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VOLUME I

Great Seal

GEOPHYSICAL FLUID DYNAMICS

WHOI

Notes on the 1970

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

Volume I   Course Lectures and Abstracts of Seminars

Volume II   Fellowship Lectures

1970

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Editor's Preface

The principal lectures of this twelfth Summer Program were given by Joseph Pedlosky of the University of Chicago. On the following page one sees Dr. Pedlosky demonstrating advanced effects caused by rotation and stratification, Only in his last few lectures do these novel phenomena emerge from the analysis, This volume contains a restatement by the Fellows of these introductory lectures,

In the first weeks of the program additional afternoon and evening lectures were given by the staff with the purpose of suggesting classes of unsolved problems and techniques to attack them. Abstracts of these lectures follow the Pedlosky notes, attesting to the enthusiasms of the staff and the blunt edges of our formal tools.

Abstracts of the summer research seminars are also recorded here. These include the contributions to a micro-symposium on Turbulence organized by Robert Kraichnan. As in past years it was concluded that turbulence theory is on the verge of resolving pressing problems in the real world,

Mrs. Mary Thayer again has done all the work in assembling and reproducing the lectures, Our debt to her skill and perseverance defies repayment.

We are also indebted to the National Science Foundation for financial support and to the Woods Hole Oceanographic Institution for encouragement and shelter,

Willem V.R. Malkus
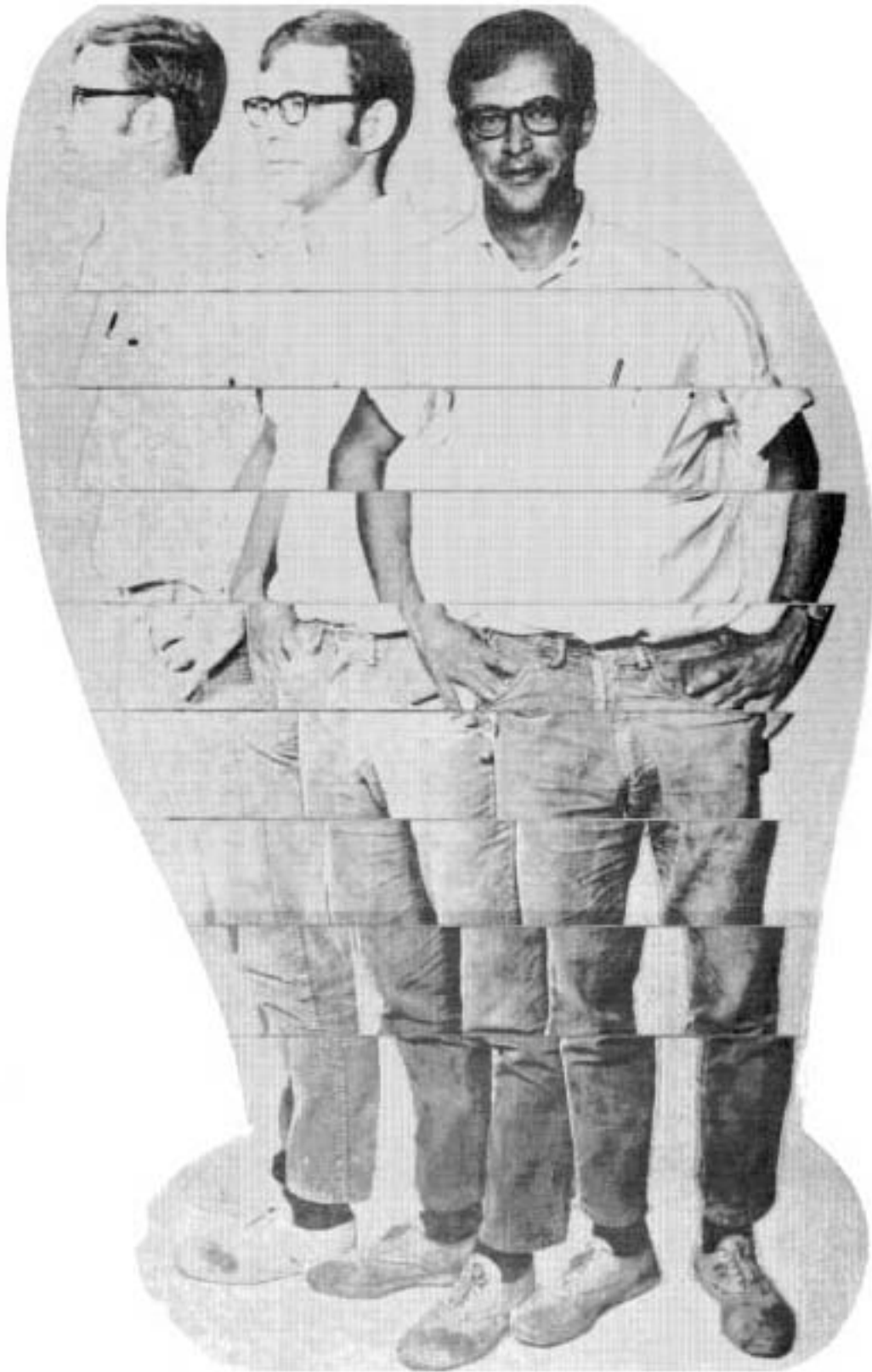
Our principal lectures, Joseph Pedlosky - rotating and stratified.

CONTENTS OF VOLUME I

Course Lectures and Seminars

Course Lectures

by

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CONTENTS OF VOLUME I   (continued)   Page No,

CONTENTS OF VOLUME I (continued)   Page No,

FLOW IN ROTATING STRATIFIED SYSTEMS

Lecture #1   Joseph Pedlosky

1. The interest in rotating stratified flows stems from the fact
that those are the flows encountered in the oceans and in the atmosphere.
To study those problems, one fruitful line of attack is to consider some
idealized models - realizable or not in the laboratory. One of the aims
of these lectures is to give through theoretical consideration an insight
into some of the mechanisms involved, Eventually consideration will be
given to a particular topic, to be studied experimentally soon in Chicago,

2. Review of the basic equations

We consider a frame of reference rotating with constant angular
velocity $\underline{\Omega} = |\underline{\Omega}| \hat{\underline{k}}$. We take the $O\underline{z}$ axis parallel to $\hat{\underline{k}}$, and, with-
out much loss of generality, consider situations where gravity is acting
along the negative $O\underline{z}$ direction: $\underline{g} = -|\underline{g}| \hat{\underline{k}}$.
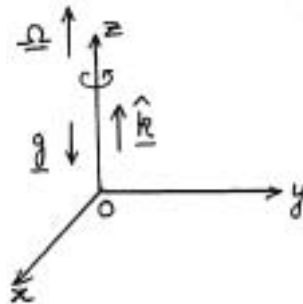
Then, denoting by $\underline{q}$ the velocity observed in the rotating frame, and with
standard notation for the other quantities, we have:

a) Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0 \qquad (1.a)$$

or $\qquad \dfrac{D\rho}{Dt} + \rho \nabla \cdot \underline{q} = 0 \qquad (1.b)$

b) Momentum:

$$\rho\left\{\left[\frac{\partial q}{\partial t}+q\cdot\nabla q\right]+2\,\underline{\Omega}\times q\right\}=-\nabla p+\rho\nabla\phi+\mu\nabla^2 q-\frac{2}{3}\mu\nabla\nabla\cdot q \tag{2}$$

ϕ is the force potential, including centrifugal potential:

$$\phi=-g\,z+\frac{1}{2}\left(\underline{\Omega}\times\underline{n}\right)^2$$

c) Energy:

$$\rho\,\frac{De}{Dt}=k\nabla^2 T+p\,\frac{D}{Dt}\left(\frac{1}{\rho}\right)+\mathcal{D} \tag{3}$$

d) State relation: usually $\rho=\rho(T,p)$; in our case $\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right]$

where $\mathcal{D}=\frac{1}{2}t_{ij}\left(q_{i,j}+q_{j,i}\right)$ is the viscous dissipation.

Although Eqs. (1), (2) and (3) constitute a closed system, it is con-
venient to introduce the following equation for the entropy Δ

$$\rho T\,\frac{D\Delta}{Dt}=\rho\,\frac{De}{Dt}-p\,\frac{D(1/\rho)}{Dt} \tag{4}$$

This, together with the thermodynamic relation

$$\delta\Delta=\frac{\partial\Delta}{\partial p}\bigg|_T\,\delta p+\frac{\partial\Delta}{\partial T}\bigg|_p\,\delta T=\frac{C_p}{T}\,\delta T-\frac{\partial\frac{1}{\rho}}{\partial T}\bigg|_p\,\delta p \tag{5}$$

enables us to write (3) in the more convenient form:

$$\rho\left[C_p\,\frac{DT}{Dt}-\frac{\alpha T}{\rho}\,\frac{Dp}{Dt}\right]=k\nabla^2 T+\mathcal{D} \tag{6}$$

$$\text{where}\quad\alpha=-\frac{1}{\rho}\,\frac{\partial\rho}{\partial T}$$

Our basic set of equations is now (1), (2) and (6) ; they are non-linear and
represent a coupling between the temperature, velocity, pressure and den-
sity fields.

3. Sealing

In order to reduce this system to a more tractable one it is
necessary to do some approximations. For this purpose we non-dimensionalize

the equations by a scaling process. We thus assume that there is a scale that characterizes the fields over most of the domain of interest (there might be some singular regions where this fails).

Let U be the velocity scale,

L be the length scale (we assume the system has only one length scale). We also write

$$T = T_0 + \underbrace{(\Delta T_v) T_s (z)}_{\text{static temp, distrib.}} + \underbrace{(\Delta T) T' (x, y, z, t)}_{\text{temp.distrib. due to motion}} \qquad (7)$$

$$P = P_0 \left[1 - \alpha (\Delta T_v) T_s + \alpha (\Delta T) T' \right] \qquad (8)$$

$$p = p_0 + p_s + P p' (x, y, z, t) \qquad (9)$$

where p_s is the static pressure:

$$\nabla p_s = \hat{k} g \alpha \Delta T_v \rho_0 T_s (z) - \rho_0 g + \rho_0 \Omega^2 \underline{R} \quad \begin{pmatrix} \underline{R} \cdot \hat{k} = 0 \\ \underline{R} \cdot \underline{R} = 1 \end{pmatrix} \qquad (10)$$

Now we write $\quad \underline{q} = U \underline{q}'$

$$\underline{r} = L \underline{r}'$$

$$t = \frac{L}{U} t'$$

Since the main feature of rotating flows is the presence of the Coriolis force and we shall consider systems where this force is of the same order of magnitude as the pressure gradient, we choose

$$P = \rho_0 \Omega U L$$

We want to relate ΔT to the external parameters, like U ; we also insist that in most regions of the flow

Coriolis force $= 0$ (horizontal pressure gradient)

Buoyancy force $= 0$ (vertical pressure gradient)$\Rightarrow \rho_0 \Omega U = \rho_0 g \alpha \Delta T$

$$\Rightarrow \Delta T = \frac{\Omega U}{\alpha g} = \frac{\Omega^2 L}{\alpha g} \varepsilon$$

where $\varepsilon \equiv \frac{U}{\Omega L}$ is the Rossby number.

If ΔT is given, one has: $\mathcal{E} \equiv \frac{\alpha g \Delta T}{\Omega^2 L}$ and $\mathcal{E}$ is thought of as the thermal

Rossby number, We can now rewrite the equations with the non-dimensional

quantities; for convenience we drop the primes, With $\theta \equiv (\Delta T_v) T_s + (\Delta T) T$,

we have

$$(1-\alpha\theta)\left[\mathcal{E}\frac{Dq}{Dt} + 2\hat{k} \times q\right] = -\nabla p + T\hat{k} + \frac{\Omega^2 L}{g} TR + \frac{\alpha \Delta T_v}{\mathcal{E}} R T_s(z) + E\nabla^2 q - \frac{2}{3} E\nabla(\nabla \cdot q) \qquad (11)$$

where $E \equiv \frac{\nu}{\Omega L^2}$ is the Eckman number, $\nu = \frac{\mu}{\rho_o}$. We shall discuss only problems

where $\mathcal{E} \ll 1$, $E \ll 1$.

$$\nabla \cdot q = (\alpha \Delta T)\frac{DT}{Dt} + w(\alpha \Delta T_v)\frac{dT_s}{dz} \qquad (w = q \cdot \hat{k}) \qquad (12)$$

$$(1-\alpha\theta)\left(\mathcal{E}\frac{DT}{Dt} + wS\frac{dT_s}{dz}\right) - \frac{\alpha T_v}{(1-\alpha\theta)^2}\left[\mathcal{E}^2\frac{\Omega^2 L^2}{C_p \Delta T}\frac{Dp}{Dt} + \mathcal{E}w\left(\frac{\alpha \Delta T_v g L T_s}{C_p \Delta T} - \frac{gL}{C_p \Delta T}\right) + \mathcal{E}q \cdot R\frac{\Omega^2 L^2}{C_p \Delta T}\right] =$$

$$= \frac{E}{\sigma}\nabla^2 T + \frac{\nu g \alpha}{C_p \Omega L}\mathcal{E}\mathcal{D} \qquad (13)$$

where $T_* = $ is the total (dimensional) temperature

$$S = \frac{g \alpha \Delta T_v}{\Omega^2 L} \quad \text{is the stratification parameter}$$

$$\sigma = \frac{\nu}{\chi} \quad \text{is the Prandtl number}$$

$$\text{and} \quad \frac{d^2 T_s}{dz^2} = 0$$

For a fluid like water C_p turns out to be $\sim 10^{-11}$ and dissipation is

unimportant for temperature changes, Also $\alpha \Delta T \lll 1$, and $\alpha \Delta T_v \lll 1$.

Then we can simplify the equations and get the Boussinesq equations:

$$\nabla \cdot q = 0 \qquad (14)$$

$$\mathcal{E}\frac{Dq}{Dt} + 2\hat{k} \times q = -\nabla p + T\hat{k} + E\nabla^2 q \qquad (15)$$

$$\mathcal{E}\frac{DT}{Dt} + wS\frac{dT_s}{dz} = \frac{E}{\sigma}\nabla^2 T \qquad (16)$$

We have also neglected terms in $\frac{\Omega^2 L}{g}$ (centrifugal effect) and in

$\frac{\alpha \Delta T_v}{\mathcal{E}}$ (slow circulation due to vertical temperature gradient), This is

equivalent to assuming <u>total</u> gravitational force is anti-parallel to $\underline{\Omega}$.

We also note that under the assumption $\frac{\Omega^2 L}{g} \ll 1$, $T_s(z)$ is solution of the static equilibrium equations only if $T_s(z) = z$. (This neglects a very slow steady convection.) Then (16) becomes:

$$\mathcal{E}\frac{DT}{Dt} + ws = \frac{E}{\sigma}\nabla^2 T \tag{16a}$$

4. Potential vorticity equation

From the preceding equations we can derive a useful relation for the vorticity $\underline{\omega} \cong \nabla \times \underline{q}$. Taking the **curl** of (15):

$$\mathcal{E}\frac{D\underline{\omega}}{Dt} = \mathcal{E}(\underline{\omega}\cdot\nabla)\underline{q} + 2\hat{\underline{k}}\cdot\nabla\underline{q} + \nabla T\times\hat{\underline{k}} + E\nabla^2\underline{\omega} \tag{17}$$

The vertical component of this is (with $\zeta \equiv \underline{\omega}\cdot\hat{\underline{k}}$)

$$\mathcal{E}\frac{D\zeta}{Dt} = \mathcal{E}(\underline{\omega}\cdot\nabla)w + 2\frac{\partial w}{\partial z} + E\nabla^2\zeta \tag{18}$$

Combining this with the z derivative of (16a):

$$\mathcal{E}\frac{D}{Dt}\left[\zeta + \frac{z}{s}\frac{\partial T}{\partial z}\right] - E\left\{\nabla^2\left(\zeta + \frac{z}{\sigma s}\frac{\partial T}{\partial z}\right)\right\} = \underbrace{\mathcal{E}(\underline{\omega}\cdot\nabla w)}_{(a)} - \underbrace{\frac{z\mathcal{E}}{s}\frac{\partial \underline{q}}{\partial z}\cdot\nabla T}_{(b)} \tag{19}$$

Evaluation of (a) by taking w from the thermal Eq.16 and of (b) by taking $\frac{\partial \underline{q}}{\partial z} = \hat{\underline{k}}\cdot\nabla\underline{q}$ from the vorticity Eq.17, shows that the right-hand side of (19) contains only terms of order $\mathcal{E}^2$ or $\mathcal{E}E$ and thus can be neglected; therefore we write:

$$\mathcal{E}\frac{D}{Dt}\left\{\zeta + \frac{z}{s}\frac{\partial T}{\partial z}\right\} = E\nabla^2\left(\zeta + \frac{z}{\sigma s}\frac{\partial T}{\partial z}\right) \tag{20}$$

The quantity $\zeta + \frac{z}{s}\frac{\partial T}{\partial z}$ is called the potential vorticity. We see that if $E \ll \mathcal{E}$, it is conserved for each fluid element. On the other hand, if $\mathcal{E} \ll E$ we have

$$\nabla^2\left(\zeta + \frac{z}{\sigma s}\frac{\partial T}{\partial z}\right) = 0 \tag{21}$$

This would be a conduction equation for the potential vorticity if $\sigma = 1$. If $\sigma \neq 1$ we consider (21) as a quasi-diffusion equation for the potential

vorticity, which will be useful in the study of very slow sready motions.

Even if ε is not small with respect to E the conduction equation may hold if the circulation is constrained by conditions of symmetry, For example, if (as in cases we shall study) the motion is symmetric about the rotation axis, and steady,

$$\frac{3}{Dt} = u \frac{\partial}{\partial n} + w \frac{\partial}{\partial z}.$$

It turns out (as will be shown later) that over most of the field

$$u = O(E), \quad w = O(E/\sigma s) \quad \text{if} \quad \sigma s > E^{1/2}$$

in which ease:

$$E \frac{D}{Dt} = O\left(\frac{\varepsilon E}{\sigma s}\right).$$

Hence the conduction equation will still hold under the weaker condition $\sigma S > \varepsilon$ instead of $E > \varepsilon$. This is an important consideration for the conduction problem is linear: we may thus say that we are dealing with two distinct classes of flow, viz:

1) Over that part of the fluid where our scaling parameters are accurate, flows with strong symmetries are often governed by the law of conduction of (essentially) potential vorticity,

2) In the absence of symmetry $\varepsilon > E$ is enough to upset this result and the principle of conservation of potential vorticity applies,

Notes submitted by

Yves J. Desaubies

EKMAN LAYER IN A STRATIFIED, ROTATING FLUID

Lecture #2   Joseph Pedlosky

References: Barcilon, V. and J. Pedlosky J.F.M. 1967 29: 1-16
              "   "   " "   "   J.F.M. 1967 29: 609-621
              "   "   " "   "   J.F.M. 1967 29: 673-690
         Veronis, G. Tellus 1967 XIX(2): 326
              "   " Tellus 1967 XIX(4): 620

Consider a cylindrical annulus filled with fluid and rotating with angular velocity Ω . The top lid of the annulus rotates differentially with respect to the sides and the bottom, The top is heated to produce stable stratification before the system is set in rotation, The side walls are insulated. Let r, θ, z be cylindrical coordinates and let u, v, w be the radial, swirl, and vertical velocity components in the rotating frame res- pectively.
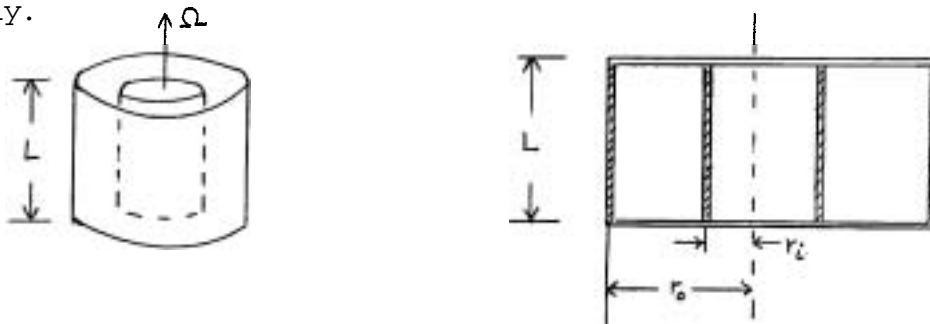
The boundary conditions on the flow are:

$(u,v,w) = 0$ on all walls except the top

$(u,v,w) = (0,v_T(r),0)$ for $z = L$

where $v_T(r)$ is the velocity of the top lid in the rotating frame.

The thermal condition for the insulating side walls is

$$\frac{\partial T}{\partial r} = 0 \quad \text{for} \quad r = r_i \text{ and } r_o$$

There are two alternatives for thermal conditions at the top and bottom. One can either impose a temperature at the top and bottom or require that

$$\frac{dT}{dz} = 0$$

there, The former condition is easily achieved experimentally but leads to
a complicated flow. The latter can be realized experimentally with some dif-
ficulty, The classical rotating annulus experiments were performed with the
temperature fixed at the inner and outer walls.

Recall the equations of motion

$$\nabla \cdot \mathbf{q} = 0$$

$$\epsilon \frac{D\mathbf{q}}{Dt} + 2\hat{\mathbf{k}} \times \mathbf{q} = -\nabla p + T\hat{\mathbf{k}} + E\nabla^2 \mathbf{q}$$

$$\epsilon \frac{DT}{Dt} + wS = \frac{E}{\sigma}\nabla^2 T$$

The length and velocity scales are L and $v_T(r_o)$ respectively.

For the problem we wish to study, $E \ll 1$ and $\epsilon \ll 1$. In regions
where the length scale is accurately represented by L the viscous effects are
negligible to $0(1)$ as far as the force balances are concerned, We anticipate
that a singular region, a boundary layer, will develop in the regions where
viscous forces are important, The first such region to be studied is the
Ekman layer on a horizontal plate, In particular, the effect of stratifi-
cation on these layers will be considered.

Let the velocity of the lower plate, z = 0, be $\mathbf{q}_L = \mathbf{q}_L(x,y)$. We impose
the condition $\nabla \cdot \mathbf{q}_L = 0$.

The approach adopted here assumes that for small ϵ the length scale
in the boundary layer can be obtained from linear theory, The linearized
equations of motion in cartesian coordinates are:

$$2_u = -p_y + E\, v_{zz}$$

$$-2_v = -p_x + E\, u_{zz}$$

$$0 = -p_z + T + E\left[w_{zz}\right]$$

$$w \, \sigma \, S = T_{zz}$$

$$u_x + v_Y + w_z = 0$$

In these equations ∇^a has been replaced by $\partial^2/\partial z^2$, the dominant part in the boundary layer. A "tracer bracket" has been placed around w_{zz} to enable us to follow the contribution of this term,

Eliminate p from the first two of these equations to obtain

$$2(u_x + v_y) = E \frac{\partial^2}{\partial z^2} (v_x - u_y)$$

Also, apply $\partial/\partial y$ to the first equation, $\partial/\partial x$ to the second to obtain

$$2(v_x - u_y) = \nabla_H^2 p - E \frac{\partial^2}{\partial z^2} (u_x + v_y)$$

Using the equation of continuity and the definition of the z-compopent of vorticity

$$\zeta = v_x - u_y$$

we obtain the following four equations in p, ζ , T, w:

$$-2 \frac{\partial w}{\partial z} = E \frac{\partial^2 \zeta}{\partial z^2}$$

$$2\zeta = \nabla_H^2 p + E \frac{\partial^3 w}{\partial z^3}$$

$$\frac{\partial p}{\partial z} = T + E \left[\frac{\partial^2 w}{\partial z^2} \right]$$

$$\sigma S w = E \frac{\partial^2 T}{\partial z^2}$$

Now successively eliminate T, p, and ζ from these equations to obtain an eauation in w

$$E^2 \left\{ \frac{\partial^6 w}{\partial z^6} + \nabla_H^2 \left[\frac{\partial^4 w}{\partial z^4} \right] \right\} + \sigma S \nabla_H^2 w + 4 \frac{\partial^2 w}{\partial z^2} = 0$$

The second term in the curly brackets is always negligible with respect to the first because of the large vertical gradients in the boundary layer, This term arises from the w_{zz} term being traced, hence this term could be neglected,

If $\sigma S = 0$ then the equation is balanced with a length scale of $\ell = E^{\frac{1}{4}}$.
As long as $\sigma S \ll E^{-1}$ the proper length scale remains $\ell = E^{\frac{1}{2}}$. (Physically,
ℓ may be interpreted as the distance over which the fluid appears homo-
geneous,) If the first and Past terms of the equation were to be balanced
the appropriate length scale would be

$$\ell = (E^{2}/\sigma S)^{1/6}$$
$$= \left(\frac{\kappa \nu}{g \propto \Delta T_{v} L^{2}}\right)^{1/6}$$
$$= R_{a}^{-1/6}$$

where R_{a} is the Rayleigh number,

The above calculations were undertaken to suggest that $E^{\frac{1}{2}}$ is the
proper length scale in the boundary layer for the full problem, Now intro-
duce the stretched variable

$$\hat{\zeta} = E^{-\frac{1}{2}} z$$

It is assumed that the velocity $q = q(x, y, \hat{\zeta})$ for the steady problem depends
explicitly on $\hat{\zeta}$ It is convenient to scale the vertical velocity accord-
ing to

$$w = E^{\frac{1}{2}} \hat{w}(x, y, \hat{\zeta})$$

The boundary layer equations of motfon can now be expressed in the form

$$\hat{u}_{x} + \hat{v}_{y} + \hat{w}_{\hat{\zeta}} = 0$$

$$\in\left[\hat{u}\hat{u}_{x} + \hat{v}\hat{u}_{y} + \hat{w}\hat{u}_{\hat{\zeta}}\right] - 2\hat{v} = -\hat{p}_{x} + \hat{u}_{\hat{\zeta}\hat{\zeta}} + E\nabla_{H}^{2}\hat{u}$$

$$\in\left[\hat{u}\hat{v}_{x} + \hat{v}\hat{v}_{y} + \hat{w}\hat{v}_{\hat{\zeta}}\right] + 2\hat{u} = -\hat{p}_{y} + \hat{v}_{\hat{\zeta}\hat{\zeta}} + E\nabla_{H}^{2}\hat{v}$$

$$\in E\left[\hat{u}\hat{w}_{x} + \hat{v}\hat{w}_{y} + \hat{w}\hat{w}_{\hat{\zeta}}\right] - E^{\frac{1}{2}}\hat{T} = -\hat{p}_{\hat{\zeta}} + E\hat{w}_{\hat{\zeta}\hat{\zeta}} + E^{2}\nabla_{H}^{2}\hat{w}$$

$$\in\left[\hat{u}\hat{T}_{x} + \hat{v}\hat{T}_{y} + \hat{w}\hat{T}_{\hat{\zeta}}\right] + E^{\frac{1}{2}}S\hat{w} = \frac{1}{\sigma}\hat{T}_{\hat{\zeta}\hat{\zeta}} + \frac{E}{\sigma}\nabla_{H}^{2}\hat{T}$$

As long as $\in \ll 1$ the $0(1)$ balance in the momentum equation is

$$-2\hat{v} = -\hat{p}_x + \hat{u}_{\zeta\zeta} \tag{2.1}$$

$$2\hat{u} = -\hat{p}_y + \hat{v}_{\zeta\zeta} \tag{2.2}$$

$$0 = \hat{p}_\zeta \tag{2.3}$$

Equation (3) implies that $\hat{p}_x$ and $\hat{p}_Y$ are independent of $\hat{\zeta}$ Thus, by differentiating Eqs. (2.1) and *(2.3)* with respect to $\hat{\zeta}$ we obtain the following differential equations for $\hat{u}$ and $\hat{v}$:

$$-2\hat{v}_\zeta = \hat{u}_{\zeta\zeta\zeta}$$

$$2\hat{u}_\zeta = \hat{v}_{\zeta\zeta\zeta}$$

If we introduce the function

$$\Upsilon = u + \iota v$$

then upon integration in $\hat{\zeta}$ we find

$$\Upsilon_{\zeta\zeta} = 2\iota\Upsilon + \Upsilon_o$$

The solution for Υ which decays into the interior is

$$\Upsilon = A e^{-(1+\iota)\zeta} - \tfrac{1}{2}\iota\,\Upsilon_o.$$

where the integration constants **A** and Υ_o are obtained from matching condieions. As $\hat{\zeta} \longrightarrow 0$ the flow must approach the interior, geostrophic flow so

$$-\tfrac{1}{2}\iota\,\Upsilon_o = -\tfrac{1}{2}\left(P_{Iy} - \iota P_{Ix}\right)$$

At $\hat{\zeta} = 0$ the flow must match the velocity of the lower plate so

$$u_L + \iota v_L = A - \tfrac{1}{2}\left(P_{Iy} - \iota P_{Ix}\right)$$

Thus, the solution for Υ becomes

$$\Upsilon = \tfrac{1}{2}\left(-P_{Iy} + \iota P_{Ix}\right) + e^{-\zeta}\cos\zeta\left[\left(u_L + \tfrac{1}{2}P_{Iy}\right) + \iota\left(v_L - \tfrac{1}{2}P_{Ix}\right)\right] + e^{-\zeta}\sin\zeta\left[\left(v_L - \tfrac{1}{2}P_{Ix}\right) - \iota\left(u_L + \tfrac{1}{2}P_{Iy}\right)\right]$$

Expressed in vector notation this is

$$\underset{\sim}{q}_H = \tfrac{1}{2}\hat{k} \times \nabla_H P_I + e^{-\zeta}\cos\zeta\left[\underset{\sim}{q}_L - \tfrac{1}{2}\hat{k} \times \nabla_H P_I\right] - e^{-\zeta}\sin\zeta\,\hat{k} \times \left[\underset{\sim}{q}_L - \tfrac{1}{2}\hat{k} \times \nabla_H P_I\right] \tag{2.4}$$

where $\underset{\sim}{q}_H$ is the horizontal velocity.

