Notes on the 1966

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

1966

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Editor's Preface

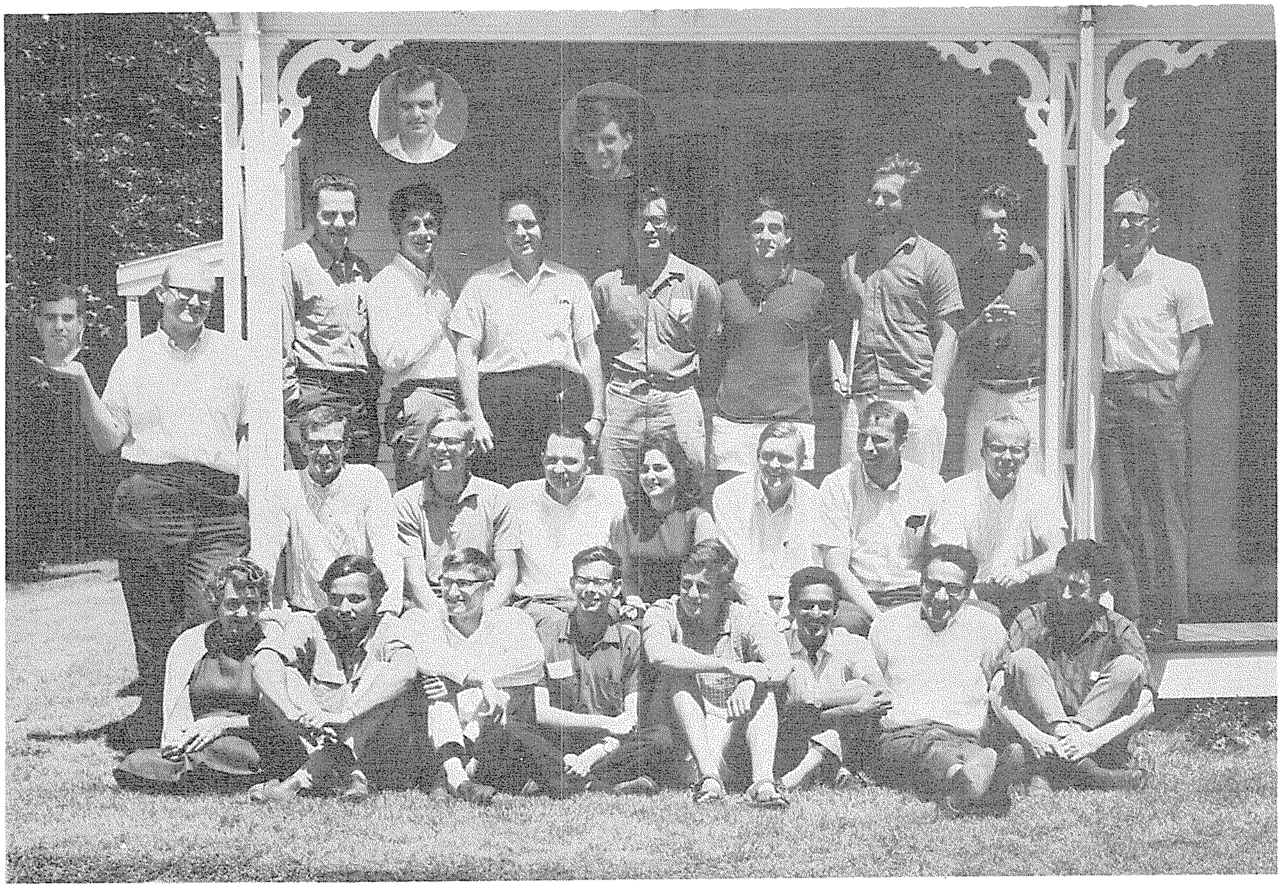
This volume contains a restatement by the Fellows of the summer program's first lecture series. The lecturers, Drs. Howard, Stern and Veronis, have introduced the participants to several aspects of geophysical fluid dynamics at the frontiers of current research. Their choice of topic and its development was to serve, on one hand, a pedagogic function and, on the other, to suggest a variety of allied unsolved problems.

