stability, and if the family of periodic solutions occurs on the unstable side ('supercritical bifurcation'), then at least close enough to the bifurcation point the periodic solutions will be stable.

This theorem is the formalization of techniques which have been used very extensively to calculate small amplitude oscillations in many kinds of systems, some of them infinite dimensional like convection in rotating systems or hydromagnetic dynamos.

The second lecture began with the presentation of a convenient way to organize the calculations required to study a Hopf bifurcation, and illustrated this with an example. After this, attention was directed to singular perturbation techniques which can be used to study certain strongly nonlinear oscillators. Two major types were distinguished, relaxation oscillators and Flatto-Levinson systems. Both refer to singular perturbation systems of the form

$$\dot{x} = f(x, y, \varepsilon)$$
$$\varepsilon \dot{y} = g(x, y, \varepsilon)$$

where x and y are vectors of dimensions n and m say. in both cases the x, y phase space contains an invariant n-dimensional submanifold, the 'slow manifold', given asymptotically for $\varepsilon \to 0$ by the equations $g(x,y,0) = 0$. Off the slow manifold y must vary rapidly. In a typical relaxation oscillator the slow manifold is folded over so that if $g(x,y,0) = 0$ is solved for y, $y = h(x)$, then for at least some range of x the solution is multiple valued - often triple valued: $y = h_+(x),\ h_0(x),\ h_-(x)$. Typically, though not invariably, the portions of the slow manifold given by $h_\pm(x)$ are attracting, while the middle sheet $y = h_0(x)$ is not. Relaxation oscilla ions may occur when the motion on a stable part of the slow manifold, described approximately by $x = f(x,h_+(x),0)$, say, always leads eventually to an edge where this portion connects to $y = h_0(x)$. When this point is reached the trajectory jumps rapidly over to the other stable branch $y = h_-(x)$ and then moves along this portion to another edge, where it jumps back to the original sheet $y = h_+(x)$. Of course this process need not always tend toward a closed orbit, but when it does we get a limit cycle of relaxation oscillator type. In many interesting examples the motions on the slow manifold can be approximately determined fairly easily, and from this a reasonably satisfactory description of the oscillator can be constructed. Such oscillations are characterized by the alternation of short periods of rapid transition ('fast phases') with longer periods of slow evolution ('relaxation phases'). The details of the rapid transition are usually none too simple, however, and a full asymptotic description is often remarkably complicated, even in the simplest examples.

The Flatto-Levinson theorem is not particularly concerned with a folded slow manifold, but with situations where on the slow manifold for one branch of it) $y = h(x)$ there is a limit cycle solution of $x = f(x, h(x), 0)$. It is also assumed that at least near this limit cycle the slow manifold has a 'hyperbolic structure', meaning that $g_y(x,h(x), 0)$ is non-singular. Thus we are here concerned with a singular perturbation of a limit cycle. The theorem asserts that for small ε there is also a periodic solution of the full system, whose orbit is close to that of the limit cycle of $x = f(x, h(x), 0)$.

The third lecture began with the presentation of the application of the Flatto-Levinson theorem to finding the limit cycle solution of a model chemical oscillator (the 'Oregonator'). This was followed by a discussion of some topological methods for showing the existence of periodic solutions, mainly the Poincaré-Bendixson theorem in the plane, and the sequential box method of showing the existence of fixed points of the Poincaré map, which has been successfully used in certain higher dimensional cases. As an example the Hastings-Murray treatment of the Oregonator was sketched. Finally certain methods for the numerical calculation of periodic solutions, and some questions of numerical analysis especially relevant to finding unstable periodic solutions and dealing with stiff systems like relaxation oscillators were touched upon.

MELTING ICEBERGS

Herbert E. Huppert

Each year 10^{17} cubic metres of ice melts into the Weddell Sea. It can be argued that the melting takes place primarily along the sides of the icebergs. Previous studies have led to two inconsistent suggestions: 1) that the relatively fresh meltwater rises in a thin boundary layer up the side of the iceberg without any significant mixing with the ocean; and 2) that the Grashof number based on the total depth of the iceberg is so large, of order 10^{17}, that the boundary layer entrains a large amount of salty ocean water and the resulting mixture rises to the surface. A study by Huppert and Turner indicates that there is a third and, in their opinion, more probable process. They argue that both of the above suggestions neglected to include the effects of the existing salinity gradient in the Weddell Sea. Modelling the melting in a series of laboratory experiments, Huppert and Turner found that the meltwater moves through the boundary layer and propagates mainly horizontally. The ambient fluid supplies the heat for the melting, sinks in the surrounding density gradient until its buoyancy force becomes zero and then turns into the interior. The melting thus generates a series of layers, of thickness h say, containing regions of inwardly flowing ambient fluid and outwardly flowing meltwater mixed with ambient fluid. A movie and some slides were shown, some of the latter being copies of figures presented in "Melting Icebergs" by H. E. Huppert and J. S. Turner, Nature, 271, 5640: 46-48, January 5, 1978, Experiments with different salinity gradients indicate that when the Grashof number lies between 10^5 and 10^8 the layer thickness is given by

$$h = 0.66\,\Delta\rho/\phi$$

where $\Delta\rho$ is the density difference between the meltwater and the ambient fluid evaluated at the mean salinity in the water column and ϕ is the vertical density gradient due to salt. It is planned to perform experiments extending the Grashof number range in the near future.

Finally, it was observed that the heat generated by the audience during the two-hour lecture would have greatly accelerated the melting of any iceberg.

SMOOTH EQUATIONS FOR ROUGH PROBLEMS

Joseph B. Keller

By a rough problem we mean a problem involving irregularly fluctuating or rapidly varying functions. Such problems arise in the analysis of wave propagation in random media, in the generation of magnetic fields by conducting fluids in turbulent motion, etc. Because of the difficulty of analysing such problems, it is desirable to replace them by problems involving only smooth functions. The resulting smooth equations can then be treated much more completely than can those of the original problem. This goal has arisen in many different contexts and has been attained by various methods. Many of them involve some kind of averaging, such as spatial, temporal, or ensemble averaging, or a combination of them. Other methods involve the introduction of multiple spatial and/or temporal scales.

In this lecture two different systematic methods are presented for obtaining smooth equations. One is the so-called smoothing method, which has been used widely for about 15 years. It is usually based upon stochastic or ensemble averaging, but it can be used also with other kinds of averaging. The basic assumption underlying it is that the fluctuations in the given functions and in the solutions are small.

The second method is that of using multiple space and time variables, each corresponding to one of the scales of variation of the solution. The assumption upon which this method is based is that there is a great disparity between these different scales. For example the given coefficients may vary rapidly, but the main part of the solution may vary slowly. Then equations for this slowly varying part are obtained. The results can also be described in terms of a suitable spatial or temporal averaging procedure.

Finally it is shown how the two methods, that of smoothing and that of multiple scales, can be combined. This combination simplifies some of the calculation in the multi-space method, and is applicable when that method is applicable.

As an example the two methods are applied to the equation governing a magnetic field in a conducting fluid undergoing turbulent motion. Each method leads to a dynamo equation for the large scale magnetic field.

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STATISTICAL DYNAMICS OF THE LORENZ MODEL

Edgar Knobloch

1. Introduction

There are two basic approaches to the problem of turbulence. In the first, one seeks to obtain statistical solutions to the equations of motion by taking repeated moments of the. equations and using some kind of a closure scheme to close the hierarchy of moment equations. tn the second, one solves nonlinear differential equations obtainable from the equations of motion, that have no explicit stochastic element in them, but that can, for certain ranges of values of the parameters, exhibit apparently random solutions owing to the appearance of a strange attractor. Both methods are assumed to be relevant to the problem of "turbulence", although the connection between the two is not immediately clear.

In the first part of this paper we shall consider the most famous example of a system of equations with a strange attractor. These are the Lorenz (1963) equations, derivable from the first nontrivial truncation of a modal expansion of the equations for Benard convection in the Boussinesq approximation. The equations may be written in the standard dimensionless form

$$\dot{x} = \sigma(y - x) \tag{1a}$$

$$\dot{y} = rx - y - xz \tag{1b}$$

$$\dot{z} = -\nu z + xy \tag{1c}$$

Here the variable x measures the vertical convective velocity! y the temperature fluctuation. and z the mean convective temperature gradient: σ is the Prandtl number of the fluid, r is a reduced Rayleigh number ($r = 1$ for the onset of convection), and ν is related to the wavenumber of the convection rolls. If σ and ν are fixed at 10 and 8/3 respectively (the values originally used by Saltzman (1962)), and r is gradually increased, it is found that at $r \simeq 24{\cdot}74$ the solutions to the equations become unstable according to the linear theory, although there exist finite amplitude instabilities already for $r > 21$. The system is then in the "turbulent" state.

In the following section we shall apply to equations (1) the techniques used in the first of the above approaches and shall compare the results of such a calculation with the numerical evaluation of certain statistical averages of the solution carried out by Lücke (1976). We shall find good agreement between the theory and the numerical results. We hope in this way to show that these two, apparently quite unrelated, approaches to turbulence theory are in fact closely related, and to suggest that both methods are useful in contributing to our understanding of the physics of turbulence.

In the second part of this work we examine the variational problem for the Lorenz model, and suggest a new and potentially very useful method for carrying out an approximate statistical mechanics of the Lorenz model and other systems with strange attractors.

II. Statistical dynamics of the Lorenz model

In this section we shall be concerned with calculating various time averages of the solution to equations (1) in the turbulent regime.

From the equations it is easy to show that

$$\langle x\rangle = \langle y\rangle = \langle xz\rangle = \langle yz\rangle = 0 \tag{2}$$

where the angular brackets denote time averages. Throughout what follows, we shall assume that the solutions of (1) in the turbulent regime are ergodic. Thus we shall assume that we may identify averages over an ensemble of realizations of the solution with time averages in any one realization. In particular we shall assume that the solutions are statistically stationary. There is ample evidence for this property from numerical investigation, but a rigorous mathematical proof of this property is not available. As a consequence of this assumption all time derivatives of averages vanish. By writing down quantities of the form

$$\frac{d}{dt}\langle A(x,y,z)\rangle = \langle \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z}\rangle = 0 \tag{3}$$

and using equations (1), it is possible to obtain an infinite number of relations between various averages. One obtains (cf. Ludke 1976)

$$\langle x^k y\rangle = \langle x^{k+1}\rangle, \quad k \text{ integer} \tag{4a}$$

$$\langle x^2\rangle = \nu\langle z\rangle \tag{4b}$$

$$\langle x^2 z\rangle - \langle xyz\rangle = (\sigma+1)[\langle y^2\rangle - \langle x^2\rangle] \tag{4c}$$

$$\langle xyz\rangle = \nu\langle z^2\rangle \tag{4d}$$

$$\langle x^2 z\rangle = \sigma[\langle y^2\rangle - \langle x^2\rangle] + (r-1)\langle x^2\rangle \tag{4e}$$

$$\nu\langle x^2 z\rangle = 2\sigma[\langle xyz\rangle - \langle x^2 z\rangle] + \langle x^4\rangle . \tag{4f}$$

These relations have been verified numerically by Lucke (1976), providing further evidence for the validity of the stationariness hypothesis. From equations

(4) one obtains an important identity

$$\frac{1}{\sigma^2}\langle \dot{x}^2\rangle = \langle (y-x)^2\rangle = \nu[(r-1)\langle z\rangle - \langle z^2\rangle]. \tag{5}$$

An equivalent result has been given by Malkus (1972):

$$\langle x^2\rangle = \nu\langle z\rangle = \nu(r-1) - \frac{1}{\sigma^2}\frac{\langle \dot{x}^2\rangle}{\langle z\rangle} - \nu\,\frac{\langle (z-\langle z\rangle)^2\rangle}{\langle z\rangle} \tag{6}$$

Since $\langle x^2\rangle = \langle x\cdot y\rangle$ is the convective heat flux, this result shows that the heat flux transported in turbulent convection has to be less than that transported in steady convection ($\dot{x} = 0$, $z = \langle z\rangle$). This important result is derived by another method in section III. In particular equations (5) and (6) show that

$$(r-1)^2 \geq (r-1)\langle z\rangle \geq \langle z^2\rangle \tag{7}$$

with equality only in steady convection.

It is to be observed that the number of relations of the type (3) is insufficient to determine all the averages. For example, the relations (4) only enable one to express the low order moments in terms of the two unknown correlations $\langle z\rangle$, and $\langle z^2\rangle$, that are constrained only by the inequality (7). Our task will be to calculate these two moments, and other such essentual moments.

We shall use the general method for solving stochastic differential equations with rapidly fluctuating coefficients (Ban Kampen 1974, 1976). Suppose that we have a stochastic differential equation of the form

$$\frac{df}{dt} = L(t)f, \tag{8}$$

where $L(t)$ is a stochastic matrix. If L_0 is the mean of L, and $L_1 = L-L_0$ is the rapidly fluctuating part of L that need not be independent of f, then the mean of he process f, $\langle f\rangle$, satisfies the equation

$$\frac{d}{dt}\langle f\rangle = \left\{L_0 + \int_0^\infty d\tau \langle L_1(t)e^{L_0\tau}L_1(t-\tau)\rangle e^{-L_0\tau}\right\}\langle f(t)\rangle \tag{9}$$

In order to apply this theory to the Lorenz model we rewrite equations (1), by eliminating y, in the form

$$\ddot{x} + b\dot{x} + (-a + \omega(t))x = 0 \tag{10}$$

where

$$a = \sigma[r-1-\langle z\rangle],\quad b = \sigma + 1 \tag{11}$$

$$\omega(t) = \sigma[z - \langle z\rangle] \tag{12}$$

and

$$\dot{z} + \nu z = \frac{d}{dt}\left(\frac{1}{2\sigma}x^2\right) + x^2 \tag{13}$$

Equation (10) is thus an equation of a linear "oscillator" with a zero-mean frequency modulation $\omega(t)$. The quantity a is always positive definite as can be seen from the inequality (7). To the author's knowledge the present method has not been applied to a damped simple harmonic oscillator with a random frequency component. The only treatment of such a system hitherto carried out has been done by Bourret (1971) using the so-called Bourret (1962) integral equation. However this equation is not a self-consistent approximation for short correlation times (Van Kampen 1974). We know from numerical studies that in the turbulent regime the quantities x and z are both rapidly fluctuating in time. This can be seen for example in the paper by Robbins (1977) dealing with the equations for the disk dynamo with a shunt, which can be transformed into the standard Lorenz form (1). $\omega(t)$ is thus a rapidly Fluctuating zero-mean random process. In order to calculate $\langle z \rangle$, we shall calculate $\langle x^2 \rangle$ from equation (10).

Writing it in the form (8), we obtain

$$f = \begin{bmatrix} x^2 \\ x\dot{x} \\ \dot{x}^2 \end{bmatrix}, \qquad L_0 = \begin{bmatrix} 0 & 2 & 0 \\ a & -b & 1 \\ 0 & 2a & -2b \end{bmatrix}, \; L_1 = -\omega \quad 0 \quad 0 \;, \tag{14}$$

In order to apply equation (9), we have to calculate the quantity exp $L_0 t$. This can be done most easily in the following way. Observe that $f = f_0 \exp L_0 t$ is a solution of the set of ordinary differential equations

$$\dot{f} = L_0 f \,. \tag{15}$$

Seeking solutions proportional to exp st, we can calculate the eigenvalues s of the system (15). They are given by

$$s = -b \;, \quad s = -b \pm \lambda \;, \quad \lambda^2 = 4a + b^2 . \tag{16}$$

The general solution to the system (15) is then a superposition of these three funcamental solutions:

$$f_1 = e^{-bt}\left(A + Be^{\lambda t} + Ce^{-\lambda t}\right) \tag{17a}$$

The corresponding expressions for f_2 and f_3 follow from equation (15)

$$f_2 = \tfrac{1}{2} e^{-bt}\left(-bA + (\lambda - b)Be^{\lambda t} - (\lambda + b)Ce^{-\lambda t}\right) \tag{17b}$$

$$f_3 = \tfrac{1}{2} e^{-bt}\left(-2 \quad + (\lambda^2 - b\lambda - a)Be^{\lambda t} + (\lambda^2 + b\lambda - a)Ce^{-\lambda t}\right). \tag{17c}$$

If the coefficients A, B, C are now eliminated in favor of $f_1(0)$, $f\,(0)$, $f_3(0)$, equations (17) can be written in the form $f_i(t) = S_{ij} f_j(0)$, where $S \equiv \exp L_0 t$ is known. The following elements of S will be required in what follows:

$$\lambda^2 S_{11} = e^{-bt}\left[2a + b\lambda \sinh \lambda t + (2a + b^2)\cosh \lambda t\right] \tag{18a}$$

$$\lambda^2 S_{12} = 2e^{-bt}\left[-b + \lambda \sinh \lambda t + b\cosh \lambda t\right] \tag{18b}$$

$$\lambda^2 S_{13} = 2e^{-bt}\left[\cosh \lambda t - 1\right] \tag{18c}$$

$$\lambda^2 S_{21} = e^{-bt}\left[ab(\cosh \lambda t - 1) + a\lambda \sinh \lambda t\right] \tag{18d}$$

$$\lambda^2 S_{22} = e^{-bt}\left[-b^2 + 4a\cosh \lambda t\right] \tag{18e}$$

$$\lambda^2 S_{23} = e^{-bt}\left[-b(\cosh \lambda t - 1) + \lambda \sinh \lambda t\right] \tag{18f}$$

The quantity exp $-L_0 t$ is obtained by changing the sign of t in the above expressions. Evaluating the expression on the right side of equation (9) using the results (14) and (18), we obtain finally the equations

$$\frac{d}{dt}\langle x^2\rangle = 2\langle x\dot{x}\rangle \tag{19a}$$

$$\frac{d}{dt}\langle x\dot{x}\rangle = \left[a + \frac{2}{\lambda^2}\left(b(\gamma-\alpha) + \lambda\beta\right)\right]\langle x^2\rangle + \left[-b + \frac{4}{\lambda^2}(\gamma-\alpha)\right]\langle x\dot{x}\rangle + \langle \dot{x}^2\rangle \tag{19b}$$

$$\frac{d}{dt}\langle \dot{x}^2\rangle = \frac{4}{\lambda^2}\left[a\gamma + (a + \tfrac{1}{2}b^2)\alpha - \tfrac{1}{2}b\lambda\beta\right]\langle x^2\rangle + 2a\langle x\dot{x}\rangle + \left[-2b + \frac{4}{\lambda^2}(\gamma-\alpha)\right]\langle \dot{x}^2\rangle, \tag{19c}$$

where

$$\alpha = \int_0^\infty d\tau\, \eta(\tau)\cosh\lambda\tau, \quad \beta = \int_0^\infty d\tau\, \eta(\tau)\sinh\lambda\tau, \quad \gamma = \int_0^\infty d\tau\, \eta(\tau), \tag{20}$$

and $\eta(\tau)$ is the autocorrelation function of the process $\omega(t)$ defined by

$$\langle \omega(t)\omega(t')\rangle \equiv \eta(t-t') \equiv \eta(\tau). \tag{21}$$

We have again assumed that the process $\omega(t)$ is stationary. In what follows we shall assume that $\omega(t)$ has an autocorrelation function that falls off sufficiently rapidly that the quantities α, β, γ are well defined. As we are assuming that the process x is also stationary we shall set the time derivatives of the correlations equal to zero. From equations (19b) and (19c) it then follows that

$$\left[a + \frac{2}{\lambda^2}\left(b(\gamma-\alpha) + \lambda\beta\right)\right]\left[b - \frac{2}{\lambda^2}(\gamma-\alpha)\right] + \frac{1}{\lambda^2}\left[2a\gamma + (2a + b^2)\alpha - b\lambda\beta\right] = 0. \tag{22}$$

An examination of the definitions (20) suggests the approximation

$$\beta, \gamma \ll \alpha, \tag{23}$$

valid for correlation times that are short, but not too short, as observed in the numerical results (Robbins 1977). It then follows that

$$\alpha = \frac{a\lambda^2}{b} \tag{24}$$

is the condition required for statistically stationary solutions. This condition gives the "strength" of the fluctuations, or the energy input required on average to counterbalance the damping term. Thus (24) is an example of *a* fluctuation - dissipation theorem for an equilibrium system. The other root of equation (22) is negative and therefore unphysical. On substituting the condition (24) into equation (19b) we find that in the stationary state

$$\langle \dot{x}^2 \rangle = a \langle x^2 \rangle . \tag{25}$$

This is the usual result that the mean kinetic energy is equal to the mean potential energy.