Since $\hat{p}_\zeta = 0$ we have

$$\hat{p}(x, y, \hat{\quad}) = p_I(x, y, 0)$$

From the equation of continuity

$$-\hat{w}_\zeta = u_x + v_y$$
$$= \nabla_H \cdot q_H$$
$$= e^{-\zeta} \sin \zeta \left[\hat{k} \cdot \nabla \times q_L - \tfrac{1}{2} \nabla_H^2 p_I \right]$$

The boundary condition $\hat{w} = 0$ for $\hat{\zeta} = 0$ enables us to evaluate w_0. Thus

$$\hat{w} = \tfrac{1}{2}\left[\hat{k}\cdot\nabla\times q_L - \tfrac{1}{2}\nabla_H^2 p_I \right] e^{-\hat{\zeta}}(\cos\hat{\zeta} + \sin\hat{\zeta}) + w_0 \qquad (2.5)$$

(But as $\hat{\zeta} \to \infty$, $\hat{w}$ approaches the interior vertical velocity, thus)

$$\hat{w} = \tfrac{1}{2}\left[\hat{k}\cdot\nabla\times q_L - \tfrac{1}{2}\nabla_H^2 p_I \right]\left[e^{-\zeta}(\cos\hat{\zeta}+\sin\hat{\zeta}) - 1 \right]$$

$$\text{and} \quad W = E^{1/2}\hat{w}$$

Now suppose q and T are partitioned into interior and boundary layer parts, namely

$$q = q_I + q_B(x, y, \hat{\zeta})$$
$$T = T_I + T_B(x, y, \zeta)$$

If $\sigma S = 0(1)$ then

$$T_B = 0\left[E^{1/2} \max(\epsilon, E^{1/4}) \right]$$

Hence, to the lowest order the boundary conditions are

i) $w_I = 0$ for $z = 0, 1$

ii) T_I satisfies the thermal conditions.

At $z = 1$ the same situation prevails, The analogous solution for the bertical velocity in the upper Ekman Payer gives

$$E^{1/2}\hat{w}(x, y, \infty) = w_I(x, y, \hat{\zeta}) = -\tfrac{1}{2}E^{1/2}\left[\hat{k}\cdot\nabla\times(q_{HI} - q_I) \right].$$

Notes submitted by

William B. Heard

FLOW IN ROTATING STRATIFIED SYSTEMS

Lecture #3   Joseph Pedlosky

The dynamics of axially symmetric motions: the interior,

The equations governing axfally symmetric, steady motions are, in the nondimensional form

r-momentum $\quad \varepsilon\left[u u_r + w u_z - \frac{v^2}{r}\right] - 2v = -p_r + E\left[\nabla^2 u - \frac{u}{r^2}\right]$ $\qquad$ (3.1a)

θ-momentum $\quad \varepsilon\left[u v_r + w v_z + \frac{uv}{r}\right] + 2u = E\left[\nabla^2 v - \frac{v}{r^2}\right]$ $\qquad$ (3.1b)

z-momentum $\quad \varepsilon\left[u w_r + w w_z\right] - T = -p_z + E \nabla^2 w$ $\qquad$ (3.1c)

continuity $\quad \frac{1}{r}(ru)_r + w_z = 0$ $\qquad$ (3. 1d)

temperature $\quad \varepsilon\left[u T_r + w T_z\right] + Sw = \frac{E}{\sigma} \nabla^2 T$ $\qquad$ (3.1e)

in which the nondimensional parameters are

$\qquad$ the Ekman number $\qquad E = \frac{\nu}{\ell L^2} << 1$

$\qquad$ the Rossby number $\qquad \varepsilon = \frac{U}{\Omega L} << 1$ $\qquad$ (3.2)

$\qquad$ the stratification measure $\quad S = \frac{g \alpha \Delta T_v}{\Omega^2 L}$

It will be assumed that in the interior the scaling of the swirl velocity, the pressure, the temperature, and the lengths are correct, i.e. the interior is not a singular region.

First it is necessary to estimate the divergence of the vertical flow in the interior. From the Ekman boundary layers studied in the previous lecture, (Lecture #2, Eq,(2.2)),

$$w = 0(E^{\frac{1}{2}}) \qquad \text{at } z = 0.1$$

and from Eq. (3.1d)

$$w_z = 0(u) \qquad\qquad (3.3)$$

an estimate for the interior vertical velocity is found

$$w = O(u, E^{1/4})$$

This estimate for w is used in Eq. (3.1b) for finding

$$u = O(\varepsilon \, E^{1/4}, E) \tag{3.4}$$

whence Eq, (3.3) becomes

$$w_z = O(\varepsilon \, E^{1/4}, E) \tag{3.5}$$

The basic dynamics at the lowest approximation are

$$2v = p_r \tag{3.6a}$$

$$T = p_z \tag{3.6b}$$

although these are not sufficient to determine the flow. Eliminating the

pressure between (3.6a) and (b) yields

$$2v_z = T_r \tag{3.6c}$$

- the thermal wind equation.

Case 1   S = O   [Homogeneous Fluid)

If the fluid is homogeneous and the boundary conditions impose no

fluctuations in temperature, there will be no fluctuations in temperature in

the fluid

$$T = 0$$

Then Eq, (3.6c) gives a partial statement of the Taylor-Proudman theorem.

There is no vertical shear in the swirl

$$v_z = 0$$

$$\text{or} \quad v = v(r) \tag{3.7}$$

Now the Ekman efflux into the interior has a magnitude $E^{1/2}$, while the

estimate (3.5) of the divergence of the vertical velocity is smaller than

$E^{1/2}$. Thus to the first approximation in w

$$w_z = 0$$

$$\text{or} \quad w = w(r) \tag{3.8}$$

Equations (3.7) and (3.8) express the strong restraint that the rotation imposes on the flow.

As the velocities v and w are independent of the vertical, they can be determined by evaluaeing them at $z = 0,1$ where they are constrained by the Ekman boundary layer matching. Lecture #2, Eq. (2.2) yields

at $z = 1$ $\qquad w(r) = \dfrac{E^{\frac{1}{2}}}{2} \cdot \dfrac{1}{r} \dfrac{\partial}{\partial r} r \left(v_T - v(r) \right)$

at $z = 0$ $\qquad w(r) = \dfrac{E^{\frac{1}{2}}}{2} \cdot \dfrac{1}{r} \dfrac{\partial}{\partial r} r v(r)$

Then eliminating w

$$\frac{1}{r} \frac{\partial}{\partial r} r v(r) = \frac{1}{2} \cdot \frac{1}{r} \frac{\partial}{\partial r} r v_T$$

i.e. $v(r) = \dfrac{v_r}{2} + \dfrac{C}{r}$ $\hspace{3cm}$ (3.9)

where C is a constant to be determined.

Consider the radial mass flux in order to find C.

	radial velocity	vertical length
	$0(1)$	$0(E^{\frac{1}{2}})$
	$0(\varepsilon E^{\frac{1}{2}}, E)$	$0(1)$
	$0(1)$	$0(E^{\frac{1}{2}})$
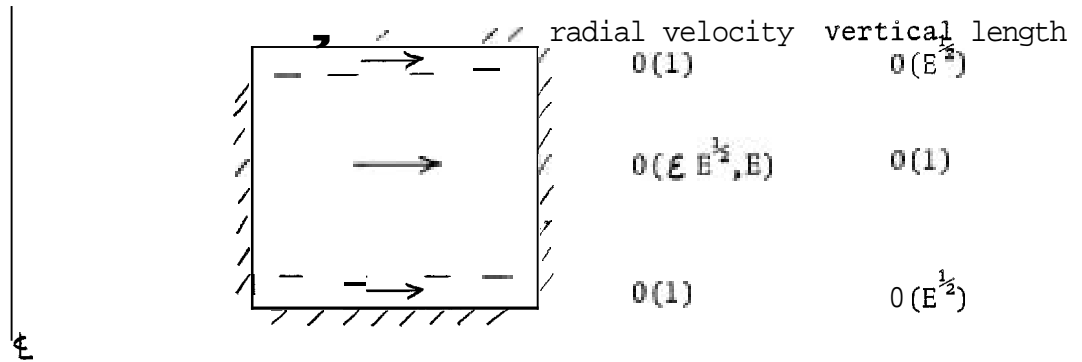
In the interior, (3.4) gives an estimate of $0(\varepsilon E^{\frac{1}{2}}, E)$ for the radial flux, while in the Ekman layers it has a magnitude of $E^{\frac{1}{2}}$. Thus the radial mass fluxes in the Ekman layers must balance, These are found using (3.9) in Lecture #2, Eq. (2.4), at the top
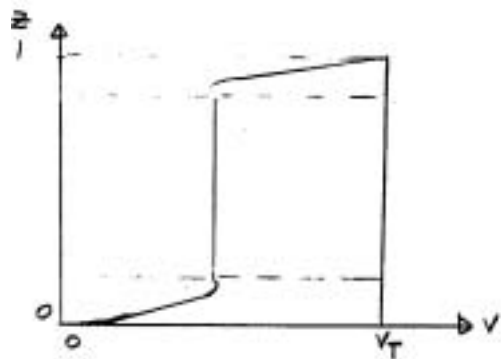
$$\int_0^{\infty} r u_T E^{\frac{1}{2}} d\zeta = E^{\frac{1}{2}} \left[\frac{r v_T}{4} - \frac{C}{2} \right]$$

at the bottom

$$\int_0^{\infty} r u_B E^{\frac{1}{2}} d\zeta = E^{\frac{1}{2}} \left[-\frac{r v_T}{4} - \frac{C}{2} \right]$$

Hence C must vanish,

Thus the interior swirl flow has been found

$$V = \frac{V_T(r)}{2} \qquad (3.10)$$

and the interior vertical flow

$$w = \frac{E^{1/2}}{4} \frac{1}{r} \frac{\partial}{\partial r}(r v_T) \qquad (3.11)$$

Meridional circulation
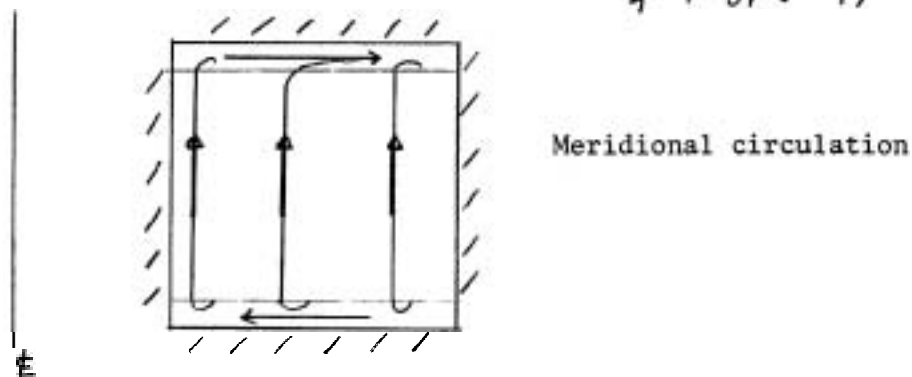
As there is no vertical shear, Eq. (3.7), then Eq. (3.1b) gives a radial flow

$$u = O(E) \qquad (3.12)$$

The detailed structure of the flow in the Ekman layers can be found by using the solution (3.10) in Lecture #2, Eqs. (2.4) and (2.5), To complete the detailed description of the flow everywhere it would be necessary to discuss $E^{1/4}$ and $E^{1/3}$ layers on the side walls.

In the Ekman layers there are radial fluxes top and bottom

$$\pm \frac{E^{1/2}}{4} r v_T(r) \qquad (3.13)$$

Unless v_T vanishes on $r = r_i$ and r_o, there will be radial fluxes in the layers,

This cannat connect across the interior, and so there must be side layers to accommodate it, These are found possible, One interesting result is that when $v_T(r)$ is specified (smooth) over the full cylinder then the flux up the layer on the inner cylinder is exactly that vertical flux in the interior of the excluded central cylindrical region. This is because the Ekman layers, due to the restraint of strong rotation, are a local phenomena: the radial fluxes in (3.13) depend only on the radius at which they are calculated, Thus at $r = r_i$ the radial flux in the Ekman layer does not know whether it came from the interior of the completed cylinder or up the side wall layer of the annulus,

Case 2 $S \neq 0$ (Stratiffed Fluid)

     The correct measure of the significant stratification can be estimated by comparing the vertical flow out of the Ekman boundary layer, assuming an 0(1) discrepancy between the interior and imposed boundary vertical components of vorticity

$$w = 0(E^{\frac{1}{2}})$$

with the vertical advection of the stratification needed to balance the diffusion of temperature, (3.1e)

$$w = 0\left(\frac{E}{\sigma} S\right)$$

Thus it can be expected that the fluid will behave differently from when it is homogeneous when

$$\sigma S \gtrsim E^{\frac{1}{2}} \tag{3.14}$$

It will be seen later (Lecture 5) that when the stratification exceeds this, '
it restricts the interior flow in such a way that the Ekman suction is not as strong as $E^{\frac{1}{2}}$.

     The linear problem is considered, a *posteriori* it is possible to find

the limitations on ε for the nonlinear terms to be ignored.

$$\varepsilon \ll S$$

The linearized equations are

$$-2v = -p_r + E\left[\nabla^2 u - \frac{u}{r^2}\right]$$

$$2u = E\left[\nabla^2 v - \frac{v}{r^2}\right]$$

$$-T = -p_z + E\nabla^2 w$$

$$\frac{1}{r}(ru)_r + w_z = 0$$

$$\sigma S = E\nabla^2 T$$

As before, assuming the sealing of the interior is correct, i.e. v, p, T,

$$ar)\frac{\partial}{\partial z}$$ are of order 1, it is possible to expand

$$v = v_I^{(0)} + \ldots$$

$$u = E u_I^{(0)} + \ldots$$

$$w = \frac{E}{\sigma S} w_I^{(0)} + \ldots$$

$$T = T_I^{(0)} + \ldots$$

$$p = p_I^{(0)} + \ldots$$

The governing equations for this first approximation to the interior then become

$$2 v_I^{(0)} = p_z^{(0)} r$$

$$2 u_I^{(0)} = \nabla^2 v_I^{(0)} - \frac{v_I^{(0)}}{r^2}$$

$$T_I^{(0)} = p_{I z}^{(0)}$$

$$\frac{1}{r}(ru_I^{(0)})_r + w_{I z}^{(0)} = 0$$

$$w_I^{(0)} = \nabla^2 T_I^{(0)}$$

This is a linear system and can thus be reduced to an equation in a single variable, e.g. pressure

$$\nabla^2\left[\nabla_H^2 p_I^{(0)} + \frac{4}{\sigma S}\frac{\partial^2 p_I^{(0)}}{\partial z^2}\right] = 0 \qquad (3.15)$$

The previous equation in the elimination is familiar:

$$\nabla^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r v_I^{(o)} \right) + \frac{2}{\sigma S} \frac{\partial T_I^{(o)}}{\partial z} \right] = 0$$

which is the conduction equation for the potential vorticity. The constraint of symmetry with respect to the azimuthal direction has made the small diffusion effects dominant in determining the steady state interior flow even though $\tilde{E}$ may exceed E.

The equation (3.14) is an "elliptic" partial differential equation, and so boundary conditions have to be imposed on both horizontal and vertical boundaries. The Ekman layers on the top and bottom have been studied already in Lecture 2. The matching conditions they impose on the interior are for the vertical velocity $w_I^{(o)}$.

$$w_I^{(o)} = \frac{\sigma S}{E^{1/2}} \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \left(v_T - v_I^{(o)} \right) \right\} \quad at \quad z = 1 \qquad (3.15a)$$

$$w_I^{(o)} = \frac{\sigma S}{E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r v_I^{(o)} \qquad\qquad at \quad z = 0 \qquad (3.15b)$$

and for the interior temperature $T_I^{(o)}$ those conditions imposed on the temperature which in their general form are

$$m_o' T_I^{(o)} + \ell_o' \frac{\partial T_I^{(o)}}{\partial z} = 0 \qquad on \quad z = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right.$$

The side boundaries are the subject of the next lecture, and the solutions for the interior flow will be presented in Lecture 5.

Notes submitted by
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VERTICAL BOUNDARY LAYERS IN A ROTATING, STRATIFIED FLOW

Lecture #4   Joseph Pedlosky

Side Wall Boundary Layers

In investigating the regions close to the inner and outer cylinder walls we follow the same procedure as in the Ekman layer analysis. The Rossby number is provisionally taken to be small so that we can look at the linear problem. From this analysis we find the scales appropriate to the linear boundary layer and from these the restriction on the size of the Rossby number.

The field variables may be written as the sum of two parts. For example:

$$u = u_I + u_B$$

where u_B is a boundary layer correction to the interior value chosen to help satisfy the boundary conditions on the wall and to vanish rapidly as we leave the layer. Because the equations are linear the equations governing these correction variables will be the same as for the total quantities, In the dissipative terms only the radial derivatives will be important and we are left with the following set of equations.

$$-2v_B = -\frac{\partial p_B}{\partial r} + E\left[\frac{\partial^2 u_B}{\partial r^2}\right] \tag{1}$$

$$2u_B = \qquad E\frac{\partial^2 v_B}{\partial r^2} \tag{2}$$

$$-T_B = -\frac{\partial p_B}{\partial z} + E\frac{\partial^2 \omega_B}{\partial r^2} \tag{3}$$

$$\frac{\partial u_B}{\partial r} + \frac{\partial \omega_B}{\partial z} = 0 \tag{4}$$

$$\sigma S \omega_B = \qquad E\frac{\partial^2 T_B}{\partial r^2} \tag{5}$$

By eliminating pressure between Eqs. (1) and (3) we get:

$$-2v_{\theta z}+T_{\theta r} = E\,\frac{\partial^2}{\partial r^2}\left\{\left[\frac{\partial u_{\theta}}{\partial z}\right]-\frac{\partial \omega_{\theta}}{\partial r}\right\} \tag{6}$$

On differentiating by z and using the continuity equation we have:

$$-2v_{\theta zz}+T_{\theta rz} = E\,\frac{\partial^2}{\partial r^2}\left\{\left[\frac{\partial^2 u_{\theta}}{\partial z^2}\right]+\frac{\partial^2 u_{\theta}}{\partial r^2}\right\}$$

where the term within the square trace brackets can now be neglected in comparison with the next term' This equation shows the departure from a thermal wind relationship within the boundary layer,

By continuing to eliminate variables we finally arrive at the sidewall boundary layer equation:

$$E^2\,\frac{\partial^6 v_{\theta}}{\partial r^6}+\sigma S\,\frac{\partial^2 v_{\theta}}{\partial r^2}+4\,\frac{\partial^2 v_{\theta}}{\partial z^2}=0 \tag{7}$$

which is reminiscent of the Ekman layer equation:

$$E^2\,\frac{\partial^6 w}{\partial z^6}+4\,\frac{\partial^2 w}{\partial z^2}+\sigma S\,\nabla_H^2\,w=0$$

The relative importance of the individual terms in this equation depends on the thickness of the boundary Payer, A balance between the first and second gives a thickness δ_1 of

$$\delta_1 = \frac{E^{1/2}}{(\sigma S)^{1/4}} = \frac{(\nu X)^{1/4}}{(g\alpha\Delta T_V L^2)^{1/4}}\left(\frac{1}{R_a}\right)^{1/4} \tag{8}$$

and makes the last term of $O(E/(\sigma S)^{3/2})$ relative to the first. Therefore

$$\sigma S > E^{2/3}$$

for this type sf Payer to exist, Balancing the second and third terms produces a Sayer of thickness

$$\delta_2 = \sqrt{\sigma S}. \tag{9}$$

The same condition $\sigma S > E^{2/3}$ is necessary to make the first term of smaller magnitude than she other two, In order to apply boundary layer techniques to this layer we need the additional condition that

$$\sigma S < 1$$

Looking back at the interior equation we lose the $\nabla_H^2 P_I^{(a)}$ term. This vorticity diffusion is then restricted to the boundary layer region.

If σS is $0(1)$ there is only one boundary layer. When $\sigma S < E^{\frac{2}{3}}$ the balance is between the first and third terms which yields an $E^{1/3}$ layer. An $E^{1/4}$ boundary layer also enters in this range of σS but the details will be ignored here, The different possible layers for the range of values of the parameter σS are represented by the chart in Fig.1. We will restrict ourselves to
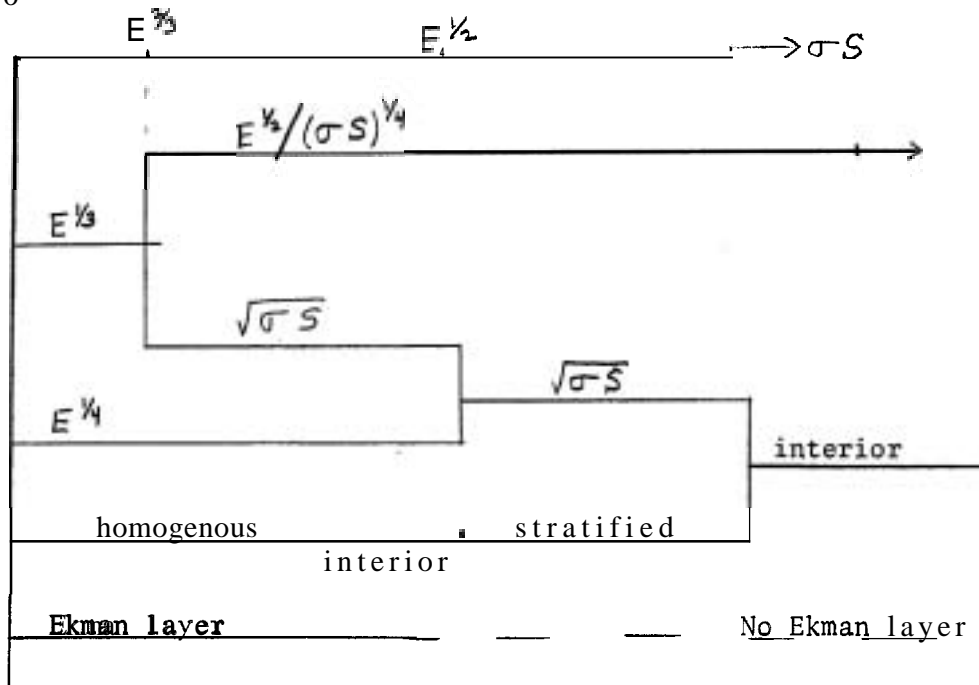
Fig.1. Features of the Dynamics,

In the consideration of the upper and lower boundary conditions on the sidewall layers it is noteworthy that these layers are all thicker than the Ekman layers. As far as the Ekman layers are concerned the sidewall layers look like an interior and we may use the Ekman layer results to determine boundary conditions, Therefore, in terms of the boundary layer variables:

on $z = 0,1$
$$w_B = \pm \frac{E^{\frac{1}{4}}}{2} \frac{\partial \tau_B}{\partial r}$$

$$= \frac{E}{\sigma S} T_{Brr} \quad \text{(from (5))}$$

$$= 2 \frac{E}{\sigma S} V_{Br z} \quad \text{(from (6))}$$

or $\pm \frac{E^{\frac{1}{4}}}{2} V_B = \frac{2E}{\sigma S} V_{Bz}$ on $z = \binom{0}{1}$, (10)

after integrating with respect to r.

We will first look in detail at the thin inner layer of thickness δ_1 - the so-called buoyancy layer.

Buoyancy Layer $(\delta_1 = E^{\frac{1}{2}}/(\sigma S)^{\frac{1}{4}})$

At $r = r_0$ let us define a stretched coordinate

$$\eta = (r_0 - r)/\delta$$ (11)

and allow all B-subscripted variables to be explicit functions of η. If we define the boundary layer temperature scale θ_B by

$$T_B = \theta_B \widetilde{T}_B(\eta)$$ (12)

where θ_B is still to be determined, and $\widehat{T}_B(\eta)$ is $O(1)$, the scales of the other variables can be established from the equations of motion in the buoyancy layer to be:

$$w_B = \frac{\theta_B}{\sqrt{\sigma S}} \hat{\omega}_B$$

$$u_B = \theta_B \frac{E^{\frac{1}{2}}}{(\sigma S)^{\frac{3}{4}}} \hat{u}_B$$

$$v_B = \theta_B \frac{E^{\frac{1}{2}}}{(\sigma S)^{\frac{3}{4}}} \hat{v}_B$$

$$p_B = \theta_B \frac{E}{(\sigma S)^{\frac{3}{4}}} \hat{p}_B .$$

We now have the following set of equations:

$$-2 \hat{v}_B = \hat{p}_{B\eta} + \sigma S \hat{u}_{B\eta\eta}$$ (13)

$$2 \hat{u}_B = \hat{v}_{B\eta\eta}$$ (14)

$$-\hat{T}_B = -\hat{P}_{Bz}\frac{E}{(\sigma S)^{3/2}} + \hat{w}_{B\eta\eta} \tag{15}$$

$$\hat{u}_{B\eta} = \hat{w}_{Bz} \tag{16}$$

$$\hat{w}_B = \hat{T}_{B\eta\eta} \tag{17}$$

As we are considering the case $\sigma S > E^{2/3}$ the pressure term in the vertical momentum balance can be safely neglected to give a balance between buoyancy and diffusion.

Equations (15) and (17) now form an fndependent set from which we can determine $\hat{T}$ and $\hat{w}$. Elimination of $\hat{T}$ yields an Ekman-like equation

$$\hat{w}_{B\eta\eta\eta\eta} + \hat{w}_B = 0 \tag{18}$$

This is a local equation with no z derivatives and therefore the solution will depend only on local quantities and we do not really need the boundary conditions determined from the Ekman layers.