Following these notes, the abstracts of the summer research seminars are recorded. These were prepared by their authors and range from brief assertions of progress to short, but complete, manuscripts. The editor's request for an extended abstract, clearly noting novel content and with proper references, has been achieved only in the mean.

Manuscript records of the research lectures of the Fellows appear in Volume II. In most instances the Fellow felt that "just a few more weeks" would have permitted him to construct a far more complete document. That is to say, Volume II is a product of enthusiams, yet to be quenched by sober afterthoughts.

Mrs. Mary Thayer has done all the work in assembling and reproducing the lectures. We are all indebted to her for her remarkable efforts in keeping the summer course running smoothly and to the National Science Foundation for its financial support of the program.

Willem V.R. Malkus



In Flight: Herring, Childress
Top Row: Orsag remnant, Toomre, Spiegel, Bisshopp, Anderson, Kraichnan, Goldreich, Thorpe, Robinson, Malkus
Middle Row: Pedlosky, Beardsley, Towne, Whitman, Howard, Field, Michie
Bottom Row: Thayer, Silk, Schmid-Burgk, Saslaw, Garrett, Rasiwala, Veronis, Morton. Absent: Finkelstein, Keller, Phillips, Stern.

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Course Lectures and Seminars

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Contents of Volume I (continued)

The Figure of the Earth (a)

Melvin E. Stern

Newton first solved the problem of calculating the eccentricity of the earth assuming the earth to be an ellipsoid of revolution. Evaluating the contour integral of the forces along two perpendicular columns centered on the minor and major axes he arrived at the figure of $\frac{1}{250}$ for the ellipticity. He did not show that the ellipsoid was an equilibrium configuration of a rotating mass of fluid. Measurements of the ellipticity yielded a figure of $\frac{1}{300}$. The difference was attributed to different moments of inertia about the two axes.

We calculate the equilibrium figure of the rotating earth assuming the density ρ to be constant. The pressure force is everywhere balanced by the gravitational force and the centrifugal force, i.e.,

$$-\nabla\left\{\frac{P}{\rho}-\Phi+\frac{\omega^2 R^2}{2}\right\}=0$$

Here P is the pressure, Φ the gravitational potential, ω the angular velocity and R the distance of a point from the axis of rotation.

On a free surface P = const.

Hence $\Phi - \frac{\omega^2 R^2}{2}$ = const. on a free surface.

We now apply an incorrect perturbation theory, neglecting the non-uniform mass distribution within the ellipsoid, and assuming the rotation rate to be small. The centrifugal acceleration is then balanced by the local gravity and the ellipticity $\frac{b-a}{b} \ll 1$,

b = major axis
a = minor axis

We then have an exact equation

$$\Phi_b - \frac{\omega^2 b^2}{2} = \Phi_a$$

and an incorrect equation

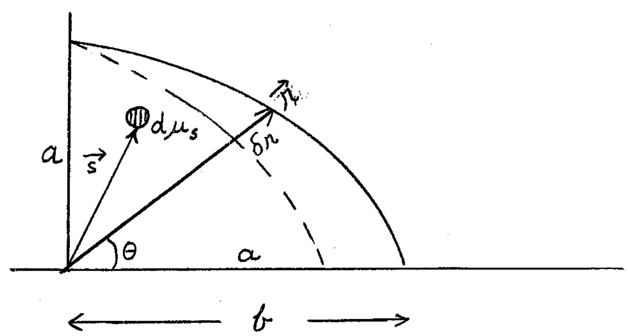
$$\Phi_b - \Phi_a \approx g(b-a)$$

The two together yield

$$\frac{b-a}{b} = \frac{\omega^2 b}{2g} \approx \frac{\omega^2 a}{2g}$$

Putting in the numerical values one obtains $\frac{b-a}{b} = \frac{1}{500}$. This shows that the change in potential between two points is not just due to their different distances from the center of mass but that the non-uniform mass distribution within the ellipsoid produces a first-order effect of the same order.