We now apply these results to the calculation of certain statistical averages. Consider first the quantity $\Omega_0^2 \equiv \langle \dot{x}^2 \rangle / \langle x^2 \rangle$. From equations (11) and (25) we obtain

$$\Omega_0^2 = \sigma \left[r - 1 - \langle z \rangle \right] \tag{26}$$

On the other hand, using equation (6), we find

$$\Omega_0^2 = \sigma^2 \left[r - 1 - \frac{\langle z^2 \rangle}{\langle z \rangle} \right] \tag{27}$$

We shall be interested in studying the quantities and defined by

$$\langle z^2 \rangle = \frac{\langle z \rangle^2}{1 - \Delta^2} , \quad 0 < \Delta^2 < 1 , \tag{28}$$

and

$$\langle z \rangle = (r-1)(1-\zeta) , \quad 0 < \zeta < 1 . \tag{29}$$

From equations (26) - (29) we obtain the relation

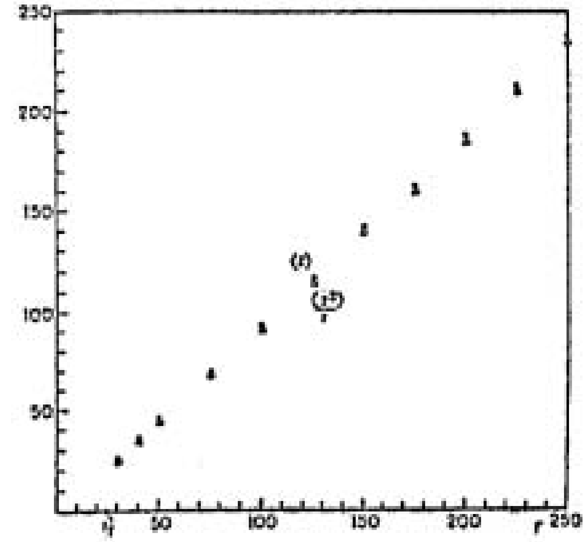
$$\zeta = \Delta^2 \sigma / (\sigma - 1 + \Delta^2) . \tag{30}$$

Lücke (1976) has carried out numerical computations of certain statistical averages of the solutions to the Lorenz equations in the turbulent regime. His graph of $\langle z \rangle$ against r for $\sigma = 10$, and $\nu = 8/3$, reproduced here as Fig. 1, shows an approximately straight line of slope slightly less than one. The slope of the line for $30 \leq r \leq 150$ is found to be 0.94, and it increases somewhat for larger r. Equation (30), with $\zeta = 0.06$, and $\sigma = 10$ then predicts that $\Delta = 0.23$, in excellent agreement with the numerical results shown here in Fig. 2. The corresponding agreement between the predicted graph of a,= 0.78 $\sqrt{r-1}$ and Lücke's result shown here in Fig.3 is not quite so good. Lücke's results show the same Rayleigh number dependence, but with a coefficient closer to unity.

We shall define the autocorrelation time τ_c of the process $\omega(t)$ by the relation $\alpha = \eta(0)\tau_c$. Substituting from equations (12), (21) and (24) we now obtain

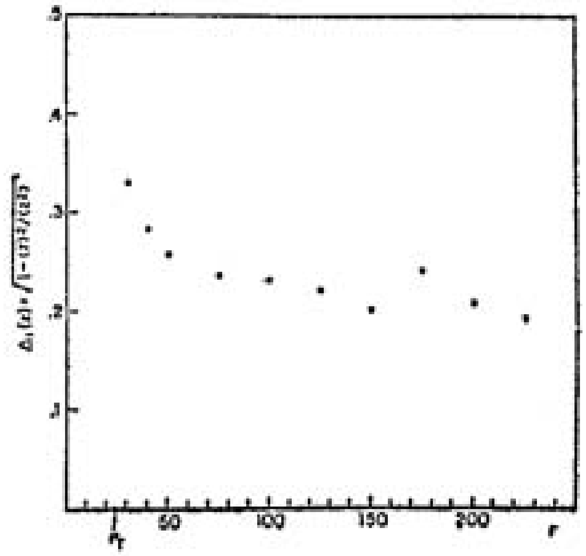
$$\zeta^2 (r-1)^2 + \frac{1}{4} \frac{(\sigma+1)^2}{\sigma} \zeta (r-1) = \frac{1}{4} \tau_c (\sigma+1) \frac{\Delta^2}{1-\Delta^2} (r-1)^2 (1-\zeta)^2 \tag{31}$$

or. using equation (30) to eliminate ζ .



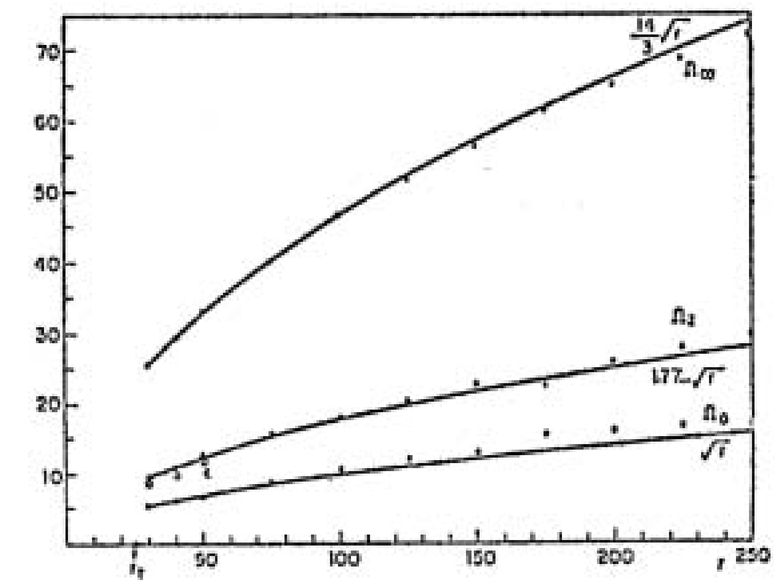
Average values (z) (dots) and $(z^2)/r$ (triangles for several values of r.

Fig. 1



Average normalized fluctuations $\Delta r(z) = \Delta(z)/\langle z^2\rangle^{1/2}$ for various values of r.

Fig.2



Characteristic frequencies (dots) of the model for various values of r. Solid lines denote the curves $\sqrt{r}$, 1.77 ...$\sqrt{r}$, and (14/3) $\sqrt{r}$, respectively. Squares represent the oscillation frequencies ϵ of $\phi_{33}(t)$, according to Fig.5.

Fig.3

(From Lücke (1976))

$$\Delta^2 = (\sigma^2 - 1)\frac{\tau_c(r-1)(\sigma-1) - (\sigma+1)}{4\sigma^2(r-1) + \tau_c(r-1)(\sigma^2-1)(\sigma-1) + (\sigma+1)^2} \tag{32}$$

Suppose that, as a first approximation, ζ is independent of r. (cf.Fig.1) It then follows that Δ is independent of r, so that from (32) it is necessary that $\tau_c r \gg 1$, with τ_c independent of r. Recall that τ_c will turn out to be a small number. Hence

$$\zeta \simeq \frac{\tau_c(\sigma^2-1)}{4\sigma + \tau_c(\sigma^2-1)} \tag{33a}$$

$$\Omega_0^2 \equiv \sigma(r-1)\zeta = \frac{\sigma(\sigma^2-1)\tau_c}{4\sigma + \tau_c(\sigma^2-1)}(r-1) \,. \tag{33b}$$

The graph $\Omega_0 = \sqrt{r-1}$ is therefore well matched by

$$\tau_c = 0.045, \qquad \zeta = 0.10, \qquad \Delta = 0.30. \tag{34}$$

This is stil in good agreement with Lucke's results. Observe that the dimensionless correlation time is indeed a small number, so that the short autocorrelation time approximation required for the derivation of equation (9) is indeed satisfied.

Thus far we have been able to obtain good values of several quantities given one numerical result. However, we have only used the equation for x in terms of z. We shall now determine the correlation time τ_c self-consistently by considering the equation for z in terms of x. We shall then have theoretical results for all the four quantities τ_c, ζ, Δ and Ω_0. In order to do the calculation we shall need to know the quantity $\langle x^2 \rangle$, which can be obtained using the same method as used above.

Writing the equations for the five fourth-order moments in the form (8), we obtain

$$f \quad \begin{bmatrix} x^4 \\ \dot{x}x^3 \\ \dot{x}^2x^2 \\ \dot{x}^3x \\ \dot{x}^4 \end{bmatrix}, \qquad L_0 = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ a & -b & 3 & 0 & 0 \\ 0 & 2a & -2b & 2 & 0 \\ 0 & 0 & 3a & -3b & 1 \\ 0 & 0 & 0 & 4a & -4b \end{bmatrix} 3 \tag{35a}$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\omega & 0 & 0 & 0 & 0 \\ 0 & -2\omega & 0 & 0 & 0 \\ 0 & 0 & -3\omega & 0 & 0 \\ 0 & 0 & 0 & -4\omega & 0 \end{bmatrix} \tag{35b}$$

Again we have to calculate the matrix exp $L_0\tau$. Proceeding as before we find that the eigenvalues S of L_0 are given by

$$S = -2b, \quad -2b \pm \lambda, \quad -2b \pm 2\lambda \tag{36}$$

where again $\lambda^2 = b^2 + 4a$. The general solution is thus

$$f_1(t) = e^{-2b}\left(A + Be^{\lambda t} + Ce^{-\lambda t} + De^{2\lambda t} + Ee^{-2\lambda t}\right), \tag{37}$$

with corresponding results for $f_2(t), \ldots f_5(t)$ obtained from the equation $\dot{f} = L_0 f$. We can write the result in the form

$$f(t) = e^{-2bt}\, T\left\{A, Be^{\lambda t}, Ce^{-\lambda t}, De^{2\lambda t}, Ee^{-2\lambda t}\right\}, \tag{38}$$

where the matrix T has the rows

$$T_{1j} = (1, 1, 1, 1, 1) \tag{39a}$$

$$T_{2j} = \left(-\frac{b}{2}, \pm\frac{\lambda}{4} - \frac{b}{2}, \pm\frac{\lambda}{2} - \frac{b}{2}\right) \tag{39b}$$

$$T_{3j} = \left(-\frac{a}{3} + \frac{b^2}{6}, \mp\frac{1}{4}\lambda b + \frac{1}{4}b^2, \qquad a \mp \frac{1}{2}\lambda b + \frac{1}{2}b^2\right) \tag{39c}$$

$$T_{4j} = \left(\frac{1}{2}ab, \mp\frac{1}{4}a\lambda \pm \frac{1}{8}b^2\lambda - \frac{1}{8}b^3, -\frac{3}{2}ab \pm \frac{1}{2}\lambda \pm \frac{1}{2}b^2\lambda - \frac{1}{2}b^3\right) \tag{39d}$$

$$T_{5j} = \left(a^2, -a^2 - \frac{1}{2}ab^2 \pm \frac{1}{2}ab\lambda, a^2 + 2ab - \frac{1}{2}b^4 \mp \frac{1}{2}b^3\lambda \mp ab\lambda\right) \tag{39e}$$

We can now determine the constants A, B, C, D, E in terms of $f_i(0)$ $(i = 1, \ldots, 5)$. We obtain

$$\lambda^4 A = 6a^2 f_1(0) - 12ab\, f_2(0) + (6b^2 - 12a) f_3(0) + 12b f_4(0) + 6 f_5(0) \tag{40a}$$

$$\lambda^4 B = (4a^2 + 2ab^2 + 2ab\lambda) f_1(0) + (-2b^3 + 4a\lambda - 2\lambda b^2) f_2(0) + (-6b^2 - 6b\lambda) f_3(0)$$
$$+ (-4\lambda - 8b) f_4(0) - 4 f_5(0) \tag{40b}$$

$$\lambda^4 D = \left(a^2 + 2ab + \frac{1}{2}b^4 + \frac{1}{2}b^3\lambda + ab\lambda\right) f_1(0) + (6ab + 2b^3 + 2a\lambda + 2\lambda b^2) f_2(0)$$
$$+ (3b^2 + 6a + 3b\lambda) f_3(0) + (2b + 2\lambda) f_4(0) + f_5(0) \tag{40c}$$

$$C(\lambda) = B(-\lambda), \quad E(\lambda) = D(-\lambda). \tag{40d}$$

The above relations determine the matrix exp $L_0 t$ as the matrix of coeffieients of $f_i(0)$ in the equations for $f_i(t)$. In order to calculate the right-hand side of equation (9), let

$$\langle L_1(t) \exp L_0\tau\, L_1(t-\tau) \exp -L_0\tau \rangle = \eta(\tau)\gamma(\tau), \tag{41}$$

where γ is a 5 x 5 matrix, and η the autocorrelation function of the process ω. We shall be interested in the correlations $\langle x^4 \rangle$ and $\langle \dot{x}^2 x^2 \rangle$. From equation (9) these are connected by the equations

$$\frac{d}{dt}\langle x^4\rangle = 4\langle \dot{x} x^3\rangle \tag{42a}$$

$$\frac{d}{dt}\langle \dot{x} x^3\rangle = \left(a + \int_0^\infty d\tau\, \eta\, \gamma_{21}(\tau)\right)\langle x^4\rangle + \left(-b + \int_0^\infty d\tau\, \gamma_{22}(\tau)\right)\langle \dot{x} x^3\rangle + \left(3 + \int_0^\infty d\tau\, \eta\, \gamma_{23}(\tau)\right)\langle x^2 \dot{x}^2\rangle . \tag{42b}$$

After a considerable amount of algebra the expressions for γ_{21} and γ_{23} reduce to

$$\gamma_{21} = \frac{-4b}{\lambda^2}\left(\cosh \lambda\tau - 1\right) + \frac{4}{\lambda}\sinh \lambda\tau, \quad \gamma_{23} = 0 \tag{43}$$

In the statistically stationary state γ_{22} will not be required, Equations (42) now give, using again the approximation (23), the relation

$$\left(\frac{4b\alpha}{\lambda^2} - a\right)\langle x^4\rangle = 3\langle \dot{x}^2 x^2\rangle . \tag{44}$$

Using the stationariness condition (24), this finally becomes

$$a\langle x^4\rangle = \langle \dot{x}^2 x^2\rangle . \tag{45}$$

From equations (la) and (4) it now follows that

$$\langle x^2 y^2\rangle = \left(\nu \quad 1 + \frac{a}{\sigma^3}\right)\left[(\sigma\nu + 2\sigma^2 + 2\sigma + \nu)(r-1)\langle z\rangle - \sigma(\nu + 2\sigma + 2)\langle z^2\rangle\right]. \tag{46}$$

In order to estimate the autocorrelation time τ_c of the process ω, we now return to equation (13), and write it in the form

$$\dot{z}' + \nu z' = A(t) \equiv \frac{1}{\sigma} x\dot{x} + x^2 - \nu\langle z\rangle , \tag{47}$$

where $z' = z - \langle z\rangle$. Thus equation (47) is a Langevin equation with $A(t)$ being a zero-mean rapidly fluctuating forcing term. From the formal solution to equation (47), we find that

$$\langle z'^2\rangle = \int_0^t\int_0^t dt'\,dt''\, e^{-\nu(t-t')} e^{-\nu(t-t'')}\langle A(t')A(t'')\rangle \simeq \frac{1}{2\nu}\tau_c'\langle A^2\rangle , \tag{48}$$

where we have assumed that the forcing term A(t) has a short autocorrelation time, τ_c', and that the process z' is statistically stationary. Thus

$$\langle z^2\rangle - \langle z\rangle^2 = 23\ \tau_c'\left[\langle x^2 y^2\rangle - \nu^2\langle z\rangle^2\right]. \tag{49}$$

We shall suppose that the correlation times τ_c' and τ_c are comparable. Then using the definitions *(28)* and (29), and the result (46), we obtain

$$2\Delta^2 + \tau_c\nu(1-\Delta^2) = \tau_c\left(1 + \frac{(r-1)\zeta}{\sigma}\right)\left\{\nu + \frac{(2\sigma+\nu)(\sigma+1)(\sigma^2-1)\tau_c}{4\sigma^2 + (\sigma-1)(\sigma^2-1)\tau_c}\right\} \tag{50}$$

Upon using the results *(30)* and *(33)* we obtain a quadratic equation for τ_c. The physically meaningful root of this equation can be obtained approximately by setting

$$r\zeta \gg \sigma, \quad \tau_c\sigma \ll 4, \quad \tau_c r \gg 1, \quad \sigma^2\tau_c \gg 2\nu . \tag{51}$$

Although these conditions are not strictly satisfied for the values of the parameters used in Lucke's numerical calculations, they do give the correct order of magnitude for the correlation time. We obtain

$$\tau_c \simeq \sqrt{\frac{8(\sigma+2\nu)}{2\sigma+\nu}}\,(r-1)^{-1/2} = \frac{2}{3}\,(r-1)^{-1/2} \tag{52}$$

for $\sigma = 10$, $\nu = 8/3$. We see now that the correlation time does have a weak Rayleigh number dependence; it varies between $0.1 > \tau_c > 0.02$ for $50 < r < 200$. This is in satisfactory agreement with the results (34) deduced on the basis of the present theory and one numerical result. We have thus shown that an entirely self-consistent theory follows from the application of stochastic differential equations to the Lorenz equations, one that not only predicts the correct functional dependence on the Rayleigh number of various statistical averages, but that also predicts quite well the numerical values of the quantities ζ, Δ and Ω_o characteristic of the solution in the turbulent regime. Moreover there is some evidence for the decrease of the correlation time with increasing r in Lücke's numerical calculations. For example, the curve of $\langle z \rangle$ vs. r in Fig.1 deviates upwards for large r from the straight line defined for smaller values of r. More convincingly perhaps, the approximate formula

$$\Delta^2 \sim \frac{\sigma\tau_c}{4+\sigma\tau_c} \tag{53}$$

shows that Δ decreases with increasing r. Indeed, Fig.2 shows a decrease consistent with the $r^{-1/2}$ behavior. We also predict that for large r, the curve of Ω_o^2 vs. r should have a slope that falls below unity.

Let us now turn our attention to the quantity Ω_z^2 also calculated by Lücke. By definition

$$\Omega_z^2 = \frac{\langle \dot z^2\rangle}{\langle (z-\langle z\rangle)^2\rangle} = \frac{1}{\Delta^2}\,\frac{\langle x^2y^2\rangle}{\langle z^2\rangle} - \frac{\nu^2}{\Delta^2}. \tag{54}$$

Using the results *(28)* and (46), the quantity (54) can be written in the form

$$\Omega_z^2 = \frac{-\nu}{\Delta^2}\Big[\nu+\Big(1+\frac{a}{\sigma^2}\Big)\sigma\,(2\sigma+\nu+2)\Big] + \frac{1-\Delta^2}{\Delta^2}\Big(1+\frac{a}{\sigma^2}\Big)\frac{\nu(r-1)}{\langle z\rangle}\,(\nu+2\sigma^2+\sigma\nu+2\sigma) \tag{55}$$

Substituting for a and $\langle z\rangle$ from equation (11) and (29), and using the result (30), we obtain the result

$$\Omega_z^2 = \frac{\nu}{\sigma-1}\big(\nu+2\sigma^2+\sigma\nu+2\sigma\big) + \frac{\nu^2}{\sigma-1+\Delta^2}\,(r-1). \tag{56}$$

Since $0<\Delta^2<1$, it follows that for large enough r,

$$\Omega_z = \Big(\frac{\nu}{\sqrt{\sigma-1}}\Big)\sqrt{r-1}, \tag{57}$$

essentially independently of the correlation time τ_c. For $\sigma = 10$, $\nu = 8/3$, Lücke finds numerically that $\Omega_z = 1.77 \ldots \sqrt{r}$. Equation (57) on the other hand predicts a slope of $8/9 = 0.888$ - exactly half of Lücke's value. We have been unable to find the reason for this discrepancy. We note here that the result (57) predicts that $\Omega_z(p=30) \simeq 10.5$ in agreement with Lucke's curve and the mode-coupling calculation that he carried out for values of σ in this region. It is at least certain that the formula for the slope in terms of σ and ν, viz. $\frac{\sigma+1}{\sigma}+\nu-2$, suggested by Lucke cannot be correct.

Finally, there appears to be no way of calculating the quantity $\Omega_\infty^2 = \frac{\langle \ddot{x}^2 \rangle}{\langle \dot{x}^2 \rangle}$, also plotted by Lücke, using only the fourth order moments calculated above.

III. Variational approach and the statistical mechanics of the Lorenz Model

Ever since the pioneering paper of Howard (1963), variational methods have assumed great importance in the study of turbulent convection. As the Lorenz equations are a truncation of the model expansion of the equations for the Bénard problem, it is of considerable interest to apply these techniques to this simpler system in the hope of testing their convergence and accuracy.

Howard maximized the heat flux transported by the fluid between two horizontal surfaces subject to two power integrals derived from the equation of motion by Malkus (1954), with the useful property that they led to a separable variational problem. In this way he was able to obtain upper bounds on the heat flux in terms of the Rayleigh number, but independent of the Prandtl number.

In the present case we are interested in finding bounds on the total heat flux, both convective and conductive, given in terms of the Lorenz variables by $\langle xy \rangle + \nu\langle z \rangle$. The power integrals are obtained by taking moments of the Lorenz equations. The first nontribial variational problem arises from the following three constraints:

$$0 = \langle x\dot{x} \rangle = \sigma[\langle xy \rangle - \langle x^2 \rangle] \tag{58a}$$

$$0 = \langle y\dot{y} \rangle = r\langle xy \rangle - \langle y^2 \rangle - \langle xyz \rangle \tag{58b}$$

$$0 = \langle z\dot{z} \rangle = -\nu\langle z^2 \rangle + \langle xyz \rangle \tag{58c}$$

We wish to maximize the functional $\mathcal{L}$ given by

$$\mathcal{L} = \langle xy \rangle + \nu\langle z \rangle + \lambda_1[\langle xy \rangle - \langle x^2 \rangle] + \lambda_2[r\langle xy \rangle - \langle y^2 \rangle - \langle xyz \rangle] + \lambda_3[\langle xyz \rangle - \nu\langle z^2 \rangle], \tag{59}$$

where λ_1, λ_2, λ_3 ate three Lagrange multipliers. The Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} = (1+\lambda_1+\lambda_2 r)y - 2\lambda_1 x - (\lambda_2-\lambda_3)yz = 0 \tag{60a}$$

$$\frac{\partial \mathcal{L}}{\partial y} = (1 + \lambda_1 + \lambda_2 r)x - 2\lambda_2 y - (\lambda_2 - \lambda_3)xz = 0 \tag{60b}$$

$$\frac{\partial \mathcal{L}}{\partial z} = -(\lambda_2 - \lambda_3)xy - 2\nu\lambda_3 z + \nu = 0 \tag{60c}$$

From Eqs. (60a) and (60b) and the constraint (58a) we find that

$$x = y, \qquad \lambda_1 = \lambda_2 . \tag{61}$$

From the constraints (58b) and (58c) it then follows that

$$x^2 = y^2 = \nu z = \nu(r-1) \tag{62}$$

with the corresponding Lagrange multipliers

$$\lambda_1 = \lambda_2 = -2\lambda_3 = \frac{2}{r-1} \tag{63}$$

The result (62) is of course the steady convective solution to the Lorenz equations (x = y = z = 0). Hence the maximum heat flux, $2\nu(r-1)$, is attained in steady convection, and in nonsteady (turbulent) convection (within the Lorenz model) the heat flux transported by the convection must be less. This is in agreement with Eq. (6). What appears to be happening is that the optimal solution becomes unstable at large enough Rayleigh numbers, resulting in slightly reduced heat flux (c.f. Eqs. (29) and (34)). It is as if the system were in fact trying to maximize the flux but was being prevented from doing so by an intrinsic instability.