Once $\hat{w}$ is determined, $\hat{T}$ is obtained from (15), $\hat{u}$ from (16), $\hat{v}$ from (14), and finally $\hat{p}$ from (13). The results are:

$$\hat{w}_B = e^{-\eta/\sqrt{2}}\left\{A(z)\cos\eta/\sqrt{2} + B(z)\sin\eta/\sqrt{2}\right\} \tag{19}$$

$$\hat{T}_B = e^{-\eta/\sqrt{2}}\left\{-A\sin\eta/\sqrt{2} + B\cos\eta/\sqrt{2}\right\} \tag{20}$$

$$\hat{u}_B = \left\{\frac{e^{-\eta/\sqrt{2}}}{\sqrt{2}}\right\}\left\{\frac{dA}{dz}\left(\sin\eta/\sqrt{2} - \cos\eta/\sqrt{2}\right) - \frac{dB}{dz}\left(\sin\eta/\sqrt{2} + \cos\eta/\sqrt{2}\right)\right\} \tag{21}$$

Now if we are fixing the temperature on the sidewalls of the annulus, then $\hat{\theta}_B$ must be 0(1) in order that $\hat{T}_B$ can correct the interior temperature field. Because w_I is 0(E/σS) we have the ratio w_I/w_B being $0(E/\sqrt{\sigma}\,S)$ which is much less than unity. This implies that w_B must be zero by itself at the boundary. On the other hand if we specify the insulating eondition:

$$O = \frac{\partial T}{\partial r} = \frac{\partial T_I}{\partial r} - \frac{\theta_B}{E^{\frac{1}{2}}}(\sigma S)^{\frac{1}{4}} \frac{\partial T_B}{\partial \eta} \tag{22}$$

on the boundary we must have θ_B of $O(E^{\frac{1}{2}}/(\sigma S)^{\frac{1}{4}})$. The ratio w_I/w_B is now $O(E^{\frac{1}{2}}/(\sigma S)^{\frac{1}{4}})$ which is still much less than unity provided that $\sigma S \gg E^{\frac{1}{2}}$. We again conclude that

$$w_B = 0 \text{ at } \eta = 0. \tag{23}$$

This implies that

$$A \equiv 0.$$

The unknown $B(z)$ can now be determined from the other boundary conditions. Choosing the insulating temperature condition of Eq. (22) we find:

$$B(z) = -\sqrt{2} \left. \frac{\partial T_I^{(\omega)}}{\partial r} \right|_{r=r_o} \tag{24}$$

From (19) we calculate a mass transport of

$$M_B = -\frac{E}{\sigma S} \left. \frac{\partial T_I^{(\omega)}}{\partial r} \right|_{r=r_o}$$

and it seems that the buoyancy layer can compensate for the interior mass flux. From Eq. (21) evaluated at the boundary:

$$\hat{u}_B(\eta = 0) = \frac{1}{\sqrt{2}} \left(-\frac{dB}{dz} \right) \frac{E}{\sigma S}$$

As the total radial velocity must vanish at $r = r_o$ we find:

$$u_I^{(\omega)}(r=r_o) = \frac{1}{\sigma S} \frac{dB}{dz} \tag{25}$$

a condition on the interior field similar, in form, to the one derived for the Ekman layer. The last condition is that the total swirl velocity go to zero at the boundary. In the interior v_I is $O(1)$ while in the buoyancy layer v_B is $O(E/(\sigma S)^{\frac{1}{4}})$ which is much less than unity. Therefore v_I must vanish by itself at the sidewall and from the thermal wind equation at $r = r_o$:

$$T_{Ir} = 2 \frac{\partial v_I}{\partial z} = 0$$

and therefore:

$$B(z) \equiv 0. \tag{26}$$

To this order there is no **buoyancy** layer!

From Eq. (25) we have the condition on the interior field that:

$$\text{at } r = r_0, \; r_i \qquad 0 = 2u_I^{(0)} = \nabla^2 v_I - v_I^2/r^2$$

$$= \frac{\partial}{\partial r} \int_I^{(0)} \tag{27}$$

where $\int_I^{(0)} = \frac{1}{2} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial p_I^{(0)}}{\partial r}$. In addition the swirl velocity vanishes so that

$$2v_I^{(0)} = \frac{\partial p_I^{(0)}}{\partial r} = 0. \tag{28}$$

On the upper and lower surface we have the following results from the Ekman layer analysis,

On $z = 0$
$$\nabla^2 \frac{\partial p_I^{(0)}}{\partial z} = \frac{\sigma S}{4 E^{1/2}} \nabla_H^2 p_I^{(0)}, \tag{29}$$

$z = 1$
$$\nabla^2 \frac{\partial p_I^{(0)}}{\partial z} = -\frac{\sigma S}{4 E^{1/2}} \left[\nabla_H^2 p_I^{(0)} - \int_T \right], \tag{30}$$

as well as the temperature conditions:

$$m_0 T_I^{(0)} + \ell_0 \frac{\partial T_I^{(0)}}{\partial z} = 0 \quad on \; z = \binom{1}{0} \tag{31}$$

where m_0 and ℓ_0 are specified functions of **r**.

The previous considerations of the Ekman regions also allow us to calculate the interior swirl velocity at the top and bottom. For $\sigma S \gg E^{1/2}$ we have from (29) and (30) using the **fact** that $2v_I^{(0)} = \frac{\partial p_I^{(0)}}{\partial r}$;

On $z = 0$:
$$\frac{1}{r} \frac{\partial r v_I^{(0)}}{\partial r} = 0$$

and on $z = 1$:
$$\frac{1}{r} \frac{\partial}{\partial r} r v_I^{(0)} = \frac{\partial}{\partial r} \int_T .$$

These may be solved **easily** to yield the results:

$$v_I^{(0)}(r, 0) = \frac{A_0}{r} \tag{32}$$

and

$$v_I^{(0)}(r, 1) = v_T + \frac{A_1}{r} \tag{33}$$

where A_0 and A_1 are constants to be determined.

Equations (32) and (33) determine the relative vorticity between the interior and the top and bottom at the boundary, Therefore we can calculate the Ekman layer quantities, In particular the radial mass flux in each layer can be found. In the top layer we find a flux of $E^{\frac{1}{2}}A_1/2$ inward and in the bottom $E^{\frac{1}{2}}A_0/2$ inward. Because of the strong stratification constraint the Ekman layers are non-divergent and ehese fluxes must balance each other, This implies that:

$$A_0 = -A_1$$

However, this flux can only enter the sidewall which we have discovered can not accept such a large flux, Therefore:

$$A_0 = A_1 = 0 \tag{34}$$

and we have the surprising result that as well as no buoyancy layer there is also no Ekman layer!

The problem for the interior is now fully posed, but for arbitrary σS is rather difficult to solve. We will look at the cases of σS small or large but still larger than the other small parameters,

For σS less than $0(1)$ the equation (3.15) (from Lecture #3, Eq.(3.15)) for the interior reduces to

$$\nabla^2 \frac{\partial^2 p_I^{(o)}}{\partial z^2} = 0$$

Integrating once with respect to z we have:

$$\nabla^2 \frac{\partial p_{(I)}^{(o)}}{\partial z} = F(r) = w_I^{(o)}(r)$$

The interior vertical velocity $w_I^{(o)}$ is only a function of r as a result of the constraint on σS.

From the hydrostatic balance in the vertical:

$$\nabla^2 T_I^{(o)} = w_I^{(o)}(r) \tag{35}$$

The Ekman conditions at top and bottom are:

$$W_I^{(\omega)}(z=0) = \frac{\sigma S}{2E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r v_I^{(\omega)}(r, 0)$$

$$W_I^{(\omega)}(z=1) = \frac{\sigma S}{2E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r \left[V_T - v_I^{(\omega)}(r, 1) \right]$$

which when added together give:

$$W_I^{(\omega)} = \frac{\sigma S}{4E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r \left[V_T - \left\{ v_I^{(\omega)}(r, 1) - v_I^{(\omega)}(r, 0) \right\} \right] \qquad (36)$$

Integration of the thermal wind equation over the fluid depth yields:

$$v_I^{(\omega)}(r, 1) - v_I^{(\omega)}(r, 0) = \frac{1}{2} \int_0^1 \frac{\partial T_I^{(\omega)}}{\partial r} dz . \qquad (37)$$

Substitution of (37) into (36) and the result into (35) produces the

"interior-interior" equation.

$$\nabla^2 T_I^{(\omega)} + \frac{\sigma S}{8E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \int_0^1 T_I^{(\omega)} dz = \frac{\sigma S}{4E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} r V_T \qquad (38)$$

Notes submitted by

Nelson G. Hogg

AXIALLY SYMMETRIC CIRCULATION IN AN ANNULUS

Lecture #5   Joseph Pedlosky

In the interior our vorticity balance is of the form

$$\nabla^2 \left[\nabla_H^2 p_I^{(\omega)} + \frac{4}{\sigma S} \frac{\partial^2 p_I^{(\omega)}}{\partial z^2} \right] = 0 \qquad (1)$$

For $\sigma S \ll 1$ we saw in the previous lectures that there is a boundary layer of $O(\sqrt{\sigma S})$ and that the remaining "interior-interior" equation is

$$\nabla^2 T_I^{(\omega)} + \frac{\sigma S}{8E^{1/2}} \nabla^2 \int_0^1 T_I^{(\omega)} dz = \frac{\sigma}{4E^{1/2}} \frac{1}{r} \frac{\partial}{\partial r} U_T \qquad (2)$$

We must now use the $\sqrt{\sigma S}$ sidewall layer to set the swirl velocity to zero.

To solve the boundary layer equation we will define a stretched variable $\eta = (r_0 - r)/\sqrt{\sigma S}$ and we will denote by bars the additive boundary

layer corrections.   To first order

$$v = v_I^{(a)} + \cdots$$
$$+ \bar{v}(\eta) \tag{3a}$$

$v_I^{(a)}$ and $\bar{v}$ are of the same order since the existence of the boundary layer is to balance $v_I^{(a)}$ at the walls.   The remaining variables will be scaled relative to $\bar{v}$, thus

$$p = p_I^{(a)} + \cdots$$
$$+ \sqrt{\sigma S} \, \bar{p}(\eta) \tag{3b}$$

$$u = E u_I^{(a)} + \cdots$$
$$+ \frac{E}{\sigma S} \bar{u} \tag{3c}$$

$$w = \frac{E}{\sigma S} w_I^{(a)} + \cdots$$
$$+ \frac{E}{(\sigma S)^{3/2}} \bar{w} \tag{3d}$$

$$T = T_I^{(a)} + \cdots$$
$$+ \sqrt{\sigma S} \, \bar{T} \tag{3e}$$

The equations of motion in the $\sqrt{\sigma S}$ layer are then

$$-2\bar{v} = \bar{p}_\eta \tag{4a}$$

$$2\bar{u} = \bar{v}_{\eta\eta} \tag{4b}$$

$$-\bar{T} = -\bar{p}_z + \left(\frac{E^{1/2}}{\sigma S}\right)^3 \bar{w}_{\eta\eta} \tag{4c}$$

$$\bar{w} = \bar{T}_{\eta\eta} \tag{4d}$$

$$\bar{u}_\eta = \bar{w}_z \tag{4e}$$

We can neglect the last term in Eq. (4c) since $\sigma S > E^{1/2}$ and we have already studied the case in which the buoyancy was balanced by the viscous stresses,   Again the boundary conditions at $z = 0,1$ are from the Ekman suction which yield the following relations:

$$\text{at } z = 0 \quad \bar{w} = \bar{T}_{\eta\eta} = -\frac{\sigma S}{E^{1/2}} \bar{u}_\eta = -2\bar{v}_\eta \tag{5a}$$

$$\text{at } z = 1 \quad \bar{w} = \bar{T}_{\eta\eta} = \frac{\sigma S}{E^{1/2}} \bar{v}_\eta = -2 v_\eta \tag{5b}$$

The last equality above is due to the thermal wind $-2\,\bar{V}_{\hat{z}} = -\overline{}$ Integrating the above two equations in η yields the following condition on the swirl velocity:

$$\bar{V}_{\hat{z}} = \pm \frac{\sigma S}{4 E^{\frac{1}{2}}} \bar{V} \quad at \quad z = \binom{0}{1} \tag{6}$$

Elimination of $\bar{T}$, $\bar{P}$, and $\bar{w}$ from the equations of motion yields:

$$\bar{V}_{\eta\eta} + 4\bar{V}_{\hat{z}\hat{z}} = 0 \tag{7}$$

Consider a solution of the form:

$$\bar{V} = \sum_{K} A_{K} e^{-\gamma_{K}\eta} \phi_{K}(z) \tag{8}$$

Plugging this back into Eq. (7) yields:

$$\frac{d^{2}\phi_{K}}{dz^{2}} + \frac{\gamma_{K}^{2}}{4}\phi_{K} = 0 \tag{9}$$

The boundary conditions (6) reduce to

$$\frac{d\phi_{K}}{dz} = \pm \frac{\sigma S}{4 E^{\frac{1}{2}}}\phi_{K} \quad at \quad z = \binom{0}{1} \tag{10}$$

Note that if we multiply Eq. (9) by ϕ_{K} and integrate in z , we get the following result:

$$-\int_{0}^{1}\left(\frac{d\phi_{K}}{dz}\right)^{2}dz + \frac{\gamma_{K}^{2}}{4}\int_{0}^{1}\phi_{K}^{2}\,dz - \left[\phi_{K}^{2}(1) + \phi_{K}^{2}(0)\right] = 0$$

Each of the terms is positive definite and this implies that γ_{K} is real and that there are no oscillations, i.e. that all solutions have the proper boundary layer character,

The solution to Eq. (9) is

$$\phi_{K} = \alpha_{K}\sin\frac{\gamma_{K}}{2}z + \beta_{K}\cos\frac{\gamma_{K}}{2}z \tag{11}$$

The boundary conditions (10) yield the following conditions:

$$\alpha_{K}\frac{\gamma_{K}}{2} = \frac{\sigma S}{4 E^{\frac{1}{2}}}\beta_{K} \tag{12a}$$

$$\frac{\gamma_{K}}{2}\left[\alpha_{K}\cos\frac{\gamma_{K}}{2} - \beta_{K}\sin\frac{\gamma_{K}}{2}\right] = \frac{-\sigma S}{4 E^{\frac{1}{2}}}\left[\alpha_{K}\sin\frac{\gamma_{K}}{2} + \beta_{K}\cos\frac{\gamma_{K}}{2}\right] \tag{12b}$$

which lead to

$$\left[\alpha_K \cos \frac{\gamma_K}{2}\right] \frac{\alpha S}{4E^{\frac{1}{2}}} = \left[\frac{\gamma_K^2}{4} - \left(\frac{\sigma S}{4E^{\frac{1}{2}}}\right)^2\right] \sin \frac{\gamma_K}{2} \tag{12c}$$

In the limit of high stratification, $\sigma S \gg E^{\frac{1}{2}}$, $\beta_K = 0$ and $\gamma_K = 2\pi n$,

n = 0,1,2. This means that $\overline{U}$ goes to zero at z = 0,1 and therefore the

Ekman layers made no contribution, which is just what we expect in the high

stratification limit. In the opposite limit $\sigma S \ll E^{-\frac{1}{2}}$, we get two types

of solutions. The first is a cosine series: $\alpha_K = 0$ and $\gamma_K = 2\eta\pi$, $\eta = 1,2,$

. . . The second is a solution for small γ_K which yields $\gamma_K \sim \frac{1}{\sqrt{2}} \frac{\sqrt{\sigma S}}{E^{\frac{1}{4}}}$.

If we substitute this expression for γ_K we really have then an $E^{\frac{1}{4}}$ layer

which is just the Stewartson layer.

We now have the swirl velocity in the $\sqrt{\sigma S}$ layer. From the thermal

wind relationship we can calculate the temperature field:

$$\overline{T} = 2 \sum_K A_K \frac{\phi_K'(z)}{\gamma_K} e^{-\gamma_K \eta} \tag{13}$$

Let us now define a streamfunction $\psi_I^{(0)}$ such that

$$\omega_I^{(0)} = \frac{1}{r} \frac{\partial \psi_I^{(0)}}{\partial r} \tag{14}$$

Recalling that $\omega_I^{(0)}$ is independent of z and reintroducing the scaling we

have

$$\psi_I = \frac{E}{\sigma S} \psi_I^{(0)}(r) \tag{15}$$

In the boundary layer

$$\overline{\omega} = -\frac{1}{r_0} \frac{\partial \overline{\psi}_0}{\partial \eta} \tag{16}$$

$$\overline{\psi} = \frac{E}{\sigma S} \overline{\psi}_0 (\eta) \tag{17}$$

From the heat equations (4d) and (16) we have

$$\overline{\psi}_0 = -r_0 \overline{T}_\eta \tag{18}$$

The boundary condition on $\bar{v}$ at the sidewalls is that the total velocity go to zero, i.e.,

$$v_{I}\left(r_{o},z\right)+\bar{v}\left(r_{o},z\right)=0$$

$$\text{or}\quad v_{I}\left(r_{o},z\right)+\sum_{k}A_{k}\,\phi_{k}=0$$

(19)

and similarly at $r=r_{i}$.

An additional boundary condition at the wall is that there be no radial flow, i.e. that the wall be a streamline, This implies that

$$\Psi_{I}^{'}\left(r_{o}\right)+2\,r_{o}\sum_{k}A_{k}\,\phi_{k}^{'}\left(z\right)=0$$

(20)

Equations (19) and (20) combine to give the boundary condition on the interior swirl velocity, namely

$$2\,\frac{\partial v_{I}^{(o)}}{\partial z}=\frac{1}{r_{o}}\,\Psi_{I}^{(o)}\left(r_{o}\right)$$

(21)

Let us now convert this relationship into a condition on the interior temperature, The transport down in the boundary layer is given by

$$\frac{E}{\sigma s}\int_{o}^{1}\bar{w}\,d\eta=T_{\eta}\,\frac{E}{\sigma s}=-2\,\bar{v}_{\theta}\left(\frac{E}{\sigma s}\right)=2\,\frac{\partial v_{I}^{(o)}}{\partial z}\left(\frac{E}{\sigma s}\right)$$

(22)

Note that since $v_{I\,z}^{(o)}$ is independent of z , the integral of $\int\bar{w}\,d\eta$ is independent of z although $\bar{w}$ itself is not. The mass transport into the $\sqrt{\sigma s}$ from the top Ekman layer is

$$M_{1}=\frac{E^{1/2}}{2}\left[v_{T}-v_{T}\left(r_{o},1\right)\right]$$

(23)

while the transport into the bottom Ekman layer is

$$M_{2}=\frac{E^{1/2}}{2}\,v_{T}\left(r,0\right)$$

(24)
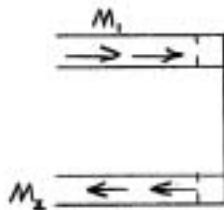
Both the transports M_1 and M_2 must equal the transport in the $\sqrt{\sigma S}$ layer, manely $\frac{2E}{\sqrt{\sigma S}} \times U_{I\mathbf{z}}^{(o)}$. This is due to the fact that the interior radial velocity is $O(E)$. Since the transport in the $\sqrt{\sigma S}$ layer is independent of z, we can add the two equalities to yield

$$4 v_{I\theta}^{(o)}(r_o) = \frac{\sigma S}{2E^{1/2}}\left[v_T - \left(v_I(r_o,1) - v_I^{(o)}(r_o,0)\right)\right]$$

or

$$\frac{2\partial T_I^{(o)}}{\partial r}(r_o) = \frac{\sigma S}{2E}\left[v_T - \frac{1}{2}\int_0^1 \frac{\partial T_I^{(o)}}{\partial r}d\mathbf{z}\right] \tag{25}$$

This is the boundary condition on the internal temperature, For the case of insulated walls it is actually unnecessary to find the boundary layer solution, We can easily find the transport in the $\sqrt{\sigma S}$ layer from the heat equation:

$$\int d\mathbf{z}\, \bar{w}_{B.L.} = \int \frac{E}{\sigma S} T_{B.L.rr}\, d\mathbf{z} = \frac{E}{\sigma S}T_{B.L.r} = -\frac{E}{\sigma S}T_{I_r}(r_o) \tag{26}$$

where the subscript B.L. denotes the unscaled boundary layer variable. We then equate this transport to the Ekman transports and proceed as above,

To solve the interior problem let us make a change of variable,

Let

$$\theta = T_I^{(o)} + \frac{\sigma S}{8E^{1/2}}\int_0^1 T_I^{(o)}d\mathbf{z} \tag{27}$$

Our differential equation (2) becomes

$$\nabla^2\theta = \frac{\sigma S}{8E^{1/2}}\frac{1}{r}\frac{\partial}{\partial r}r v_T \tag{28}$$

with

$$\frac{\partial \theta}{\partial r} = \frac{\sigma S}{4E^{1/2}}v_T \quad \text{on } r = r_o \tag{29}$$

To find $T_I^{(o)}$ from θ consider the following

$$\int_0^1 \theta\, d\mathbf{z} = \left(1 + \frac{\sigma S}{8E^{1/2}}\right)\int_0^1 T_I^{(o)}d\mathbf{z}$$

and thus

$$T_I^{(o)} = \theta - \frac{\frac{\sigma S}{8E^{1/2}}}{\left(1 + \frac{\sigma S}{8E^{1/2}}\right)}\int_0^1 \theta\, d\mathbf{z}$$

The solution to Eq. (28) is

$$\theta = \frac{\sigma S}{4E^{1/2}} \int_0^r \mathcal{V}_T(r')\, dr' + \bar{\Phi}(r,z)$$

where $\nabla^2 \bar{\Phi}(r,z) = 0$ and $\frac{\partial \bar{\Phi}}{\partial r} = 0$ on r_o, r_i

The boundary conditions on temperature are

$$m_o T_z^{(\omega)} + \ell_o \frac{\partial T_z^{(\omega)}}{\partial r} = 0 \quad \text{on} \quad z = \binom{0}{1} \tag{30}$$

These conditions determine $\bar{\Phi}$:

$$\bar{\Phi} = \sum_m \left[\frac{J_o(k_m r/r_o)}{J_1(k_m r_i/r_o)} - \frac{Y_o(k_m r/r_o)}{Y_1(k_m r_i/r_o)} \right] \times \left[a_n \sinh \frac{k_m}{r_o}\left(z - \tfrac{1}{2}\right) + b_n \cosh \frac{k_m}{r_o}\left(z - \tfrac{1}{2}\right) \right] \tag{31}$$

J_1, J_0, Y_1, Y_0 are Bessel functions, the k_m are determined from the re-
quirement that $\frac{\partial \bar{\Phi}}{\partial r} = 0$ on r_o and r_i and the a_n and b_n are determined from
Eq. (30). Let us call the first bracket in (31) $\emptyset_n(r)$.

Now that we have $T_z^{(\omega)}$ we can write $\mathcal{V}_z^{(\omega)}$ from the thermal wind rela-
tionship. Letting $\lambda = \sigma S/E^{1/2}$ we have

$$\mathcal{V}_z^{(\omega)} = \frac{\mathcal{V}_T(r)}{2} + \frac{\lambda}{8(1 + \lambda/8)} \mathcal{V}_T(r)\left(z - \tfrac{1}{2}\right) +$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} \frac{\partial \emptyset_n}{\partial r} \left[a_n \left\{ \cosh \frac{k_n}{r_o}\left(z - \tfrac{1}{2}\right) - \cosh \frac{k_n}{2 r_o} \right\} \frac{r_o}{k_n} + \right.$$

$$\left. + b_n \left\{ \sinh \frac{k_n}{r_o}\left(z - \tfrac{1}{2}\right) - \frac{\lambda(z - \tfrac{1}{2})}{4(1 + \lambda/8)} \sinh \frac{k_n}{2r_o} \right\} \right] \tag{32}$$

In the ease of no vertical heat flux, due to the temperature produced by
the mechanical driven flow, i.e. $m_0 = m_1 = 0$ all of the a_n and b_n are zero.
As we increase the stratification, λ , the swirl velocity goes from a con-
stant value, $\mathcal{V}_T/2$, coupled to Ekman suction, to a linear function of z in
which the Ekman suction is choked off. In the more general case where a_n
and b_n are not zero we get the same effect on the Ekman layer in the limit
of large λ , i.e. $\mathcal{V}(r,1) = \mathcal{V}_T(r)$ and $\mathcal{V}(r,0) = 0$.

The radial transport, in the upper Ekman layer is
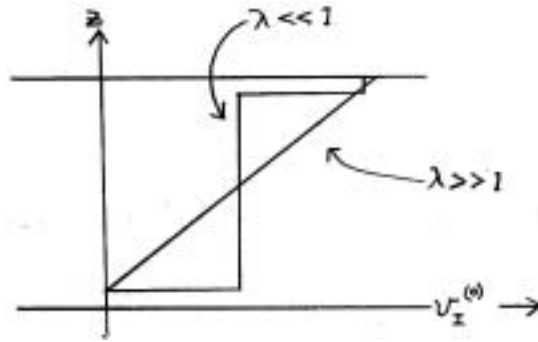
$$M_E = \frac{E^{1/2}}{4}\left[\frac{1}{1 + \frac{\sigma S}{8 E^{1/2}}}\right] V_T(r)$$

and therefore M_E is $O(E^{1/2})$ for $\sigma S < O(E^{1/2})$ but only $O(E/\sigma S)$ for $\sigma S > E^{1/2}$. Therefore the mass transport in the $\sqrt{\sigma S}$ sidewall layer is independent of the inner radius r_i just as in the homogeneous case.

The circulation has the following structure
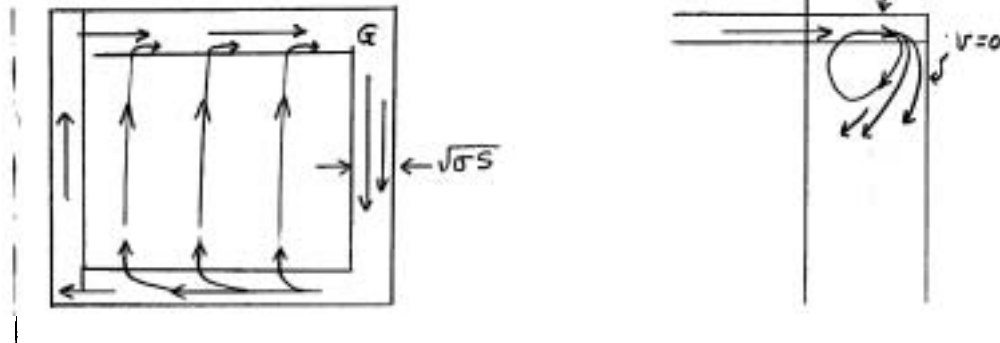
Due to the discontinuity in the boundary condition there is a recirculating eddy in the upper corner region.

Finally let us look at the total dimensional temperature field T^*. For the case of a rotating top with angular velocity $\epsilon \Omega$ we have
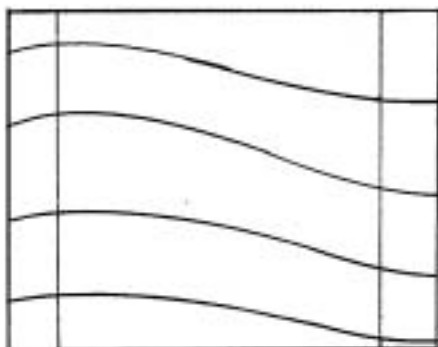
$$T^* = T_0 + \Delta T v \frac{z^*}{L} + \frac{\epsilon \Omega}{\alpha g L}(r^{*2} - r_i^2)$$

when $T_z = 0$ at $z = 0,1$ and the temperature contours have the following appearance.

Notes submitted by
Joel E. Hirsh

LINEAR STABILITY OF THE SYMMETRIC FLOW (INVISCID THEORY) I,

Lecture #6.   Joseph Pedlosky

Some additional remarks concerning the symmetric flow.

   1. In the case where $\sigma S \gg 1$, in the **interior** u = 0(E) and

w = $0(E/\sigma S)$ and therefore w $\ll$ u. Physically, this means that the con-

straint of stratification is much stronger than the rotational constraint,

To lowest order the continuity equation becomes

$$\frac{1}{r} \frac{\partial}{\partial r} r u_I = 0,$$

and the boundary conditions

$$u_I = 0 \quad \text{on} \quad v = r_o, \; r_i$$

imply that u_I = 0 in the interior, Then the equation for the swirl velocity

$$\nabla^2 v_I - v_I/r^2 = 0 .$$

If the annulus is shallow this equation shows that the flow is essentially

Couette and the plate velocity is transmitted to the interior of the **fluid**

by viscous stresses,

     The temperature (obtained from the thermal wind equation) will not

satisfy the boundary conditions at the top and the bottom. Thus there is a

thermal boundary layer of thickness $0 \left(1/\sqrt{\sigma S} \right)$ in which the thermal conditions

are satisfied, In these boundary layers the thermal wind balance still ap-

plies and a small correction to the swirl velocity of $0 \left(1/\sqrt{\sigma S} \right)$ gives an

0(1) change in T.