We now consider an ellipsoid of revolution with a non-uniform mass distribution:



The potential at a point $\vec{r}$ on the surface of the ellipsoid is given by

$$\Phi(\vec{r}) = -G \iiint \frac{d\mu_s}{|\vec{r}-\vec{s}|}$$

where $d\mu_s$ = an element of mass with radius vector $\vec{s}$. The volume integral can be separated into an integral over the sphere of radius a and an integral due to the "bulge":

$$\Phi(\vec{r}) = -G \underset{\substack{\text{sphere}\\ \text{of}\\ \text{radius } a}}{\iiint} \frac{d\mu_s}{|\vec{r}-\vec{s}|} - G \underset{\text{bulge}}{\iiint} \frac{d\mu_s}{|\vec{r}-\vec{s}|}$$

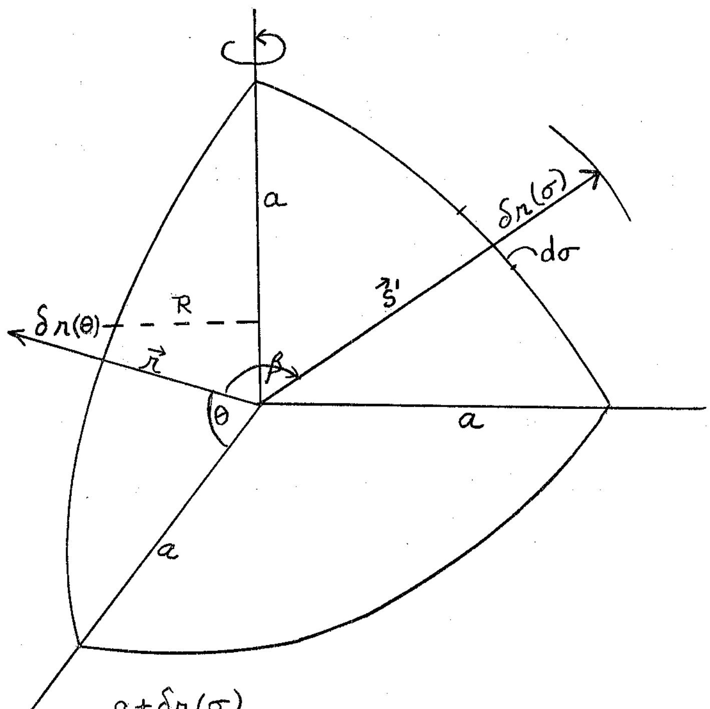
We now assume that $r(\theta) = a + \delta r(\theta)$, where θ = latitude.

Then
$$-G \iiint \frac{d\mu_s}{|\vec{r} - \vec{s}|} = - \frac{\frac{4\pi}{3} a^3 G \rho}{a + \delta r}$$

Expanding $\frac{1}{1 + \frac{\delta r}{a}}$ and neglecting terms of the order $\left(\frac{\delta r}{a}\right)^2$ we have:

$$G \iiint_{\text{sphere}} \frac{d\mu_s}{|\vec{r} - \vec{s}|} = - \frac{4\pi}{3} \rho G a^2 + \frac{4\pi}{3} \rho G a^2 \left(\frac{\delta r}{a}\right)$$

<u>Evaluation of the bulge-integral B.I.</u>



$$\text{B.I.} = -\rho G \iint_{\text{sphere}} d\sigma \int_a^{a+\delta r(\sigma)} \frac{ds'}{|\vec{r} - \vec{s}'|}$$

where $d\sigma$ = element of area on the surface of the sphere

$\vec{s}'$ = variable vector and with $|\vec{s}'| > a$

Now
$$\int_a^{a+\delta r(\sigma)} \frac{ds'}{|\vec{r} - \vec{s}|} = \int_a^{a+\delta r(\sigma)} \frac{ds'}{\sqrt{(\vec{r} - \vec{s})^2}}$$
$$= \frac{\delta r(\sigma)}{\sqrt{2a^2(1 - \cos\beta)}}$$

expanding $\frac{1}{\sqrt{(\vec{r}-\vec{s})^2}}$ with $r = a + \delta r$ and $s' = a + \delta s$ up to the first order.