Because the heat flux is globally bounded by the steady convection, an improved bound on the nonsteady convection cannot be achieved by adding more constraints - all the infinity of constraints derivable from the Lorenz equations by taking higher moments are identically satisfied by the steady solution.

Two alternative approaches suggest themselves: either bounding from below, or using alternative constraints to begin with. In what follows we shall adopt the latter approach, since it leads naturally to a new and interesting way of looking at the Lorenz model.

The choice of constraint is dictated by the desire that the 'steady solution not be the whole story. We therefore introduce constraints quadratic in x, y, z, so that the corresponding Euler-Lagrange equations are now time-dependent. Let

$$\begin{aligned} \mathcal{L} = \langle xy\rangle + \nu\langle z\rangle &+ \lambda_1\left[\langle \dot{x}^2\rangle - r^2\langle x^2\rangle + 2\sigma^2\langle xy\rangle - \sigma^2\langle y^2\rangle\right] + \lambda_2\left[\langle \dot{y}^2\rangle - r^2\langle x^2\rangle\right. \\ &\left. - \langle y^2\rangle - \langle x^2 z^2\rangle + 2r\langle xy\rangle + 2r\langle x^2 z\rangle - 2\langle xyz\rangle\right] + \lambda_3\left[\langle \dot{z}^2\rangle - \nu^2\langle z^2\rangle\right. \\ &\left. + 2\nu\langle xyz\rangle - \langle x^2 y^2\rangle\right], \end{aligned} \tag{64}$$

where $\lambda_1, \lambda_2, \lambda_3$ are three Lagrange multipliers. The Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{65}$$

can be written in the form

$$\lambda_1 \ddot{x} = -\frac{\partial V}{\partial x} \tag{66a}$$

$$\lambda_2 \ddot{y} = -\frac{\partial V}{\partial y} \tag{66b}$$

$$\lambda_3 \ddot{z} = -\frac{\partial V}{\partial z} \tag{66c}$$

where

$$V(x,y,z) = (\sigma^2\lambda_1 + \lambda_2)x^2 - (1 + 2\lambda_1\sigma^2 + 2r\lambda_2)xy + (\lambda_1\sigma^2 + \lambda_2)y^2 + \lambda_3 x^2 y^2$$
$$-\nu z + \lambda_3 \nu^2 z^2 - 2r\lambda_2 x^2 z + 2(\lambda_2 - \nu\lambda_3)xyz + \lambda_2 x^2 z^2 \tag{67}$$

Thus the Euler-Lagrange equations contain no dissipative terms, and form a Hamiltonian system describing the motion of a particle in a complicated potential $V(x,y,z)$. The potential V is of sufficiently high order in x,y,z that we may expect the kind of chaotic behavior of a particle in this potential that is described by the Kolmogorov-Arnold-Moser theory for such systems. We may therefore attempt to describe the statistical mechanics of particles in such a potential. Since it is known that the bounding solutions (such as the steady convection) are not bad bounds for the unsteady convection, we may hope that the present system approximates the Lorenz attractor quite well. By understanding the equations (66) it is hoped to gain some understanding of those features of the attractor that do not depend on the details of its topology.

Multiplying Eqs. (66) by x, y, z respectively and adding, we find that the system (66) has a first integral

$$E(x,y,z) \equiv \lambda_1 \dot{x}^2 + \lambda_2 \dot{y}^2 + \lambda_3 \dot{z}^2 + V(x,y,z) = K \text{ (a constant).} \tag{68}$$

If we now average Eq. (67) and use the constraints, we find that

$$2\left[\lambda_1 \langle \dot{x}^2\rangle + \lambda_2 \langle \dot{y}^2\rangle + \lambda_3 \langle \dot{z}^2\rangle\right] = K + \langle xy\rangle + \nu\langle z\rangle \tag{69}$$

This can be considered to be the analog of the virial theorem, and is a general property of the Euler-Lagrange equations for constraints of the form

$$\langle \dot{x}^2\rangle = \sum \langle x^m y^n z^k\rangle .$$

In order to compute the Lagrange multipliers, we take x,y,z moments of Eqs. (66a) - (66-c) respectively and use the assumed stationariness of the solution to observe that $\langle x\ddot{x}\rangle = -\langle \dot{x}^2\rangle$ etc. Solving the resulting equations and using the constraints we obtain

$$2\lambda_1\sigma^2\left[\langle y^2\rangle - \langle xy\rangle\right] = \left[\langle xy\rangle + \nu\langle z\rangle\right]\left[1 + r\,\frac{r\langle x^2\rangle - \langle x^2 z\rangle - \langle xy\rangle}{\langle x^2 z^2\rangle - r\langle x^2 z\rangle + \langle xyz\rangle}\right] \tag{70a}$$

$$2\lambda_2\left[\langle x^2 z^2\rangle - r\langle x^2 z\rangle + \langle xyz\rangle\right] = \langle xy\rangle + \nu\langle z\rangle \tag{70b}$$

$$2\lambda_3[\langle x_2 y_2\rangle - \nu\langle xyz\rangle] = \langle xy\rangle \tag{70c}$$

In order to study the statistical mechanics of particles moving in the potential $V(x,y,z)$, we define the probability distribution function

$$p(x,y,z;\dot{x},\dot{y},\dot{z}) \equiv e^{-\beta E} / \int d\Gamma\, e^{-\beta E}, \tag{71}$$

where Γ is the 6-dimensional phase space, and β is the inverse of a "temperature" that can be related to the Rayleigh number r by the relation

$$\frac{\int E e^{-\beta E} d\Gamma}{\int e^{-\beta E} d\Gamma} = K = \frac{3}{2\beta} \tag{72}$$

The probability distribution p is well defined only when the "kinetic energy" and the potential are both positive definite. It is therefore important to evaluate the Lagrange multipliers (70), by using the prescription

$$\langle\ldots\rangle = \int d\Gamma\,(\ldots)\, e^{-\beta E} / \int d\Gamma\, e^{-\beta E} \tag{73}$$

The resulting multipliers are then functions of β ; Eq. (72) shows that β is known if the constant of motion K is known. This can be obtained from Eq. (69) by evaluating the averages in (69) using the prescription (71) and the results for the Lagrange multipliers. Equations (69) and (72) are thus an integral equation for β in terms of the parameters of the Lorenz model, in particular the Rayleigh number r. Unfortunately it appears that K cannot be unambiguously determined from the steady convection solution because of divergence in the Lagrange multipliers. These arise because the constraints become meaningless for the steady convection. The next step in the calculation is the investigation of the existence of a possible phase transition at some critical β (corresponding to a critical Rayleigh number) by means of the usual thermodynamic relations (Landau and Lifshitz, 1969)

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0, \qquad (T \equiv \beta^{-1}), \tag{74}$$

where the "pressure" p has to be calculated from the free energy F given in terms of the partition function $Z = \int e^{-\beta E} dP$ by

$$F = -\frac{1}{\beta} \ln Z \tag{75}$$

Unfortunately due to the difficulty of calculating the partition function and other ensemble averages (73) arising from the high degree of the potential V we have thus far been unsuccessful in carrying through this program. This difficulty is inherent in the problem because lower order potentials will not have the stochastic behavior predicted by the Kolmogorov-Arnold-Moser theory. It is also possible that the bounding equations are too poor at low Rayleigh numbers, so that no discontinuity in the statistical properties is required by the sudden appearance of the strange attractor in the Lorenz system.

IV. Discussion

In this paper we have seen that some systems with strange attractors, such as the Lorenz model, can be treated by standard statistical methods that are used in treating "noisy" systems. We have seen that these methods predict the correct functional dependence of certain statistical averages on the Rayleigh number r, as well as giving the correct amplitude to a good accuracy. In this way we have shown that strange attractors and "noisy" systems, while apparently dissimilar, can have a good deal in common if one is interested only in their statistical properties. Indeed it is likely that the origin of many stochastic systems lies in hidden nonlinear systems with strange attractors, and it is therefore necessary to know that the details of the process producing the fluctuations are irrelevant for the gross properties of the system.

In the second part of the paper we have suggested a new way of looking at the Lorenz model and related systems with strange attractors. Although we have thus far been unable to carry out the details of the calculations, we have seen that such systems can be approximated by conservative systems for which the whole machinery of equilibrium statistical physics can be employed. In this way it appears possible to calculate an approximate probability density distribution $P(\Gamma)$, which can be used to calculate any desired statistical property. Thus a wealth of new statistical properties could be investigated that is inaccessible to both numerical analysis, and the method described in the first part of this work.

In conclusion we discuss the implications of the calculations presented here for the Lorenz model to real turbulent convection. The first question that is well illustrated by the Lorenz model is whether the heat flux in convection is maximized by a laminar flow (Busse 1969, 1970; Howard, 1972). This thus far unresolved question has important consequences because it appears that in astrophysical situations the optimal heat fluxes are never realized. The question then arises whether due to instabilities of the laminar solutionsa finite gap is created between the maximal heat flux and that achieved by turbulent motions.

A promising aspect of the discussion in section III lies in its application to Bénard convection, described by the equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} + \frac{1}{\rho}\nabla p - \alpha g T \underset{\sim}{k} = \nu\nabla^2\underset{\sim}{u} \tag{76}$$

$$\nabla\cdot\underset{\sim}{u} = 0 \tag{77}$$

$$\frac{\partial T^*}{\partial t} + \underset{\sim}{u}\cdot\nabla T^* = \kappa\nabla^2 T^*, \tag{78}$$

where $T^* = T + \overline{T^*}$, β is constant, and the remaining symbols have their usual meanings (Howard 1963). The boundary conditions are $T^*(o) = T_o$, $T^*(d) = T_o - \Delta T$, $\underset{\sim}{u}(o) = \underset{\sim}{u}(d) = 0$. From Eqs. (76) - (78) the following two-dimensionless constraints may be derived

$$\langle \dot{\underset{\sim}{u}}^2 \rangle = \sigma R \langle T \dot{w} \rangle \tag{79}$$

$$\langle \dot{T}^2 \rangle = -\langle \dot{T} \underset{\sim}{u} \cdot \nabla T \rangle + \langle \dot{T} w \rangle + \langle \dot{T} w \rangle \langle T w \rangle - \langle \overline{\dot{T} w}\, \overline{T w} \rangle \tag{80}$$

where $\underset{\sim}{u}$ has been scaled by k/d, t by d^2/k, T by Δ T, and

$$\sigma = \nu/k, \qquad R = \frac{d^3 \alpha g \Delta T}{k \nu} \tag{81}$$

Here w is the vertical convective velocity, and the angular brackets denote an average over the whole convecting layer, while the horizontal overbar indicates a horizontal average. In what follows we shall find it useful to include in both averages also time averages. We note here the appearance of the parameter σR rather than just R as in the variational problem considered by Howard (1963). This appears to be the first time that it may be possible to obtain rigorous bounds on the heat flux as a function of the Prandtl number σ, a result that would be of considerable importance, particularly for astrophysical convection. For maximum Nusselt number

$$N = 1 + \langle w T \rangle \tag{82}$$

we obtain the Euler-Lagrange equations

$$2\lambda_1 \ddot{u} - \lambda_2 \dot{T} \frac{\partial T}{\partial x} = 0 \tag{83a}$$

$$2\lambda_1 \ddot{v} - \lambda_2 \dot{T} \frac{\partial T}{\partial y} = 0 \tag{83b}$$

$$2\lambda_1 \ddot{w} - \lambda_2 \dot{T} \frac{\partial T}{\partial z} + \dot{T}\left(-\lambda_1 \sigma R + \lambda_2 + \lambda_2 \langle T w \rangle - \lambda_2 \overline{T w}\right) + T\left(-1 + \lambda_2 \langle \dot{T} w \rangle - \lambda_2 \overline{\dot{T} w}\right) = 0 \tag{83c}$$

$$2\lambda_2 \ddot{T} + \lambda_2 \frac{\partial}{\partial t}(\underset{\sim}{u} \cdot \nabla T) + \dot{w}\left(\lambda_1 \sigma R - \lambda_2 - \lambda_2 \langle T w \rangle + \lambda_2 \overline{T w}\right) + w\left(-1 + \lambda_2 \langle \dot{T} w \rangle - \lambda_2 \overline{\dot{T} w}\right) + \lambda_2 \underset{\sim}{u} \cdot \nabla \dot{T} = 0 \tag{83d}$$

The Lagrange multipliers are given by

$$\lambda_1 \sigma R\left[\langle \dot{T}^2 \rangle + \langle w \dot{T} \rangle \langle w T \rangle - \langle \overline{w T}\, \overline{w \dot{T}} \rangle - \langle w \dot{T} \rangle\right] \langle w \dot{T} \rangle = -\langle w T \rangle \left[\langle \dot{T}^2 \rangle + \langle w \dot{T} \rangle \langle w T \rangle - \langle \overline{w T}\, \overline{w \dot{T}} \rangle + \langle w \dot{T} \rangle\right] \tag{84a}$$

$$\lambda_2\left[\langle \dot{T}^2 \rangle + \langle w \dot{T} \rangle \langle w T \rangle - \langle \overline{w T}\, \overline{w \dot{T}} \rangle - \langle w \dot{T} \rangle\right] = 2 \langle w T \rangle. \tag{84b}$$

Equations (83) can be combined to give

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla\right] \lambda_2 \dot{T}^2 + \frac{\partial}{\partial t}\left[\lambda_1 \dot{u}^2 + w T\left(-1 + \lambda_2 \langle \dot{T} w \rangle - \lambda_2 \overline{\dot{T} w}\right)\right] = 0, \tag{85}$$

which can be written in the current conservation form

$$\left[\frac{\partial}{\partial t} + \underset{\sim}{u}\cdot\nabla\right] Q = 0. \qquad (B6)$$

Thus the problem of convection is not too different from the Lorenz model, although the presence here of partial differential equations greatly complicates the calculation.

Acknowledgments

I am indebted to Professor Edward Spiegel for enlightening me about strange attractors, and for bringing Lücke's work to my attention. Without his insight section III would not have taken its present form. I would also like to thank the Geophysical Fluid Dynamics Staff and my fellow summer fellows for making my stay at Woods Hole so pleasant and rewarding.

Appendix

The Lorenz model is a special case of a particle moving according to the equation

$$\ddot{x} + (\sigma+1)\dot{x} = -\frac{\partial V}{\partial x}, \qquad (A1)$$

where V is a fourth order potential in x, with coefficients that obey an equation of the form

$$\dot{\lambda} + \kappa\lambda = g(x), \qquad (A2)$$

so that the system is provided with a feedback (Spiegel, 1978). The generic (in the sense of Thom fourth order potential is

$$V = \frac{1}{4}x^4 - \frac{1}{2}\lambda x^2 - kx \, . \qquad (A3)$$

The Lorenz model can be shown to be equivalent to the above with $k = 0$. It is of some interest to know whether the methods described in section 11 can be applied to the generic potential (A3).

In this case Eq. (10) becomes

$$\ddot{x} + (1+\sigma)\dot{x} + \left[-a + \omega(t)\right]x = k \, . \qquad (A4)$$

Now $\langle x \rangle$ no longer vanishes; writing (A4) in the form

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a-\omega & -b \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ k \end{pmatrix} \qquad (A5)$$

and proceeding as before, we obtain

$$\langle \ddot{x} \rangle + \left(b - \frac{\gamma-\alpha}{2\lambda^2}\right)\langle \dot{x} \rangle + \left[-a - \frac{b}{4\lambda^2}(\gamma-\alpha) - \frac{\beta}{2\lambda}\right]\langle x \rangle = k \, . \qquad (A6)$$

For a stationary system, with the approximation (23)

$$\left[a - \frac{\alpha b}{4\lambda^2}\right] \langle x \rangle \simeq -k. \qquad (A7)$$

Similarly the second moments satisfy the equations

$$(2a + b^2)\alpha \langle x^2 \rangle = (b\lambda^2 + 2\alpha) \langle \qquad (A8)$$

$$\langle \dot{x}^2 \rangle + \left(a - \frac{2b\alpha}{\lambda^2}\right)\langle x^2 \rangle + k \langle x \rangle = 0 \qquad (A9)$$

One can now proceed as before; and get a necessary condition for stationariness by eliminating $\langle \dot{x}^2 \rangle$, $\langle x^2 \rangle$ and $\langle x \rangle$ from the above equations. That condition is now much more complicated than the condition (24), and does not lead to a simple quadratic equation for $\langle z \rangle$ like Eq.(31). Nevertheless the method can be relatively straightforwardly adapted to this generic case.

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INVISCID EQUILIBRATION

Joseph Pedlosky

This seminar first briefly reviewed the general nature of the weakly nonlinear, finite amplitude dynamics of unstable waves.. A general distinction was drawn between two classes of problems. In the first for a given spatial wavenumber, the linear stability threshold of the basic state is given by a critical parameter Γ_c which represents the point where energy extraction by the instability just balances dissipation acting directly on the unstable mode. The slightly nonlinear extension of linear theory then gives the Landau equation for the disturbance amplitude A, as

$$\frac{dA}{dt} = \sigma A - N_D A^3$$

where σ is the linear growth rate and N_D is a number (in general complex) determined by the theory (see Stuart (1960) and Watson (1960)). The amplitude evolution equation is not reversible in time and if Real (N) 0 the amplitude evolves monotonically to a steady state.

In the second class the linear stability threshold is determined by a balance between a stabilizing inviscid constraint and the destabilizing inviscid mechanism associated with the instability. In this case the energy extraction is proportional to the rate of change of the disturbance amplitude rather than the amplitude itself. The generic form for the supercritical inviscid amplitude evolution equation is then

$$\frac{d^2A}{dT^2} = \sigma^2 A - N_I A B(A)$$

where again, σ is the linear growth rate. B(A) is a quadratit Function of A and N_I is the inviscid equilibration coefficient. For an example of meteorological importance see Pedlosky (1970).

An example of thermal convection in the absence of dissipation was discussed. The convection is inhibited entirely by a uniform, horizontal magnetic field B_0 and for the purposes of illustration the convective motion was assumed to occur in rolls oriented at right angles to the mean field (although rolls along the field would in fact be unaffected by the field). If

$$\Gamma = \frac{B_0^2}{\mu \, \partial\rho/\partial z \, L^2}$$

where B_0 is the mean field, L the layer depth, $\partial\rho/\partial z$ the unstable mean gradient, and μ the magnetic permeability, then the critical value of Γ for a cell of wavelength $2\pi/K$ is

$$T_c^{-1} = k^2 + m^2\pi^2$$

where m is the vertical mode number. For slightly supercritical states

$$T = T_c(1-\Delta)$$

the amplitude of the convective cell was shown to satisfy

$$\frac{d^2B}{dt^2} - \sigma^2 B + NB\left(|B|^2 - |B(0)|^2\right) = 0$$

where

$$\sigma^2 = \frac{k^2}{(k^2+m^2\pi^2)}\Delta$$

$$N = \frac{k^2}{(k^2+m^2\pi^2)} m^2\pi^2(3m^2\pi^2 - k^2),\ B(0) = (B)_{t=0}$$

so that the magnetic field of both the mean and the perturbation undergo long period oscillations if $N > 0$ $(k^2 < 3m^3\pi^3)$.

The presence of a small amount of dissipation was shown to lead to a third order set of equations, which it was pointed out, can be transformed to the set first discovered by Lorenz (1963) and which allow both stable limit cycles and persistent aperiodic motions.

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MAGNETIC FLUX ROPES AND CONVECTION

Michael R.E. Proctor

in order to understand the observed intermittency of the magnetic flux structures (usually called ropes or bundles) that tread the solar convection zone, it is necessary to take account of the mutual interaction between fields and flows. The kinematic aspects of flux concentration by persistent inexorable motions are well understood (Weiss 1966, 1977; Busse 1975): the. flux is expelled from regions with closed streamlines and concentrated in their boundary layers of B thickness $R_m^{-1/2} l$, where $R_m = UL/\eta \gg 1$ is the magnetic Reynolds number (U and L are velocity and length scales for the convection $\propto$ and η is the magnetic diffusivity). Since flux is conserved in the concentration process, the peak field B_m^* in the ropes is of order $B_0 R_m$ where B_0 is the mean vertical field. Such strong fields are certainly dynamically important: indeed, near the surface of the photosphere B^* appears to be so strong as to be approaching the upper limit $B_p \sim \sqrt{2\mu p_e}$ (where μ is the permeability and p_e the external pressure) set by consideration of normal stress across the edge of the flux rope. Deeper down in the convection zone, though, pressure differences between the rope and its surroundings are less significant and another type of dynamical effect is possible. The Lorentz $(\underset{\sim}{j} \wedge \underset{\sim}{B})$ forces, where $\underset{\sim}{j}$ is the current density, can act to impede the flow near the flux rope, so that the local magnetic Reynolds number (and so the amplification of the fluid) is less than for weak fields. It seems possible that the rate of amplification will decrease so fast with increasing flux that the peak field cannot exceed some global maximum B_m as a function of the flux. To investigate this, Galloway, Proctor and Weiss (1978) considered a simple problem for an axisymmetric cylindrical geometry in which a basic incompressible flow is confined to a cylinder of height d and radius of order, d and is driven by a prescribed body force. If there is no field $\underset{\sim}{U} = \underset{\sim}{U}_0$ and the configuration is defined by the two dimensionless parameters

$$R_m = \frac{(U_0)L}{\eta} \quad \text{and} \quad Q = \frac{B_0^2 L^2}{\mu \rho \nu \eta}$$

where L is of the same order and ρ is the density and ν the kinematic viscosity of the fluid. R_m is large and the flow is such as to concentrate flux at the base $z = 0$ of the cell is a measure of the amplification in the kinematic limit $(Q \rightarrow 0)$. The problem can. be solved exactly in this limit since the axisymmetric flux rope formed on the axis only affects the flow in its immediate vicinity provided that Q is not so large that the rope is no longer thin. The main result of the analysis is an explicit form for B^* the peak field at $z = 0$, the base of the rope, namely

$$B^* = B_0 R_m \cdot \frac{2}{\Lambda} e^{2d^2/\Lambda} \int_0^d f_0(y) e^{-2y^2/\Lambda} dy$$

where $\Lambda = Q\ell \; R_m^{1/2}$ and $f_0(y)$ is the basic vertical flow on the axis of symmetry.