     2. In the situation where σS = 0(1) the existence of boundary

layers depends critically on the side wall thermal conditions. In contrast

to the case discussed above, when T = 0 on the side walls we have both **buoy-**

ancy Payers and Ekman layers carrying a mass flux of $0(E^{\frac{1}{4}})$.

Linear stability of the symmetric flow.

Inviscid theory.

It has been shown that the potential vorticity of the motion is governed by the equation

$$\epsilon \frac{D}{Dt}\left(\zeta + \frac{2}{S}\frac{\partial T}{\partial z}\right) + E \nabla^2 \left(\zeta + \frac{2}{\sigma S}\frac{\partial T}{\partial z}\right) = 0$$

In the annulus the constraint of symmetry restricts the class of motions in such a way that even for $\epsilon > E (\epsilon \ll 1)$, the potential vorticity is (except in any boundary layers) governed by the conduction type equation

$$\nabla^2 \left(\zeta + \frac{2}{\sigma S}\frac{\partial T}{\partial z}\right) = 0.$$

The possibility of non-symmetric motions removes this restriction and an asymmetric perturbation to the symmetric flow may produce an entirely different balance in the interior of the fluid.

In order to study the stability of the symmetric flow we write the symmetric state as

$$\vec{q} = \vec{q}_s (r,z)$$
$$= \left(E u_s(r,z),\ V_s(r,z),\ E/\sigma_s\, w_s(r,z)\right),$$
$$T = T_s(r,z).$$

The perturbation velocities are scaled in the same way as the symmetric flow and it is assumed that Rossby and Ekman numbers of the perturbation are small, i.e. the perturbations are in hydrostatic and geostrophie balance to the lowest order. It is further assumed that the time scale of the perturbations is of the same order as the advective time scale of the flow, We write

$$\vec{q}' = \left(u'(r,\theta,z,t),\ v'(r,\theta,z,t),\ \epsilon w'(r,\theta,z,t)\right).$$

(For $\epsilon \gtrsim E^{\kappa}$, accelerations of the $0(1)$ horizontal velocities give departures

from geostrophy sf $0(\epsilon)$. This implies that the perturbation vertical velocity is $0(\epsilon)$ in the interior.)

Up to this point no restrictions have been placed on the size of the perturbation velocities relative to the symmetric flow velocities, However, the restriction $\epsilon \ll 1$ implies that the azimuehal length scale of the perturbations is $0(1)$, The case $\epsilon \ll E^{\lambda}$ will not be eonsidered here as all perturbations are found to decay in that case.

We write the total velocity and temperature fields as

$$\vec{q} = \bar{q}_s(r,z) + \vec{q}'(r,\theta,z,t),$$
$$= \hat{r}\left[Eu_s + u'(r,\theta,z,t)\right] + \hat{\theta}\left[v_s + v'\right] + \hat{z}\left[E/\sigma_s w + \epsilon w'\right],$$
$$T = T_s + T'$$

Denoting by ζ the vertical component of the total vorticity, we have that the equation of motion for the total field in the interior is

$$\frac{D}{Dt}\left(\zeta + \frac{2}{\sigma s}\frac{\partial T}{\partial z}\right) = 0$$

Here

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}rv_s + \frac{1}{r}\frac{\partial}{\partial r}rv' - \frac{1}{r}\frac{\partial u'}{\partial \theta},$$

and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(Eu_s + u'\right)\frac{\partial}{\partial r} + \left(\frac{\quad}{r}\right)\frac{\partial}{\partial \theta} + \left(\frac{E}{\sigma s}w_s + \epsilon w'\right)\frac{\partial}{\partial z}.$$

Substituting for ζ and T we have

$$\left[\frac{\partial}{\partial t} + \frac{v_s}{r}\frac{\partial}{\partial \theta}\right]\left[\zeta' + \frac{2}{s}\frac{\partial T'}{\partial z}\right] + u'\frac{\partial}{\partial r}\left[\zeta_s + \frac{2}{s}\frac{\partial T_s}{\partial z}\right] + u'\frac{\partial}{\partial r}\left[\zeta' + \frac{2}{s}\frac{\partial T'}{\partial z}\right] +$$
$$+ \frac{v'}{r}\frac{\partial}{\partial \theta}\left[\zeta' + \frac{2}{s}\frac{\partial T'}{\partial z}\right] = 0\left(E, \epsilon, E/\sigma_s\right). \tag{1}$$

In the limit $\epsilon \ll 1$, $E \ll 1$ the right-hand side is effectively zero. If the perturbation were symmetric the above equation would place no eonstraint on it and it would be necessary to go back to the conduction equation.

In the region of parameter space mentioned above

$$u' = -\frac{1}{2\nu}\frac{\partial p'}{\partial \theta} + O(\epsilon),$$

$$v' = \frac{1}{2r}\frac{\partial p'}{\partial r} + O(\epsilon),$$

$$T' = +\frac{\partial p'}{\partial z} + O(\epsilon).$$

Substituting into (1) we have

$$\left[\frac{\partial}{\partial t} + \frac{v_s}{r}\frac{\partial}{\partial \theta}\right]\left[\nabla_H^2 p' + \frac{4}{s}\frac{\partial^2 p'}{\partial z^2}\right] - \frac{1}{r}\frac{\partial p'}{\partial \theta}\left[\zeta_s + \frac{2}{s}\frac{\partial T_s}{\partial z}\right]_r =$$

$$= \frac{1}{2}\left\{\frac{1}{r}\frac{\partial p'}{\partial \theta}\left[\nabla_H^2 p' + \frac{4}{s}\frac{\partial^2 p'}{\partial z^2}\right]_r - \frac{1}{r}\frac{\partial p'}{\partial r}\left[\nabla_H^2 p' + \frac{4}{s}\frac{\partial^2 p'}{\partial z^2}\right]_\theta\right\}.$$

We now derive the boundary conditions on the flow.

Vertical walls

On $r = r_o, r_i$

$$u = 0 \implies u' = 0 \implies \frac{\partial p'}{\partial \theta} = 0.$$

Horizontal walls

The thermal equation is

$$w' = -\frac{1}{s}\left[\frac{\partial T'}{\partial t} + u'\frac{\partial T'}{\partial r} + u'\frac{\partial T_s}{\partial r} + \frac{v'}{r}\frac{\partial T'}{\partial \theta} + \frac{v_s}{r}\frac{\partial T'}{\partial \theta}\right].$$

It is necessary to have Ekman layers top and bottom to satisfy the no-slip condition on the perturbation. The vertical velocity pumped out by such an Ekman layer is

$$= \pm\frac{E^{1/2}}{2}\text{ (vorticity of the perturbation) on } z = \binom{0}{1}.$$

This implies

$$w' = \begin{cases} \dfrac{E^{1/2}}{2\epsilon}\zeta'(r, \theta, 0, t), \\[2mm] -\dfrac{E^{1/2}}{2\epsilon}\zeta'(r, \theta, 1, t). \end{cases}$$

This is a non-linear boundary condition, We see that the effect of viscous dissipation depends on the ratio $E^{1/2}/\epsilon$.

An additional constraint on the flow

The azimuthal equation of motion may be integrated around the annulus to give

$$\epsilon \int_0^{2\pi} \frac{\partial v'}{\partial t}\, d\theta + \epsilon \int_0^{2\pi} u' \frac{\partial}{\partial r}(v_s + v')\, d\theta + \int_0^{2\pi} 2u'\, d\theta = 0.$$

This holds for all r and in particular on the side walls. But $u' = 0$ on the vertical walls (to lowest order) and it seems likely that

$$\int_0^{2\pi} 2u'\, d\theta = 0$$

to $0(\epsilon)$. In other words, we postulate that the side walls boundary layers cannot accept a mean radial flux of $0(\epsilon)$. Then

$$\frac{\partial}{\partial t} \oint v'\, d\theta = 0 \quad \text{on} \quad r = r_o, r_i.$$

Notes submitted by
Paul F. Linden

QUASI-GEOSTROPHIC POTENTIAL VORTICITY EQUATION

Lecture #7   Joseph Pedlosky

The equation governing non-steady, non-symmetric perturbations when $\epsilon > E$ is the conservation of potential vorticity - hereafter referred to as the quasi-geostrsphie potential vorticity equation (Q.G.P.V.E.) because u and v are evaluated geostrophically.

We can introduce a horizontal stream function ψ such that

$$p' = 2\psi$$
$$u' = -1/r\ \partial\psi/\partial\theta$$
$$v' = \partial\psi/\partial r$$
$$T' = 2\ \partial\psi/\partial z$$

with this definition, the Q G P.V.E. becomes

$$\frac{D}{Dt}\left[\nabla_H^2 \psi + \frac{4}{5}\frac{\partial^2 \psi}{\partial z^2}\right] - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\left[\zeta_s + \frac{1}{5}\frac{\partial \zeta_s}{\partial z}\right] = \left[\frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial}{\partial \theta}\right]\left[\nabla_H^2 \psi + \frac{4}{5}\frac{\partial^2 \psi}{\partial z^2}\right]$$

where $\frac{D}{\partial t} = \left[\frac{\partial}{\partial t} + \frac{v_s}{r}\frac{\partial}{\partial \theta}\right]$, the linearized total derivative and $\zeta_s = \pm \frac{\partial}{\partial r} r v_s$, the vorticity of the basic flow. The boundary conditions in terms of ψ are

$$\frac{\partial \psi}{\partial \theta} = 0 \quad \text{on} \quad r = r_i , r = r_o$$

$$\frac{\partial}{\partial t} \oint \frac{\partial \psi}{\partial r} d\theta = 0 \qquad r = r_i , r = r_o$$

and $\frac{D}{Dt}\frac{\partial \psi}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial v_s}{\partial z} + \left[\frac{\partial \psi}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial}{\partial r}\right]\frac{\partial \psi}{\partial z} = \pm \frac{E^{1/2}_s}{4\epsilon}\nabla_H^2 \psi$

$$\text{on} \quad z = \begin{pmatrix}1\\0\end{pmatrix}$$

expanding ψ in a small parameter $a \ll 1$ where a can be regarded as the relative amplitude of the perturbation at t = 0 to the amplitude of the swirl flow. Then $\psi = a\psi_1 + a^2 \psi_2 + a^3 \psi_3 + \ldots$ Inserting this expansion into the equation for ψ and considering those terms proportional to a, we can investigate the stability of the symmetric flow; and the initial growth rate and initial strucutre of the perturbation if unstable.

For ψ_1 :

$$\frac{D}{Dt}\left[\nabla_H^2 \psi_1 + \frac{4}{5}\frac{\partial^2 \psi_1}{\partial z^2}\right] - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial \Pi_s}{\partial r} = 0$$

where Π_s is the potential vorticity of the basic, symmetric state.

Boundary Conditions:

$$\frac{D}{Dt}\frac{\partial \psi_1}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\frac{\partial v_s}{\partial z} = \pm \frac{E^{1/2}_s}{4\epsilon}\nabla_H^2 \psi_1 \qquad \text{at} \quad z = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\frac{\partial \psi_1}{\partial \theta} = 0 \qquad \text{at} \quad r = r_o , r_i$$

Since the coefficients are independent of θ and t we can Pook for a solution of the form

$$\psi_1 = \text{Re}\,\Phi(r, z)e^{i(m\theta - \sigma t)}$$

which amounts to specifying wavelike initial conditions, In reality this is not restricting the problem because when one considers a more general initial

condition, the non-wavelike portion always decays.

Using Ψ_1 above, we obtain an equation for Φ

$$\left(\frac{v_s}{r} - \frac{\sigma}{m}\right)\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r} + \frac{4}{s}\frac{\partial^2\Phi}{\partial z^2} - \frac{m^2}{r^2}\Phi\right] - \frac{1}{r}\Phi\frac{\partial\pi_s}{\partial r} = 0$$

with boundary conditions

$$\Phi = 0 \quad \text{on} \quad r = r_i, r_o$$

$$\left(\frac{v_s}{r} - \frac{\sigma}{m}\right)\frac{\partial\Phi}{\partial z} - \frac{1}{r}\Phi\frac{\partial v_s}{\partial z} = \mp\frac{iE^{1/2}}{m4\epsilon}\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Phi}{\partial r} - m^2\Phi\right] \quad \text{at} \quad z = \binom{1}{0}$$

Considering σ to have a real and an imaginary part

$$\sigma = \sigma_r + i\sigma_i$$

The condition for instability is $\sigma_i > 0$.

Inviscid Linear Problem

If $E^{1/2}/\epsilon \ll 1$ we can set the right-hand side of the boundary conditions at $z = 0$ and $z = 1$ to zero. Before specifying v_s and π_s several general conclusions can be arrived at,

Suppose $\sigma_i \neq 0$

Since the complex conjugate of σ will also be solution, $\sigma_i \neq 0$ insures an unstable mode,

Multiplying the equation for Φ by $\dfrac{r\Phi^*}{\left(\frac{v_s}{r} - \frac{\sigma}{m}\right)}$,

where Φ^* is the complex conjugate of Φ, and integrating

$$-\int_{z=0}^{1}\int_{r_i}^{r_o} r\,dr\,dz\left[\left|\frac{\partial\Phi}{\partial r}\right|^2 + \frac{m^2}{r}|\Phi|^2 + \frac{4}{s}\left|\frac{\partial\Phi}{\partial z}\right|^2\right] + \frac{4}{s}\int_{r_i}^{r_o} r\Phi^*\frac{\partial\Phi}{\partial z}dr\Big|_{z=0}^{z=1} - \int\int_0^r\frac{|\Phi|^2\partial\pi_s/\partial r}{r\left(v_s - \frac{\sigma r}{m}\right)}r\,dr\,dz$$

but $\dfrac{\partial\Phi}{\partial z} = \dfrac{\Phi}{v_s - r\frac{\sigma}{m}}\dfrac{\partial v_s}{\partial z}$ at $z = 0,1$

Substituting

$$-\int_0^1\int_{r_i}^{r_o}\left[r\,dr\,dz\,(\text{real, positive definite})\right] +$$

$$+\frac{4}{s}\int_{r_i}^{r_o}\frac{\frac{\partial v_s}{\partial z}|\Phi|^2 r\,dr}{\left(v_s - \frac{r\sigma}{m}\right)}\Big|_{z=0}^{z=1} - \int\int\frac{|\Phi|^2\partial\pi_s/\partial r}{\left(v_s - \frac{r\sigma}{m}\right)}r\,dr\,dz = 0$$

we can write

$$\frac{1}{v_s - \frac{r\sigma}{m}} \quad \frac{1}{\left|v_9 - r\frac{\sigma}{m}\right|^2} \left(v_s - \frac{r\sigma_*}{m}\right)$$

Then separating real and fmaginary

$$-\iint r\,dr\,dz \; a \qquad \text{positive definite}]+. \quad \blacksquare \; \blacksquare \; +$$

$$+\frac{i\sigma_i}{m}\left[\frac{4}{s}\int_{r_i}^{r_o}\frac{r^2|\phi|^2 dr}{\left|v_9 - \frac{r\sigma}{m}\right|^2}\frac{\partial v_s}{\partial z}\Big|_{z=0}^{z=1} - \iint\frac{|\phi|^2\frac{\partial \pi_s}{\partial r}r\,dr\,dz}{\left|v_s - \frac{r\sigma}{m}\right|^2}\right] = 0$$

Since $\sigma_i \neq 0$, we can conclude:

$$\frac{4}{s}\int_{r_i}^{r_o}\frac{r|\phi|^2 dr}{\left|v_s - \frac{r\sigma}{m}\right|^2}\frac{\partial v_s}{\partial z} - \iint_o^1\frac{dz\,r\,dr|\phi|^2\frac{\partial \pi_s}{\partial r}}{\left|v_s - \frac{r\sigma}{m}\right|^2} = 0$$

1. If T is fixed at $z = 0$ and $z = 1$, then $\frac{\partial T}{\partial r} = 0$ on $z = 0$, $z = 1$.

$\therefore \frac{\partial v_s}{\partial z} = 0$ on $z = 0$ and $z = 1$ from the thermal wind equation. The second integral must vanish, which means that $\frac{\partial \pi_s}{\partial r}$ must equal 0 somehwere in the fluid in order to have instability,

2. We can also consider the case where $\frac{\partial \pi_s}{\partial r} = 0$ everywhere. Then the first integral requires that the radial temperature gradient be of the same sign at $z = 0$ and $z = 1$,

It is possible to prove a second theorem which is a "semicircle theorem" for the eigenfrequencies,

Define $\omega = \sigma/m$.

Then $\quad (v_s - \omega r)\left[\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial \phi}{\partial r} + \frac{4}{s}\frac{\partial^2 \phi}{\partial z^2} - \frac{m^2}{r^2}\phi\right] - \phi\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}rv_s\right] + \frac{4}{s}\frac{\partial^2 v_s}{\partial z^2}\right\} = 0$

but we can write $\quad \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}rv_s = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}(v_s - \omega r) - \frac{(v_s - \omega r)}{r^2}$

and $\quad \frac{\partial^2 v_s}{\partial z^2} = \frac{\partial^2}{\partial z^2}(v_s - \omega r)$

Then for $\sigma_i \neq 0$ write: $\quad \phi = WF(r,z); \quad W = v_s(r,z) - \omega r$

Insert into D.E. $\quad \frac{1}{r}\frac{\partial}{\partial r}\left[W^2 r\frac{\partial F}{\partial r}\right] + \frac{4}{s}\frac{\partial}{\partial z}W^2\frac{\partial F}{\partial z} - (m^2 - 1)\frac{W^2 F}{r^2} = 0$

with boundary conditions

$$F = 0 \quad \text{an} \quad r = r_o, \; r_1$$

$$\frac{\partial F}{\partial z} = 0 \qquad z = 0, 1$$

Multiply D.E. by rF^* and integrate in z and r

$$\int_0^1 dz \int_{r_1}^{r_o} r\, dr\, w^2 \left\{ \left|\frac{\partial F}{\partial r}\right|^2 + \frac{4}{3}\left|\frac{\partial F}{\partial z}\right|^2 + \frac{m^2-1}{r^2}|F|^2 \right\} = 0$$

where the $\left|\frac{\partial F}{\partial r}\right|^2$ and $\left|\frac{\partial F}{\partial z}\right|^2$ terms we obtained by integrating by parts and using the boundary condition ar $z = 0$ and $z = 1$. The quantity inside $\{\}$ is positive definite, call it $Q > 0$. We can write

$$w^2 = \left(v_s - r\,\omega_{real} - ir\,\omega_i\right)^2 = \left(v_s - r\omega_r\right)^2 - r^2\omega_i^2 - i2r\omega_i(v_s - r\omega_r)$$

Then for the imaginary part

$$\omega_i \iint r^2\, dz\, dr\, (v_s - r\omega_r)\, Q = 0$$

or

$$\omega_i \iint r^3\, dz\, dr\, \left(\frac{v_s}{r} - \omega_r\right) Q = 0$$

which means that the real part of the angular velocity of the wave pattern is equal to the angular velocity of the symmetric flow field at some point in the fluid. This result can also be interpreted as showing that the time derivative of the wave field is sf the same order as the advective time scale,

Turning our attention to the real part, we have

$$\iint r\, dr\, dz\, \left[v_s^2 - 2r\omega_r v_s + r^2\omega_r^2 - r^2\omega_i^2\right] Q = 0$$

we just showed that $v_s = r\omega_r$

therefore we find

$$\iint r^3\, dr\, dz\, \left(\frac{v_s}{r}\right) Q = \iint r^3\left(\omega_r^2 + \omega_i^2\right) Q\, dr\, dz$$

v_s/r is always finite in the annulus, therefore we can write $\Omega_b < v_s/r < \Omega_a$

and

$$0 \geq \iint \left(\frac{v_s}{r} - \Omega_a\right)\left(\frac{v_s}{r} - \Omega_b\right) r^3 Q \, dr \, dz = \iint Q r^3 dr \, dz \left[\left(\frac{v_s}{r}\right)^2 - (\Omega_a + \Omega_b)\frac{v_s}{r} + \Omega_a \Omega_b\right] =$$

$$= \iint Q r^3 dr \, dz \left[(\omega_r^2 + \omega_i^2) - \omega_r(\Omega_a + \Omega_b) + \Omega_a \Omega_b\right] =$$

$$= \iint Q r^3 dr \, dz \left[\left\{\omega_r - \frac{(\Omega_a + \Omega_b)}{2}\right\}^2 + \omega_i^2 - \left(\frac{\Omega_a - \Omega_b}{2}\right)^2\right]$$

which leads to a condition

$$\frac{\Omega_a - \Omega_b}{2} \geq \left\{\omega_r - \frac{\Omega_a + \Omega_b}{2}\right\}^2 + \omega_i^2$$

This condition can be represented graphically

for $\omega_i > 0$
ω must lie inside semicircle
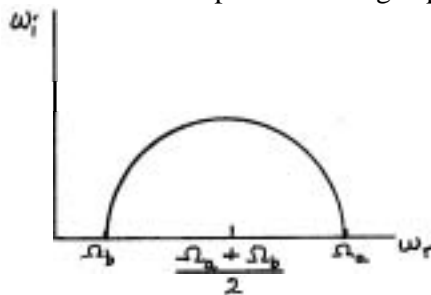
This result shows that the energy available to the perturbation depends on the velocity gradients.

Also shows that the advective time scale of the basic state and the time derivative of the wave are of the same order.

We have found several necessary conditions for instabiliry. We now proceed to solve the problem in order to find sufficient conditions, The problem we will solve is the one in which we have insulating boundary eon- ditions at z = 0 and z = 1 and the mechanical driving is produced by a solidly rotating top

$$\frac{\partial T}{\partial z} = 0 \quad \text{on} \quad z = 0, \ z = 1$$

with $\sigma S \leq 1$

$$V_s = \frac{v_T(r)}{2} + \frac{\sigma S / 8 E^{1/2}}{1 + \sigma S / 8 E^{1/2}} v_T(r) \left[z - \frac{1}{2}\right]$$

where $v_T(r) = r$.

We define δ , a measure of the shear

$$\delta = \frac{\sigma S/_8 E^{1/2}}{1 + \sigma S/_8 E^{1/2}}$$

Also when $u_T(r) = r$, $\frac{\partial \pi_s}{\partial r} = 0$

For this specific case, the equation for conservation of potential vorticity becomes

$$\left[\tfrac{1}{2} + \delta(z - \tfrac{1}{2}) - \omega\right]\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} + \frac{4}{s}\frac{\partial^2 \Phi}{\partial z^2} - \frac{m^2}{r^2}\Phi\right] = 0$$

with boundary conditions $\Phi = 0$ $r = r_o, r_i$

and

$$\left[\tfrac{1}{2} + \delta(z - \tfrac{1}{2}) - \omega\right]\frac{\partial \Phi}{\partial z} - \delta \Phi = 0 \quad \text{on } z = 0 \text{ and } 1 \quad \text{when } \epsilon \gg E^{1/2}$$

Choose

$$\Phi = \phi_{mn}(r) D(z)$$

then

$$\phi_{mn} = \frac{J_m(K_{mn} r/r_o)}{J_m(K_{mn})} - \frac{Y_m(K_{mn} r/r_o)}{Y_m(K_{mn})}$$

with the condition

$$0 = J_m(K_{mn}\alpha) Y_m(K_{mn}) - J_m(K_{mn}) Y_m(K_{mn}\alpha)$$

where $\alpha \equiv r_i/r_o$, J_m — Bessel function of first kind

Y_m — Neumann function.

$D(z)$ must satisfy

$$\frac{d^2 D}{dz^2} - K_{mn}^2 \frac{s}{4} D = 0$$

then $D(z) = A \sinh K_{mn} \frac{s^{1/2}}{2}(z - \tfrac{1}{2}) + B \cosh K_{mn} \frac{s^{1/2}}{2}(z - \tfrac{1}{2})$

with boundary conditions

$$\left[\tfrac{1}{2} + \tfrac{\delta}{2} - \omega\right]\left[\frac{K_{mn} s^{1/2}}{2}\left\{A \cosh \frac{K_{mn} s^{1/2}}{4} + B \sinh \frac{K_{mn} s^{1/2}}{4}\right\}\right] =$$
$$= \delta\left[A \sinh \frac{K_{mn} s^{1/2}}{4} + B \cosh \frac{K_{mn} s^{1/2}}{4}\right] \text{ at } z = 1$$

and $\left[\tfrac{1}{2} - \tfrac{\delta}{2} - \omega\right]\left[\frac{K_{mn}}{2} s^{1/2}\left\{A \cosh \frac{K_{mn} s^{1/2}}{4} - B \sinh \frac{K_{mn} s^{1/2}}{4}\right\}\right] =$
$$= \delta\left[-A \sinh \frac{K_{mn} s^{1/2}}{4} + B \cosh \frac{K_{mn} s^{1/2}}{4}\right] \text{ at } z = 0.$$

In order to satisfy the boundary conditions

$$\omega = \frac{1}{2} \pm \frac{2\delta}{K_{mn}S^{1/2}} \left[\left\{ \frac{K_{mn}S^{1/2}}{4} - \tanh \frac{K_{mn}S^{1/2}}{4} \right\} \left\{ \frac{K_{mn}S^{1/2}}{4} - \coth \frac{K_{mn}S^{1/2}}{4} \right\} \right]^{1/2}$$

Since $\dfrac{K_{mn}S^{1/2}}{4} \geq \tanh \dfrac{K_{mn}S^{1/2}}{4}$ from $x \geq \tanh x$

the criteria for instability (ω complex) is

$$\frac{K_{mn}S^{1/2}}{4} < \coth \frac{K_{mn}S^{1/2}}{4}$$

We note that stability depends on the parameter S not Δ while the rate of growth does depend on Δ. Neutral behavior occurs when

$$\frac{S^{1/2}}{4} = \coth \frac{K_{mn}S^{1/2}}{4}$$

the intersection being $\dfrac{K_{mn}S^{1/2}}{4} = 1.199$.

The stability criteria can be illustrated by considering $r_i/r_o = 0.8$

Then $K_{11} \approx 10$ which results in an unstable mode when $S < \frac{1}{6}$. Similarly, if $r_i/r_o = 0.4$ instability occurs for $S < 1.2$

One last comment can be made. $\omega = \frac{1}{2}$ at the marginal point, which follows from the dispersion relationship. From the boundary conditions at $z = 0$ and $z = 1$ with $\omega = \frac{1}{2}$ when $\dfrac{K_{mn}S^{1/2}}{4} = \coth \dfrac{K_{mn}S^{1/2}}{4}$ we find $A = 0$

Therefore $\Psi_1 = Re \phi_{mn} \cosh \frac{K_{mn}S^{1/2}}{2} (z - \frac{1}{2}) e^{i(m\theta - t/2)}$. With Ψ_1 in this form, the non-linear terms in the quasi-geostrophic potential vorticity equation vanish and thus Ψ_1 is a solution of the complete equation without regard to initial amplitude.

Notes submitted by

Frank M. Richter

LINEAR STABILITY OF THE SYMMETRIC FLOW (VISCOUS THEORY)

Lecture #8   Joseph Pedlosky

To repeat last lecture's results recall we assumed that $E^{\frac{1}{2}}/\epsilon \ll 1$, in which case the stability of the flow depended entirely on $S = \dfrac{g\alpha(\Delta T_v)}{\Omega^2 L}$. Stability results if S exceeds a certain value, depending on the aspect ratio of the tank. The condition for instability was

$$\frac{k_{mn} S^{\frac{1}{2}}}{4} \le 1.2$$

$$\text{or} \quad S^{\frac{1}{2}} \le \frac{4 \times 1.2}{k_{mn}}$$

And the lowest mode $k_{\parallel}$ will give the least restrictive condition on S:

$$S^{\frac{1}{2}}_{crit} \le \frac{1.2 \times 4}{k_{\parallel}} \quad \text{(unstable)}$$

Now interpret this in terms of the Rossby radius of deformation

$$R \equiv \sqrt{\frac{g\alpha(\Delta T_v)L}{\Omega^2}} \quad .$$

Then $S = R^2/L^2$, and $S_c \le$ (number) R^2/L^2 will give instability, i.e. the length scale L must exceed the Rossby radius of deformation for instability. This agrees with yesterday's result that the flow may be stabilized by making the annulus narrower.

Note also that the stability did not depend on the magnitude of the shear, $\delta = \dfrac{\sigma^{S}/_{8} E^{1/2}}{1 + \sigma^{S}/_{8} E^{1/2}}$ The mean flow was $\dfrac{u_{\bar{\imath}}}{r} = \frac{1}{2} + \delta\left(\bar{\imath} - \frac{1}{2}\right)$ and note last lecture's result that when $\delta \to 0$ $\omega = \frac{1}{2}$, which just means that any perturbation on the interior flow is just carried around in the fluid interior,

Next we consider effects of viscosity, considering ϵ, $E^{\frac{1}{2}}$ to be of the same order, To see what this means,

$$\epsilon/E^{1/2} = \frac{u}{\Omega L} \cdot \frac{L\sqrt{\Omega}}{\sqrt{\nu}} = \frac{u/L}{\sqrt{\nu\Omega}/L} = \frac{\text{Ekman dissipation time}}{\text{circulation time}}$$

$$\text{or} = \frac{\text{spin up time}}{\text{circulation time}}$$

This parameter is therefore a measure of the viscous dissipation rate compared to the energy input rate for a perturbation.   Refer to the beginning of Lecture #7 for the derivation of the equations of motion in which we seek solutions for the stream function, $\psi \equiv \frac{1}{2}p'$, of the form

$$\left[\psi = \mathcal{R}e\, \Phi(r,z)\, e^{i(m\theta - \sigma t)} \;,\; \omega = \frac{\sigma}{m} \right.$$

$$m = \text{integer}$$

$$\left. \text{periodic in } \theta \;,\; \text{the azimuthal angle} \right]$$

in the viscous, linear equation

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} - \frac{m^2}{r^2}\Phi \times \frac{4}{S}\frac{\partial^2 \Phi}{\partial z^2} = 0$$

with boundary conditions

$$\left[\frac{1}{2} + \frac{\delta}{2} - \omega\right]\frac{\partial \Phi}{\partial z} - \delta\Phi = -\frac{iE^{1/4}S}{m4E}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} - \frac{m^2}{r^2}\Phi\right] \quad (z=1)$$

$$\left[\frac{1}{2} - \frac{\delta}{2} - \omega\right]\frac{\partial \Phi}{\partial z} - \delta\Phi = +\frac{iE^{1/4}S}{m4E}\left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \Phi}{\partial r} - \frac{m^2}{r^2}\Phi\right] \quad (z=0)$$

$$\Phi = 0 \quad \text{on } r_o, r_i \quad .$$

As before it is separable $\Phi(r,z) = \varphi_{mn}(r)D(z)$ with

$$D(z) = A\sinh \frac{k_{mn}S^{1/2}}{2}\left(z - \frac{1}{2}\right) + B\cosh \frac{k_{mn}S^{1/2}}{2}\left(z - \frac{1}{2}\right)$$

and some algebra yields the following eigenvalue equation

$$\omega - \frac{1}{2} = -\frac{in}{2q}\coth 2q \pm \frac{1}{2}\left\{\frac{-n^2}{q^2\sinh^2 2q} + \frac{\delta^2}{q^2}(q - \coth q)(q - \tanh q)\right\}^{1/2}$$

where we have defined

$$q \equiv \frac{k_{mn}S^{1/2}}{4}$$

$$n \equiv \frac{E^{1/2}}{\epsilon}\frac{Sk_{mn}}{4m}$$

Note here that when $n = 0$ $\left(\text{or } \dfrac{E^{1/2}}{\epsilon} \ll 1\right)$ this result is the same as last time and viscosity only adds these $n \neq 0$ terms. Note also that the conditions for instability ~~will~~ depend this time on the shear δ .

Consider what happens when $\delta = 0$, then

$$\omega - \tfrac{1}{2} = \begin{cases} -\dfrac{in}{2q}\,\tanh q \,, & (+)\text{ root} \\[2mm] -\dfrac{in}{2q}\,\coth q \,, & (-)\text{ root} \end{cases}$$

Thus $\omega_i < 0$ in both cases and recalling that $\dfrac{U_s}{r} = \tfrac{1}{2} + \delta\left(z - \tfrac{1}{2}\right)$ this just says that any perturbation just rides with the fluid interior and decays slowly.

Next we would like to find the condition on δ which __just__ makes $\omega_i = 0$ (i.e. that amount of shear which causes the flow to become unstable). For marginal stability then

$$\omega_i = 0$$

$$= -\frac{n}{2q}\coth 2q + \frac{1}{2}\sqrt{\frac{-n^2}{q^2\sinh^2 2q} + \frac{\delta^2}{q^2}\left(q - \coth q\right)\left(q - \tanh q\right)}$$

$$= \frac{n^2}{4q^2}\underbrace{\left(\frac{\cosh^2 2q}{\sinh^2 2q} + \frac{-1}{\sinh^2 2q}\right)}_{=1} + \frac{\delta^2}{4q^2}\left(q - \coth q\right)\left(q - \tanh q\right) = 0$$

or $\quad \delta^2 = -n^2 \big/ \left(q - \coth q\right)\left(q - \tanh q\right)$

or $\quad \epsilon\,\delta = \dfrac{E^{1/2}Sk_{mn}}{4m}\,\dfrac{1}{\sqrt{\left(\coth q - q\right)\left(q - \tanh q\right)}}$

Plot
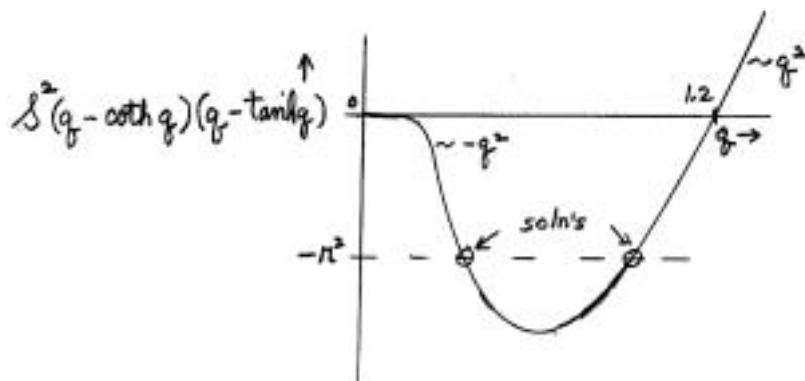
and thus
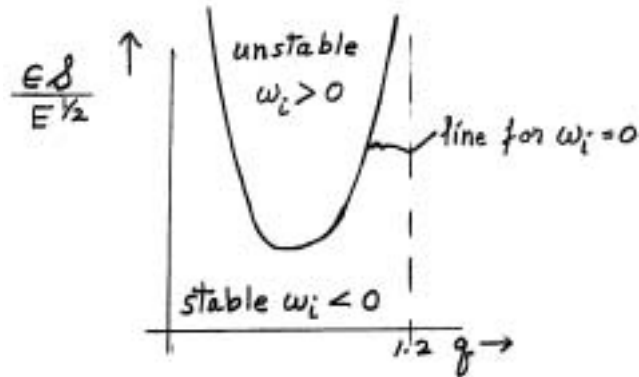
where every scale on this curve is $O(1)$. Thus the minimum value of shear $\mathcal{S}$ for instability is $O(E^{\frac{1}{2}})$. The Λ terms in the eigenvalue equation appeared due to the boundary conditions we had to satisfy. These boundary conditions give rise to a perturbation-caused Ekman boundary layer where the basic flow had none. **As** $\Lambda \to 0$ **the** instable region fills all of the area $q \lesssim 1.2$.
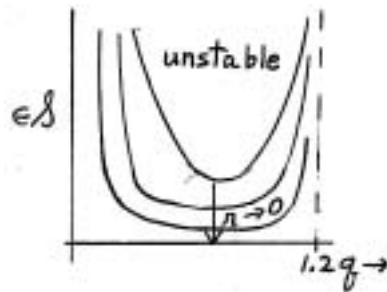
Now we will investigate the narrow gap case.
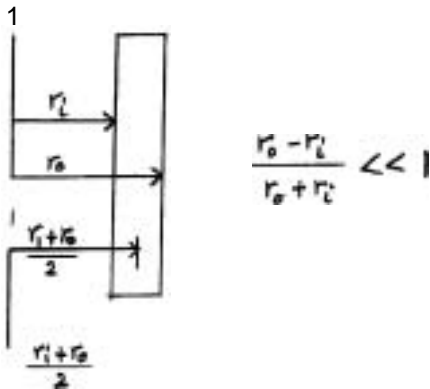
$$\frac{r_0 - r_i}{r_0 + r_i} << 1$$

Re-scale the horizontal coordinates:

define
$$\chi \equiv \frac{r_0 + r_i}{2}\,\theta \cdot \left(\frac{1}{r_0 - r_i}\right)$$

$$y \equiv \frac{r_0 - r}{r_0 - r_i} \qquad 0 \le y \le 1$$
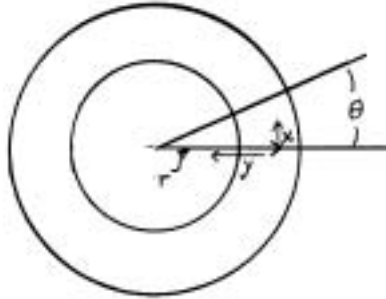
to be a right-handed system

Then $\Psi = \mathcal{R}e\,\Phi(y,z)\,e^{i(kx - \sigma t)}$ where $k = \underbrace{()\!\!-\!\!}\ (2m)$, will replace the former $\mathcal{R}e\,\Phi(r,z)\,e^{i(m\theta - \sigma t)}$, and the equation of motion will become

$$\frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial \Phi}{\partial r} \Rightarrow \frac{\partial^2 \Phi}{\partial y^2} + \mathcal{O}\!\left(\frac{r_0 - r_i}{r_0 +}\right)$$

Finally note that the notation will change here for u, v, w:
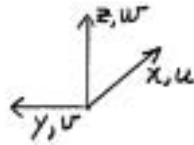
We restate the problem (linear, viscous) with these new coordinates:

$$\left(u_s(y,z) - C\right)\left[\frac{\partial^2 \Phi}{\partial y^2} - k^2\Phi + \frac{1}{G}\frac{\partial^2 \Phi}{\partial z^2}\right] - \Phi\left[\frac{\partial^2 u_s}{\partial y^2} + \frac{1}{G}\frac{\partial^2 u_s}{\partial z^2}\right] = 0$$

where
$$u_s = \frac{1}{2} + \delta\left(z - \frac{1}{2}\right)$$

$$G = \frac{g\alpha(\Delta T_v)L}{4\Omega^2(r_{ov} - r_{iv})}$$

which takes into account the different horizontal and vertical scaling,

With boundary conditions

$$\left(u_s(z=0,1) - C\right)\frac{\partial \Phi}{\partial z} - \Phi\frac{\partial u_s}{\partial z} = \mp\frac{n}{k}\left[\frac{\partial^2 \Phi}{\partial y^2} - k^2\Phi\right]; \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\quad \eta = \left(\frac{\nu}{\Omega}\right)^{1/2}\frac{1}{L}\frac{\Omega(r_{ov} - r_{iv})}{u}\,G$

Now the results of this problem are essentially unchanged since the struc-
ture of the problem remains the same. Instead of Bessel function solutions,

we get $\sin(n\pi y)$ eigenfunctions in the narrow gap.

We wish to investigate the <u>mechanism</u> for instability in the narrow gap problem, The equations of motion are

$$\epsilon\left\{\frac{\partial u}{\partial t}+u_s\frac{\partial u}{\partial x}\right\}-v=-p_x \tag{8.1}$$

$$\epsilon\left\{\frac{\partial v}{\partial t}+u_s\frac{\partial v}{\partial x}\right\}+u=-p_y \tag{8.2}$$

$$T=-p_z \tag{8.3}$$

with $\epsilon=\frac{u}{2\Omega\ell}, f=2\Omega, \ell=r_{out}-r_{in}, \Delta T=\frac{U\ell(2\Omega)}{g\alpha L}$, giving a slight scale change,

Let $w=\epsilon\hat{w}$, then

$$\left\{\frac{\partial T}{\partial t}+u_s\frac{\partial T}{\partial x}+v\frac{\partial T_s}{\partial y}\right\}+\epsilon\hat{w}=0 \tag{8.4}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\epsilon\frac{\partial\hat{w}}{\partial z}=0 \tag{8.5}$$

Take ((1) x u) + ((2) x v) and integrate over volume:

$$\epsilon\frac{\partial}{\partial t}\iiint\frac{u^2+v^2}{2}dxdydz=-\iiint(p_x u+p_y v)\,dxdydz,$$

by parts: $=\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dxdydz$

continuity: $=-\epsilon\iiint p\frac{\partial\hat{w}}{\partial z}dxdydz$

$$=-\epsilon\iiint\left[\frac{\partial}{\partial z}(p\hat{w})-\hat{w}\frac{\partial p}{\partial z}\right]dxdydz$$

(3) $\rightarrow$ $=\epsilon\iiint(\hat{w}T)dxdydz-\epsilon\iint dxdy\left[p(xy1)\hat{w}(xy1)-p(xy0)\hat{w}(xy0)\right]$

It should be remembered here that the $\hat{w}$'s refer to the <u>interior</u> problem and are not necessarily zero at z = 0,1. To investigate the kinetic energy in the perturbation

$$<K>\equiv\iiint\frac{u^2+v^2}{2}dxdydz$$

$$\epsilon\frac{\partial<K>}{\partial t}=\epsilon\iiint(\hat{w}T)dxdydz+\frac{E^{1/2}}{2}\iint dxdy\left[p(1)\zeta(1)+p(0)\zeta(0)\right]$$

$$\boxed{\frac{\partial<K>}{\partial t}=\frac{E^{1/2}}{2\epsilon}\iint dxdy\left[p(1)\zeta(1)+p(0)\zeta(0)\right]+\iiint(\hat{w}T)dxdydz}$$

Now, the geostrophic relation tells us that lower pressures will exist where $\zeta > 0$.

So Ekman suction is forcing fluid out of the Payer and from Bower to higher pressures:

low pressures

(i.e. this does work, which damps the perturbing flow, and the first term on the right is negative.) The correlation of $\hat{\omega}$ and T term must certainly be positive to get unstable flow.

Now investigate the $\langle \hat{\omega} T \rangle$ term. Multiply (4) x $\dfrac{T}{G}$

$$\left[\frac{1}{G} \langle \hat{\omega} T \rangle \quad - \frac{1}{G} \langle T \frac{DT}{Dt} \rangle \right]$$

giving
$$\frac{\partial}{\partial t} \left[\langle K \rangle + \frac{\langle T \rangle^2}{2G} \right] = \iiint dx\, dy\, dz \left[\frac{v T}{G} \left(- \frac{\partial T_s}{\partial y} \right) \right] + \begin{pmatrix} \text{same} \\ \text{Ekman} \\ \text{term} \end{pmatrix}.$$

(the $u_s \frac{\partial T}{\partial x}$ term integrates to zero since u_s is independent of x in the symmetric basic flow),

Using $- \frac{\partial T_s}{\partial y} = \frac{\partial u_s}{\partial z}$ (thermal wind), consider $\iiint v T \left(\frac{\partial u_s}{\partial z} \right)$: we see that the correlation $\langle v T \rangle$ must be > 0 as well as $\langle \hat{\omega} T \rangle > 0$ in order to have energy in the perturbation increase, These correlations tell how the perturbation wave must move the fluid to gain energy: warmer fluid must be moved to a cooler region and vice versa,

Now let $\hat{\zeta} \equiv$ Lagrangian displacement in z (not vorticity)

$\eta \equiv$ Lagrangian displacement in y

then
$$\left(\frac{\partial}{\partial t} + u_s \frac{\partial}{\partial x} \right) \begin{pmatrix} \hat{\zeta} \\ \eta \end{pmatrix} = \begin{pmatrix} \hat{\omega} \\ v \end{pmatrix}$$

$$\left[\frac{\partial}{\partial t} + u_s \frac{\partial}{\partial x} \right] \left[T + \eta \frac{\partial T_s}{\partial y} + \hat{\zeta} G \right] = 0 \quad \text{(heat equation)}$$

with solution $T = -\eta \frac{\partial T_s}{\partial y} - \hat{\zeta} G$.

Put this back into the integral

$$\iiint (\hat{w} T)\, dx\, dy\, dz = \iiint dx\, dy\, dz \left[\frac{d\hat{\zeta}}{dt}\left(-\eta\, \frac{\partial T_s}{\partial y} - \hat{\zeta} G\right)\right]$$

$$= -\iiint dx\, dy\, dz\; G\hat{\zeta}\, \frac{d\hat{\zeta}}{dt} \left\{ 1 + \frac{\eta}{\hat{\zeta}}\, \frac{\partial T_s/\partial y}{G} \right\}$$

Thus we need $\left\{ 1 + \frac{\eta}{\hat{\zeta}}\, \frac{\partial T_s/\partial y}{G} \right\} < 0$ for instability.

But $\dfrac{\partial T_s/\partial y}{G} \sim \dfrac{\partial T_s/\partial y}{\partial T_s/\partial z}$ = slope of isotherms in the basic symmetric flow,

Putting this back in terms of dimensional (*) quantities,

$$\frac{\zeta^*}{\eta^*} < \left.\frac{\partial z^*}{\partial y^*}\right|_{T = const.}$$

isotherms of the symmetric
basic flow
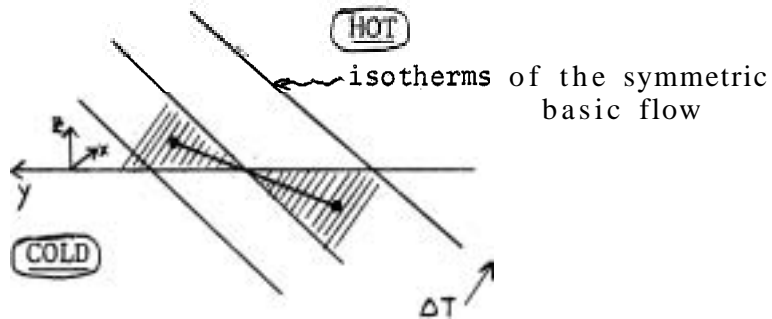
In order for the perturbation wave to take energy from the basic flow, it must move in the above cross-hatched wedge: - e.g. hotter fluid must rise, move in and into a cooler region to give energy up,

　　　Recall that T determines $\Delta \rho$ or the buoyancy. Thus as the hot and cold fluid elements get interchanged within the above wedge, the potential energy of the system on a whole is lowered,　If G is too large this can't be done - for example from the geometric constraint of too narrow an annulus. Finally recall that we showed earlier that the Ekman boundary layer dissipation term must not be too great or motion even in this wedge will be damped,

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Notes submitted by
Dennis Randolph Watts

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NON-LINEAR STABILITY OF THE SYMMETRIC FLOW

Lecture #9   Joseph Pedlosky

Reference: J. Pedlosky, 1970, "Finite Amplitude Baroclinic Waves",
                    J.A.S. 27; 15-30.

   We want to find   (1) the final amplitude of a wave if it equilibrates

                      (2) how long it takes to reach final state

                      (3) the manner in which it reaches equilibration,

The basic flow is

$$u_s(z) = \tfrac{1}{2} + \Delta \left(z - \tfrac{1}{2}\right).$$

The full (non-linear) equation for the perturbation is

$$\left(\frac{\partial}{\partial t} + u_s(z)\frac{\partial}{\partial x}\right)\left(\nabla_H^2 \psi + \frac{1}{G}\psi_{zz}\right) + J\left(\varphi, \nabla_H^2 \psi + \frac{1}{G}\psi_{zz}\right) = 0$$

with boundary conditions:

$$\text{at } z = \binom{1}{0}: \left[\frac{\partial}{\partial t} + \left(\tfrac{1}{2} \pm \tfrac{\Delta}{2}\right)\frac{\partial}{\partial x}\right]\frac{\partial \psi}{\partial z} + J(\psi, \psi_z) - \Delta\frac{\partial \psi}{\partial x} = \pm n\nabla_H^2 \psi$$

$$\text{at } y = 0,1: \frac{\partial \psi}{\partial y} = 0 \ \text{ and } \ \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{\partial \psi}{\partial y}\,dx = 0$$

where
$$J(a,b) = a_x b_y - a_y b_x$$

$$n = \frac{E^{\frac{1}{2}}G}{\epsilon} = \frac{1}{L}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{\Omega \ell}{U}G$$

$$G = \frac{g\alpha(\Delta T)_y L}{4\Omega^2 \ell^2}$$

$$\ell = r_{0*} - r_{i*}$$

The integral $\int_{-\infty}^{+\infty} dx$ may be taken over a wavelength.

   We can't do the problem in general, But by allowing small non-

linearity and slow growth from linear destabilization, the two can balance

to give equilibration,

<u>Inviscid Case</u>: $E^{\frac{1}{2}} \ll \epsilon$, i.e. $A = 0$.

   For a given k , linear theory yields a critical value of G, i.e. G_c,

for instability. Therefore let

$$G = G_c + A \quad \text{where } \triangle \ll 1 .$$

The results of the linear problem were

$$\psi = Re \left\{ e^{i(kx - \sigma t)} \sin m\pi y \left[A \cosh a_m (z - \tfrac{1}{2}) + B \sinh a_m (z - \tfrac{1}{2}) \right] \right\}$$

where $\quad \sigma = \dfrac{k}{2} + \dfrac{k}{a_m} \left\{ \left(\dfrac{a_m}{2} - \coth \dfrac{a_m}{2} \right) \left(\dfrac{a_m}{2} - \tanh \dfrac{a_m}{2} \right) \right\}^{1/2}$

and $\quad a_m = \sqrt{(m^2 \pi^2 + k^2) G}$

The critical value for a_m was

$$a_m < 2.4 \quad \text{for instability}$$

so this gives G_c as a function of k :

$$G_c = \frac{2.4}{m^2 \pi^2 + k^2}$$

This gives a "neutral" curve for fixed $m \neq 0$:
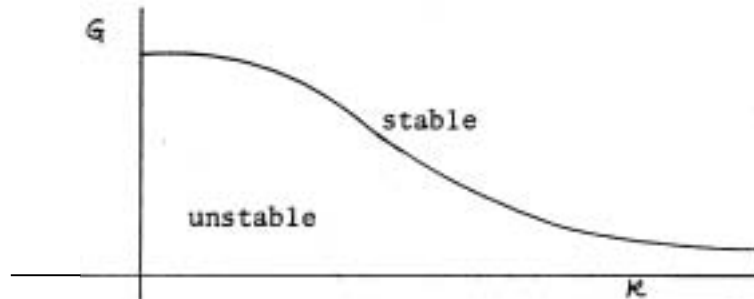
For a particular margfnal wave number, say k , the level of G is determined from $G_c(k)$. For this level of G other waves may be unstable, but we restrict the initial conditions so that only this marginal wave is present.

The linear problem gives

$$B = \frac{\left(\tfrac{1}{2} + \tfrac{\delta}{2} - \tfrac{\sigma}{k} \right) \left(a_m \sinh \tfrac{a_m}{2} - \delta \cosh \tfrac{a_m}{2} \right) A}{\sinh \tfrac{a_m}{2} - \left(\tfrac{1}{2} + \tfrac{\delta}{2} - \tfrac{\sigma}{k} \right) \tfrac{a_m}{2} \cosh \tfrac{a_m}{2}}$$

but for the marginal wave $\begin{cases} \dfrac{\sigma}{k} = c = \tfrac{1}{2} \\ \dfrac{a_m}{2} = \coth \dfrac{a_m}{2} \end{cases}$ therefore, B = 0.

We can also see from the linear results that the growth rate for

$G = G_c + \Delta$ will be the order of $\Delta^{1/2}$. Thus there are two natural time

scales in the problem:

1. a rapid scale from the real part of σ

2. a slow scale upon which the amplitude is growing,

The non-linear effects are weak and we expect them to "come in" on the longer

time scale,

Assuming that we can separate the dependence on these time scales, we

write

$$\psi = \psi(x, y, z, t', T)$$

where $t' = t$

$$T = |\Delta|^{1/2} t$$

Then the time derivatives must be transformed since

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + |\Delta|^{1/2} \frac{\partial}{\partial T}$$

by the chain rule,

The function is expanded

$$\psi = a\,\psi_1 + a^2\psi_2 + a^3\psi_3 + \ldots$$

where a is a scale for the magnitude of ψ and will be assumed small,

a could be a measure of the initial amplitude of a wave-like disturbance.

If one does the strictly linear problem with this slow growth rate, one finds

an amplitude equation in which

$$\sigma_i^2 a \sim a \Delta$$

where Δ is the amount of destabilization, The non-linear contribution

will result from the interaction of the wave with the change in the mean

flow and will be proportional to $a \times a^2 = a^3$. Therefore, if (a^3) balances

$(a\ A)$, then $a = 0(|\Delta|^{1/2})$, and we can write

$$\psi = |\Delta|^{\frac{1}{2}} \psi_1 + |\Delta| \psi_2 + |\Delta|^{\frac{3}{2}} \psi_3 + \ldots$$

Substitution of this expansion into the equations leads to a sequence of problems according to the orders of Δ :

The $0(|\Delta|^{\frac{1}{2}})$ problem:

This is the linear, marginal problem; the solution is

$$\psi_1 = Re\left\{A(T)e^{ik(x-ct)} \sin m\pi y \cosh a_m \left(z - \tfrac{1}{2}\right)\right\}$$

where $\quad a_m^2 = G_c\left(k^2 + m^2\pi^2\right)$

$$c = \frac{\sigma}{k} = \frac{1}{2}$$

The function $(A(T)$ must be determined from the higher order problems.

The $0(|\Delta|)$ problem:

The equation for ψ_2 is

$$\left\{\frac{\partial}{\partial t} + \left[\tfrac{1}{2} + \delta\left(z - \tfrac{1}{2}\right)\right]\frac{\partial}{\partial x}\right\}\left(\frac{\partial^2 \psi_2}{\partial z^2} + G_c \nabla_H^2 \psi_2\right) = -\frac{\partial}{\partial t}\left[G_c \nabla_H^2 \psi_2 + \frac{\partial \psi_1}{\partial z^2}\right] - J\left(\psi_1, G_c \nabla_H^2 \psi_1 + \frac{\partial^2 \psi_1}{\partial z^2}\right)$$

Both terms on the right-hand side are zero since the marginal wave, i.e. ψ_1 , has zero potential vorticity, Hence, a particular solution for ψ_2 has the same form as the linear solution:

$$\psi_2 = Re\left\{e^{ik(x-ct)} \sin m\pi y\left[A_2 \cosh a_m\left(z - \tfrac{1}{2}\right) + B_2 \sinh a_m\left(z - \tfrac{1}{2}\right)\right]\right\}$$

The boundary conditions for ψ_2 are

$$\left[\frac{\partial}{\partial t} + \left(\tfrac{1}{2} + \delta\left(z - \tfrac{1}{2}\right)\right)\frac{\partial}{\partial x}\right]\frac{\partial \psi_2}{\partial z} - \delta \frac{\partial \psi_2}{\partial x} = -\frac{\partial}{\partial T}\frac{\partial \psi_1}{\partial z} \quad \text{at } z = 0, 1$$

Using the above solution, the boundary conditions become

at $z = 1$: $\dfrac{ik\delta}{2}\left[A_2 a_m \sinh \dfrac{a_m}{2} + B_2 a_m \cosh \dfrac{a_m}{2}\right] - ik\delta\left[A_2 \cosh \dfrac{a_m}{2} + B_2 \sinh \dfrac{a_m}{2}\right] =$

$$= -\left(a_m \sinh \dfrac{a_m}{2}\right)\dfrac{dA}{dT}$$

at $z = 0$: $-\dfrac{ik\delta}{2}\left[-A_2 a_m \sinh \dfrac{a_m}{2} + B_2 a_m \cosh \dfrac{a_m}{2}\right] - ik\delta\left[A_2 \cosh \dfrac{a_m}{2} - B_2 \sinh \dfrac{a_m}{2}\right] =$

$$= \left(a_m \sinh \dfrac{a_m}{2}\right)\dfrac{dA}{dT}$$

For rhe marginal wave, $\frac{a_m}{2} = \coth \frac{a_m}{2}$, and these two equations become re-
dundant., In face, $\%$ drops out altogether, and then we can solve for B_2 in
terms of $\frac{dA}{dT}$. A_2 is arbitrary but inclusion of it would merely reproduce
the basic linear solution, If we assume that the amplitude of the basic wave
is given entirely by $A(T)$, we may set $A_2 = 0.$ This is simply a normalization
condition, Thus

$$B_2(T) = \frac{i\, a_m \sinh^2 \frac{a_m}{2}}{k\, \delta} \frac{dA}{dT}$$

There is another solution for Ψ_2 that must be included at this order:
any function of y, z and T only, say $\phi_2(y,z,T)$. This is a slowly-varying
mean flow that will be determined in the next order problem.