From this it can be seen that B^* reaches its maximum B_m as a function of Q when $Q = O(1/\ln R_m^{1/2})$. These results, while paving the way for the solution of more complicated problems (see elsewhere in these notes), can also be used to give rough estimates of the sizes of field to be expected at various depths of the convection zone. (Galloway, Proctor, and Weiss 1977.)

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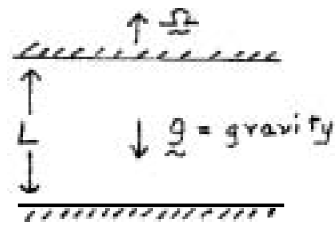
A CONVECTIVELY DRIVEN DYNAMO (Lecture #1)

Andrew Soward

There are reasons to believe that the geodynamo is the result of thermal convection in the Earth's liquid core. Perhaps the simplest model which isolates the influence of rotation on convection, is that of a Boussinesq fluid confined in a plane layer of width L, rotating with angular velocity $\underset{\sim}{\Omega}$ about a vertical axis, heated from below and cooled from above. When the Ekman number,

$$E = \nu/\Omega L^2,$$



where ν is the kinematic viscosity, is small, the onset of instability is characterized by convection cells with short horizontal length scale of order $\ell = E^{1/3} L$. Owing to the two length scale separation ($\ell \ll L$), this model provides a convenient starting point for the analytic development of a hydromagnetic dynamo.

With a suitable choice of boundary conditions the linear solution describing steady convection can be represented as the sum of N-rolls, for which the vertical velocity takes on the simple form

$$w = \sum_{n=1}^{N} w^{(n)} \sin \pi z \, e^{i \underset{\sim}{k}^{(n)} \cdot \underset{\sim}{x}} + \text{c.c.} \qquad \left(|\underset{\sim}{k}^{(n)}| = k_c \right).$$

Here z is the vertical coordinate, $k^{(n)}$ is a horizontal wave number, k_c is the critical wave number describing the onset of instability and c.c. denotes the complex conjugate of the expression preceeding it. Finite amplitude solutions of this type together with their stability have been discussed by Küppers and Lortz (1969) for the case of infinite Prandtl number.

When the fluid is electrically conducting the development of a skewed, horizontal magnetic field $\underline{B}_H(z,t)$ is governed by the dynamo equation

$$\frac{\partial \underline{B}_H}{\partial t} = \hat{\underline{z}} \times \frac{\partial}{\partial z}\left(\underline{\underline{\alpha}} \cdot \underline{B}_H\right) + \eta \frac{\partial^2 \underline{B}_H}{\partial z^2},$$

where t is the time, $\hat{\underline{z}}$ is the unit vector in z-direction, η is the magnetic diffusivity and the components of the tensor $\underline{\underline{\alpha}}$ are

$$\alpha_{ij} = -\left\{\sum_{n=1}^{N} \left(H^{(n)}/\eta k_c^4\right) k_i^{(n)} k_j^{(n)}\right\} \sin 2\pi z .$$

Here $H^{(n)} \sin 2\pi z$ is the contribution made by the nth roll to the horizontal average of the helicity $\underline{u}_0 . \mathrm{curl}\ \underline{u}$. In our model, the boundaries are supposed to be perfectly conducting and so the dynamo equation is solved subject to

$$\int_0^L \underline{B}_H \, dz = 0 \quad \text{and} \quad \partial \underline{B}_H / \partial z = 0 \quad \text{on} \quad z = 0, L.$$

Provided that there is more than one roll $(N > 1)$ and that the motion is sufficiently vigorous magnetic field regeneration is possible. The results of Childress and Soward (1972) and Soward (1974) indicate that, once the influence of the ensuing weak Lorentz force is taken into account, the preferred mode of convection is a roll whose axis is normal to (wave vector $\underline{k}$ is parallel to) a weighted z-average of $\underline{B}_H$. The resulting α-effect tends to regenerate new magnetic field in the direction perpendicular to the original field (i.e. in the direction $\hat{3} \times \underline{k}$). Consequently the orientation of both the magnetic field and the most vigorously convecting rolls tends to rotate on the magnetic diffusion time scale. In this way the system operates as an efficient dynamo.

In a limited parameter range corresponding to very weak magnetic fields Soward (1974) demonstrated the existence of stable hydromagnetic dynamos. For stronger magnetic fields, however, Childress (1976) has found that the dynamo is unstable. The reason can be traced to the well-known result that, when the magnetic field is uniform, the critical Rayleigh number R_c for the onset of convection in a rapidly rotating system initially decreases with increasing field strength. For the hydromagnetic dynamo problem the implication is that as magnetic field grows so does the vigour of the convection. Consequently the = -effect becomes more intense and the magnetic field grows at an ever-increasing rate. One may speculate, therefore, that the dynamo can only equilibrate when the Coriolis and Lorentz forces are comparable. By contrast, Busse (1975) has developed a similar dynamo model in an annulus rather than a plane layer in order to represent more faithfully geometrical constraints imposed by the spherical shape of the Earth's liquid core. In this case R_c initially increases with increasing magnetic field strength and so the stability of the dynamo is assured for weak magnetic fields.

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A KINEMATIC THEORY OF LARGE MAGNETIC REYNOLDS NUMBER DYNAMOS
(Lecture #2)

Andrew Soward

When a magnetic field permeates an incompressible, perfectly conducting fluid, the field lines are frozen to and move about with the fluid. If initially the magnetic field at position $\mathbf{x}$ is $\mathbf{b}(\mathbf{x})$, then later when the fluid particle has moved to $\tilde{\mathbf{x}}(\mathbf{x})$ the magnetic field is

$$\tilde{\mathbf{b}} = \mathbf{b}\cdot\nabla\tilde{\mathbf{x}} \qquad (1)$$

If instead the fluid has large but finite electrical conductivity, the above result only provides a correct first approximation for a limited time, since the error increases indefinitely. To avoid this secular behaviour it is necessary to allow the reference field $\mathbf{b}$ to evolve slowly on the magnetic diffusion time scale. Any rapid development of the field caused by the advection of the field lines is accommodated by the strain tensor $\partial\tilde{x}_i/\partial x_j$ in (1). A further generalisation of this Lagrangian description is made by introducing a reference fluid velocity $\mathbf{u}(\mathbf{x})$ so that the actual fluid velocity at $\tilde{\mathbf{x}}$ is

$$\tilde{\mathbf{u}} = \partial\tilde{\mathbf{x}}/\partial t + \mathbf{u}\cdot\nabla\tilde{\mathbf{x}}. \qquad (2)$$

Here $\tilde{\mathbf{x}}$, $\mathbf{u}$ (as well as $\mathbf{b}$) depend on both $\mathbf{x}$ and time t. As a result of the transformations, the magnetic induction equation becomes

$$\partial\mathbf{b}/\partial t = \nabla\times(\mathbf{u}\times\mathbf{b}) + \eta\nabla^2\mathbf{b} - \eta\nabla\times\boldsymbol{\mathcal{E}}, \qquad (3a)$$

where η is the magnetic diffusivity,

$$\mathcal{E}_i = \epsilon_{ipq}(\alpha_{kp} - \delta_{kp})\,\partial b_q/\partial x_k - \mu_{ij}b_j, \qquad (3b)$$

$$\alpha_{kp} = \tilde{\nabla}x_k\cdot\tilde{\nabla}x_p, \quad \mu_{ij} = -\partial\tilde{\mathbf{x}}/\partial x_i\cdot(\tilde{\nabla}\times\partial\tilde{\mathbf{x}}/\partial x_j) \qquad (3c,d)$$

and $\tilde{\nabla}$ denotes the gradient with respect to $\tilde{\mathbf{x}}$.

This formulation provides an especially convenient method for considering Braginsky's (1964 a,b) nearly axisymmetric dynamo model (see Soward (1972), Moffatt (1978)). In essence we consider an axisymmetric reference flow,

$$\underline{u} = U\hat{\underline{\phi}} + \epsilon^a u_M, \qquad (\epsilon << 1) \tag{4a}$$

where $\hat{\underline{\phi}}$ denotes the unit vector in the azimuthal direction and the suffix M denotes the meridional component. Asymmetries of the real flow are accommodated by the small displacement,

$$\epsilon \underline{\eta} = \tilde{\underline{x}} - \underline{x}, \tag{4b}$$

of fluid particles from their mean trajectories, which according to the assumed form (4a) are almost circular. Direct substitution of (4) into (3) shows that the reference magnetic field is almost axisymmetric and has the form

$$\underline{b} = B\hat{\underline{\phi}} + \epsilon^2 \underline{b}_M + O(\epsilon R^{-1}) \tag{5}$$

Here the asymmetric part of $\underline{b}$ is represented by the error term and R is the magnetic Reynolds number which is assumed to be large.

On the basis of the scaling in (4), we may take the # -average of (3) and legitimately neglect all averages of products of fluctuating quantities with the exception of

$$\mathcal{E}_\phi = -\Gamma B, \tag{6a}$$

where

$$\Gamma = -\frac{\epsilon^2}{2\pi}\int_0^{2\pi}\left(\frac{\partial \underline{\eta}}{\partial \phi} \cdot \nabla \times \frac{1}{s}\frac{\partial \underline{\eta}}{\partial \phi}\right) d\phi \tag{6b}$$

and s is the distance from the axis of symmetry. Provided,

$$\epsilon^2 = O(R^{-1})$$

there is the possibility that the resulting α-effect is sufficiently large to prevent the otherwise inevitable collapse of the meridional magnetic Field.

It should be emphasized that $\underline{u}_M$ differs significantly from the ϕ-average of the actual (as opposed to the reference) meridional flow velocity. Indeed when the latter average is zero, we may identify $\underline{u}_M$ with the systematic meridional flow of fluid particles (this is the phenomena of Stokes drift). The difference between the averages of actual and reference quantities accounts for Braginsky's (1964 a,b) use of "effective" variables, which are simply $\underline{u}_M$ and $\underline{b}_M$ introduced in (4a) and (5).

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A STRANGE ATTRACTOR

Edward A. Spiegel

This discussion, based on work done with C. Marzec and with J.-P.Poyet is aimed at exploring aperiodic behavior in the solutions of the dynamical system

$$\ddot{x} = -\frac{\partial V}{\partial x} - \epsilon\mu\dot{x} \tag{1}$$

where $x = x(t)$ and the potential is a polynomial:

$$V = V(x;\lambda) = \frac{1}{m}x^m - \sum_{k=1}^{m-1}\alpha_k(\lambda)\frac{x^k}{k}. \tag{2}$$

The differentiation of V in (1) is for fixed λ ; but λ has time dependence which may be specified explicitly or implied by a differential equation. We consider the specific form

$$\dot{\lambda} = -\epsilon\left[\lambda + g(x)\right], \tag{3}$$

with constant ϵ and μ. It then remains to furnish the $\alpha_k(\lambda)$ and $g(x)$. A particular choice corresponds to the model studied by Lorenz[1]. Here, we concentrate on the example[2,3]

$$\alpha_1 = \lambda,\ \alpha_2 = \delta = \text{const.},\ g = x - x^3,\ \mu = 0. \tag{4}$$

The equilibrium surface $\partial V/\partial x = 0$ plays an important role in the dynamics. In the case of (4) this describes a pleated surface[4] in (x, λ, δ) space. The projection of the pleat onto the $\lambda - \delta$ plane is delineated by the cusp, $\lambda = 2\delta^{3/2}/3^{3/2}$; this shows the interest in the parameter[3] $B = \sqrt{3}\,\lambda\,\delta^{-3/2}$.

Equation (1) can be replaced by the two equations, $\dot{x} = y, \dot{y} = -V_x - \epsilon\mu\dot{x}$. The flow divergence, $\partial\dot{x}/\partial x + \partial\dot{y}/\partial y + \partial\dot{\lambda}/\partial\lambda$, has the constant value $-\epsilon$. Swarms of representative points in (x, y, λ) - space will condense down to zero volume exponentially in time. The sets of points onto which these swarms ultimately condense are called attractors. Attractors may be stable critical points or stable limit cycles, for example. When they are sufficiently complex they are called strange, to use the term suggested by Ruelle and Takens[5]. Normally, strange attractors associated with ODE's are found by numerical integration. But astronomers know that you do not need an ephemeris to study the form of an orbit; it is often valuable to get the time out of the problem.

Let λ be taken as independent variable: $x(t) = \tilde{x}(\lambda)$,

$$t = \int \frac{d\lambda}{\lambda + g(\tilde{x})} \tag{5}$$

Then if prime denotes differentiation w.r.t. λ, (and we drop the tildes) we find

$$\epsilon^2\left[(\lambda+g)^2 x'' + (1-\mu)(\lambda+g)x' + g_x(\lambda+g)x'^2\right] = -\frac{\partial V}{\partial x}. \tag{6}$$

We obtain a standard-looking problem in matched asymptotic expansions whose study reveals much about the role of $g(x)$ in these affairs. Unfortunately, the inner problem is not easy; in one of the simpler cases[1] (6) reduces to the equation for the 2nd Painlevé transcendant. This is just the way to describe the transition from one trough to another in V in terms of nonlinear turning point theory (the inner equation is a nonlinear version of Airy's equation).

The traditional asymptotic methods, such as the method of averaging or two-timing, are also very enlightening. For $\epsilon << 1$ the method of averaging for the example (4) describes periodic orbits 3. When $\delta <$ .62 these are stable but they lose stability when δ increases through .62, to be replaced by quasiperiodic orbits. When δ increases through .75, the method no longer works easily but by combining it with results from (6) we may be able to extend it.

The "transition" at δ = .62 suggested by the asyrnptotics is reflected in a corresponding change in the character of the numerical solutions. As δ is increased through .62 the period of the limit cycle starts a series of doublings corresponding to what are called pitchfork bifurcation[6]s. When δ reaches .625 this doubling is over and the behavior seems genuinely aperiodic; a strange attractor seems to have formed. In fact, the object which corresponds to the attractor appears to exist even for $\delta <$ .62, but it is not their attractor.

We have studied the form of the attractor mainly in (E, B, s)-space, where

$$s = (3/\delta)^{1/2} x,\ B = \lambda(3/\delta^3)^{1/2},\ E = 3\delta^{-2}(\dot{x}^2/2 + V). \tag{7}$$

To examine the solutions, we construct a Poincaré map, or surface of section, in which successive crossings of the E-B plane with $\dot{s} > 0$ are marked by a point, and many such points are accumulated. For δ = .625 the surface of section is that shown in Fig.1 for all initial conditions we have tried, apart from differences in transients. If we look at the very tip of one of the "leaves" dangling from the attractor shown in Fig.1 and magnifyit manyfold, we obtain Fig.2. The numerical integrations available thus far are not sufficient to warrant another blowup,

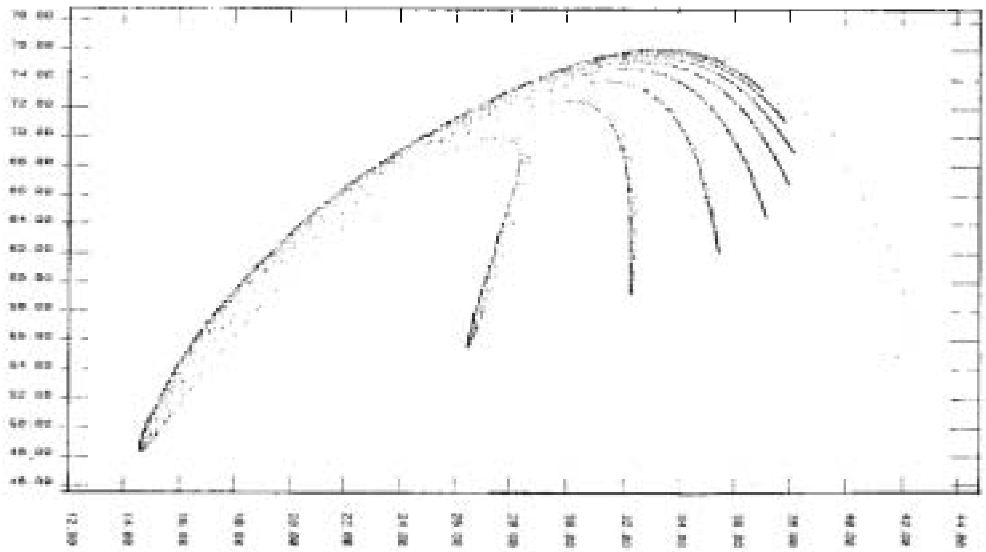

Fig.1. Surface of section (s=0) in (E,B,s) space of the attractor of (1)-(4) for δ = .625.

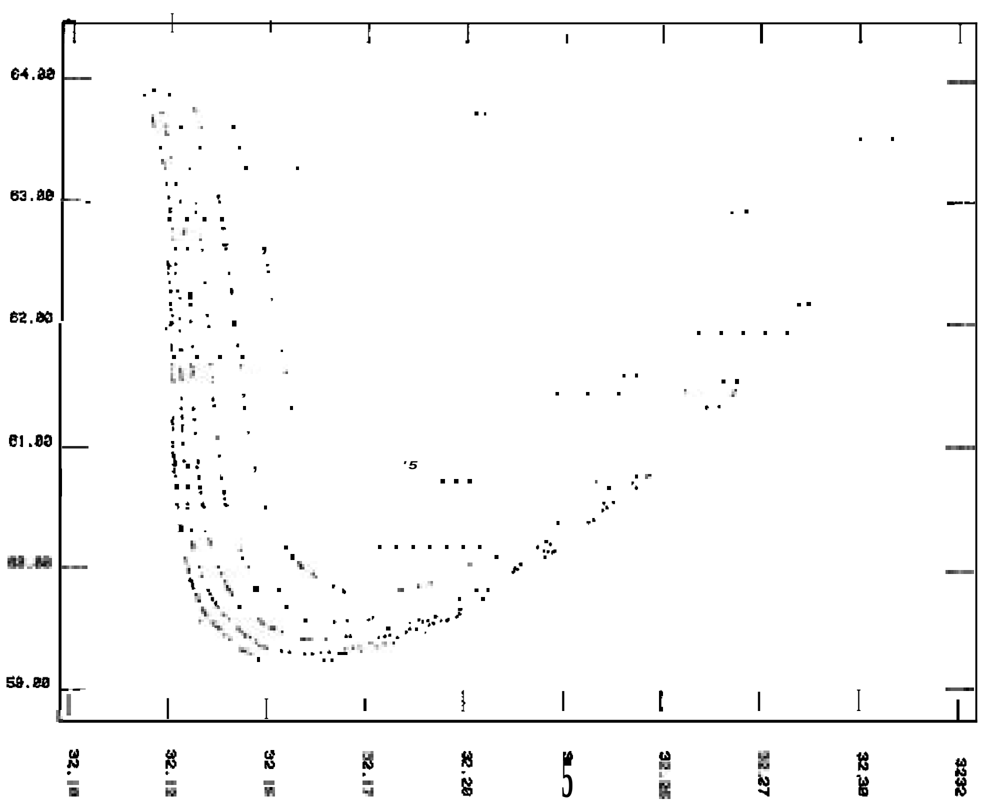

Fig.2. A blowup of the tip of one leaf of Fig. 1, showing fine structure.

but we conjecture that this would only show a repeat, on a finer scale, of the structure of Fig.2. In other words we presume that the attractor has the structure of a Cantor set in one of its dimensions, much as Henon's[7] well-known model does.