The total solution to this order is

$$\psi = Re\left\{ |\Delta|^{\frac{1}{2}} \sin m\pi y \, e^{ik(x-ct)} \left[A(T) \cosh a_m(3-\tfrac{1}{2}) + \frac{|\Delta|^{\frac{1}{2}}}{k} i \frac{dA}{dT} a_m \sinh \frac{2a_m}{2} \sinh a_m(3-\tfrac{1}{2}) \right] \right\} +$$
$$+ |\Delta| \phi_2(y,3,T) + O(|\Delta|^{\frac{3}{2}})$$

We note that the harmonic contribution that has arisen at this order
gives a phase shift to the wave that is proportional to the rate of change
of amplitude. In the linear problem, this phase shift is constant since the
growth rate is constant. (That is, for the linear wave, $\frac{1}{A} \frac{dA}{dT} =$, a
constant,) But for the non-linear problem, $\frac{1}{A} \frac{dA}{dT}$ is not constant and so
the phase of the wave will vary slowly with time.

We still have not determfned $A(T)$ so we must press on to the next
order,

The $O(|\Delta|^{\frac{3}{2}})$ problem.

Now the perturbation in G, i.e. Δ , enters - also the non-linear
interaction between the wave and the mean flow, ϕ_2 .

The equations for Ψ_3 , after the non-linear products have been calculated, are

$$\left\{\frac{\partial}{\partial t}+\left[\frac{1}{2}+\delta\left(3-\frac{1}{2}\right)\right]\frac{\partial}{\partial x}\right\}\left(\frac{\partial^2 \Psi_3}{\partial 3^2}+G_c \nabla_H^2 \Psi_3\right)=-\frac{\partial}{\partial T}\left(G_c \nabla_H^2 \phi_2+\frac{\partial^2 \phi_2}{\partial 3^2}\right)+$$

$$+\frac{a_m}{G_c}\frac{\Delta}{|\Delta|}\delta\left(3-\frac{1}{2}\right)\frac{ik}{2}e^{ik(x-ct)}\sin m\pi y\ A(T)\cosh a_m\left(3-\frac{1}{2}\right)+\text{complex conjugate} \qquad \text{(A)}$$

at $z = 1$: $\left(\frac{\partial}{\partial t}+\left(\frac{1}{2}+\frac{\delta}{2}\right)\frac{\partial}{\partial x}\right)\frac{\partial \Psi_3}{\partial 3}-\delta\frac{\partial \Psi_3}{\partial x}=-\frac{\partial}{\partial T}\frac{\partial}{\partial 3}\phi_2-a_m^2\frac{m\pi}{4}\sinh^2\frac{a_m}{2}\frac{\sin 2m\pi y}{\delta}\frac{d}{dT}|A|^2-$$

$$-\frac{1}{2k}\frac{d^2 A}{dT^2}\frac{a_m^2}{\delta}\sinh^2\frac{a_m}{2}\cosh\frac{a_m}{2}\sin m\pi y\ e^{ik(x-ct)}-\left(\Psi_{1x}\phi_{2xy}-\phi_{2y}\Psi_{1xx}\right)+\text{complex conjugate} \qquad \text{(B)}$$

plus a similar equation at $z = 0$.

Eliminating resonance in Eq. (A) we get an equation for ϕ_2 :

$$\frac{\partial}{\partial T}\left[\frac{\partial^2 \phi_2}{\partial 3^2}+G_c\frac{\partial^2 \phi_2}{\partial y^2}\right]=0$$

Similarly, we have boundary conditions on ϕ_2 from Eqs. (B) :

$$\frac{\partial}{\partial T}\frac{\partial \phi_2}{\partial 3}=-a_m^2\frac{m\pi}{4}\sinh^2\frac{a_m}{2}\frac{\sin 2m\pi y}{\delta}\frac{d}{dT}|A|^2 \quad \text{at } z=0,1$$

Or, integrating with respect to T and using $\phi_2 = 0$ at T = 0 as an initial condition on ϕ_2 :

$$\phi_{2_3}=-a_m^2\frac{m\pi}{4}\sinh^2\frac{a_m}{2}\frac{\sin 2m\pi y}{\delta}\left[|A|^2-|A_o|^2\right] \quad \text{at } z=0,1$$

where $A_o = A(0)$.

On the side walls there is another condition on ϕ_2 :

$$\frac{\partial}{\partial t}\frac{\partial \phi_2}{\partial y}=0 \quad \text{at } y = 0,1$$

These equations give a solution for

$$\phi_2(y,3,T)=-\left[\frac{|A|^2-|A_o|^2}{\delta}\right]\sinh^2\frac{a_m}{2}\left[\frac{\sinh 2m\pi G_c^{1/2}\left(3-\frac{1}{2}\right)\sin 2m\pi y}{2m\pi G_c^{1/2}\cosh m\pi G_c^{1/2}}\right]+\chi_2(y,3)$$

where χ_2 is a solution of $\frac{\partial^2 \phi_2}{\partial 3^2}+G_c\frac{\partial^2 \phi_2}{\partial y^2}=0$ needed to satisfy the boundary conditions,

Substituting for Ψ_1 and ϕ_2 in Eqs. (B) and removing secular terms leads to the amplitude equation:

$$\frac{d^2A}{dT^2} + \frac{\Delta}{|\Delta|} \sigma_{oi}^2 A + k^2 N A \left[|A|^2 - |A_0|^2 \right] = 0$$

The coefficient of the second term is just the linear growth rate squared. Note that Δ is negative for lfnear instability. The constant factor N in the non-linear term has been found to be always positive,

If we let $A(T) = R(T) e^{i\phi(T)}$, where $R = |A|$, substitute for A, and separate real and imaginary parts, two equations for R and ϕ emerge:

$$R^2 \frac{d\phi}{dT} = \text{constant} = L, \text{ say}$$

$$\frac{d^2R}{dT^2} = -\frac{\Delta}{|\Delta|} \sigma_{oi}^2 R - k^2 N R \left[R^2 - R_0^2 \right] + \frac{L^2}{R^3}$$

We have a second order equation for A so we need initial conditions on A and $\frac{dA}{dT}$. Let us consider the case where the wave conforms to the linear unstable solution at $T = 0$. Then $\frac{1}{A}\frac{dA}{dT}$ would be real to $T = 0$. But

$$\frac{1}{A}\frac{dA}{dT} = \frac{1}{R}\frac{dR}{dT} + i\frac{d\phi}{dT}$$

Therefore we have $\frac{d\phi}{dT}(0) = 0$. This gives us $L = 0$ so the equation for R is

$$\frac{d^2R}{dT^2} = -\frac{\Delta}{|\Delta|} \sigma_{oi}^2 R - k^2 N R \left[R^2 - R_0^2 \right]$$

A first integral can be found:

$$\frac{1}{2}\left(\frac{dR}{dT}\right)^2 + V(R) = E, \text{ a constant}$$

where $V(R) = -\frac{k^2}{2}\left[\frac{\sigma_{oi}^2}{k^2} + N^2 R_0^2\right]R^2 + \frac{k^2 N}{4} R^4$

$$E = \frac{1}{2}\left[\frac{dR}{dT}(0)\right]^2 - \frac{\sigma_{oi}^2}{2} R_0^2 - \frac{k^2 N}{4} R_0^4$$

after taking $\Delta = -|\Delta|$.

This gives us the following qualitative picture of the behavior of R:

R will oscillate with time between the maximum and minimum amplitudes which are determined by the initial "energy" E. Note that the amplitude can become less than the initial amplitude. Note also that the shape of the curve, and thus the minimum point, depends on the initial amplitude R_o.

R as a function of T can be calculated in terms of elliptic integrals. Qualitatively, it looks like

The period can also be calculated. The limiting amplitudes are given by

$$R^2_{\genfrac{}{}{0pt}{}{(max)}{(min)}} = \frac{1}{N} \frac{\sigma_{io}^2}{k^2} + R_o^2 \pm \frac{\sigma_{io}^2}{Nk^2}\left[1 + \frac{2Nk^2}{\sigma_{io}^2}R_o^2\right]$$

The energy transfer process is as follows: As the wave grows, $\frac{1}{A}\frac{dA}{dT}$ does not remain constant. Initially $\frac{1}{A}\frac{dA}{dT} = \sigma_{io}$ and the wave extracts energy from the mean field as in linear theory. As the wave grows and its amplitude departs from its initial value, A_o, the zonal flow is altered. When $R = R_e$ the altered zonal flow is stable according to linear theory but

$\frac{1}{A}\frac{dA}{dT} \neq 0$. Hence the wave still possesses a phase tilt with height (Pines of constant p sloping upstream with height) which allows continued extraction of energy from the zonal flow. The wave continues to grow, its phase shift decreasing as it approaches its maximum amplitude R_m. At this point $\frac{1}{A}\frac{dA}{dT} = 0$, the wave no longer can gain energy from the mean flow (whish is now quite stable), The wave amplitude then decreases with time, the phase shift is reversed and the wave returns its energy to the mean flow until the wave achieves its minimum amplitude (which is less than A(0)) at which point the wave proceeds to grow again starting another cycle of the oscillation.

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Notes submitted by
Michael A, Weissman

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NON-LINEAR INSTABILITY OF SYMMETRIC FLOW, concluded:
THE EFFECTS OF DISSIPATION

Lecture #10   Joseph Pedlosky

In all the previous analysis, the effects of viscosity have been treated as negligible. We now ask what happens if a little dissipation is present, and in answer we shall quote a few results without derivation, These results do not apply directly to the continuously stratified Payer, but are from a two-layer model in which dissipation occurs at the rigid top and bottom. In this model, the interface tilts in the same sense as do the isotherms of the continuously stratified model, and the two models are indeed essentially similar in ail imperrant respects except tractability.

For the two-layer model, let

$\Delta \rho$ be the density difference,

be the layer height, and

ℓ be the gap width.

The important parameter is then $F = \dfrac{4 \Omega^2 \ell^2}{g \frac{\Delta \rho}{\rho} L}$, and the form of the amplitude

equation is the same for both models. The models differ slightly in the

values of the coefficients σ_{oi}^2 and N.

We shall first discuss the ease of large dissipation $\left(\epsilon = O\left(E^{1/4}\right)\right)$.

We have

$$\frac{dA}{dT} - \sigma(r)A + N(r)A|A|^2 = 0, \tag{1}$$

$$\text{where} \quad r = E^{1/2}/\epsilon$$

$$\text{and} \quad T = |\Delta| t.$$

As in the inviscid case, the first two terms of this equation are the

linear results, and the third term is the non-linear effect, which here

leads to equilibration, In the inviscid case, two modes grew slowly, Here,

however, the order of the equation is reduced (phase shift is proportional

to amplitude rather than rate of change of amplitude), and only one mode grows,

The graph depicts initial growth via linear theory leveling off to a steady

finite-amplitude solution $|A_\infty|^2 = \sigma(r)/N(r)$ which is independent of ini-

tial conditions, The behavior is thus very different from that of the in-

viscid ease, There the wave "remembers" its initial amplitude; here its

initial phase。

We now consider an intermediate case. A critical value is $r = 0(A)^{\frac{1}{2}}$, for which viscosity and growth have comparable effects on phase, With $T = |\Delta|^{\frac{1}{2}}t$, the equations are

$$\frac{d^2 A}{dT^2} + \frac{3}{2}\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{dA}{dT} - \sigma_{oi}^2 A - \frac{k^2(U_1 - U_2)}{2(m^2\pi^2 + k^2)}A\int_0^1 \sin^2 m\pi y \frac{d^3\Phi_2(T)}{dy^3}dy = 0 \qquad (2)$$

$$\frac{\partial}{\partial T}\left[\frac{\partial^2\Phi}{\partial y^2} - (m^2\pi^2 + k^2)\Phi_2\right] + \frac{r}{|\Delta|^{\frac{1}{2}}}\frac{d^2\Phi_2}{dy^2} = \frac{(k^2 + m^2\pi^2)}{2(U_1 - U_2)}\left[\frac{d}{dt}|A|^2 + \frac{2r}{|\Delta|^{\frac{1}{2}}}|A|^2\right]\sin 2m\pi y \qquad (3)$$

This may be integrated for $r = 0$. The problem reduces to the inviscid one, If $\frac{r}{|\Delta|^{\frac{1}{2}}}$ is large, we obtain the small-r limit of Eq.(1). In general, one can calculate the equilibrium value A_∞ by solving (2) neglecting T derivatives, using Φ from (3) viz.:

$$|A_\infty|^2 = \frac{\Delta}{|\Delta|}(U_1 - U_2)^2/m^2\pi^2(k^2 + m^2\pi^2)$$

To gain insight into (2), we relax the boundary conditions on the part of the flow that is independent of x. Then, assuming a natural ($\sim \sin 2m\pi y$) periodicity in y, we solve for Φ in terms of A and obtain the following equation for A:

$$\frac{d^2 A}{dT^2} + \frac{3}{2}\frac{r}{|\Delta|^{\frac{1}{2}}}\frac{dA}{dT} - \sigma_{oi}^2 A + PA\left[|A|^2 - |A(o)|^2 e^{-\gamma T} + \frac{2r}{|\Delta|^{\frac{1}{2}}}\int_0^T \frac{k_m^2 + 2m^2\pi^2}{k_m^2 + 4m^2\pi^2}|A|^2 e^{-\gamma(T-T')}dT'\right] = 0, \qquad (4)$$

where $\qquad P = \dfrac{m^4\pi^4 k^2}{2(4m^2\pi^2 + k_m^2)}$

and $\qquad \gamma = \dfrac{r}{|\Delta|^{\frac{1}{2}}}\dfrac{4m^2\pi^2}{k_m^2 + 4m^2\pi^2}$,

and $\qquad k_m^2 = m^2\pi^2 + k^2$

From (4), we see that the wave gradually forgets its initial conditions as time goes on and the integral causes the memory of distant past history to fade. The early growth is oscillatory, and the final steady state in the small-r limit of the previously calculated $|A_\infty|^2 = \sigma(r)/N(r)$. We note that she final state A, is independent of the initial conditions

and may, therefore, actually be smaller than the initial amplitude.
Finally, we reluctantly remark that numerical solutions of (4) should be
informative."

To summarize briefly all the lectures, we recall that we first ex-
amined the importance of potential vorticity, its conservation and its dif-
fusion, Next we studied the importance of boundary layers in determining
the interior flow of symmetric states. Lastly, we determined that such
states may be unstable (for sufficiently small Ekman number) to wave states
which can grow via baroclinic instability and equilibrate via non-linearity.

All of these results have been found in a very simple thought experi-
ment. In real geophysics, the same mechanisms are present, but in vastly
more complicated forms. If we actually perform the experiment, we shall ex-
pect to find confirmation of these results, Being experienced in this field,
we shall expect surprises too.

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*Calculations done since have shown that when $k^2 \geqslant 3 \, m^2 \pi^2$ the possible
steady state is in fact unstable and then a limit cycle rather than a fixed
amplitude is obtained for large T when $r/|\Delta|^{1/2}$ is small.

Notes submitted by
Richard C. J. Somerville

ABSTRACTS OF SEMINARS

INSTABILITIES OF CONVECTION ROLLS IN A HIGH PRANDTL NUMBER FLUID

Friedrich Busse

It has been shown by Schlüter, Lortz and Busse (1) that rolls represent the only stable form of convection in a fluid layer heated from below provided that the fluid satisfies the Boussinesq approximation and that the Rayleigh number exceeds the critical value by a sufficiently small amount. The range of wave number for which rolls are dynamically possible solutions of the basic equations increases strongly with R. The range is bounded by the curve R_0 in Figure 1. Only for a relatively small sub-range of wave numbers α, however, rolls are stable. The region of sea-bility is roughly centered around the critical wave number α_c and is bounded by the curves B and C which correspond two different instability mechanisms: Rolls with a too small wave number become unstable to the "zig-zig instability". By bending the original rolls this instability leads to a pattern of shorter wave length. Rolls with a sufficiently high wave number become unstable to the "cross roll instability" which induces rolls perpendicular to the original ones. For Rayleigh numbers above 8000 the latter instability is also responsible for the boundary on the left side of the stability region in Fig.1.

The characteristic wave number of the marginal cross roll instability at the stability boundary is shown by the dashed line in Fig. 1. The fact that the Pine crosses the stability boundary at $R \approx 15000$ indicates that the cross roll instability can not induce rolls as stable stationary solution for $R \gtrsim 15000$. Instead a three-dimensional form of convection, called bimodal convection, is induced. Figure 2 shows the streamlines for the horizontal component of the velocity of bimodal convection near the boundary,

Fig. 1

Stability region.

Fig. 2

Streamlines for the horizontal component
of the velocity of bimodal convection near the boundary.

Experimental evidence suggests that bimodal convection represents the only stable form of stationary convection for R $\geq$ 22600 when rolls are no longer stable, An experimental investigation of the instabilities of convection rolls has been carried out recently by Busse and Whitehead (2) in extension of the earlier work by Chen and Whitehead (3). The observations show reasonable agreement with the theoretical predictions displayed in Fig.1 which are taken from (4).

(1)   Schlüter, A,, D. Lortz and F. H. Busse 1965 J.Fluid Mech,, 23: 129,

(2)   Busse, F. H., and J. A. Whitehead 1970 submitted to J.Fluid Mech.

(3)   Chen, M. M. and J. A. Whitehead 1968 J.Fluid Mech., 31: 1,

(4)   Busse, F. H. 1967 J.Math. and Phys. 46: 140.

ON THE EARTH'S CORE-MANTLE INTERFACE

Raymond Hide

The tightness of the coupling between the core and mantle of the Earth that is implied by the occurrence of irregular fluctuations of up to about 5 x 10^{-3}s in the length of the day on time scales of a few years and upwards raises important questions concerning the nature of the horizontal stresses at the core-mantle interface. Electromagnetic coupling does not suffice unless very rapid magnetic fluctuations in the core - on time scales well below the "skin effect" cut-off due to the mantle - are invoked, An alternative suggestion is that the core-mantle interface is bumpy, typical undulations being a kilometre or so in height, and recent laboratory experiments on "Spin-up" in irregular containers remove one theoretical objection to this suggestion. Planetary-scale undulations of the core-mantle interface would contribute significantly to the Earth's regional gravita-

tional field, and they would also distort the magnetic field. The recent discovery of evidence of a previously unsuspected correlation between global features of the Earth's gravitational and magnetic field lends weight to the suggestion that undulations are present on the care-mantle interface. The study of theoretical models of interactions between these undulations and magnetshydrodynamic motions in the core will be a useful development in geophysical fluid dynamics,

Reference, R. Hide, "On the Earth's core-mantle interface", Quart.J. of the Royal Meteorological Society (Oct. 1970).

THE PERIODS OF FINITE-DISK GENERATED GRAVITY WAVES

Louis N. Howard

Gravity waves in a cylindrical tank of water excited by vertical oscillation of a centrally placed disk exhibit resonant frequencies different from those of the cylindrical tank with a completely free surface, The lowest axisymmetric resonanee has been found experimentally by Kaiser and Murty (Phys.Fl. 12: 1144(1969)) to be nearly the same as the computed natural frequency of an annulus whose inner eylinder lies just below the exciting disk.

In this lecture it is shown that resonances observed in this manner should be accurately given by natural frequencies of free surface modes in a cylinder with a partial rigid cover coinciding in position with the forcing disk, Using the general variational characterization of free surface waves, the lowest axisymmetric frequency of the "capped" cylinder is estimated, and it is found to be bounded from below by that sf the annulus. A slightly more complicated but basically similar estimate from above shows that in cir-

eumstances like those of the experiments, the eapped cylinder frequency also cannot be more than a few percent above the annulus frequency, even when the "cap" is as large as 3/4 of the tank radius and the annulus frequeney differs markedly from that of the free cylinder. This gives a theoretical rationalization of Kaiser and Murty's observation.

SOAP FILMS

Louis N. Howard

Primarily a popular talk centered around demonstrations of large soap films $(O(1\ m^2.))$, this lecture dealt with some less obvious but everyday manifestations of surface tension phenomena, a few mathematical curiousities regarding minimal surfaces and surfaees of constant mean curvature, and the role of soap in the actual production of films, With respect to the latter, emphasis was placed on the necessity of <u>variations</u> in surface tension with macroscopic decay times, both for the existenee and the stability of films. A simple experimental demonstration of this variation in a vertical soap film together with a discussion of the physical mechanism by which it can be produced with the aid of soap, were presented,

INERTIAL TAYLOR COLUMNS

Andrew P. Ingersoll

A homogeneous fluid is bounded above and below by horizontal plane surfaees in rapid rotation about a vertical axis, An obstaele is attached to one of the surfaces, and at large distances from the obstacle the flow is uniform and horizontal. Steady solutions are obtained as power series ex-

pansions in the Rossby number, uniformly valid as the Ekman number approaches zero,

If the height of the obstacle is greater than the Rossby number times the depth, there is a region of closed streamlines in the vicinity of the obstacle. The effect of viscosity, however, is to prohibit flow within such regions in the steady state. Thus a stagnant region (Taylor column) forms over the obstacle. Outside this region the fluid obeys inviscid equations of motion, and there is a net circulation in a direction opposite the rotation.

The shape of the Taylor column is not known *a priori*. This shape is determined from the free surface boundary condition on the critical streamline: free shear layers do not exist when the slope of the obstacle is much less than one. Velocity is therefore continuous in the inviscid limit, Thus velocity must vanish on the closed portion of the critical streamline. This condition applied to the inviscid exterior flow uniquely determines the shape of the Taylor column and the circulation round it in the steady state.

Reference: J.Atmos.Sci., 26: 744 (1969).

VENUS: THE GREENHOUSE THAT RAN AWAY

Andrew P. Ingersoll

Venus, Mars and the earth have nearly the same average density, which suggests that they have nearly the same average composition, Nevertheless, the atmospheres of the three planets do not resemble each other in any obvious way. A reconciliation is possible if one assumes that large amounts of H_2O and CO_2 have been exhaled from the interiors of all three planets, and that

differences in their atmospheres have evolved because of differences in the amount of solar heating.

On Mars, the temperatures are so low that both H_2O and CO_2 exist mainly in solid form, The partial pressure of these gases in the Martian atmosphere is controlled by vapor-solid equilibrium at the Martian surface, On the earth the temperatures are such that H_2O exists mainly in liquid form, and the partial pressure of H_2O in the atmosphere is controlled mainly by vapor-lfquid equilibrium at the surface. CO_2 in the earth's atmosphere is controlled by equilibrium with dissolved CO_2 in the oceans, which in turn is controlled by equilibrium with metallic carbonates in sedimentary rocks.

On Venus, the situation is less clear. Both H_2O and CO_2 can only exist in the vapor phase, but there appears to be very little H_2O on Venus at present. One explanation is that H_2O has been destroyed by ultraviolet light from the sun at a rate which is several orders of magnitude greater on Venus than it is on the earth or Mars, Such a circumstance could arise if H_2O were once the major constituent of the Venus atmosphere. The quantum efficiency for destruction of H_2O on Venus would then have been close to one,

On the earth the quantum efficiency is about 10^{-5}, since H_2O is only a minor constituent of the earth's atmosphere, If large amounts of H_2O have been dissociated on Venus, the hydrogen must have escaped into space, and the oxygen must have gone into oxidation of surface rocks.

The remainder of the argument is to show why H_2O should have been a major constituent of the early Venus atmosphere, although it is a minor constituent of the earth's atmosphere. Water vapor is the major source of

infrared opacity in the earth's atmosphere, and therefore the depth of the radiating layers, where the infrared optical depth is less than one, is determined by the amount of water vapor present. However, if the amount of water vapor is determined by the vapor pressure of the liquid, then the temperature of the radiating layers cannot rise above a certain point. This means that equilibrium with the sun is impossible if the sunlight absorbed exceeds a critical value, as long as the water vapor pressure is controlled by the liquid. Calculations indicate that Venus and the earth are on opposite sides of this equilibrium, and therefore H_2O could never have existed in liquid form on Venus, Thus H_2O may have been the major constituent of the early Venus atmosphere, and this may account for the absence of H_2O on Venus today.

Reference: J.Atmos.Sci., 26: 1191 (1969)

TURBULENT BURSTS

Mårten T. Landahl

Shear flow turbulence is discussed starting from the mathematical model

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\nabla^2 v - U''\frac{\partial v}{\partial x}-\nu\nabla^4 v + C(v) = g + G(v)$$

Here u, v, w are the fluctuating velocity components of a parallel shear flow of velocity $U(y)$ and

$$g = \frac{\partial^2}{\partial x_\ell \partial x_\ell}\left[\frac{\partial}{\partial x_j}\left(\overline{u_j u_2}-u_j' u_2'\right)\right]-\frac{\partial}{\partial x_2}\left[\frac{\partial^2}{\partial x_i \partial x_j}\left(\overline{u_j u_i}-u_j' u_i'\right)\right]$$

$$\left(u_1 = u,\ u_2 = v,\ u_3 = w\right)$$

may be considered a source term due to the fluctuating Reynolds stresses (see Landahl, J.F.M. 29: 441). Except for the term $C(v)$, the left-hand

side is the Orr-Sommerfeld operator, The model may be described as "linear ringing" (the modified Orr-Sommerfeld response) caused by "non-linear banging" (the fluctuating Reynolds stress terms). The operator $C(v)$, is added to include in some fashion the scattering effect of the background turbulence on the linear response (predominantly damped Orr-Sommerfeld waves, see J.F.M. 29: 441). The most important terms in q (i.e. the ones containing the highest derivatives in y) are

$$q \simeq \frac{\partial^2}{\partial y^2} \left[\frac{\partial}{\partial x} (uv) + \frac{\partial}{\partial z} (vw) \right]$$

These can be expected to be large wherever there is a locally high growth rate of fluctuations, Such is known to occur in turbulent spots, It is therefore of great importance to understand the conditions leading to bursts.

The observations by Klebanoff *et al.* (J.F.M. (1962), 12: 1) of bursts in a laminar boundary layer undergoing transition are, so far, the best ones to use as a basis for discussion as the experimental situation was very well controlled. These experiments showed that the bursts occurred very suddenly once a critical value of the amplitude of the primary wave is exceeded. The growth rate of the fluctuation amplitude is far too high to be explained by secondary instability, A different mechanism must therefore be at work, The explanation proposed is based on a discussion of the relationship between the wave propagation velocities of the primary and the secondary waves, When the primary wave amplitude increases, the group velocity of the secondary wave will (probably) decrease. A criticality condftion is reached when the group velocity of the secondary wave becomes equal to the phase velocity of the primary one somewhere along the primary wave, At that condition, secondary dfsturbances travelling into the primary

wave are trapped and must eventually cause complete breakdown of the primary wave to remove the critical flow region. An analogy with choking in a Laval nozzle, in which deceleration from supersonic to subsonic flow is attempted, can be drawn. A correlary of the proposed explanation is that criticality should depend exclusively on the local instantaneous velocity profile, which is in excellent accord with the observations. Furthermore, the criticality condition is calculable on basis of linear stability theory for a parallel flow.