The asymptotic study of the case (4) revealed that in (E,B,λ)-space the curve defined by $\ddot{s} = 0$, $\dot{s} = 0$, seemed to play a special role. This is true also for the appearance of strange behaviour in the numerical solutions. These statements find a congenial expression in the language of catastrophe theory[4]. We may introduce a superpotential[1]

$$\mathcal{U} = \frac{x^{m+1}}{m+1} - \sum_{1}^{m-2} \frac{m\alpha_k(\lambda)}{k(k+1)} x^{k+1} - m\,\mathcal{E}x \tag{8}$$

where $\mathcal{E}$ is to be thought of as a parameter on the same footing as λ. Now the condition $\partial\mathcal{U}/\partial x = 0$ is easily seen to be equivalent to $\mathcal{E} = \mathcal{V}$, hence we have also that $\ddot{x} = 0$ and $\partial\mathcal{V}/\partial x = 0$. These two conditions define the catastrophe set[4] of U which, for m = 4, is called a swallow tail. We find that motion of a system point through the catastrophe set generally involves erratic behavior even when there is not a strange attractor. When there is one, the attractor lies near the catastrophe surface, as in Fig.3, where the attractor's surface

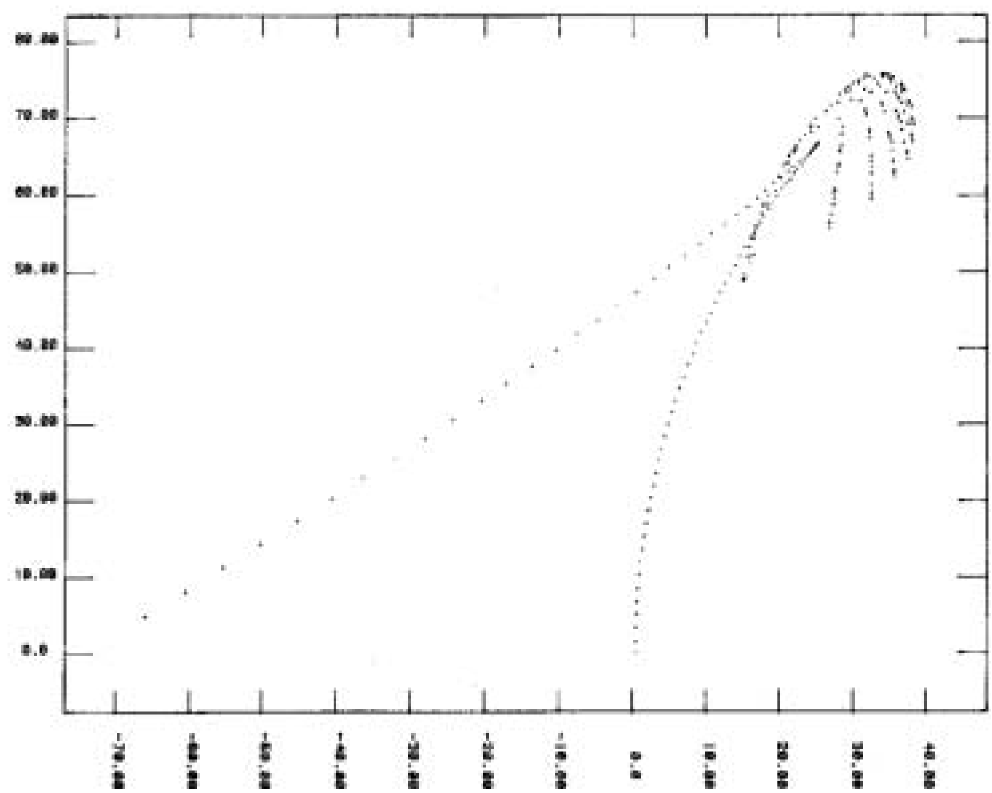

Fig.3. Same as Fig.1 but showing the catastrophe set of $\mathcal{U}$

of section in the E-B plane is shown with the catastrophe set of V for the example of (4). The tip of the swallow-tail is at B = 2/3, which defines the catastrophe set of V. These results seem not to depend sensitively on g, provided that the choice of g is not trivial.

We believe that the system (1) - (3) provides a class of strange attractors when the oscillation in λ engendered by (3) takes the system near the catastrophe set of V. A guide to the behavior also seems to involve the superpotential U. Examples in GFD are discussed in these proceedings by Childress and Pedlosky.

The potential V has m-2 parameters and we have made them all depend on λ. We could add more free parameters and in addition to (3) we could introduce m-3 additional equations of this form. A variety of interesting examples may be constructed in this way.

Of course, one of the main interests in the present kind of study is the possible clarification of turbulence that it may provide. To approach this problem systematically, we wish to extend the highly truncated Fourier analysis of the fluid equations on which Lorenz's[1] and related investigations are based[8,9]. We obtain equations like (1) - (3), with additional terms of a form not included there, but also where X becomes a vector. For this reason, the important generalizations correspond to similar equations with more degrees of freedom. Already with two degrees of freedom we can study in a given system an illustration of KAM theory[10] when $\epsilon = 0$ and a strange attractor, when $\epsilon \neq 0$. How do the two problems come together? That is a problem we are trying to understand at present.

Is the bearing on turbulence theory of this kind of study more direct than these vague analogies suggest? To look into this, suppose that at $t = 0$, $X = a$. Consider a swarm of system points with only one system point at each X at $t = 0$. Let the initial velocity field be such that orbits do not cross for a nonzero interval of time. The orbit of any system is $x = X(t,a)$.

Now $\dot{x} \equiv [\partial x/\partial t]_{a \text{ fixed}} \equiv DX/Dt = v(t,a)$.

For some time, according to our assumptions, we can solve for $a = A(t,x)$ and hence may write $v(t,a) = v(t, A(t,x)) = u(x,t)$. Equations (1) - (3) become, with $\lambda = \lambda(x,t)$.

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial V}{\partial x} - \epsilon\mu u \tag{9}$$

$$\frac{D\lambda}{Dt} = \frac{\partial \lambda}{\partial t} + u\frac{\partial \lambda}{\partial x} = -\epsilon g - \epsilon\lambda \tag{10}$$

This is a sort of Burgers description of convection - not unlike one contemplated many years ago by D.W.Moore - and has many of the features of the usual Burgers description. It generalizes readily to two or more dimensions. In that case, it may pay to consider using some modern methods of diffraction theory[11].

Let

$$u = \frac{\partial S}{\partial x}, \quad \lambda = \frac{\partial \Phi}{\partial x}. \tag{11}$$

Suppose also that

$$\alpha_k = a_k \lambda + b_k, \tag{12}$$

where a_k and b_k are constants. Then we find

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \epsilon \mu S + V = -\sum_{k=1}^{m-a} \frac{x^k}{k} \Phi. \tag{13}$$

The equation is less edifying, But the qualitative impression is that the complex nature of certain caustics[12] may have a family relation to the structure of strange attractors.

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MAGNETOCONVECTION

Nigel O. Weiss

Discrete ropes of flux seem to be characteristic of magnetic fields in the sun's convection zone, and these flux ropes should be included in any detailed dynamo model for the solar cycle. As a preliminary, we can study Boussenesq convection in the presence of an externally imposed magnetic field. Numerical integration of the nonlinear equations makes it possible to explore the dynamical interaction between magnetic flux ropes and convection.

For simplicity let us consider two-dimensional convection with the velocity and the magnetic field confined to'the xz-plane and independent of y, where the z-axis points upwards. We assume that the boundaries of the region $\{0<x<\lambda d;\ 0<z<d\}$ are stress-free and that the total flux is equal to that for a uniform vertical field B_0. Then a particular configuration is characterized by the Reyleigh number R, by the parameter $Q = B_0^2 d^2/\mu\rho\nu\eta$, by the Prandtl numbers $p_1 = \nu/\chi$, $p_2 = \nu/\eta$ and by the aspect ratio λ (cf. Chandrasekhar 1961, Weiss 1977). Other useful parameters are $p_3 = \chi/\eta$ and Q/p_3, (which, like R, contains χ and ν in the denominator). In all the computations $p_1 = 1$. $\lambda = 1$ except where stated otherwise.

If $p_3 < 1$ convection sets in as a direct instability, when

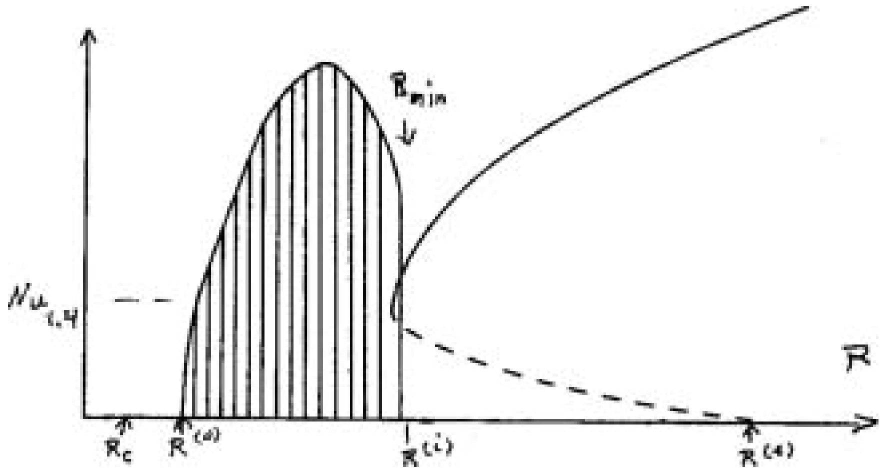
$$R = R^{(e)} = R_c + 2\pi^2 Q,$$

where $R_c = 8\pi^4$ is the critical Rayleigh number in the absence of a magnetic field. For $Q \gg p_3 \gg 1$ (the astrophysically relevant case) convection first appears as an overstable mode when $R = R^{(o)} \sim \pi^2 Q/p_3$ and there is a transition from oscillatory to direct modes at $R = R^{(1)} \sim 2\pi^2 Q/p_3$. Busse (1975) showed that for $p_3 \gg 1$ and Q sufficiently small, finite amplitude steady convection first appeared when

$$R = R_{min} = R_c + 61.0\,(Q/p_3^{1/2})^{4/5}$$

The nonlinear results for $p_3 > 1$ show finite amplitude oscillations when $R > R^{(o)}$. For $1 \ll Q/p_3^{1/2} \lesssim 100$, steady convection appears at the Rayleigh number predicted by Busse. The field is concentrated into ropes with a Gaussian profile and the horizontal velocity varies linearly across these ropes. When $Q \gg p_3^{1/2}$ the situation changes: motion is excluded from the flux ropes, which are almost stagnant. The value of R_{min} is close to $R^{(1)}$ and independent of p_3

for $p_3 >> 1$. Within the rope the field is nearly uniform, with a narrow current sheet separating the flux rope from the convective eddy.



Sketch showing the Nusselt number N_u as a function of R for $Q >> p_3 >> 1$. The shaded region shows the peak value of Nu for oscillatory convection. The unstable steady solution branch from R_{min} to $R^{(e)}$ is conjectural.

Calculations with $R = 10^4$ and different values of Q show that steady convection in the dynamic regime is possible for $p_3 R \lesssim Q \lesssim p_3^{1/2} R^{1/3}$. The transition from a dynamic to a kinematic (weak field) regime occurs when the concentrated field is no longer strong enough to exclude the motion from the flux rope (Galloway, Proctor and Weiss 1978). At higher Rayleigh numbers narrower cells are preferred; for $R = 10^5$, square cells broke up into cells with $\lambda = 1/2$. In the dynamic regime a second solution appears, with most of the flux concentrated on one side of the convective cell. Symmetrical cells are apparently unstable to perturbations which develop into these lopsided cells, though the latter transport slightly less heat.

The dynamical importance of flux ropes is clear from these numerical experiments. There are also indications that a few large ropes may be preferred to many small ones. Galloway and Moore (1978) have obtained similar results for axisymmetric cells, where flux concentration is much more potent. On the other hand, nonlinear thermohaline convection (Huppert and Moore, 1977) apparently shows no analogue of the high Q dynamical regime.

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HYDROMAGNETIC PLANETARY WAVES

Willem V.R. Malkus

The secular variations of the earth's magnetic field appear to be wave-like processes in an underlying dynamo. Models of the geodynamo suggest that there is a strong toroidal field (approximately 100 gauss). Here we discuss several aspects of global hydromagnetic oscillations in rotating systems both stratified and unstratified.

In an early paper, Malkus (1967), an idealization was found in which all the modes of a hydromagnetic oscillation of a rotating spheroid could be determined. By good fortune, the choice of a uniform electric current to define the basic magnetic field led to a modified Poincaré eigenvalue problem. Due to the work of Cartan (1922), Roberts and Stewartson (1963 a,b,c) and Greenspan (1964, 1965), many of the properties of the Poincaré problem are understood. Here several dispersion relations are established determining that in the hydromagnetic case, modes of the system have phase velocities both East to West. For small rotation rates the lowest non-axisymmetric modes are unstable -- for rotation rates of geophysical interest all normal modes are stable. It is found that the zonal phase velocities of fast magneto-hydrodynamic and slow hydromagnetic waves can be of either sign. From the unstable normal modes of this problem, it was consluded that selective excitation of the observed westward motion may be a consequence of shear or buoyancy instability. More recent studies by Acheson (1972) confirm that most unstable modes of the large scale slow hydromagnetic sort do move towards the West. However, an important class of "shellular" modes was found (Malkus (1967), Stewartson (1967) to move to the East. The addition of stratification added a whole new class of interesting problems including that of magnetic buoyancy. Recent studies by Parker (1977) and Acheson (1978) discuss the various instabilities of rotating magnetic systems which could lead to westward phase velocities. Perhaps the most interesting of these has to do with the destabilizing effects of ohmic, thermal and viscous diffusions. The criteria for these instabilities are derived and Tabulated.

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MINI-SYMPOSIUM ON MAGNETOHYDRODYNAMICS AND DYNAMO THEORY

Abstracts of Seminars

MAGNETIC PROBING OF EARTH'S LIQUID CORE

Edward R. Benton

Consideration is given to the following question: What can be inferred, theoretically, about earth's liquid core using measurements of only the geomagnetic field at earth's surface: We discuss how, in principle, the following four quantities of interest can be obtained from a primitive "first order" model of the earth.

(a) Depth or radius of the liquid core (a recent result of Hide)

(b) Depth at which substantial vertical motion and intense electric current begin to flow

(c) Horizontal fluid motion adjacent to the core-mantle boundary

(d) Rate of increase with depth of the azimuthal field at the surface of the core;

Consistent with the present state of geomagnetic observations relevant to this problem, we adopt only a simple model of the earth based on the following physical assumptions:

(i) The mantle is a smooth spherical annulus without ellipticity or topography.

(ii) The mantle is either an insulator or at most a weak spherically symmetric conductor.

(iii) On the decade time scale the core fluid moves like an inviscid Boussinesq liquid of nearly uniform density and perfect conductivity, stirred by radial gravitational forces (thermal, compositional, or phase-change in origin).

The data needed for practical application to this work (not presently available in adequate form) consist of global measurements of the three-component vector geomagnetic field of internal origin as seen at earth's surface for two different epochs separated by a few decades in time. Alternatively, use could be made of B and B (secular variation) at a single epoch but this is regarded as more difficult, observationally, and is also the harder to utilize.

It is essential for these purposes to devise schemes that fit the data not so as to best reproduce field values at specified locations, but rather inversely to give most accurate locations at which the field takes on prescribed values (i.e. contour curves of the field need to be accurately located). The classic (former) problem is linear in the Gauss coefficients, the latter, highly nonlinear, so interesting developments are to be expected.

The results obtained can also be used to provide new constraints on secular variation models.

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MAGNETOHYDRODYNAMIC MODELS OF PLANETARY DYNAMOS

Friedrich H. Busse

There are two reasons for the study of the magnetohydrodynamic dynamo problem in order to understand the origin of planetary magnetism. First, kinematic dynamo models do not provide sufficient information to isolate a particular dynamo mechanism. Widely different velocity field can give rise to the same observed magnetic field. Secondly, the kinematic dynamo problem does not determine the equilibrium amplitude of the magnetic field which is the most important parameter of physical interest. Moreover, the systematic variation of the strength of the magnetic field of different planets provides the most stringent test for any theory of planetary dynamos.

The analysis of the magnetohydrodynamic dynamo problem is complicated by numerous nonlinear effects that can occur, some of which are discussed on the basis of dynamo models of Busse (1973, 1975).

(I) Lenz' rule: The normal effect of the nonlinear Lorentz force is to alter the velocity field (mainly by decreasing its amplitude) such that the growth of the magnetic field is terminated and an equilibrium amplitude is achieved, at least in the time average sense.

(2) The Lorentz force may enhance dynamo action and equilibrium amplitudes for the magnetic field may be found for less than the initial value of the magnetic Reynolds number (Busse, 1977).

(3) When the equilibrium solution is unstable, nonlinear oscillations can occur. This situation occurs, for example, when the α-effect

decreases with increasing magnetic Reynolds number. This property is caused by flux expulsion from the velocity eddies (Roberts 1972, Busse 1973) and is typical of dynamos with nearly steady motion.

(4) The Lorentz force may release dynamic constraints, in particular the constraints of the Coriolis force. It is this effect which is the basic physical reason for the generation of magnetic fields in rotating planetary cores. The opinions only differ on the particular way in which this release is accomplished. Since there are no dynamo models with strong Lorentz forces available, the subject is speculative. One such speculation is that an upper bound on the magnetic field provided by the condition for the existence of the hydrodynamic branch of convective solutions in the annulus model (Busse, 1975),

$$B_o \leq \Omega r_o \left(\frac{\rho_o \mu \chi}{\lambda} \right) \frac{\eta}{a^2}$$

when B_o is the field strength in the planetary core, Ω, r_o and ρ_o are rotation rate, radius and density of the core. λ and χ are magnetic and thermal diffusivities, and μ is the magnetic permeability. η is a geometric factor of the order 1/2 and a is the typical wavenumber of the convection columns in the core based on the radius r_o as length scale. Using a lower bound on a of the order 10 suggested as a condition for dynamo action by the numerical experiments of Bullard and Gubbins (1977) the upper bound (*) appears to give remarkably good fit to the observed amplitudes of planetary magnetic fields (Busse, 1976).

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A PRELIMINARY REPORT OK PROGRESS IN MODELING THE SOLAR DYNAMO

Peter A. Gilman

For the past several years, we have been developing a numerical model for a full MHD dynamo in a rotating spherical shell. The motions responsible for the induction are convective flows, driven by uniform heating at the bottom of the shell, together with the differential rotation these motions drive. The motivation for the model is the solar dynamo, although the model physics is still considerably simpler than the solar case.

Our strategy has been first to develop a model for convection and differential rotation, which produces surface differential rotation much like that of the sun, even in quantitative terms. We then study the dynamo properties of this "best" solution. The full j x B feed-backs of the induced field on the flow are included.

Briefly, the model physics is as follows:

1. Boussinesq fluid
2. Linear diffusion of temperature, momentum, and magnetic field
3. Central gravity ($1/r^2$)
4. Stress free top and bottom, constant heat flux bottom, constant temperature top
5. Perfect electrical conductor bottom, radial magnetic field or perfect conductor top (no potential field)
6. Shell depth arbitrary

The solution technique is:

1. All dependent variables are expanded in fourier series in longitude
2. Resulting amplitude functions are solved for on an energy con-' serving, staggered grid in the meridian plane.
3. Leap-frog time differencing is used
4. Pressure is found by solving Poisson type equation from divergence.

5. Two components of the induction equation are solved, plus $\nabla \cdot B = 0$, with the third used as a check.
6. The calculation is currently started from random numbers in the temperature field, and in the seed toroidal field.

So far, a small number of limited experiments have been performed, with 13 points in the radial direction, 61 points from pole to pole, and between 2 and 11 wave numbers in longitude (always including wave number 0 for the axisymmetric flow and field). No symmetries about the equator are assumed, because we wish to look for any tendency for symmetry selection.

We have most extensively studied the case with Rayleigh number $= 8 \times 10^4$, Taylor number $= 2 \times 10^5$, Prandtl number P = 1 with shell depth 1/3 of the outer radius. For this case we get the following preliminary results.

1. Dynamo action is sustained for magnetic Prandtl number $P_M = \eta/\kappa \lesssim 0.25$ for 2 modes, $\eta/\kappa \lesssim 0.2$ for 11 modes, so convergence seems good. These P_M's correspond to internal magnetic Reynolds numbers in the range 150-200.
2. The feed-backs on the motion are quite large if P_M is much below the critical value. For example, differential rotation energy can be reduced by a factor of two, before statistical equilibrium is reached, compared to the same case without magnetic fields. Approximate equipartition between the field and flow is rather easily achieved.
3. We do get field reversals, whose period is not sharply regular. The typical period is shorter than for the sun by a factor of 20-30, for surface differential rotation of the same size as the sun has. Migration of torodial fields both toward the poles and equator is seen, so the "butterfly diagram" for the model would be more complex than on the sun.
4. No clear symmetry selection mechanism has been found so far. That is, symmetric (quadripolar) and antisymmetric (dipolar) magnetic fields have roughly equal amplitudes, on the average. This is despite the fact that the motion fields turn out to be strongly biased towards symmetry about the equator (meaning east-west and radial motions are symmetric). It is possible that the model has

to be run longer (at least from random numbers) to establish this property.

5. For this case, the dynamo appears to be more "α^2" like, rather than "$\alpha - \omega$" like, in that the maintenance of the axisymmetric toroidal field is due primarily to induction by the nonaxisymmetric convection, rather than due to shearing of the axisymmetric poloidal field by differential rotation.

Both the short period of the model compared to the sun, and the dominance of convection rather than differential rotation in maintaining the toroidal field, may be explained by the fact that the helicity of the convection is two or three orders of magnitude larger than has been previously assumed in simpler dynamo models in order to get the right solar period. In other words, convection sufficiently influenced by rotation in the model to drive the right differential rotation for the sun has much more helicity than apparently is felt by the solar magnetic field. Reducing the rotational influence (reduced Taylor number, or increases Rayleigh number) to get the right magnitude helicity does not work, because the equatorial acceleration is lost. We doubt that the addition of compressibility to the model will help, because it would have to destroy most of the helicity of the convection, while still retaining its Reynolds stresses to maintain the right differential rotation.

Instead, we suspect that the ability of the solar magnetic fields to concentrate into tight flux tubes, around which the plasma may flow, is crucial. The model shows some tendency to do this. That is, we find a greater fraction of magnetic energy is in the high wave numbers than for the kinetic energy and the toroidal field is more highly structured than the differential rotation. Unfortunately, the cost of computing with much higher resolution makes it very difficult to represent the concentration mechanism very well. On the sun, perhaps the concentration of the field into small flux tubes we see at the surface extends throughout the convection zone, with convective flows, and differential rotation, slipping around the tubes. This should reduce the net helicity felt. The period of reversal then may be a nonlinear function of the fraction of the total volume occupied by flux tubes. The magnitude of the reaction of the field upon the global flow should also be reduced for flux tubes occupying a small fraction of the volume.

If these concentration effects are fundamental to the solar case then a serious problem for the future is how to represent such a small scale process accurately in global dynamo models in which the hydrodynamics is explicitly calculated.

In the light of our results, albeit preliminary, recent "success" by Yoshimura in modeling the solar cycle and its envelope using effects of global convection and differential rotation, is probably the result of fortuitously compensating over-simplifications of the hydrodynamics, induction, and nonlinear feed-backs, which have resulted in enough free parameters, when allowed to take mutually inconsistent values, to "model" almost any nonlinear system.

TURBULENT DIFFUSION OF MAGNETIC FIELDS

Edgar Knobloch

In studying the turbulent diffusion of magnetic fields, we are interested in calculating the statistical properties of the magnetic field $\underset{\sim}{B}(\underset{\sim}{x}, t)$ in terms of the statistical properties of the turbulent velocity field $\underset{\sim}{u}(\underset{\sim}{x}, t)$. In general this problem leads naturally to non-. linear coupled stochastic differential equations. Here we shall restrict ourselves to the discussion of the diffusion of passive magnetic fields by a prescribed (in a statistical sense) incompressible turbulent velocity field in the high magnetic Reynolds number regime appropriate to the sun. In this case the problem reduces to the study of the stochastic induction equation, which may be written in the general form

$$\left[\frac{\partial}{\partial t} + L(\underset{\sim}{x}, t)\right] f(\underset{\sim}{x}, t) = 0 \qquad (1)$$

where $L(x, t)$ is a stochastic operator, independent of f. This equation can be solved for the ensemble average of f, $\langle f \rangle$ or $\bar{f}$, by eliminating f', the fluctuating part of f, from the equation (Knobloch, 1977). If we assume that at time $t = 0$, $f'(0) = 0$, or that it is uncorrelated subsequently with the velocity field, the exact solution can be written as the integro-differential equation

$$\left[\frac{\partial}{\partial t} + \bar{L}\right]\bar{f} = \int_{0}^{t} dt' \langle L'(t) \exp_0\left\{-\int_{t'}^{t} dt_1 u_0(t, t_1)(1+A)L'(t_1)\right\} u_0(t_1, t') L'(t') \rangle \bar{f}(t'), \qquad (2)$$

where $\bar{L} \equiv \langle L \rangle$, $L' \equiv L - \bar{L}$, U_0 is the Green's operator for the equation

$$\left[\frac{\partial}{\partial t} + \bar{L}\right]\bar{f} = 0,$$

and A is a projection operator that takes an ensemble average of everything following it. The subscript 0 on the exponential indicates a time-ordered exponential. In what follows we shall restrict ourselves to homogeneous turbulence, and shall therefore assume that $\langle \underset{\sim}{u} \rangle = 0$. In the high Reynolds number limit the diffusion term in the induction equation may be omitted, so that now $\bar{L} = 0$ and $U_0 = 1$. In this case equation (2) may be cast into a differential equation for $\bar{f}$,

$$\frac{\partial \bar{f}}{\partial t} = \left[\sum_{m=2}^{\infty} K_m\right]\bar{f}, \qquad (3)$$

where the operators K_m involve $m-1$ integrations over the cumulants of L'. For this reason, the result (3) is an expansion in powers of the autocorrelation time τ_c of the stochastic operator L' (Van Kampen 1974, Terwiel 1974). The incorrectly time-ordered terms in the cumulants correct for the memory effects lost in pulling $\bar{f}(t')$ from under the integral signs in equation (2).

The simplest approximation arises when the autocorrelation time τ_c of the turbulent velocity field is short. Then the first term on the right side of (3) dominates, and the approximate equation may be written as

$$\frac{\partial \bar{f}}{\partial t} = \left\{\int_0^{\infty} \langle L'(t)\, L'(t-\tau) \rangle d\tau\right\} \bar{f}(t) \qquad (4)$$

The resulting equation for $\bar{\underset{\sim}{B}}$ becomes for isotropic helical turbulence

$$\frac{\partial \bar{\underset{\sim}{B}}}{\partial t}(\underset{\sim}{x}, t) = \eta_1^2 \nabla \times \bar{\underset{\sim}{B}}(\underset{\sim}{x}, t) + \eta_2^2 \nabla^2 \bar{\underset{\sim}{B}}(\underset{\sim}{x}, t) \qquad (5)$$

where

$$\eta_1^2 = -\frac{1}{3}\int_0^{\infty} \langle \underset{\sim}{u}(\underset{\sim}{x}, t) \cdot \nabla \times \underset{\sim}{u}(\underset{\sim}{x}, t-\tau) \rangle d\tau,\ \eta_2^2 = \frac{1}{3}\int_0^{\infty} \langle \underset{\sim}{u}(\underset{\sim}{x}, t) \cdot \underset{\sim}{u}(\underset{\sim}{x}, t-\tau) \rangle d\tau. \qquad (6)$$

Here the subscript denotes the coefficient of ∇, and the superscript the number of velocities entering in the definition. Equation (5) in the usual dynamo equation; the first term on the right side represents the α-effect, responsible for field amplification by helical turbulence. It represents the statistical effect of the term $\underset{\sim}{B} \cdot \nabla \underset{\sim}{u}$ in the induction equation. The coefficient η_2^2 is the turbulent diffusivity,[3] and is positive in this approximation.

For longer autocorrelation times the result (4) gives rise to the expression

$$\frac{\partial \bar{B}}{\partial t} = \left\{ (\eta_1^2 + \eta_1^3 + \ldots) \nabla \times + (\eta_2^2 + \eta_2^3 + \eta_2^4 + \eta_2'^4 + \ldots) \nabla^2 + (\eta_3^3 + \ldots) \nabla^2 \nabla \times + \ldots \right\} \bar{B} . \quad (7)$$

The inclusion of each higher term in the cumulant expansion has two distinct effects. First, a term with a higher derivative of $\bar{B}$ is introduced; such terms are negligible when the mean magnetic field is large-scale. Second, each cumulant adds a contribution to each of the preceding transport coefficients. As a result the transport coefficients may be said to be renormalized by the higher order terms:

$$\alpha \equiv \sum_{n=2}^{\infty} \eta_1^n \quad , \quad \beta \equiv \sum_{n=2}^{\infty} \eta_2^n = \ldots \quad (8)$$

Each transport coefficient is an infinite series in powers of $R \equiv \bar{u}\tau_c/\ell$, where ℓ is of order λ , the Taylor microscale (Knobloch, 1978). In stationary turbulence, $R \simeq (\bar{u}\tau_c/L)(uL/\nu)^{1/2} \sim R_L^{1/2}$, where L is the eddy correlation length ("typical" eddy size), and the eddy correlation time is approximately $L/\bar{u}$ for realistic turbulence. Here $\bar{u}$ is the r.m.s. turbulent velocity and R_L is the turbulent Reynolds number. For fully developed turbulence the transport coefficients therefore formally diverge, but their value could be estimated using, for example, Padé approximants. The condition that the dynamo equation be valid (i.e. that the higher derivative terms are negligible) is $K \equiv \bar{u}\tau_c k << 1$ when k is the wavenumber of $\bar{B}$. This condition is equivalent to $k << L^{-1}$.

The above Eulerian results can be shown to be formally identical with the Lagrangian results of Moffatt (1974). The divergence of the expressions (8) is related to the use of a Taylor expansion when converting Lagrangian variables to Eulerian ones.

Because of the presence of the term $\eta_2'^4$ in equation (7), the diffusion of the magnetic field will differ from that of a scalar field. To lower order in τ_c the diffusion of a scalar and magnetic field by non-helical turbulence is the same. However, for realistic turbulence these higher order terms may not be neglected, and since both η_2^4 and $\eta_2'^4$ (related to mean square shear and helicity, respectively) provide negative contributions to the sum, the possibility arises that the turbulent diffusivity of the mean magnetic field could be negative (Kraichnan, 1976). This may be related to the steepening of gradients (and the expulsion of flux in the presence of small molecular resistivity) by eddies with long correlation times (Weiss, 1966). On the other hand, for small τ_c an eddy would have no time to affect such an expulsion and when it was replaced by an uncorrelated one, any such tendency would be on average reversed. Such a situation would correspond to a positive eddy diffusivity, as in equation (6).

For a more complete statistical description of the diffusing magnetic field, the above method may be adapted to calculating higher moments of the field. For example, the mean magnetic energy $\langle B^2 \rangle$ is an important quantity, particularly if β is indeed negative.

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NEGATIVE DIFFUSIVITY

Robert H. Kraichnan

In this talk the origin of the negative diffusivity which turbulence in a conducting fluid can exert on weak magnetic fields when ther are substantial fluctuations of helicity about a zero mean is reviewed, and some speculations are given about the persistence of this phenomenon into the strong-magnetic field regime.

Turbulence exerts a positive diffusivity on a passive scalar field advected by the motion in consequence of the random walk executed by fluid elements. More complicated and intuitively surprising things happen when a magnetic field is frozen in a moving, conducting fluid because the magnetic field is changed in direction and intensified by stretching as well as displaced.

Following Moffatt, the time derivative of the scalar or magnetic field, after averaging over ensemble, can be expressed as an infinite expansion involving ascending space derivatives of the field and ascending cumulants of the distribution of fluid-element displacements. This follows formally from Cauchy's integral solution of the advection equations and suitable spatial Taylor expansions. For times of evolution such that a typical fluid element migrates a distance which is small compared to the 'scale of spatial variation of the mean scalar or magnetic field, the expansions are dominated by terms containing only first and second space derivatives of the mean field.

In the scalar case, these leading terms give

$$\partial\langle\phi(x,t)\rangle/\partial t-\kappa(t)\nabla^2\langle\phi(x,t)\rangle=0,\qquad \kappa(t)=\tfrac{1}{2}d\langle\xi_i^2\rangle/dt \tag{1}$$

where $\langle\phi(x,t)\rangle$ is the mean scalar field, ξ_i is the displacement of the fluid element which arrives at x, t, and for simplicity in writing we take isotropic turbulence. The displacement is measured from time $t=0$, when the turbulence is switched on. In the magnetic case

$$\frac{\partial\langle B(x,t)\rangle}{\partial t}=-\alpha(t)\nabla\times\langle B(x,t)\rangle+\eta(t)\nabla^2\langle B(x,t)\rangle \tag{2}$$

where

$$\alpha(t)=d\gamma(t)/dt,\qquad \gamma(t)=\langle\xi_3\frac{\partial\xi_2}{\partial a_1}\rangle,$$

$$\eta(t)=\kappa(t)+\tfrac{1}{2}\frac{d}{dt}\left\{\langle\xi_1^2\frac{\partial\xi_2}{\partial a_2}\rangle+[\gamma(t)]^2\right\},$$

and a_i is the initial position of the fluid element which arrives at x, t.

If the turbulence is statistically stationary, $\kappa(t)$ is a positive constant for times long compared to the turbulence correlation time. $\alpha(t)$ is zero if there is no helicity and approaches a constant (either sign) if there is a constant mean helicity everywhere. In the latter case, the final term $[\gamma(t)]^2$ in (3) grows like t^2. But an alternative, Eulerian calculation of $\eta(t)$, accurate for short enough turbulence correlation time, shows that $\eta(t)$ also approaches a constant, positive value for helical turbulence with simple statistics. It follows that, for helical turbulence,

$$\langle\xi_1^2\partial\xi_2/\partial a_2\rangle$$

becomes negative and grows like $-t^2$ for t large compared to the correlation time. This final fact is the origin of the negative diffusivity for turbulence with zero mean helicity.

Suppose now that the turbulence has zero mean helicity but that there are fluctuations of helicity such that the helicity keeps the same sign over regions which are several correlation lengths of the turbulence in extent and several correlation times in duration. Since the mean helicity is zero,

$\gamma(t)$ vanishes. But for times short enough that a typical fluid particle does not migrate out of the region of helicity fluctuation in which it starts, the

$$\left\langle \xi_1^2 \frac{\partial \xi_2}{\partial a_3} \right\rangle$$

term in (3) is nearly unaltered in value from what it would be if the helicity were uniformly nonzero. Since that term goes negative regardless of the sign of helicity, so does $\eta(t)$. Clearly we can make $\eta(t)$ as negative as desired by making the helicity fluctuations sufficiently extensive and persistent.

All the properties inferred above have been verified by computer simulations, and by analytical model cases (Kraichnan, Parker).

If the magnetic field is strong, how do Lorentz forces affect the phenomenon of negative diffusivity? In the extreme strong-field case, the turbulence is replaced by random Alfven waves propagating on the lines of force, Preliminary analysis suggests that the negative diffusivity phenomenon persists, with the typical Alfven period playing the role of effective turbulence correlation time.

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A NEW THEORY OF THE SOLAR CYCLE*

David Layzer

Following Cowling, nearly all modern workers attribute the cyclical variation of sunspot fields to the quasi-periodic reversal of a submerged toroidal field, from which the surface fields are assumed to derive through some kind of hydromagnetic instability. It is also generally agreed that the toroidal field is generated by differential rotation acting on the poloidal component of a weak large-scale field. The real difficulty lies in understanding the origin of the weak poloidal field and the mechanism for reversal of the toroidal field derived from it. In regenerative-dynamo theories the poloidal field is derived from the toroidal field itself through processes in which helical turbulence (or convection) and turbulent diffusion play crucial roles. The regenerated poloidal field changes sign every half-cycle. In the alternative theory sketched in IV, V, the nonconvective core of the Sun contains an irregular large-scale magnetic field. The toroidal field is generated by torsional oscillations in a transition layer between the uniformly rotating, nonconvective, magnetized core and the non-uniformly rotating, nonmagnetized convective envelope. The submerged field is the remnant of a much stronger, irregular field that was generated during the pre-main-sequence phase of solar evolution and mediated the process of spindown.

In comparing the two hypotheses we may conveniently distinguish three sorts of questions: those relating to internal consistency and the validity of specific mathematical or physical assumptions; those concerning the ability of each hypothesis to provide an adequate framework for interpreting observations of solar magnetic fields; and those concerning the relation of each hypothesis to the broader problem of the origin of stellar and inter-stellar magnetic fields.

Internal consistency. The theoretical cornerstone of regenerative dynamo theory is the dynamo equation (3.4). Although this equation is valid for flows with low Reynolds number, we have seen (§ III *b*) that the approximations on which it rests are not valid under conditions prevailing in the solar atmosphere. Under these conditions the coefficients and are given either by formally divergent expansions (Knobloch 1978b) or by oscillatory integrals whose convergence in the limit $t \to \infty$ is problematical (Moffatt 1974). There is no known theoretical or experimental basis for the assumption (Parker 1971) that the turbulent diffusivity of a passive magnetic field has a well-defined value comparable to -- or even with the same sign as -- the turbulent diffusivity of a passive scalar field (§ III *c*). Finally, we have argued (§ III *d*) that turbulent-dynamo theories do not correctly describe the way in which differential rotation and turbulent motions jointly act on the magnetic field. Mathematical models of $\alpha\omega$ dynamos unjustifiably omit terms that describe the interaction between differential rotation and the fluctuating component of the magnetic field.

The alternative hypothesis (§§ IV,V) does not invoke rapid merging of small-scale magnetic fields. It does postulate (*a*) that certain kinds of large-scale flows occur during a critical period of solar evolution, (*b*) that a remnant of the strong,.irregular, large-scale, magnetic field generated by these motions would have persisted to the present day, and (c) that a nonuniformly rotating, magnetized layer separates the uniformly rotating, convective envelope. These postulates seem to be consistent with present physical and astronomical knowledge but need to be made more precise and secure by detailed studies.

Interpretation of observed solar magnetic fields. The dynamo theory evolved during a period when observational evidence seemed to indicate that the Sun has a weak poloidal field that reverses quasi-periodically. At present there is no direct observational evidence for the existence of such a physical field. Leighton (1964) has argued that an average poloidal field results from the breakup of sunspot fields and random horizontal motions of their components. But there are no known theoretical or observational reasons (apart from the requirements of the dynamo theory) for asserting that the residual sunspot fields merge to form a large-scale field, rather than remaining fragmented throughout their decay (Stenflo 1976). Finally, the absence or near absence of sunspots during extended periods (Eddy 1976) presents a serious and as yet unmet challenge to turbulent-dynamo theories. The normal modes of a regenerative dynamo are exponential. While it is easy to understand how the amplitude of an exponentially growing mode can be limited by nonlinear effects, there is no obvious reason why an exponentially decaying mode should not disappear altogether. Leighton's (1969) numerical simulations suggest that this is indeed what happens to such a mode.

The alternative theory relates the variability of the solar cycle to the variable rate at which magnetic flux in the radiative core penetrates the convective envelope. The observed correlation between the rise-time of the sunspot number and the total sunspot number in a given half-cycle is explained by the fact both quantities increase monotonically with the thickness Δr of the transition layer.

Solar and stellar magnetic fields. The dynamo theory does not explain the origin of a large-scale solar magnetic field; it postulates that a large-scale field was present in the material from which the Sun formed. The only known process for the spontaneous generation of large-scale magnetic fields under astronomically relevant physical conditions is Biermann's mechanism, which operates in any differentially rotating, partially ionized gas-cloud. Thus large-scale magnetic fields may be expected to develop spontaneously in gaseous protostars as well as in larger self-gravitating gas-clouds which spin up as they contract. Soon after the Biermann field has begun to grow in a contracting gas-cloud, it will be amplified by fluid motions (§ IV). The resulting complex fields mediate the transfer and loss of angular momentum, enabling protostars to contract to stellar dimensions.

The subsequent evolution of the magnetic field depends on the extent and disposition of convective regions in the star. In stars with convective envelopes the convection zone tends to exclude the submerged field, but also interacts with it in a more or less narrow transition layer. We suggest, as a working hypothesis, that this coupling between the submerged magnetic field and the overlying convective envelope has two observable effects: (1) It mediates the outward transfer (and eventual loss) of angular momentum. (2) It gives rise to torsional oscillations of the transition layer which produce strong toroidal fields.

In stars whose outer convection zone is weak or absent the submerged field will penetrate the visible layers. We suggest that the fields of magnetic A stars may be interpreted in this way. A complete theory along these lines would, of course, need to explain other conspicuous properties of the magnetic A stars -- in particular, their slow rotation. These and related questions lie outside the scope of the present discussion. The existence of stars with large-scale but distinctly irregular magnetic fields does however bear directly on the present. theory, which predicts that such a field is present in the Sun's nonconvective core.

*From a paper submitted to the Astrophysical Journal. References in the text are to this paper.

SPECULATIONS ON THE THERMAL STATE OF THE CORE

David E. Loper

It is argued that the most plausible source of power for the geodynamo is gravitational energy released by the growth of the solid inner core. Results of model calculations by Lopes (1978a) show that the power available to drive the dynamo by this mechanism is linearly related to the density jump at the inner-outer core boundary and can be as large as 1.25×10^{12}W if $\Delta\rho = 2.63 \times 10^3 kg/m^3$. This can sustain a toroidal field as large as 10^3, gauss.

The thermal regimes of the outer core which are possible if the dynamo is gravitationally powered are studied. The regimes are defined by the ordering of the magnitudes of the gradients of the adiabat T_A', the liquidus T_L' and the conduction temperature T_C'. It is argued that regimes $T_L' < T_A' < T_C'$ and $T_L' < T_C' < T_A'$ are not possible since they result in a solid outer core. The regime for which $T_A' < T_C' < T_L'$ is the simplest and possesses no unusual features. The second regime with $T_A' < T_L' < T_C'$ is similar to the first except that a slurry layer must occur at the bottom of the outer core. The thermodynamics of such a slurry have been studied by Loper and Roberts (1978). In each of these regimes the fluid is both thermally and compositionally unstable. This is in contrast to the third regime, $T_C' < T_A' < T_L'$, in which thermal gradients tend to stabilize the fluid. However, it assumed that overturning is driven by the stronger compositional buoyancy. This introduces the possibility that heat may be transported radially inward by the convection driven by compositional buoyancy. Consequently there is no direct relation between the rate that heat is conducted outward in the outer mantle and the rate of heat transfer to the mantle. The fourth regime, $T_C' < T_L' < T_A'$, allows compositionally driven convection provided the thermal conductivity is sufficiently large that $T_C' < T' < T_L' < T_A'$ where T' is the actual temperature gradient. This possibility appears to have been overlooked by Higgins and Kennedy (1971). It is argued that a slurry in the bulk of the outer core as envisaged by Busse (1972) and Malkus (1973) is incompatible with overturning because transport processes produce both thermal and compositional gradients which tend to stabilize the fluid.

The possibility that the core fluid may be less metallic than the eutectic as Braginsky (1963) suggested is considered and it is shown that a layer of variable composition must form at the bottom of the outer core. Difficulties associated with the removal of heat from this layer leads to the conclusion that a metal-poor composition for the core is unlikely. The thermal evolution of the earth is discussed and it is noted that if $T_A' < T_C'$, the heat transfer problems for the core and mantle are decoupled with conditions in the core leading to a prescribed temperature at the base of the mantle and the mantle in turn prescribing the heat flux which must emanate from the core. It is found that if $T_A' = T_L'$, a significant flux of heat may flow from the core to the mantle with virtually no change in temperature. For detailed discussions of these ideas see Loper (1978b).