Some consequences of the proposed model were discussed. The Reynolds stresses in the flow will be produced almost exclusively by the bursts whereas the linear random waves will contribute very little. Hence a reasonable assumption to determine approximately the operator $G(v)$ is to assume that the fluctuatfons in a burst are simply convected in a "frozen" manner by the wave. This assumption leads to

$$G(v) = \frac{\partial^3}{\partial x^2 \partial y} (\delta \overline{v^2}) + \frac{\partial^3}{\partial x \partial y^2} \left[\delta \frac{d(\overline{uv})}{\partial y} \right]$$

where δ is the streamline displacement by the wave which is connected to v by

$$v = \frac{\partial \delta}{\partial t} + U \frac{\partial \delta}{\partial x}$$

and the bar denotes average.

TURBULENT SHEAR FLOW STRUCTURE

Mårten T. Landahl

Some model problems believed relevant for understanding the mechanics of turbulent shear flow are proposed and discussed. Recent experimental investigations, primarily by Kline's group at Stanford, indicate that the strong turbulence production that takes place in the wall layer is associated with

local inflexional instability of strongly concentrated shear layers pro-
duced by local interaction of longitudinal vortices and the fluctuations
due to the large eddies, As inflexional instability is predominantly an
inviscid phenomenon, it is proposed that the non-linear behavior of a
strongly unstable inflexional region can be modelled by a distribution of
concentrated potential vortices interacting with each other in the pre-
sence of a rigid wall, The simple case of a concentrated two-dimensional
shear layer was set up on a computer, and the development of the vortex
layer following an initial small deformation in form of a bump studied,
The numerical results show that a "hole" in the shear layer develops very
rapidly and that it starts to roll up on each side of the hole very much in
the fashion of a roller curtain. Such behavior might be consistent with
the recent observations by Corino and Brodkey (J.F.M. 37: 1) of turbulent
breakdown showing the formation of vertical jets on the edge of the low-
speed flow region resulting in violent turbulent mixing with the higher
velocity fluid above.

The "passive" region of the turbulent boundary layer is believed to
react in a linear fashion to the perturbations excited by the local break-
downs in the active layer, Dominating statistically will be wave propagation
modes (see Landahl, J.F.M. 29: 441). Of importance is to determine the ef-
feet of background turbulence on the wave propagation. The formulation of
this problem is considered and some possible approximations discussed,

O D E   T O   T U R B U L E N C E

by Mårten T. Landahl

(sung to the Marine Hymn)

FROM THE LAMINAR SOLUTION

INTO FINITE AMPLITUDE

WE SHALL STRIVE FOR UNDERSTANDING

WITH MATHEMATICS NEAT OR CRUDE

BUT IF THE FLUID BECOMES TURBULENT

WITH EDDIES AND RANDOM BURSTS

WE SHALL RALLY 'ROUND THE PHYSICS

AND LET COMPUTORS DO THEIR WORST,

"WHITHER SST - BOOM OR BUST?"

Mårten T. Landahl

The main economic-engineering argument for the supersonic transport is explained and aerodynamic consequences for the design demonstrated, The basic fluid dynamic reasons for the sonic boom are reviewed, and the parameters influencing its strength discussed, The physics of the phenomenon makes the chances of a substantial reduction of boom strength limited. The results of sociological investigations show quite clearly that the annoyance effect, in particularly the startle (illustrated), is so large that it is highly unlikely that any democratic government would ever allow regular supersonic flights over inhabited territory, Thus, present proposed designs are intended to operate supersonically over water, only. Some of the implications of this limitation are discussed.

CYTHERIAN ENERGETICS: HOW DOES VENUS KEEP HER COOL?

Willem V. R. Malkus

Recent work on the atmosphere of Venus is reviewed.   The observed four-day circulation of the visible clouds led Schubert and Whitehead to suggest that, given a sufficiently small effective Prandtl number, the zonal flow could be a Halley circulation.   In 1686 Halley proposed that a heat source moving over a layer of fluid would induce an average horizontal flow, Stern and Davey have constructed quasi-linear theories supporting this proposal, Here, the work of Davey is extended to the non-linear regime by the construction of an Oseen-type solution, Quite fortuitously, it is found that the maximum zonal velocity is insensitive to the atmospheric mixing processes, depending on the one-sixth power of the Prandtl number.   The computed velocity for Venus is in keeping with the observations.   Other aspects of the theory relate potential observables, such as the solar-antisolar temperature contrast and the depth of the high velocity region, to the unknown magnitudes of the turbulent transport of heat and momentum.   The inference from available data is that an effective thermal diffusion coefficient for Venus is orders of magnitude larger than previously suggested, Among the unresolved problems which emerge from this study is the thermal-tidal resonant interaction.   Such interaction could produce long term torques on the planet, perhaps explaining her retrograde rotation.

THE STRUCTURE AND STABILITY OF VORTEX RINGS

Tony Maxworthy

A series of observations on experimentally produced vortex rings was described. Velocity and vorticity fields were observed using dye and hydrogen bubble techniques. It was concluded that stable rings are unique entities which grow in such a way that their vorticity is always contained within the fluid mass moving with the ring. As the vorticity diffuses out of the ring into the outer irrotational fluid it causes this fluid to be entrained into the interior of the bubble. It is possible that an exponentially small amount of vorticity escapes reentrainment and is shed into a very small wake behind the moving fluid mass. It has been found that the velocity of these stable rings varies as t^{-1}; where t is the time measured from a virtual origin of the motion at downstream infinity. A simple entrainment model containing all of these features of the observed flow predicts such a behaviour,

Sufficiently energetic rings become unstable and significant vorticity is then shed into a wake. Under some circumstances a new more stable vortex emerges from this shedding process and continues with less vorticity than before, Eventually, all motion should cease as the ring circulation decays to zero under the diffusing action of viscosity.

LES TOURBILLONS DES FUMEURS

Tony Maxworthy

A lecture honouring Prof. Henri Bouasse who came close to correctly describing the flow field of non-buoyant vortex rings, This presentation described work extending the previous lecture on "smoke rings". An entrainment model was presented (Fig.1) which correctly predicts the behaviour of

Fig.1

vortex rings, As vorticity diffuses from the main ring structure it is picked up by irrotational fluid flowing around the ring; some is entrafned into the ring the rest is shed as a wake. The amount of fluid being en- trained is:

$$\frac{d\,a^3}{dt} \sim U a \left(\nu \frac{a}{U} \right)^{1/2}$$

We can show that at high Reynolds number the impulse is closely con- stant even though a small amount of momentum is shed into the wake; whence $I \sim U a^3 =$ const. Substitution gives

$$a^3 \sim (I \nu)^{1/2} t$$

and

$$U \sim \left(\frac{I}{\nu} \right)^{1/2} t^{-1}$$

The decrease in ring circulation can be accounted for by the loss of vorticity to the wake,

The model can be extended to buoyant rings, for which:

$$\alpha^3 \sim (\nu F)^{1/2} t^{3/2}$$
$$\nu \sim \left(\frac{F}{\nu}\right)^{1/2} t^{-1/2}$$

and the circulation is constant, i.e. as much vorticity is being shed into a wake as is being produced by buoyancy forces, F is the total buoyancy force acting on the ring,

Alternative explanations for this behaviour have been given by J. S. Turner using similarity arguments. More refined experiments are needed in order to distinguish between the two models,

The lecture concluded with demonstrations of the major features of vortex rings, In particular it was shown that two rings of closely equal initial velocity do not pass through each other, as commonly believed, but the slower entrains the faster and they combine,

TOPOGRAPHIC EFFECTS ON THE WIND-DRIVEN OCEAN CIRCULATION

Elliott E. Schulman

We consider the linear wind-driven flow of a homogeneous ocean in which bottom friction is the dominant mechanism of dissipating the wind-induced vortieity. The bottom topography varies only along latitude lines and the effects of an oceanic ridge, plateau and continental rise are studied, Solutions are obtained both for the inviscid case and those in which the bottom frictional parameter are small,

We find that a mid-oceanic ridge (such as the Mid-Atlantic Ridge) tends to reduce the Sverdrup transport in the basin and direct the westerly drift to the south. The diversion is the strongest where the bottom frictional parameter is the smallest. In general, closed streamline patterns

can be found over variable topography without diffusing vorticity to the ocean floor. As the ocean rises onto a plateau (such as the Blake Plateau) the circulation is blocked from riding onto the plateau, A western continental rise (such as the rise from the Blake Plateau to the eastern coast of North America) strongly reduces the magnitude of the westerly intensified northward flow at the coast; the maximum transport of the northward current appears at the foot'of the continental rise.

HEAT TRANSFER IN STEADY TWO-DIMENSIONAL BÉNARD CONVECTION

Richard C. J. Somerville

Consider the following dimensionless Boussinesq system (Chandrasekhar, 1961):

$$\nabla \cdot W = 0 \tag{1}$$

$$W \cdot \nabla W + \nabla p - \sigma R T \, \hat{k} - \sigma \nabla^2 W = 0 \tag{2}$$

$$W \cdot \nabla T - \nabla^2 T = 0 \tag{3}$$

Here σ is the Prandtl number, R is the Rayleigh number, $\hat{k}$ is the vertical unit vector, and W, p, T are scaled velocity, pressure, and temperature variables respectively. This system may be solved numerically in two space dimensions (x, z) subject to the boundary condition

$$W = 0, \quad T = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{on} \quad z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}$$

and. the periodicity condition

$$\begin{pmatrix} W \\ T \end{pmatrix}(x, z) = \begin{pmatrix} W \\ T \end{pmatrix}(x + \lambda, z) \tag{5}$$

where λ is a parameter.

To each solution of this system there corresponds a value of the Nusselt number N:

$$N(R, \sigma, \lambda) = \frac{1}{\lambda} \int_0^\lambda \left(T \mathbf{m} \cdot \mathbf{IV} - \frac{\partial T}{\partial z} \right) dx \qquad (6)$$

Physically, N is a heat transport, which is invariant with depth z, as may be seen by integrating (3), subject to (1) and (5). In experiments (e.g., Krishnamurti, 1970), N and λ have been measured in approximately steady roll convection in the parameter ranges

$$\sigma = 0.7 \text{ (air)} \quad \text{and} \quad \sigma \gtrsim 6.8 \text{ (water, oils)} \qquad (7)$$

$$R_c = 1708 < R \lesssim 22,600, \qquad (8)$$

The upper bound in (8) is from Busse's (1967) theory. Observed values of λ depend on R, σ, the history of the flow, and several experimental details; but seem to lie between 2 (at $R = R_c$) and values slightly in excess of $\lambda_{max}(R)$, Busse's maximum permitted wavelength for infinite σ:

$$2 < \lambda \leq \lambda_{max}(R) \qquad (9)$$

The purpose of this note is to give an accurate explicit formula for N(R, σ, λ), based on numerical solutions of the above system, so as to facilitate comparisons with experimental and theoretical determinations of N.

$$\text{Let } \delta(\sigma) = \begin{cases} 1 & \text{for } \sigma = 0.7 \\ 0 & \text{for } \sigma \gtrsim 6.8 \end{cases}$$

Then the Nusselt number N(R, σ, λ), as determined by numerical solutions of (1) - (6) in the parameter ranges (7) - (9), is given by

$$N(R, \sigma, \lambda) = \left(\frac{R}{R_c} \right)^{0.385 + 0.9 \frac{R_c}{R}} \cdot + 0.01 \frac{R}{R_c} \left\{ \delta(\sigma) + 2.3 \left[2 - \lambda \right] \left[1 + \frac{\delta(\sigma)}{3} \right] \right\} \qquad (10)$$

The formula (10) may be misleading outside the parameter ranges (7) - (9). It is meaningless, for example, that the exponent $\longrightarrow 0.385$ as $R \longrightarrow \infty$. Inside the ranges (7) - (99, however, the formula and the numerical models

used to solve (1) - (5) are sufficiently accurate so that the error in the formula (10) is at most about 2%.

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Busse, F. H., J.Math.Phys. 46: 149 (1967)
Chandrasekhar, S. Hydrodynamic and Hydromagnetic Stability, Oxford (1961)
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Lipps, F. B. and R. C. J. Somerville, to appear.

PREFERRED MODES IN CONVECTION

Richard C. J. Somerville

This study is concerned with steady, two-dimensional (roll) convection in a Boussinesq fluid layer bounded by rigid horizontal plates maintained at different constant temperatures. This regime of Bénard convection is experimentally observed to occur for Rayleigh number R and Prandtl number σ in the ranges $1708 < R \lesssim 20,000$ and $\sigma \gtrsim 1$, in accordance with a theory of Busse (1967) for $\sigma \to \infty$. This theory also predicts that rolls are stable only if their wavelength λ lies inside a certain closed region (interior to the neutral. curve of marginal stability theory) in the R, λ plane, Many experiments (e.g., Krishnamurti, 1970) at finite σ, while differing in detail, confirm this prediction, but also show that:

(a)   the wavelength exceeds its critical value ($\lambda > \lambda_c$,

where $\lambda_c \fallingdotseq 2.016 \fallingdotseq \lambda$ at R $\fallingdotseq R_c \fallingdotseq 1708$);

(b)   the excess increases with Rayleigh number $\left(\frac{\partial \lambda}{\partial R} > 0\right)$;

(c)   this increase becomes more marked as σ is decreased$\left(\frac{\partial}{\partial \sigma}\left(\frac{\partial \lambda}{\partial R}\right) < 0\right)$.

In the absence of adequate theoretical explanations for these properties, the author and F. B. Lipps (ESSA) have sought numerical solutions of the Boussinesq equations, When initial-value problems are integrated in a domain of horizontal dimension L $\gg \lambda_c$, we find:

(1)   Integrations in two space dimensions do not display these properties;

(2)   Preliminary results in three space dimensions display at least property (a).

These results not only indicate that the wavelength selection mechanism is intrinsically three-dimensional, but also suggest that it exists within the standard theoretical framework, and that it is not necessary to invoke effects of sidewalls, imperfectly conducting boundaries, or other experimental "contaminants". We have also studied the parametric dependence of two-dimensional solutions on λ (forcing any desired stable value of λ by fixing initial conditions and/or L) and find:

(3)   No obvious extremum principle leads to properties (a,b,c). Maximum heat transport, for example, violates (a);

(4)   Nusselt number N depends sufficiently strongly on λ to explain much of the discrepancy between experimental determinations of N(R) and many previous numerical determinations which assumed $\lambda = \lambda_c$, thus ignoring (a) and (b);

(5) Result (4) and property (c) explain why early numerical studies also failed to find the well-known experimental dependence of N on σ .

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Acknowledgement:

ESTIMATE OF CONVECTIVE TRANSFER

Edward A, Spiegel

The following results are offered for the Nusselt number in Boussinesq convection. They have been obtained in collaboration with D. O. Gough and J. Toomre.

Let the function f_i^d (x,y) be a convective plan form for a horizontal wave number a_i ; thus

$$ \nabla_i^2 f_i^d = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} \right) f_i^d = -a_i^2 f_i^d . \tag{1} $$

For a given a_i the f_i^d span the appropriate subspace of allowable plan forms. Let f_i be the most general linear combination of the f_i^d for fixed i .

Note that for $a_i \neq a_j$, $\overline{f_i f_j} = 0$ where the bar denotes horizontal average. Normalize so that

$$ \overline{f_i f_j} = \delta_{ij} . \tag{2} $$

Let us write

$$w(x,y,z,t) = \sum_i f_i(x,y) W_i(z,t)$$

$$\theta(x,y,z,t) = \sum_i f_i(x,y) \Theta_i(z,t), \text{ etc.}$$

where w is vertical velocity component and $\theta = T - \overline{T}$. Then the Boussinesq equations become

$$\frac{1}{\sigma} \mathcal{D}_k \frac{\partial W_k}{\partial t} + \frac{1}{\sigma} \sum_{\ell,m} \frac{C^{k\ell m}}{(a_\ell a_m)^2} \left[a_{mk\ell} \mathcal{D}_\ell \frac{\partial W_\ell}{\partial z} + (a_{mk\ell} + a_{k\ell m}) \frac{\partial W_m}{\partial z} \Theta_\ell W_\ell \right] =$$

$$= -R a_k^2 \Theta + \mathcal{D}_k^2 W_k , \qquad (4)$$

$$\frac{\partial \Theta_k}{\partial t} + \sum_{\ell,m} \frac{C^{k\ell m}}{(a_k a_m)^2} \left[a_{k\ell m} \frac{\partial W_m}{\partial z} + 2 a_k^2 a_m^2 W_m \frac{\partial \Theta_\ell}{\partial z} \right] = \beta W_k + \mathcal{D}_k \Theta_k , \qquad (5)$$

$$\frac{\partial \overline{T}}{\partial t} + \sum W_m \Theta_h = \frac{\partial^2 \overline{T}}{\partial t -} \qquad (6)$$

where

$$\mathcal{D}_k = \frac{\partial^2}{\partial z^2} - a_k^2 , \quad a_{k\ell m} = a_k^2 (a_\ell^2 + a_m^2 - a_k^2), \quad C^{ijk} = \frac{1}{2} \overline{f_i f_j f_k} \qquad (7)$$

and, as usual,

$$\sigma = \frac{\nu}{k} , \quad R = \frac{g \alpha}{k \nu} \Delta T d^3 . \qquad (8)$$

We have proceeded by retaining only a few terms in the expansion (**3**) and solving (49 - (6) numerically, In the case where only one term is retained, a fairly simple set of equations, similar to one reported previously, is found, These one-mode equations have been solved numerically and for a variety of initial conditions the solutions tend to a steady state. Accordingly, the steady equations have been examined in detail. These are

$$(D^2 - a^2)^2 W = R a^2 \Theta + \frac{C}{\sigma} \left[2(DW)(D^2 - a^2)W + W(D^2 - a^2)DW \right], \qquad (9)$$

$$(D^2 - a^2) \Theta = -\beta W + C \left[2 W D \Theta + \Theta DW \right], \qquad (10)$$

$$\beta + W \Theta = N_1 , \qquad (11)$$

where

$$D = \frac{d}{dz}, \quad \beta = -D\bar{T}, \quad C = C^{111}, \quad a = a_1, \quad W = W_1, \quad \Theta = \Theta_1 \tag{12}$$

and N_1 is the one-mode Nusselt number, or non-dimensional total heat transfer. These equations have been obtained by Roberts using a method of Prigogene. If simple approximations for f_1 are used C can be explicitly found. For rolls and rectangles, C = 0; for hexagons $C = 6^{-\frac{1}{2}}$.

For $R \to \infty$ and a, C, σ all O(1) we find from matched asymptotic expansions that

$$N_1 \sim \left[\frac{3}{5} \left(\frac{2}{\pi} \right)^6 \frac{\sigma}{\sigma + 4C/\pi^2} Ra^2 \left(\ln Ra^2 - 9 \ln \ln Ra^2 \right) \right]^{\frac{1}{6}}. \tag{13}$$

Further study reveals that this approximation holds for large a so long as a does exceed $qR^{\frac{1}{4}}$ where the number $q \sim \frac{1}{2}$. Thus the most heat transport obtained from one mode varies like $R^{.3}(\ln R)^{.2}$, which is reasonably representative of some experiments. The choice of $a = O\left(R^{\frac{1}{4}}\right)$, though it leads to a fair R-dependence of N, does not seem to be indicated in other respects by experiments, though the evidence is not convincing. However, at low R, a seems, if anything, to decrease with increasing R. With $a = O(1)$ we have $N_1 \sim R^{.2}(\ln R)^{.2}$ which does not accord with experiments at large R. It seems reasonable to expect to remedy this deficiency with further modes. Let us do this in an approximate way.

We take $a = O(1)$ (for example $= a_c$).

For one mode at large R we have

$$, \quad = \propto \left(R \ln R \right)^{\frac{1}{6}} \tag{14}$$

where $\propto$ is given in (13). Introduce successively more modes taking at each step the mode that will most modify N. The modes that do this will be those which can break down the thermal boundary layer. Such modes will be excited both by the fluctuating and mean field interactions of (4) - (69.

If the viscous boundary layer is thinner than the thermal boundary Payer, the fluctuating terms will probably excite small scale modes that can do this effectively when the Reynolds number of the first mode is high enough. If the thermal boundary layer is thinner, the mean-field term, βW, will be the crucial one. The latter case occurs at large enough σ and let us consider only that. We may then proceed with the help of Malkusian arguments. The second mode sees $\alpha\,\overline{T}$ like:

Thus it derives its input from layers which in total are N^{-1} the total layer thickness. It therefore sees an effective Rayleigh number

$R_{eff.} = R/N_1^3$. We use (14) and find a heat transport for 2 modes like

$$N_2 = N_1 \times \left[\propto \left(R_{eff} \ln R_{eff} \right)^{\frac{1}{3}} \right] = \propto^{1 + \frac{2}{3}} R^{\frac{1}{3}\left(1 + \frac{2}{3}\right)} \left(\ln R \right)^{\frac{1}{3}\left(1 + \frac{2}{3}\right)} . \tag{15}$$

Likewise a third mode sees $R_{eff} = R/N_2^3$ so for 3 modes

$$N_3 = \propto^{1 + \frac{2}{3} + \frac{4}{25} } R^{\frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{25}\right)} \left(\ln R \right)^{\frac{1}{3}\left(1 + \frac{2}{3} + \frac{4}{25}\right)} \tag{16}$$

Evidently, for n modes,

$$\blacksquare \qquad \left(\propto^5 R \ln R \right)^{\frac{1}{5}\left[1 - \left(\frac{2}{5}\right)^n \right]} \tag{17}$$

Of course, excitation of an infinite number of modes by mean fields will not be possible. We can only have n modes where for the $(n+1)^{th}$ mode

$$R_{eff} = \frac{R}{N_n^3} \lesssim R_c \approx 1708. \tag{18}$$

This leads us to

$$\left(\propto^5 R \ln R \right)^{\frac{2}{5}^n} \cong \propto^5 R_c \ln R. \tag{19}$$

and hence to

$$N \cong \left(\frac{R}{R_c}\right)^{\frac{1}{3}} \approx \left(\frac{R}{1708}\right)^{\frac{1}{3}},$$

a result which enjoys a certain popularity. It is worth noting that the a's seem reasonable as well. The a of the first mode, call it 4, is arbitrarily taken $O(1)$, (a_c say). For the second mode the unit of length was seen to be $O(N^{-1})$. Hence for the second mode, $a_2 = N_1 a_1 = O(R^{\frac{1}{3}}(\ell n \bar{R})^{\frac{1}{6}})$ etc. For the nth mode $a_n \cong N_{n-1} a_1 = O(R^{\frac{1}{3}})$ for large R. These results should not yet be compared with experiment since (13) holds only for R much larger than is usually attainable in experiment. For this purpose we must use our numerical solutions of (9) - (11). These details will be presented elsewhere *in extenso* if Douglas and Jüri can ever be distracted from their more pressing responsibilities.

THERMODYNAMICS AND COSMOLOGY

Edward A. Spiegel

In cosmology one considers in the simplest models a perfect fluid in homogeneous, isotropic expansion. The state of the system is then characterized by a single scalar function of the cosmic time, t. Thus, if at $t = t_o$ two infinitesimal elements of the cosmic fluid are a distance r_o apart, at a later time their separation is

$$r = \frac{R(t)}{R(t_o)} r_o$$

where $R(t)$ is the scale factor of the universe. The evolution of R is a dynamical problem of cosmology and evidently this depends on your theory of gravity. Even the interpretation of R depends on the theory. Here I have given a Newtonian interpretation but R appears also in relativistic cosmology

with a different interpretation, For the purposes of this discussion we shall just assume that R is a given function, Evidently with the interpretation given here, the cosmic density varies like

$$\rho(t) = \frac{R^3(t_o)}{R(t)} \rho(t_o).$$

A question of some interest is, What can be said of the thermodynamics of this cosmic fluid? Of course, this can be discussed as realistically as you wish but with no guarantee of success. Here I want to mention some problems that arise in one of the most idealized models, namely, that of a gas of structureless particles which collide elastically. The problems that arise even in this case have been discussed by only a few authors and Engelbert Schüking and I have been puzzling over them for a while (Comments on Astrophysics and Space Physics, 11: 121 (1970)). Some people do not agree that these problems are to be worried about, but here they are,

For a (special) relativistic gas which is not expanding, one can use the usual equilibrium distribution

$$f = A e^{-\beta E}$$

where $\beta = (kT)^{-1}$ and E is the energy of a particle. The relativity enters in the formula

$$E = \sqrt{m^2 + p^2}$$

where the units are chosen so that c = 1 and where m is the rest mass of a particle and p is its momentum. In that case the normalization becomes

$$A = \frac{N}{4\pi m^2 kTK_2(m/kT)}$$

where N is the particle density and K_2 is a special Hankel function. In the limit $p^2 < m^2$ we find

$$f = \frac{N}{(2\pi mkT)^{3/2}} e^{-\beta p^2/2m},$$

which is the classical expression.   In the opposite limit we have

$$f = \frac{\pi^2 N}{(2\pi kT)^3} e^{-\beta |p|} .$$

This would be the correct distribution for photons were it not for E-B

statistics,

Now we turn to the expanding case with R(t) specified. In a

Minkowski space expanding like R(t) we let p be the momentum relative to

the expanding coordinates. If you want to visualize the situation kin-

ematically picture two sets of observers, One set remains fixed with respect

to each other; the members of the other set move apart from each other with

their mutual distances increasing like R(t). Let p be the momentum of a

free particle moving relative to the expanding set of observers. It can be

shown that for such a particle $|p| R$ is conserved.

In cosmology this result is used as follows: For the classical gas

equilibrium is maintained if $\beta E \propto |p|^2 / T$ is invariant. Since $|p| \sim R^{-1}$ we

have $T \propto R^{-2}$ which describes the adiabatic cooling. $N/T^{3/2}$ then is also con-

stant since $N \propto R^{-3}$. Likewise for an ultrarelativistic gas to have $\beta |p|$

invariant we need $T \propto R^{-1}$ and this ensures N/T^3 is invariant. Thus in cos-

mology a classical gas is said to cool with $T \propto R^{-2}$ while a photon gas

with $T \propto R^{-1}$. This is a primitive description, but it can be made precise

with the use of the relativistic Boltzmann equation,

What of the relativistic gas? There seems to be no choice of T(R)

which makes βE invariant. Hence - and there are more rigorous arguments -

it has been concluded that an expanding relativistic gas has no equilibrium

in the quasistatic sense described here, It follows that the expanding rela-

tivistic gas is dissipative. The assumption of a perfect fluid for the cos-

mological problem would then be unjustified, though it is probably a good

approximation most of the time, Macroscopically the dissipation must result from bulk viscosity since the fluid is homogeneous and shear viscosity and conduction do not occur. You can think of an origin of this bulk viscosity like that for a classical fluid. The usual reason for bulk viscosity is that different degrees of freedom of the particles adjust to changes on different time scales. If the classical and ultrarelativistic particles can be visualized in this way the bulk viscosity seems more plausible.

Still we may wonder whether the bulk viscosity is really an artifact of the Boltzmann equation for the relativistic case. Suppose we widen the Boltzmann notions a bit. There, the distribution depends only on the momenta. But there is another property of the particles that differs from particle to particle. This is the proper time.

As an example consider a relativistic gas in a box. At $t = 0$ give each particle a clock reading 0. Clearly at any later time t the clock readings will not all be the same since the particles move differently. Indeed, you can ask what the distribution of clock readings is at any time, t For an ergodic gas, as $t \to \infty$ the dispersion of clock readings tends to zero since all particles have the same average history in some sense.

Since the ages or proper times differ for different particles we may ask what happens if we include this parameter as a statistical variable. This complicates things a bit but one simple fact emerges.

Consider the model

$$R(t) = R_0 (1 + \alpha t)$$

where R_0 and α are constants. Let τ be the age of a particle reckoned from 0 at $t = 0$. Then for a free particle in this space

$$I = \frac{R}{R_o} E - m \alpha \mathcal{T}$$

is conserved, Note also that if two particles collide and if the collision

lasts no time then the sum of their I's is conserved since the sum sf the

E's is conserved and the $\mathcal{T}$'s do not change.

Thus

$$f = A e^{-\beta I}$$

is the solution of the appropriate "Boltzmann equation", though the details

are messy, Note that in the non-expanding ease I = E and the usual equili-

brium is recovered. Thus there does seem to be a sort sf equilibrium for

a relativistic expanding gas, but we are nor sure we understand it at all,

One curious point is that in a vague sense this kind of thing is

not without precedent, In viscoelasticity it is known that an alternate

way to discuss dissipative effects is through history terms. History terms

also play a role in two component fluids such as dusty gases, Can there be

a significance to these vague similarities?