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ELECTROHYDRODYNAMICS AND MAGNETOHYDRODYNAMICS

James R. Melcher

Beginning with a brief review of the contrasts between electroquasistatic and magnetoquasistatic approximations, and between the electric and magnetic force densities, a pair of case studies are developed that illustrate analogies between phenomena in the two areas. Von Quincke's rotor consists of an insulating cylindrical rotor having radius b and permittivity ε_b immersed in a liquid having permittivity ε_a and ohmic conductivity σ. Plane electrodes on either side of the rotor in the liquid impose an electric field $\mathcal{E}(t)$ that is uniform far from the cylinder and directed perpendicular to the axis of rotation. The equations of motion, which are useful in understanding convection in many electrohydrodynamics systems, have a form which in the limit $f \equiv (\varepsilon_b - \varepsilon_a)/(\varepsilon_b + \varepsilon_a) \rightarrow 0$ are the same as for a-6-c convection.

$$P_e^{-1}\dot{\underline{\Omega}} + \underline{\Omega} = \underline{E}\,\underline{P}_y$$
$$\dot{\underline{P}}_x + \underline{\Omega}\underline{P}_y + \underline{P}_x = H_e^2\,\underline{E}\,\underline{\Omega}$$
$$\dot{\underline{P}}_y - \underline{\Omega}\underline{P}_x + \underline{P}_y = f\,H_e^2\,\underline{E}\,\underline{\Omega}$$

Here, Ω is the rotor angular velocity and P_x and P_y are proportional to the polarizations per unit length in the x and y directions. Variables are normalized so that

$$\underline{t} = t/\tau_e \equiv (\varepsilon_a + \varepsilon_b)/\sigma$$

$$\underline{E}(t) = E(t)/\mathcal{E}$$

$$\underline{\Omega} = \Omega \tau_e ; \underline{P}_{\binom{x}{y}} = \frac{2\varepsilon_a \pi \tau_e \mathcal{E}}{B} P_{\binom{x}{y}}$$

Analogous to the Rayleigh number is the square of the electric Hartmann number, H_e (the square root of the ratio of the charge relaxation time τ_e to an electroviscous time) while the role of the Prandtl number is played by (the ratio of the charge relaxation time to a viscous diffusion time).

$$H_e = \frac{2\varepsilon_a \pi b^2 \tau_e}{B} \mathcal{E} ; P_e = \tau_e / (I/B)$$

Here, I and B are respectively the rotor moment of inertia per unit length and viscous damping coefficient per unit length. If I is the rotor inertia alone. $I = \pi b^4 \rho/2$ while (for complete viscous diffusion in a liquid extending to infinity), $B = 4\pi b^2 \eta$ where 4 and ρ are viscosity and mass density respectively. Thus, $H_e^2 \equiv \tau_e/\tau_{EV}$; $\tau_{EV} \equiv 2\eta/\varepsilon_a \mathcal{E}^2$ and $P_e \equiv \tau_e/\tau_v$; $\tau_v \equiv \rho b^2/8\eta$.

There are familiar magnetohydrodynamic phenomena having features in common with this rotor model. A limiting form of one of these is discussed to motivate a continuum model for instabilities observed in liquid metals as they are shaped or levitated by high frequency alternating magnetic fields. The rotor model consists of a conducting shell with an imposed high frequency magnetic field that is uniform far from the cylinder. Incipience of instability is governed by a parameter $M = B_o^2/\mu_o \eta \omega$ (B_o and ω the peak mag-. netic flux density of the applied field and its angular frequency respectively) which is the reciprocal of the magnetoviscous time-frequency product. It is found from a continuum theory based on magnetic stresses averaged on the time scale $1/\omega$ that a planar liquid layer supporting a uniform skin current and peak surface magnetic pressure B_o^2/μ_o, is unstable for $M > 67$. Thermal convection terms are added to the rotor model to motivate explanations of why the predicted incipience of instability correlates with experiments, but the growth rate predicted by the theory is far longer than that observed.

A natural electrohydrodynamic dynamo is the thunderstorm. The film "Electric Fields and Moving Media" is used to show electrohydrodynamic dynamos involving falling water drops. These are the "Kelvin Dripper" and the Euerle 3-phase dynamo.

THE OXYMORONIC ROLE OF MOLECULAR DIFFUSIVITY IN THE DYNAMO PROCESS

H. Keith Moffatt

The delicate question concerning the behavior of the regeneration coefficient α and the turbulent diffusivity β in the limit of vanishing molecular diffusivity $(\eta \to 0)$ in helical turbulence is discussed, in the light of an exact result of Bondi & Gold (1950) viz. that when $\eta = 0$ the external dipole moment of a current distribution in a sphere is permanently bounded.

1. The oxymoron is a figure of speech which embodies an apparent contradiction; e.g. creative destruction, relaxed tension, devastating triviality, etc. The oxymoronic role of molecular diffusivity $\eta\,(=(\mu_0\sigma)^{-1})$ is this: that while non-zero diffusivity $(\eta > 0)$ is directly responsible for the natural ohmic processes of dissipation and decay, it is also indirectly responsible for the means of regeneration of the magnetic field; the dynamo process may be described as a process of 'regenerative decay' or perhaps better 'reinvigorating dissipation'[1].

2. Consider the dipole moment $\mu(t)$ associated with a current distribution $\mathbf{j}(\mathbf{x}, t) = \mu_0^{-1} \nabla \wedge \mathbf{B}$ in a conducting sphere $V: r < a$. This is given by various alternative expressions:

$$8\pi \mu(t) = \mu_0 \int_V \mathbf{x} \wedge \mathbf{j}\, dV = 3\int_V \mathbf{B}\, dV = 3\int_S \mathbf{x}\,(\mathbf{B}\cdot\mathbf{n})\, dS, \qquad (1)$$

where S is the surface $r = a$; and its rate of change is given by

$$\frac{8\pi}{3}\frac{d\mu}{dt} = \int_V \frac{\partial \mathbf{B}}{\partial t}\, dV = \int_S (\mathbf{n} \wedge \mathbf{E})\, dS. \qquad (2)$$

With $\mathbf{E} = -\mathbf{u} \wedge \mathbf{B} + \eta \nabla \wedge \mathbf{B}$,,and $\mathbf{u}\cdot\mathbf{n} = 0$ on S, this gives

$$\frac{8\pi}{3}\frac{d\mu}{dt} = \int_S \mathbf{u}\,(\mathbf{n}\cdot\mathbf{B})\, dS - \eta\int \mathbf{n} \wedge (\nabla \wedge \mathbf{B})\, dS. \qquad (3)$$

The first term on the right describes the mechanism identified by Bondi & Gold (1950) for increase of the dipole moment; field sweeping towards the magnetic poles (defined by the instantaneous direction of the vector μ)

can increase $|\mu|$, but, as emphasised by Bondi & Gold, this mechanism is strictly limited when $\eta = 0$, since $|\mu|$ then attains a finite maximum when all the flux of $\mathbf{B}$ is concentrated at opposite ends of a diameter of the sphere (as in an elementary bar magnet). To see this explicitly from the above equations, let $S_{\pm}$ denote those parts of S on which $\mathbf{n}\cdot\mathbf{B} >$ or < 0 , respectively, and let

$$\boldsymbol{\mu}_{\pm} = \frac{3}{8\pi}\int_{S_{\pm}} \mathbf{x}\,(\mathbf{n}\cdot\mathbf{B})\,dS \tag{4}$$

so that $\boldsymbol{\mu} = \boldsymbol{\mu}_{+} + \boldsymbol{\mu}_{-}$. We then have

$$|\mu_{+}| \leq \frac{3}{8\pi}\,a\,\Phi, \qquad |\boldsymbol{\mu}_{-}| \leq \frac{3}{8\pi}\,a\,\Phi, \tag{5}$$

where

$$\Phi = \int_{S+}(\mathbf{n}\cdot\mathbf{B})\,dS = -\int_{S-}(\mathbf{n}\cdot\mathbf{B})\,dS. \tag{6}$$

Now, when $\eta = 0$, Φ is constant, since flux through every closed material circuit is conserved, and so

$$|\boldsymbol{\mu}| \leq |\boldsymbol{\mu}_{+}| + |\mu_{-}| \leq \frac{3}{4\pi}\,a\,\Phi, \tag{7}$$

the maximum being attained only when the flux is entirely concentrated at the poles, as mentioned above.

3. There can therefore be no doubt that, when $\eta = 0$, exponential increase of the dipole moment is impossible, no matter what the complexity (laminar or turbulent) of the velocity field in V may be. The situation is transformed if $\eta > 0$, because then diffusive increase in the dipole moment (represented by the second term of **(3)** is possible, provided the velocity field is such as to maintain a field with a suitably negative gradient near the boundary $r = a$.

4. The impossibility of sustained dynamo action (in the sense of an exponentially increasing external dipole moment) applies equally to such basic systems as the homopolar disc dynamo. If the disc conductivity is infinite, then the magnetic flux across it cannot change with time, and exponential growth of the magnetic field associated with the device is impossible no matter how fast we rotate the disc or how ingeniously we twist the wire, and whatever conventional wisdom may tell us to the contrary. In terms of growth rate, if, in general, $\mathbf{B} \propto e^{pt}$, then p must depend on the disc Reynolds number in the manner indicated in Fig. 1. It is reasonable to conjecture that fluid dynamos also must behave in this manner.

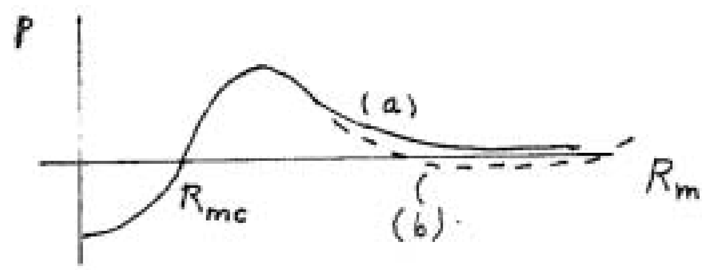

Fig. 1. Possible dependence of p on R_m for homopolar disc dynamo.
(a) wire resistance zero; (b) wire resistance non-zero.
In either case, $p \to 0$ as $R_m \to \infty$.

5. Consider now the situation in mean-field electrodynamics, in which, in convectional notation,

$$\mathcal{E}_i = \langle \underset{\sim}{u} \wedge \underset{\sim}{b} \rangle_i = \alpha_{ij} B_{0j} + \beta_{ijk} \, \partial B_{0j} / \partial x_k + \cdots , \qquad (8[$$

where $\underset{\sim}{B}_0(\underset{\sim}{x}, t) = \langle B(\underset{\sim}{x} \cdot t) \rangle$ is the large-scale (mean) field, and $b = B - B_0$. Under first-order smoothing theory (Moffatt 1978 - hereafter referred to as M - chap. 7) we have the results

$$\alpha \equiv \tfrac{1}{3} \alpha_{ii} = - \tfrac{1}{3} \eta \iint \frac{k^2 F(k,\omega)}{\omega^2 + \eta^2 k^4} \, dk \, d\omega, \qquad (9)$$

$$\beta \equiv \tfrac{1}{6} \epsilon_{ijk} \beta_{ijk} = \tfrac{2}{3} \eta \iint \frac{k^2 E(k,\omega)}{\omega^2 + \eta^2 k^4} \, dk \, d\omega, \qquad (10)$$

where $F(k, w)$, $E(k, \omega)$ are the helicity and energy spectrum functions of the random $\underset{\sim}{u}$-field. If

$$F(k,\omega) = O(\omega^2), \; E(k,\omega) = O(\omega^2) \text{ as } \omega \to 0, \qquad (11)$$

then clearly

$$\alpha \sim \alpha_0' \eta, \quad \beta \sim \beta_0' \eta \text{ as } \eta \to 0, \qquad (12)$$

where α_0' and β_0' are in general non-zero constants $(\beta_0' > 0)$. This is clearly the situation when the $\underset{\sim}{u}$-field is a field of random waves with no zero-frequency ingredients. In this case, the regenerative process normally associated with the pseudo-scalar α vanishes as $\eta \to 0$, consistent with the remarks of §1. It may be noted that the theory of Braginskii (M. chap. 8) gives an expression for the regenerative coefficient very similar to (93, and again with the property $\alpha = O(\eta)$ as $\eta \to 0$.

6. Difficulties arise however if the $\underset{\sim}{u}$-field has non-zero spectral density at $\omega = 0$, as is the case for conventional turbulence. The zero-

frequency ingredients of the turbulence are precisely those that are responsible for the dispersion of particles in a turbulent flow, and they are of vital importance also in the field-line - stretching context. It must be noted however that results such as $\langle \xi^2 \rangle \sim 2Dt$ for the relative dispersion of two particles separated by vector distance $\xi(t)$ is ultimately limited by the physical dimensions of the fluid domain; and care may then be needed in carrying over asymptotic results from strictly homogeneous turbulence to turbulence in a finite domain, particularly when these results are sensitive to the limiting ($t \to \infty$) behavior.

7. When $\eta = 0$, there is an alternative approach to the determination of the coefficient α and β using Lagrangian averages. If at some instant $t = 0$, the u and b fields are uncorrelated, then α and β are functions of t (which clearly vanish at $t = 0$). The Lagrangian procedure (M. § 7.10) leads to the expressions

$$\alpha(t) = -\frac{1}{3}\int_0^t \langle v(t)\cdot \nabla_a \wedge v(\tau) \rangle \, d\tau, \qquad (13)$$

$$\beta(t) = \frac{1}{3}\int_0^t \langle v(t)\cdot v(\tau)\rangle d\tau + \int_0^t \alpha(t)\,\alpha(\tau)\,d\tau,$$
$$+\frac{1}{t}\int_0^t\int_0^t \langle v(t)\cdot v(\tau_2)\nabla_a \cdot v(\tau_1) - (v(t)\cdot\nabla_a v(\tau_1))\, v(\tau_2)\rangle \, d\tau_1 \, d\tau_2 \qquad (14)$$

where $v(t)$ is the velocity of the fluid particle initially at position a. The difficulty here is to determine how these expressions behave for a typical field of homogeneous turbulence as $t \to \infty$. Kraichnan (1976 a, b) has argued that, in the case of turbulence with non-zero helicity,

$$\alpha(t) \sim \alpha_0, \; \beta(t) \sim \beta_0 \quad \text{as } t \to \infty, \qquad (15)$$

the apparent positive divergence in the second term of (14) being cancelled by an equal negative divergence in the third term (which involves the awkward triple Lagrangian correlations). Kraichnan's arguments rest in part on comparison with the results of first-order smoothing theory in situations where both approaches (first-order smoothing and Lagrangian) may be expected to be valid, and in part on numerical evaluation of $\alpha(t)$ and $\beta(t)$ for velocity fields with prescribed Eulerian statistics. Further numerical experimentation is needed however, before the results (15) can be regarded as absolutely and definitively established. Let us nevertheless accept (15), and pursue the consequences in the context of α^2- and $\alpha\omega$-dynamo models.

8. For an α^2-dynamo in a sphere $r < a$ (M. chap. 9), the growth rates have the form

$$p = \frac{\eta_e}{a^2} F(R_\alpha) \tag{16}$$

where $\eta_e = \eta + \beta$,and

$$R_\alpha = |\alpha|\, a^2/\eta_e , \tag{17}$$

and dynamo action occurs when $F(R_\alpha) > 0$. This generally occurs for the simplest mode of dipole symmetry when

$$R_\alpha > R_{\alpha c} , \tag{18}$$

where $R_{\alpha c}$ is a positive number of order unity which depends on the precise assumption made about any large-scale variation of α throughout the sphere. Let us suppose that, as $\eta \to 0$, the relevant behavior of α and β (cf. 15) is

$$\alpha \sim \alpha_0 , \; \beta \sim \beta_0 \quad \text{as} \quad \eta \to 0 . \tag{19}$$

Then (16) becomes

$$p \sim \frac{\beta_0}{a^2} F(\bar{R}_\alpha) , \; \bar{R}_\alpha = \frac{|\alpha_0|\, a^2}{\beta_0} \tag{20}$$

The condition $\bar{R}_\alpha > R_{\alpha c}$ is certainly satisfied if a is large enough, and then p tends to a strictly positive value as $\eta \to 0$, implying exponential increase of the mean field, and in particular of the external dipole moment. This appears to be in fundamental conflict with the Bondi & Gold result (7), which applies when $\eta = 0$ whatever the complications of the . velocity field, and whether laminar or turbulent.

The conflict does not arise under the alternative limiting behavior **(12). In** this case,

$$p \sim \frac{\eta(1+\beta_0')}{a^2} F\left(\frac{|\alpha_0'|\, a^2}{1+\beta_0'}\right) \to 0 \text{ as } \eta \to 0 \tag{21}$$

and the dipole moment does not grow exponentially in the limit $\eta \to 0$.

9. For dynamos of $\alpha\omega$-type, growth rates are generally given by

$$p = \frac{\eta_e}{a^2} F(X) , \; X = \frac{|\alpha|\, G a^3}{\eta_e^2} , \tag{22}$$

where G is a measure of the shear associated with differential rotation. The condition for dynamo action is now of the form

$$X > X_c , \tag{23}$$

where χ_c is model-dependent, but generally of order unity. Again under the behavior (19), as $\eta \to 0$,

$$\chi \to \bar{\chi} = |\alpha_0| G a^3/\beta_0^2 , \qquad (24)$$

and

$$p \sim \frac{\beta_0}{a^2} f(\bar{\chi}) > 0 \quad \text{if } \bar{\chi} > \chi_c \qquad (25)$$

and we encounter the same fundamental conflict with the Bondi 4 Gold result.

Under the alternative behavior (12),

$$p \sim \eta \frac{(1+\beta_0')}{a^2} F \frac{|\alpha_0'| G a^3}{\eta(1+\beta_0')^2} . \qquad (26)$$

To determine the behavior of p as $\eta \to 0$, we need to know the behavior of $F(\chi)$ as $\chi \to \infty$. If $F(\chi) = o(\chi)$ as $\chi \to \infty$, then $p \to 0$ as $\eta \to 0$, and conflict with Bondi & Gold is avoided. The asymptotic behavior of $F(\chi)$ as $\chi \to \infty$ does not appear to have been investigated for $\alpha\omega$-dynamos in a spherical geometry. A clue is however provided by the results for an $\alpha\omega$-dynamo in a Cartesian geometry (modelling the galactic disc). For this case, which can be solved completely (M. § 9.9),

$$f(\chi) \sim \log \chi \quad \text{as } \chi \to \infty , \qquad (27)$$

and so $p \to 0$ as $\eta \to 0$ as required.

10. It is hard to escape the conclusion that the result (19) cannot be correct, or that, if it is correct in homogeneous turbulence, it is, for some deep reason, not applicable when the turbulence is confined to a finite region (see the remarks of § 6).

Acknowledgement

I am grateful to John Chapman who helped me to sort out the argument of § 2.

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A FLUX-LINE METHOD FOR NUMERICAL STUDIES OF KINEMATIC DYNAMOS

Peter Olson

The sprectrum of the geomagnetic field and its rapid secular variation suggests that large magnetic Reynolds number conditions exist in the earth's core. ($R_m = 4\pi\sigma v L$, σ = conductivity, U and L are velocity and length scales). However, there exists at present no acceptable method for solving the induction equation under these conditions.

A method useful at large R_m is developed and applied to some likely flows. A new feature is the use of random walks to simulate the diffusion of magnetic field due finite electrical conductivity. The method is based on a solution to the vorticity equation proposed by Chorin (1973). Given the induction equation

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) + 1/R_m \nabla^2 \underline{B} \tag{1}$$

in which $\underline{B}(x, t = 0)$ and $\underline{v}(\underline{x}, t)$ are specified; the solution B at some later time t is required.

First, the initial data $\underline{B}(\underline{x}, 0)$ is partitioned into a number of slender flux ropes, idealized as curves locally parallel to the field, the density of curves proportional to the field's intensity. These curves are then represented by a number of sample points $\underline{x}^i$ along their length, the distance between adjacent sample points l^i, , and the average field intensity between points B^i .

At each time step, Δt, the sample points are advanced according to

$$\underline{x}(t+\Delta t) = \underline{x}(t) + \int_t^{t+\Delta t} \underline{V}(\underline{x}, t')\, dt' + \hat{i} N(0, 2\Delta t/R_m) + \hat{b} N(0, 2\Delta t/R_m) \tag{2}$$

where N is a normally distributed random variable with zero mean and standard derivation $\frac{2\Delta t}{R_m}$, and $\hat{i}$ and $\hat{b}$ are unit vectors along the curve's principal normal and binormal.

The field intensity between each point is then recomputed using

$$B^i(t+\Delta t) = B^i(t)\, \frac{l^i(t+\Delta t)}{l^i(t)} \tag{3}$$

The first term on the right hand side of (2) solves $\frac{\partial B}{\partial t} = \nabla \times (v \times \underline{B})$ while the second and third terms in (2) solve $\frac{\partial B}{\partial t} = 1/R_m \nabla^2 \underline{B}$ by exploiting the formal connection between random walks and diffusion from a line source. Because of the random component in (2), the value of $\underline{B}$ at a point becomes uncertain; however, dynamo calculations usually require knowledge of global functionals only, such as spherical harmonic coefficients.

The alogrithm defined by (2) and (3) applied to a large number of curves permits these average quantities to be computed as a function of time, to within a statistical error. The difficulties associated with finite difference representations of the Laplace operator are avoided, and in addition the sample points tend to accumulate in those regions where the greatest computational effort is needed.

As an application, induction by the nearly-geostrophic flow proposed by Busse (1975) is studied. This flow is characterized by a primary geostrophic circulation about columns erected parallel to the rotation axis, with Ekman suction providing flow along the columns and with it a non-zero helicity. The domain is taken to be an isolated conducting sphere in which eight columns (4 pairs) are arranged in a ring centered about the axis. The initial field is an axial dipole.

With no helicity, the dipole field decreases with time for all investigated (R_m = 100 to 1000, based on the sphere radius). The mean field in the core of each column decreased rapidly toward zero, a result which may be interpreted as "flux expulsion."

With helicity, growing fields occured for $R_m \geq 250$, although the computations have yet to be carried out sufficiently far in time to show true exponential behavior.

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MAGNETOCONVECTION AT HIGH MAGNETIC REYNOLDS NUMBER

Michael R. E. Proctor

This paper investigates steady finite amplitude solutions of the equations of a Bussinesq fluid heated uniformly from below in the presence of an imposed vertical magnetic field. Several previous studies have concentrated on the linear stability problem [Chandrasekhar 1961, Danielson 1961), in which all quantities are only slightly perturbed from the basic state. Thus the magnetic Reynolds number $R_m = UL/\eta$ is small, where U and

L are velocity and length scales and η is the magnetic field is pushed in to flux ropes and sheets as the convection becomes more vigorous and becomes large (Weiss, 1966). The dynamical effect of this intermittent flux structure is then quite different from the small case in particular, the dynamical effect of two-dimensional flux <u>sheets</u> is quite different from that of axisymmetric <u>ropes</u>, although the linearised problem is independent of the convection planform. Busse (1975) has investigated the two-dimensional problem when the total flux threading the system (measured by $Q = B_0^2 L^2/(\mu \rho \nu \eta)$ where B_0 is the mean vertical field, μ the permeability and ρ and ν the density and kinematic viscosity) is small. He finds that finite amplitude convection can occur for values of the Rayleigh number R (measuring the temperature difference across the layer) much less than that necessary for linear instability. Proctor and Galloway (1978) have investigated the analogous problem in an axisymmetric geometry. The analysis is simplified considerably compared to the two-dimensional case, chiefly due to the fact that the axisymmetric flux rope that forms only exerts a very localised dynamical effect. The analysis can be performed for all Q for which the flux rope remains thin. This gives a limit of order $R \gtrsim / \ln R_m$ or Q, which is not severe. The methods used are those of Galloway, Proctor, and Weiss (1978) (see also Proctor, these notes). The results have some rather unusual features. In particular, the finite amplitude solution is supercritical for very small Q, but becomes sub-critical for all sufficiently large Q! It is also interesting that R is close to R_c, the value for a set of instability in the absence of a magnetic field, even for large values of Q. The results suggest that there may be a region of steady finite amplitude behaviour even when linear they would suggest that instability would appear as oscillations; although no firm conclusions can be drawn within the confines of the analysis.

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A VON KÁRMÁN DISK DYNAMO

Kay A. Robbins

The homopolar disk dynamo with shunt exhibits nonperiodic reversals which resemble those of the earth's field. In this model the distortion of an initial poloidal field by the moving disk gives rise to a toroidal field (7). The poloidal field is reinforced by an equivalent alpha effect supplied by a shunt resistance. As the fields grow the Lorentz force slows the disk, and the fields decay when ohmic dissipation dominates the driving forces. The Lorentz force then decreases, the disk speeds up and the fields grow. Reversals occur when the toroidal and poloidal fields are out of phase.

Studies of disk dynamos are suggestive, but cannot determine whether this mechanism is responsible for oscillations and reversals in fluid. dynamos. This abstract describes some initial efforts in modeling this phenomenon in a fluid dynamo. A major difficulty is the dominance of Coriolis and Lorentz forces over inertial forces. Consider an infinite conducting disk of thickness $\bar{\ell}$ which is in contact with a conducting fluid (figure 1). The disk is free to rotate under a local applied torque $\bar{\ell}\rho T r^2$. The motion of the disk is opposed by the Lorentz force due to the currents in the disk. Viscous stresses may also be added. A mechanism for oscillation similar to that of the disk dynamo is thus provided (figure 2). Let the magnetic field in the fluid, $\underline{B} = ra\,\hat{r} + rb\,\hat{\phi} + c\,\hat{z}$ and the velocity $\underline{u} = ru\,\hat{r} + rv\,\hat{\phi} + w\,\hat{z}$ where a, b, c, u, v, w, depend on z. $\hat{r}$, $\hat{\phi}$, are unit vectors in a cylindrical coordinate system. Similarly in the disk $\underline{B} = rf\,\hat{r} + rg\,\hat{\phi} + h\,\hat{z}$. The equations can then be written

$$u_t = v^2 - \Omega_0^2 - u^2 - wu_z + [a_z c - 2b^2]/(\rho\mu) + \nu u_{zz}$$

$$v_t = -[2uv + wv_z] + [2ab + cb_z]/(\rho\mu) + \nu v_{zz}$$

$$a_t = -[wa - uc + \alpha b]_z + \lambda a_{zz}$$

$$b_t = cv_z - wb_z + (\alpha a)_z + \lambda b_{zz}$$

$$c_z = -2a \qquad w_z = -2u$$

where an alpha effect, α, has been assumed to provide poloidal field regeneration. Subscripts are used to denote partial differentiation.

In the disk

$$\Omega_t = T + \frac{2}{\ell \rho u}\int_{-\ell}^{0}(hg_z - gh_z)\,dz,\quad g_t = \bar{\lambda} g_{zz},\quad h_t = \bar{\lambda} h_{zz},\quad h_z = -2f.$$

Ω is the angular velocity of the disk and Ω_o is the rotation rate of the fluid at infinity. ν is the fluid viscosity. λ and $\bar{\lambda}$ are the magnetic diffusivities of the fluid and disk respectively.

At the disk fluid interface:

$$u = w = 0,\ v = \Omega,\ a = f,\ b = g,\ c = h,\ \bar{\sigma} a_z = \bar{\sigma} h_z,\ \bar{\sigma} b_z = \sigma g_z.$$

At the disk-insulator interface: $f = g = 0$. At infinity $u = a = b = 0$, $v = \Omega_o$. The insulating boundary conditions at infinity have been chosen to insure that all dynamics occur near the disk.

Linear Steady State

Following Loper (5) we can write:

$$\Omega = \Omega_0(1+\varepsilon),\ v = \Omega_o(1+\varepsilon v),\ w = \varepsilon w \Omega_o (\nu/\Omega_o)^{1/2},\ u = \varepsilon \Omega_o u$$
$$a = \varepsilon B_o (\nu\Omega_o)^{1/2}\mu\sigma a,\ b = \varepsilon B_o(\nu\Omega_o)^{1/2}\mu\sigma b,\ c = B_o + \varepsilon B_o \nu\mu\sigma c$$
$$f = \varepsilon B_o(\nu\Omega_o)^{1/2}\mu\sigma f,\ g = \varepsilon B_o(\nu\Omega_o)^{1/2}\mu\sigma g,\ h = B_o + \varepsilon B_o \nu\mu\sigma h$$
$$z = (\nu/\Omega_o)^{1/2}\zeta = \bar{\ell}\xi,\ \eta = (B_o^2\sigma)/(2\rho\Omega_o),\ T = \varepsilon T\Omega_o^2,\ t = \Omega_o^{-1} t,$$
$$\phi = \bar{\sigma}\bar{\ell}(\Omega_o/\nu)^{1/2}/\sigma,\ \phi_1 = \bar{\rho}\bar{\ell}(\Omega_o/\nu)^{1/2}/\rho,\ \alpha = (\nu\Omega_o)^{1/2}\alpha.$$

If prime and dot denote differentiation with respect to and respectively, then to zeroth order in ε:

$$P'' + 2\eta Q' + 2iP = 0$$
$$P - i\alpha Q + Q' = 0$$
$$\ddot{R} = 0$$

where

$P = u - iv$, $Q = a - ib$, $R = f - ig$ and α is assumed to be constant.

For small α the solutions are

$$P = ie^{-\lambda_1\zeta},\ Q = \frac{i\lambda_1^*}{|\lambda_1|^2}e^{-\lambda_1\zeta} + iQ_2 e^{-\lambda_2\zeta},\ R = -ig_o(\xi+1),\ \eta = -\tfrac{1}{2}T\phi_1 g_o,$$

$$Q_2 = \left[\phi - \frac{\lambda_1^*}{|\lambda_1|^2}\right]/[1+\phi\lambda_2],\ g_o = -\left[\frac{\lambda_1}{|\lambda_1|^2} + Q_2\right],\ \lambda_1 = \left[\eta + \sqrt{\eta^2+1}\right]^{1/2} + i\left[-\eta + \sqrt{\eta^2+1}\right]^{1/2},$$
$$\lambda_2 = -\alpha(i+\eta)/(\eta^2+1).$$

Thus for each T and each value of the disk velocity, a steady state value of B_o is determined. For a unique steadysolution a nonlinear balance similar to that given for the outer solution of Chawla (3) or Loper (6) must be assumed. The appearance of two rates of exponential decay in the magnetic field indicates a two layer structure to the solution. When $\alpha \to 0$ the second term in the Q expression approaches a constant. Then $c = 2\,\mathrm{Re}\{\int_{\zeta}^{0} Q\, d\zeta\} \sim O(\zeta)$ as $\zeta \to \infty$. This also indicates that there is an outer layer which provides a transition between the Ekman-Hartmann layer and the inviscid, current-free fluid at ∞. This outer nonlinear layer is the magnetic diffusion region (MDR) discussed in (1, 2, 4). A more complete description of the possible steady states and the important question of whether or not such a model can exhibit reversals will be addressed in future studies. The model is kinematic in the sense that the alpha effect is specified rather than derived from dynamical considerations. To complete the connection between the idealized model and the geodynamo, a plausible poloidal regeneration mechanism, such as that furnished by an underlying small scale turbulent velocity field, is needed.

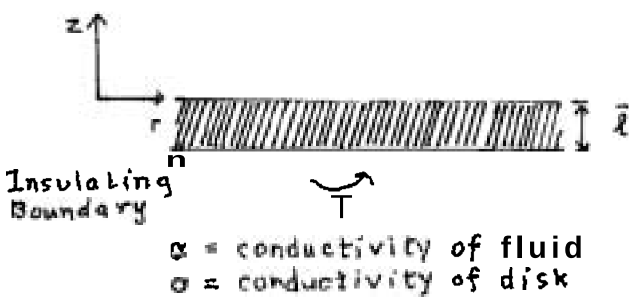

Fig. 1

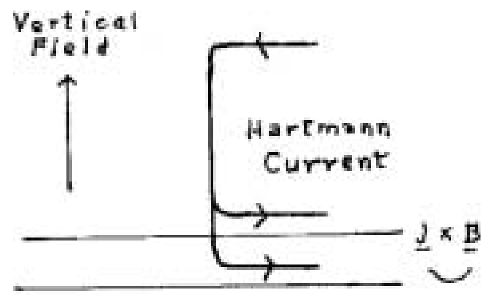

Fig. 2

Acknowledgements

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DIFFUSIVE INSTABILITIES IN MAGNETO-CONVECTION

Paul H. Roberts

It is now well established that even a uniform magnetic field can facilitate thermal convection in a rapidly-rotating fluid. A non-uniform field can, however, introduce a new class of 'magnetic instabilities' which are driven by curvature of the field lines, variation in magnetic intensity, or both. It is of interest to study the interrelation between such magnetic modes and the better understood 'convective instabilities'.

If diffusion effects are ignored a sufficient condition for convective instability is that

$$R_1 = g d \alpha \Delta T / A^2$$

attain some O(1) value. Here g is the acceleration due to gravity, A is the Alfvén velocity, α the coefficient of volume expansion, ΔT the temperature contrast, and d the length-scale. Once diffusive effects are added, the criterion is changed to $\mathcal{R} = O(1)$ where

$$\mathcal{R} = g d \alpha \Delta T / 2 \Omega \chi ,$$

Ω is the angular velocity and κ the thermal diffusivity which we suppose small compared with η, the magnetic diffusivity. (Viscosity is ignored except in thin boundary layers.) When $A^2/2\Omega\chi \gg 1$, as is for example true in the Earth's core, $R_1 \ll \mathcal{R}$ and convection occurs first through the action of diffusion. If ΔT is fixed, the A minimuzing the critical value, $\mathcal{R}_c$, of $\mathcal{R}$ at which convection first occurs is characterized by $\mathcal{E} = O(1)$, where

$$\mathcal{E} = A^2/2\Omega\eta$$

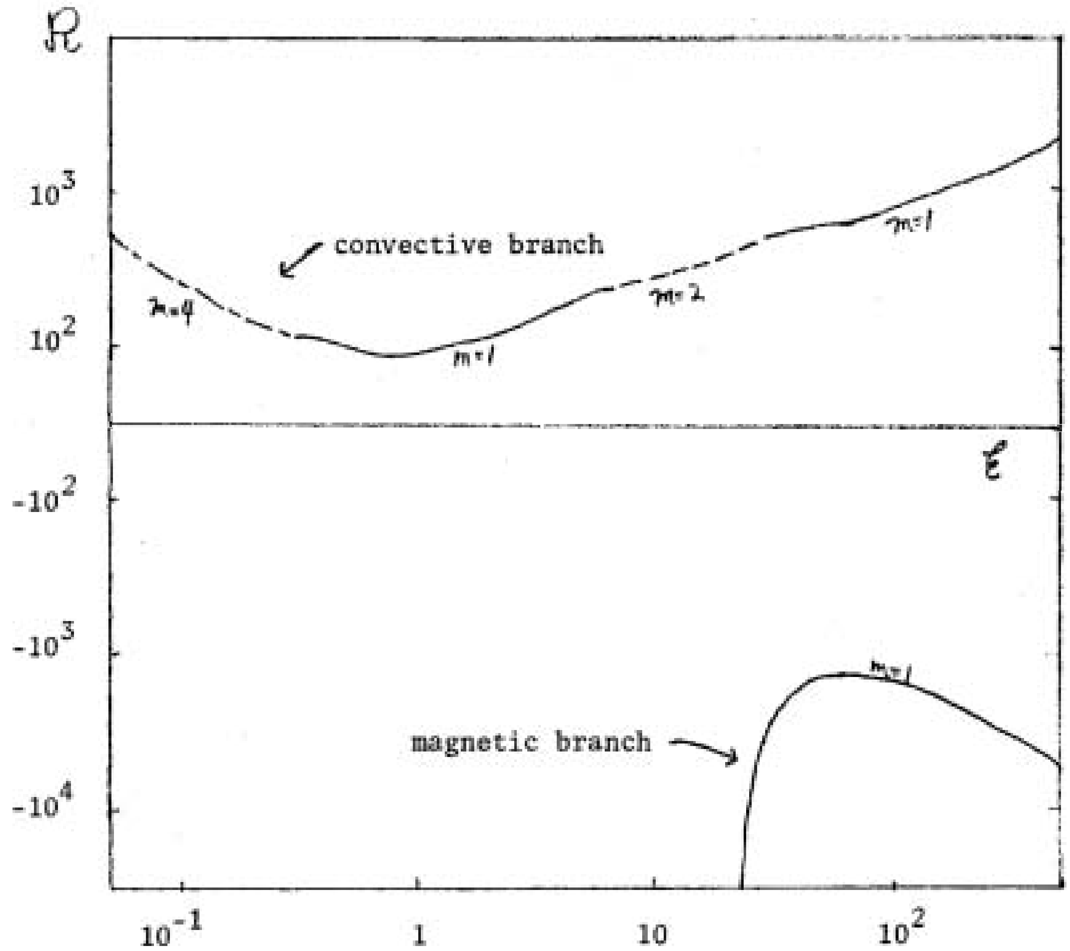
is the Elsasser number.

For simplicity, magnetic instabilities are usually studied in the absence of buoyancy forces, and typically in the context of the westward drift of the main geomagnetic field. If diffusion effects are ignored a sufficient condition for magnetic instability is that

$$\mathcal{L} = (A/2\Omega d)^2$$

attains some $O(1)$ value. Once diffusive effects are added, this criterion may be changed to $\mathcal{E} = O(1)$, as recent work by Roberts and Loper (1978) shows. When $\eta/2\Omega d^2 \ll 1$, as is for example true in the Earth's core, $\mathcal{L} \ll \mathcal{E}$ and magnetic instability occurs first through the action of diffusion.

A particularly interesting example is Malkus' (1967) model as generalized by Eltayeb and Kumar (1977). A uniform current flows through a conducting sphere of radius d parallel to the axis O_z of rotation, so that the magnetic field B is zonal and proportional to distance s from O_z; to provide buoyancy, a gravitational field directed towards, and proportional to distance r from, O is added together with a uniform distribution of heat sources, O being the centre of the sphere. Quantities such as R_1, $\mathcal{R}$, $\mathcal{E}$, are computed using equatorial values of g, A and temperature gradient $\Delta T/d$. Roberts and Loper (1978) found that, although no purely magnetic instabilities occurred, magnetic instability could be promoted by the addition of a bottom-heavy density distribution, i.e. by making $\mathcal{R}$ negative. Very recent numerical results by Fearn (to appear in 1979) exhibit this and are shown on the following figure, in which m is the preferred zonal wavenumber. His results strongly resemble those of Soward's (1978) plane layer, curved field line, model. The physical explanation of the paradoxical role of buoyancy on the magnetic mode is still lacking.



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NONLINEAR ASPECTS OF CONVECTION WITH STRONG MAGNETIC FIELDS

Andrew M. Soward

An electrically conducting Boussinesq fluid, conductivity σ, kinematic viscosity ν, density ρ is confined between two horizontal planes distance d apart. The fluid is permeated by a strong uniform horizontal magnetic field $\underset{\sim}{B}_0$ and the entire system rotates rapidly about a vertical axis with angular velocity $\underset{\sim}{\Omega}$. The fluid is heated from below and cooled from above so that in the absence of motion there is an adverse temperature β across the layer. The boundaries are rigid and perfect conductors of both heat and electricity. Attention is restricted to small values of the Ekman number E and the ratio q of the thermal and magnetic diffusivities κ and η respectively;

$$E = \nu/\Omega d^2 \ll 1 \qquad q = \kappa/\eta \ll 1.$$

In this parameter range marginal convection is steady and its character depends upon the relative sizes of the Coriolis and Lorentz forces, which is measured by

$$\lambda = 2\rho\Omega/\sigma B_0^2.$$

For order one values of λ, the critical Rayleigh number is large, specifically

$$R = g\alpha\beta L^4/\nu\kappa = O(E^{-1}),$$

where g is the acceleration due to gravity and α is the coefficient of expansion. When $\lambda \geqslant 2/\sqrt{3}$, motion consists of a single roll, whose axis is perpendicular to the applied magnetic field. On the other hand, when $\lambda < 2/\sqrt{3}$, two distinct rolls are possible: the axis of each roll lies oblique but makes an equal angle to the applied magnetic field. Only the latter case is discussed here.

The above linear results are well known (see Eltayeb (1972), Roberts and Stewartson (1972)). For the particular case of slippery boundaries, the stability of a set of oblique rolls to perturbations of the other set has also been considered by Roberts and Stewartson (1975). When $q \ll 1$, they found that both sets of rolls are unstable in the approximate range

$$1.0796 > \lambda > 2/\sqrt{3}.$$

The objective of the present analysis is to clarify the nature of the instability by considering the case of rigid boundaries. Though the convection rolls themselves are largely uneffected by this modification, any geostrophic flow alighed with the applied magnetic field, which was previously arbitrary, is now damped by Ekman suction. The latter effect is central to our treatment of the finite amplitude stability problem.

As the Rayleigh number is increased above its critical value, only one of the two sets of single rolls remains stable. The amplitude of the stable rolls increase with $\mathcal{R}$ until a second critical Rayleigh number is reached at which the system becomes unstable to unidirectional geostrophic flow. Whether the instability sets in as a steady or oscillatory shear flow depends on the importance of damping by Ekman suction. [Note that, as a result of approximations based upon small q, Alfven waves have been filtered out). In either case, the roll amplitude remains largely unaltered to further increase in the Rayleigh number with the consequence that the geostrophic flow is stabilised. On the other hand, the amplitude of the shear flow increases with $\mathcal{R}$ in a way which ensures the stability of the convection rolls. For the particular case of a steady geostrophic flow, a third critical value of $\mathcal{R}$ is isolated at which this shear becomes overstable to small amplitude perturbations.

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THE STRUCTURE AND STABILITY OF VORTEX FILAMENTS

Sheila E. Widnall

Some models for the structure of a vortex filament are reviewed. Several physical processes that result in vortex filaments are discussed and some experimental measurements of vorticity distribution within a vortex ring are shown (ref. 1).

The self-induced motion of vortex filaments is discussed and it is shown how the method of matched asymptotic expansions can be used to remove the logarithmic singularity in the classic cut-off formula for self-induced motion of a curved filament to obtain the correct result for a filament with arbitrary distribution of swirl and axial velocities (ref. 2).

$$\vec{V}_i(\vec{y}) = \int_{-\infty}^{-l} + \int_{l}^{\infty} - \frac{\Gamma}{4\pi} \frac{\vec{\xi}_1 \times d\vec{y}_1}{|y - y_1|^3}$$

where l is chosen such that $\ln l = \ln \frac{a}{2} + \frac{1}{2} - A - C$

where

$a \sim$ vortex core size

$A \sim$ swirl parameter

$$A = \lim_{r/a \to \infty} \int_0^{r/a} \bar{r} v_0^2 \, dr - \ln r/a$$

$v_0 \sim$ nondimensional swirl velocity; (if vorticity is uniform $A = \frac{1}{4}$)

and

$$C = \frac{2}{a^2} \int_0^{\infty} r \, w_0^2 \, dr \sim \text{the nondimensional axial momentum flux}$$

A vortex filament with axial flow "slows down" (ΔU) until the Kutta-Joukowski lift force $\rho(\Delta U)\Gamma$ is sufficient to balance the axial momentum flux in the curved filament.

Several configurations of vortex filaments exist such that the self-induced motions preserve their form. Examples are the ring, the helix, and various combinations of line filaments. We have investigated the stability of several of these self-preserving forms to long bending wave disturbances. (The asymptotic result for self-induced motion can be used for long waves). The Helical filament is unstable, ref. 3, the vortex pair is unstable, ref. 4, 5. The instability of the vortex ring is more difficult since the observed instability is a short-wave with a complex modal structure in the core. This instability has been extensively discussed (ref. 2, 6, 7) as has the corresponding instability for the single straight line filament in the presence of a straining flow (ref. 8). The physical mechanism of the instability of both the long and short wave is similar: vortex filaments see a background flow that corresponds to a straining or stagnation point flow; displacements along the diverging part of the flow in this field will diverge. Self-induced rotation is a stabilizing effect enabling the vortex to move into converging (stable) portions of the strain. If self-induced rotation is weak (long waves) or absent (short waves at critical values of wavenumber) bending wave displacements diverge exponentially.

The presentation included a lecture demonstration of an unstable vortex ring in a water tank made visible by hydrogen bubbles.

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