GENERALIZATIONS OF THE ROTATING FLAME EFFECT
WITH APPLICATION TO TORNADO GENESIS

Melvin E. Stern

Heat sources and sinks propagate in an azimuthal direction with fre-

quency (ω) relative to a basic fluid state of vertical thickness (h),

static stability (s) and rotation rate (f/2). If $(f^2 - \omega^2)(\omega^2 - gs)^{-1} > 0$

the forced azimuthal perturbations pump kinetic energy and angular momentum

(ω-directed) radially outwards, Thus the Reynolds stress, computed from

linear theory, leads to a mean relative spin of the air in the vicinity sf

the heat sources and this is in the opposite sense to the relative motion of

the sources. In this generalization of the "rotating flame effect" viscosity

is neglected, It is suggested that such a horiacntal vortex can be ampli-
fied as the air "passes" through a vertically convecting cumulonimbus cloud,
The kinetic energy and heat generated by the mother cloud is pumped ineo
the far-field by the azimuthal perturbations of the vortex, and angular momen-
tum is thereby separated, It is suggested that the angular momentum accumu-
lating within the radius of the mother cloud is dissipated by the smaller
scale tornado funnel.

PHASE BEHAVIOR OF LIGHT GAS MIXTURES AT HIGH PRESSURES

William B. Streett

If solid surfaces exist beneath the visible clouds of the major
planets, they may be expected to exist at depths and pressures at which the
component gas mixtures solidify under their own weight. An understanding
of the solid-fluid phase behavior in mixtures of light gases at high pressures
is therefore essential to the solution of the problem of deep atmosphere
structures in these bodies, For Jupiter and Saeurn, the mixture of primary
interest is hydrogen-helium, Although experimental results for this system
are limited to low pressures, several possible extrapolations of the phase
diagram are suggested by the results of high pressure experiments on other
helium binary systems, The suggested phase diagrams have been used to dev-
elop a structural model for a hydrogen-helium atmosphere, In this model,
gravitational separation of coexisting phases results in a layered structure,
and it is shown that masses of hydrogen-rich solid can exist in dynamic and
thermodynamic equilibrium with a fluid layer of equal density but higher
helium content, This picture may be relevant to the floating raft concept
of Jupiter's Great Red Spot,

MOTION OF JUPITER'S RED SPOT
(OCEANOGRAPHY OF JUPITER)

George Veronis

A thermodynamic justification can be made for the hypothesis that a layer of hydrogen-rich solid can form at a level of a few thousand km below the visible surface of Jupiter and that the solid is gravitationally stable (in the dynamic sense) at the level of formation, If a topographic or other such feature of the solid surface of Jupiter causes an accumulation of this material into a massive object and if the motion of this stable Cartesian diver is analyzed, the observed motion of the Red Spot can be rationalized in terms of the model, Vertical oscillations of the Cartesian diver are accompanied by longitudinal changes of position because of the associated changes in the angular momentum of the object. Speculation about the fluid motions caused by the motion of the diver leads to predicted changes in the size and location of the Red Spot which agree with observed changes。

THE MICRO-OCEANOGRAPHY OF BERMUDA

Carl Wunsch

An attempt to measure internal waves in the main thermocline at Bermuda (Wunsch and Dahlen, 1970) leads us to some peculiar physical processes that appear to go on around this mid-ocean island, The island has been explored in 3 cruises (ATLANTIS II, Cruise 47 in October 1968, GOSNOLD Cruise 144 in July 1969, and GOSNOLR Cruise 147 in July 1970), using an STD, current meters on buoys, and parachute drogues,

The internal wave measurements indicated no detectable coherence between temperature sensors 1 km apart for periods shorter than about 8-10

hours, a disconcerting feature when one calculates theoretical wavelengths exceeding 1 km for internal waves of periods longer than 30 min. The data show no inertial peak, no diurnal peak, a strong semi-diurnal tide with random phase and large amplitude (about 20 m displacement of isotherms in a small frequency band around 12.42 hours) and an unexplained strong peak at about 18-20 hours period.

The hydrographic surveys of the island revealed that the temperature and salinity microstructure is a strong inverse function of distance from the island, apparently accounting for the low internal wave coherence. It was also found that the large scale current field around Bermuda is very strong, with high shear and very time-dependent. In the first two cruises it was noted that the mixing region (defined as the area of maximum microstructure) was strongest on the northern side of Bermuda. On the second cruise, the current field (basically from the west) was also much more intense there. The buoy data was also quite perplexing as two current meters placed a mile apart at the same depth show notably different mean displacements. **Thus,** the whole island area appears to be exceedingly complex and variable.

Some progress toward understanding the problem of an island can be obtained by two hypotheses. We can postulate an internal wave field propagating into the island and breaking (Cacchione, 1970) so as to generate a microstructure. The corresponding change in the mass field generates a geostrophic current penetrating into the interior, a distance given by the baroclinic Rossby radius of deformation. A mechanism like this was used by Wunsch, 1970, to explain the large scale ocean mixing. The Reynolds stress effects of the incoming internal waves can then also generate apparent mean

Eulerian velocities leading to Barge differences between current meters not very Ear apart, (Wunsch, 1971). The intensification of the microstructure and currents on the north is then accounted for by postulating a stronger internal wave field from the worth, a rather unattractive *ad* hoe hypothesis, However, estimates of the energy flux into the island based on the temperature measurements give reasonable agreement with the change in potential energy in the current near the island, and the estimated time a given water parcel is in the vicinity of the island,

A perhaps more attractive, if less analytically tractable problem, is to hypothesize that the current shear flow past the island becomes unstable, The density gradients are then such that the mixing that occurs through the instability tends to intensify the (geostrophic) current when the island is to the right, and weakens it when the island is to the left. Thus, a current from the west would be strongest on the north side, and weakest on the south side, with the reverse being true for a flow from the east. The hypothesis is consistent with the flow features in July, 1970, and 1969,

The 18-20 hour spectral peak is difficult to explain, An analysis of an infinite cone in a rotating stratified ocean showed no non-singular trapped modes for periods between the inertial and Brunt, with many modes above and below this range, The inertial period at Bermuda is 22.3 hours, Analysis of the tide gauge records at Bermuda shows no sign of the inertial resonsnce predicted by Longuet-Higgins, and the drogue data show no discernible circulation around the island in the presence of the mean flow,

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MICRO-SYMPOSIUM ON TURBULENCE

MODEL CONSTRUCTION

Robert H. Kraichnan

A number of statistical approximations proposed for turbulence in recent years can be characterised by generalized Langevin-type equations for a model velocity field. These model equations have in common that the actual nonlinear terms of the Navier-Stokes equation are replaced by a dynamical damping term (generalized eddy viscosity), and a random forcing term, acting on each Fourier mode (or other appropriate mode) of the system. The damping and forcing terms are determined wholly by ensemble averages of the turbulent velocity field and thus are insensitive to the fluctuations in any one realization. As a result, the model amplitude equation, in contrast to the Navier-Stokes equation, is effectively linear in stochastic quantities. Thereby it leads to closed statistical equations which determine first and second order moments.

A model representation of the direct-interaction approximation is obtained by a particular choice of random forcing and damping terms, such that the Batter describes a damping with memory. In the case of isotropic turbulence, this model is most simply stated as follows. The Fourier-space form of the Navier-Stokes equation is

$$(\partial/\partial t + \nu k^2)u_i(\underset{\sim}{k},t) = -\frac{i}{2} P_{ijm}(\underset{\sim}{k}) \sum_{\underset{\sim}{p}+\underset{\sim}{q}=\underset{\sim}{k}} u_j(\underset{\sim}{p},t)u_m(\underset{\sim}{q},t), \tag{1}$$

where

$$P_{ijm}(\underset{\sim}{k}) = k_m P_{ij}(\underset{\sim}{k}) + k_j P_{im}(\underset{\sim}{k}), \qquad P_{ij}(\underset{\sim}{k}) = \delta_{ij} - k_i k_j/k^2.$$

The direct-interaction model equation is

$$(\partial/\partial t + \nu k^2)u_i(\underset{\sim}{k},t) + \int_0^t \eta(k,t,s)u_i(\underset{\sim}{k},s)ds = -iP_{ijm}(\underset{\sim}{k}) \sum_{\underset{\sim}{p}+\underset{\sim}{q}=\underset{\sim}{k}} \xi_j(\underset{\sim}{p},t)\xi_m'(\underset{\sim}{q},t), \tag{2}$$

where the damping-with-memory function is

$$\eta(k,t,s) = k\iint_\Delta b_{kpq} G(p,t,s)U(q,t,s)pqdpdq. \tag{3}$$

In these equations, $\underset{\sim}{\xi}$ and ξ' are random solenoidal vector fields statiseitally independent of each other and of the initial velocity field $\underset{\sim}{u}(k, t=0)$, but constrained by the requirement

$$\left\langle \xi_i(\underset{\sim}{k},t)\xi_j^*(\underset{\sim}{k},t')\right\rangle = \left\langle \xi_i'(\underset{\sim}{k},t)\xi_j'^*(\underset{\sim}{k},t')\right\rangle = \left\langle u_i(\underset{\sim}{k},t)u_j^*(\underset{\sim}{k},t')\right\rangle \tag{4}$$

G and U are respectively the response and covariance scalars of $u_i(k,t)$, $\iint_\Delta$ denotes integration over all p,q which form a triangle with k, and $b_{kpq} = (p/k)(xy + z^3)$, where x,y,z are the cosines of the internal angles opposite k,p,q respectively.

Given the requirement (4) on the random forcing term (the right-hand side of (2)), the form of (k,t,s) is uniquely determined to be (3) (apart from a slight ambiguity in the value of b_{kpq}) by the requirement that the ensemble-averaged energy input from the random forcing balance the drain from the damping when summed over all $\underset{\sim}{k}$, thus preserving the conservation property of the Navier-Stokes equation. Equations (2)-(4) easily yield the closed direct-interaction equations for G and U.

Simpler model amplitude equations, in which the damping term has no memory, are obtained by making the random forcing term have the same general form as in (2) but, in contrast to (4), requiring it to be a white noise in time and, again, fixing the damping term by requiring energy conservation. A particular model of this kind yields Edwards' turbulence theory and extends the latter to non-steady-state turbulence. In contrast to the direct-interaction model, these white-noise models do not give a qualitatively

faithful representation of the full space-time covariance of the $\underline{u}$ field,
However, they embody more flexibility and can be constructed to yield the
Kslmogorov inertial range spectrum, which the direct-interaction model
does not,

The generalized Langevin-type models promise to be useful in three
ways, First, they can serve directly as approximations for low-order
statistical properties of turbulence, Here the fact that a model amplitude
equation underlies the eventual statistical equations insures important
consistency properties (under suitable precautions), Second, they can serve
as the zeroth approximation in a systematic expansion scheme for the exact
statistical dynamics, for example by introducing a parameter whose value
measures the proportions of a mixture of the true right-hand side (1) and
the model right-hand side (2). Here the fact that a model amplitude equa-
tion exists makes the analyticity properties of an expansion in the parameter
accessible, Third, the models can be used to give representations of sub-
grid motions in turbulence simulations, with the advantage, over simple
eddy-viscosity devices, that energy conservation is maintained,

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STOCHASTIC MODELS

Cecil E. Leith

An important **class** of turbulence approximations is based on the construction of **stochastic** models. We examine such models **for** a quadratically non-linear system of N variables evolving according to

$$\dot{u}_\alpha(t) = \sum_{\beta,\gamma} A_{\alpha\beta\gamma} u_\beta(t) u_\gamma(t) \qquad (1)$$

with constraints on the non-linear interaction coefficients $A_{\alpha\beta\gamma}$ to provide conservation laws. The **single-time** statistical properties of ensembles of **solutions** of **Eq.(1)** is completely determined by a Liouville equation. for the ensemble probability distribution $P(\underset{\sim}{u},t)$ defined in the phase space of the system, namely,

$$\frac{\partial}{\partial t} P(\underset{\sim}{u},t) = -\sum_{\alpha\beta\gamma} A_{\alpha\beta\gamma} u_\beta u_\gamma \frac{\partial P(\underset{\sim}{u},t)}{\partial u_\alpha} \qquad (2)$$

In practice for large N we cannot solve this linear differential equation and must seek approximations.

A stochastic model for the non-linear system **(1)** is a system evolving according to the random linear equation

$$\dot{u}_\alpha(t) = -\int_0^t \gamma_\alpha(t,s) u_\alpha(s)\,ds + f_\alpha(t) \qquad (3)$$

where $\gamma_\alpha(t,s)$ is a non-random eddy visco-elasticity and $f_\alpha(t)$ a random Gaussian eddy forcing term with moments $\langle f_\alpha(t) \rangle = 0$ and $\langle f_\alpha(t) f_\alpha(s) \rangle = F_\alpha(t,s)$ specified. The functions $\gamma_\alpha(t,s), F_\alpha(t,s)$ are chosen to simulate as well as possible the non-linear coupling between modes.

The direct-interaction approximation uses the choice

$$F_\alpha(t,s) = 2\sum_{\beta,\gamma} A_{\alpha\beta\gamma}^2 \, U_\beta(t,s) \, U_\gamma(t,s) \qquad (4)$$

$$\gamma_\alpha(t,s) = -2\sum_{\beta\gamma} A_{\alpha\beta\gamma}\left[A_{\beta\gamma\alpha} g_\beta(t,s) U_\alpha(t,s) + A_{\gamma\alpha\beta} g_\gamma(t,s) U_\beta(t,s)\right] \qquad (5)$$

where $g_\beta(t,s)$ is the Green's function for Eq. (3) and $U_\beta(t,s) = \langle u_\beta(t) u_\beta(s) \rangle$.

The eddy-damped Markovian **approximation** is based on the even cruder stochastic model in which the random **forcing** is white and the eddy visco-elasticity becomes an eddy viscosity, The model equation becomes

$$\dot{u}_\alpha(t) = -\tfrac{1}{2} Y_\alpha(t) u_\alpha it) + f_\alpha(t) \tag{6}$$

with

$$F_\alpha(t) = 4 \sum_{\beta\gamma} D_{\alpha\beta\gamma}(t) A_{\alpha\beta\gamma}^2 U_\beta(t) U_\gamma(t) \tag{7}$$

$$Y_\alpha(t) = -4 \sum_{\beta,\gamma} D_{\alpha\beta\gamma}(t) A_{\alpha\beta\gamma} \left[A_{\beta\gamma\alpha} U_\gamma(t) + A_{\gamma\alpha\beta} U_\beta(t) \right] \tag{8}$$

where $D_{\alpha\beta\gamma}(t)$ is an eddy-damping time for triple moments. Here $F_\alpha(t)$ measures the strength of the white **forcing**, $\langle f_\alpha(t) f_\alpha(s) \rangle = F_\alpha(t) \delta(t-s)$.

The Markovian stochastic model Eq. (6) induces the random Liouville equation

$$\frac{\partial}{\partial t} P(\underset{\sim}{u},t) = \sum_\alpha \partial \left\{ \left[-Y_\alpha(t) u_\alpha + f_\alpha(t) \right] P(\underset{\sim}{u},t) \right\} / \partial u_\alpha \tag{9}$$

but since $f_\alpha(t)$ is white the average $\langle P(\underset{\sim}{u},t) \rangle$ over the $f_\alpha(t)$ ensemble satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t} \langle P(\underset{\sim}{u},t) \rangle = \sum_\alpha \frac{\partial}{\partial u_\alpha} \left\{ \left[Y_\alpha(t) u_\alpha + \tfrac{1}{2} F_\alpha(t) \frac{\partial}{\partial u_\alpha} \right] \langle P(\underset{\sim}{u},t) \rangle \right\} . \tag{10}$$

NUMERICAL SIMULATION OF INCOMPRESSIBLE TURBULENCE

Steven A, Orszag

The numerical simulation of turbulence usually involves simulation of complicated flows in simple geometries with simple boundary conditions, In this case, it is found that Galerkin approximation procedures have some important advantages over finite-difference approximations, Two cases of importance are examined: flows in rectangular geometries using Fourier expansions, and flows in spherical geometries using surface harmonic expansions, Transform tricks are described that make the number of numerical operations involved in evaluation of the Galerkin approximations competitive with those involved in finite-difference approximations (cf. Orszag, Phys.Fluids Suppl. 12: 250 (1969) for the rectangular case; Orszag, J.Atmos. Sci., Sept.1970 for the spherical ease),

The principal advantage of the Galerkin approximations over finite-difference approximations is the absence of spatial-difference phase-speed errors and aliasing errors, It is shown that to attain the same accuracy with (high-order accurate) finite-difference methods as with the Galerkin method requires that about 3^n times as many degrees of freedom (where n is the number of space dimensions] be retained in the finite-difference simulations as in the Galerkin simulation, Roughly, the argument is that finite difference simulations of a wave with wavenumber k on a uniform grid spaced by Δx requires $k\Delta x \sim 1$ in order that phase speed errors not be large. In other words, there should be at least 6 grid points per wavelength, while a Fourier representation requires only two degrees of freedom (the real and imaginary parts of the complex wave) to represent the same wave without

error, Several less trivial examples were cited to substantiate the claim of increased accuracy of the Galerkin approximations. Remarkably, the Galerkin approximations even do a better job of describing the evolution of localized flow structures than do finite-difference approximations, despite the fact that each of the Galerkin basis functions is non-local in physical space,

In three space dimensions, the Galerkin methods require about 10-30 times fewer degrees of freedom to describe accurately the evolution sf a given complicated flow.

Finally, the limitations of numerical simulation are discussed, An 'honest' simulation requires that all scales through the dissipation scale be described accurately. It is shown that this requires that the number of arithmetic operations involved in numerical simulation of a flow with Reynolds number R_λ (λ = Taylor microscale) scales as $R_\lambda^6 \log R_\lambda$, implying that numerical simulation of high Reynolds number flows is not likely to be feasible on digital computers in the near future, It is suggested that 'honest' numerical simulations are useful for [a] testing theories of turbulence at low and moderate Reynolds numbers, (b) investigating properties of moderate Reynolds number flows which are then applied by assuming [hoping] for Reynolds number independence in extrapolating to high R_λ , and (c) visualizing features of turbulence as in a laboratory experiment, with the advantage that the numerical experiments may be see up for more idealized flow situations than laboratory fluids permit and the results may be analyzed more fully because numerical simulations provide complete detailed flow fields,

UPPER BOUNDS

Willem V. R. Malkus - Louis N. Howard

Various studies seeking upper bounds on heat transport by turbulent (or non-turbulent) convection are reviewed, with primary emphasis on the recent thesis of S.-K. Chan (Massachusetts Institute sf Technology, January, 1970). The Patter treats the case sf infinite Prandtl number, and uses the full momentum equation as well as the continuity equation and the entropy flux integral as constraints in the bounding variational problem. Results are obtained by boundary layer methods analogous to those used by Busse in the case of the power integral constraint. With a single horizontal wave number the asymptotic bound is proportional to $R^{3/10}$ (log $R)^{1/5}$, and the overall bound is $0.152R^{1/3}$. The optimal fields also bear a striking resemblance to observed profiles of mean temperature and r.m.s. temperature fluctuations.

CONVECTION AT ZERO PRANDTL NUMBER

Jackson Herring

At low Prandtl number, σ , a hot blob of fluid loses its temperature excess rapidly to the ambient fluid by conduction, Consequently, convection is not an efficient mode of heat transport even though the temperature stratification is so unstable as to cause vigorous motion. In the limit $\sigma \rightarrow 0$, it seems likely that rapid conductive losses will entirely prevent motions from distorting the temperature profile, The gross energetics of statistically steady flow is then a balance of energy input at low wave number and viscous dissipation at large wave number, The non-linear advection term breaks up the large scale modes, and copservatively transfers their energy to small scales where viscous dissipatfon takes effect, Equations appropriate to this regime have been derived by Ledoux, Schwarzschild, and Spiegel (1961), who expand the velocity-temperature field in a power series of σ 。 Suitably non-dimensionalized, they are,

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\vec{v} = -\nabla p - (\vec{v}\cdot\nabla)\vec{v} - \hat{k} R \frac{1}{\nabla^2} w,$$
$$\nabla \cdot \vec{v} = 0,$$
$$w = \hat{k}\cdot\vec{v}. \tag{1}$$

and

Here, σ is the Prandtl number, R the Raylefgh number, $\hat{k}$ the vertical unit vector, and $\left(1/\nabla^2\right)$ is the inverse operator to ∇^2 for free boundaries。

We have begun a numerical investigation of this regime of thermal convection, and report here on some preliminary results for the free boundary initial value problem, both for the amplitude problem and the statistical moment problem. The particular statistical approximations considered

are the direct interaction approximation [Kraichnan, 1964], the self-con-
sistent field approximation (Herring, 1966), and one of the Markovian type
approximations considered earlier this week by Dr. Kraichnan, In the
present calculations R ranged from $R_c \cong 657$ to $R = 10^4$. In the numerical
simulations up to 5 vertical wave numbers and 24 horizontal wave number
vectors were used to represent the velocity field. The horizontal wave
vectors were chosen by the rule, $\vec{\alpha} = \hat{\imath}\, n\, \alpha_o + \hat{\jmath}\, m\, \alpha_o, ((n,m) = \pm(0,1,2,\cdots))$,
$\alpha_o \cong 2^{-1/2}$, with any $|\vec{\alpha}|$ greater than a cut-off value α_M discarded. In
most calculations $\alpha_M \leqq 3\alpha_o$, although for two dimensional flows $\alpha_M = 10\,\alpha_o$
was used. Initial values for the Fourier components of $\vec{v}$, $\vec{v}_{M,\vec{\alpha}}$, were
picked either by a Gaussian random number generator, or assigned values
which represented the regular plan forms, squares and rolls.

In the statistical theories, only two vertical wave numbers and
three horizontal wave number bands were used to represent the velocity co-
variance, Horizontal homogeneity and isotropy were assumed, These results
are to be compared to the randomly excited amplitude experiments in the
limit of large numbers of $\vec{\alpha}$'s .

Results for the randomly excited amplitude experiment suggest a
runaway behavior for the system. Thus the total kinetic energy, after an
initial decrease, eventually increases without apparent bound, We were
able to follow the energy amplification by about three orders of magnitude
with good accuracy. A similar runaway behavior was also found for rolls,
The square plan-form, on the other hand, stabilized at a finite value of
energy: however, these static solutions are not stable with respect to the
runaway type if they are perturbed by a three dimensional disturbance,

The D.I. approximation also has a runaway behavior, Both the total kinetic energy and the Green's function which had a positive growth rate eventually grew indefinitely, For the two other seatistical theories considered, the S.C.F. and the Markovianized model the behavior was different; the energy tended to stabilize at a finite value, although the linearly unstable Green's function remained finite at large times,

It then appears that the D.I. approximation faithfully reproduces the qualitative behavior of this free boundary truncated system, whereas the other methods do not, We have observed a similar behavior for isotropic turbulence in which a region of (small) wave numbers destabilized by negative viscosities,

With regard to the runaway character of these free boundaries solutions, it should be observed that Eq.(1) does not prohibit such behavior, This may be seen by examining its energy equation:

$$\dot{E} \equiv \frac{d}{dt} \langle v^2 \rangle = - \langle (\nabla \vec{v})^2 \rangle + R \langle w \frac{1}{\nabla^2} w \rangle \tag{2}$$

Here, the angular brackets indicate a total volume integration, If v becomes large there is no obvious way in which one can show $\dot{E} < 0$. This contrasts with the finite σ convection equations in which the last term in (2) is replaced by $R \langle w \theta \rangle$, where θ is the temperature field. One may show from the equation of motion for θ that it is bounded by its initial distribution. Hence as $|\vec{v}|$ becomes large, $\langle w \theta \rangle \sim \sqrt{\langle w^2 \rangle}$, and $\dot{E} < 0$ at least in a large region of v- phase space,

The detailed dynamical reason for the runaway character may be due to the fact that for free boundaries the "linear harmonics" non-linear terms i e , terms that couple $\vec{v}_{m,\vec{\alpha}}$ to $\vec{v}_{vm,2\vec{\alpha}}$) make no contribution

to the equation of morfon for $\vec{\mathcal{V}}_{m,\vec{a}}$. This has the consequence that an exponentially growing roll is an exact solution to Eq.(1):

$$\vec{\mathcal{V}}(t) = Ae^{\lambda t}\left\{\hat{\imath} \frac{\pi}{a} \cos ax \cos \pi z + \hat{k} \sin ax \sin \pi z\right\}$$

$$\nabla^2\left(p + e^{\lambda t}\left(\frac{-\pi^2}{2a^2}\sin^2 ax + \frac{1}{2}\sin^2 \pi z\right)\right) = 0, \quad \frac{\partial p}{\partial m} = 0, \tag{3}$$

$$\lambda = -(\pi^2 + a^2) + \frac{Ra^2}{(\pi^2 + a^2)^2}.$$

The face that the roll solutions runaway may be understoad in terms of the **dynamical stability** sf **(3)**. These **roll type** solutions are not, however, stable with respect to the class of three dimensional motions, but these three dimensional solutions also runaway,

Finally we note thar qualitatively different results may be obtained for rigid boundaries, Fop this ease, the exponentially growing roll is not a solution to Eq.(1) and the linear harmonies manifest themselves in non-linear terms in rhe $\vec{\mathcal{V}}_{m\vec{a}}$-equation. Thus it may be possible that (1) is sensible for rigid boundaries but not for free **boundaries**.

These numerical results and **conclusions** based on them must be **considered** preliminary until **calculations** using a **much larger** set of wave number points confirm or negate them. We **plan** to improve this aspect of the **calculation** soon,

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FINITE AMPLITUDE EXPANSIONS

Bernard J. Matkowsky

We treat a number of stability problems governed by non-linear equations by a new method proposed by the author (1, 2, 3). The method, which is formal, yields the complete dynamical description of the system, and is believed to be the first to treat arbitrary initial perturbations. As an example we consider the initial-boundary value problem

$$-u_t + u_{xx} = \lambda f(u) \qquad 0 < x < \pi, \ t > 0 \qquad (1)$$

$$u(0,t) = u(\pi,t) = 0 \qquad (2)$$

$$u(x,0) = h(x) \qquad (3)$$

which is a mathematical model for the temperature distribution in a bar with a non-linear heat source (sink) of magnitude $-\lambda f(u)$. We study the stability of the equilibrium solution $u \equiv 0$, and obtain a complete description of the dynamical behavior of the system uniformly for x in $(0,1)$ and $t \geq 0$, valid in a neighborhood of $u = 0$ and λ near a critical value λ_c. Here λ_c is the value of λ , which when exceeded, yields the onset of instability. It is in addition the first bifurcation point of the associated non-linear eigenvalue problem

$$v_{xx} = \lambda f(v) \qquad 0 < x < \pi \qquad (4)$$

$$v(0) = v(\pi) = 0 \qquad (5)$$

whose solutions are stationary (time independent) solutions of (1) - (2). As a consequence of obtaining the complete dynamical description, we are able to trace the evolution in time of arbitrary initial perturbations until they either decay or grow into (new) stationary solutions of (1) - (2).

The stationary states for this problem were described asymptotically by Millman and Keller (4). We therefore see how these stationary states are generated from initial perturbations. Another problem touched upon is the Bénard problem. The asymptotic analysis for the stationary problem was treated by Malkus and Veronis (5). We discuss the time evolution to these states as well. In addition the stability or instability of the new equilibrium solutions is determined, without the necessity of performing an additional perturbation analysis.

As a final example, we discuss the nsn-linear **stability** problem for the buckling of a **compressed** elastic column (9). This example illustrates the importance of treating arbitrary initial data, since the resulting motion of the **column** depends very strongly on the character of the initial data. The essential feature of the theory is a scaling of the time variable. Then, using a two-time method, **we** systematically derive the asymptotic expansion sf the solution in power of a small parameter ϵ, which is alternatively a measure sf a **norm** of u or the nearness of λ to λ_c. The method seems to enjoy certain computational **advantages** over existing methods (6, 7, 8) and these are compared and contrasted.

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