Hence

$$\text{B.I.} = -\frac{\rho G}{\sqrt{2}} \oiint_{\text{sphere}} \frac{\left(\frac{\delta r}{a}\right) d\sigma}{\sqrt{1-\cos\beta}}$$

We also have $\quad \frac{\omega^2 R^2}{2} = \frac{\omega^2 a^2 \cos^2\theta}{2} + \omega^2 \cdot \delta r \cdot a \cos\theta$

up to the first order.

$\Phi - \frac{\omega^2 R^2}{2} =$ const. then gives the following integral equation:

$$\boxed{\frac{\omega^2 a^2 \cos^2\theta}{2} = \frac{4}{3}\pi\rho G a^2 \left(\frac{\delta r}{a}\right) - \frac{\rho G}{\sqrt{2}} \oiint \frac{\left(\frac{\delta r}{a}\right) d\sigma}{\sqrt{1-\cos\beta}}}$$

neglecting terms of the second order.

The first term in this equation represents the variation of the gravitational potential with latitude, the second term the part of this potential due to the local displacement from the center of mass and the third term the correction due to the non-homogeneous mass distribution.

These notes submitted by

Moiz Rasiwala.

The Figure of the Earth (b)

Melvin E. Stern

Solution of the integral equation:

We solve the integral equation by reducing it to an eigenvalue equation. The bulge is pressed back on the sphere and we look for the inhomogeneous mass distribution $K(\theta)$ on the surface of a sphere of radius a. We write

$$K_i(\theta) = \lambda_i \oint\!\!\oint \frac{K_i(\sigma)\, d\sigma}{\sqrt{1-\cos\beta}} ,$$

this being Laplace's equation on the surface of a sphere. λ_i is a constant. Having the mass distributions $K_i(\theta)$ we may write

$$\frac{\delta r}{a} = \sum A_i K_i(\theta), \quad A = \text{const.}$$

If ψ_i is the potential corresponding to K_i, we have

$$\nabla^2 \psi_i = 0 \quad \text{outside the surface, and}$$

$$\left[\frac{\partial \psi_i}{\partial r}\right]_{a-}^{a+} = 4\pi G K_i(\theta) \text{ on the surface.}$$

This latter equation is equivalent to the condition:

$$\psi_i = -\frac{G}{a\sqrt{2}} \oint\!\!\oint \frac{K_i(\sigma)\, d\sigma}{\sqrt{1-\cos\beta}} \tag{I}$$

We separate the variables:

$$\psi_i = R_i(r)\, F_i(\sin\theta) \text{ where the } F\text{'s are Legendre}$$

polynomials.

Then
$$K_i(\theta) = \frac{F_i(\sin\theta)}{4\pi G} \left[\frac{dR_i}{dr}\right]_{a-}^{a+}$$

Writing explicitly Laplace's equation:

$$\frac{\partial}{\partial r} r^2 \frac{\partial \Psi_i}{\partial r} + \frac{1}{\cos\theta} \frac{\partial}{\partial \theta} \cos\theta \frac{\partial \Psi_i}{\partial \theta} = 0$$

we have

$$\frac{1}{R_i} \frac{d}{dr} r^2 \frac{dR_i}{dr} = -\frac{1}{F_i} \frac{d}{d\zeta}(1-\zeta^2)\frac{d}{d\zeta} F_i = \Lambda_i$$

where Λ_i = const.

and ζ = sin θ.

Now,

$$F_i(\zeta) = a_0^i + a_1^i \zeta + a_2^i \zeta^2 + a_3^i \zeta^3 + \ldots$$

$$F_i'(\zeta) = a_1^i + 2a_2^i \zeta + 3a_3^i \zeta^2 + \ldots$$

$$F_i''(\zeta) = + 2a_2^i + 6a_3^i \zeta + \ldots$$

$$\therefore -\Lambda_i \left[a_0^i + a_1^i \zeta + a_2^i \zeta^2 + \ldots\right] = -2\zeta\left[a_1^i + 2a_2^i \zeta + 3a_3^i \zeta^2 + \ldots\right] +$$

$$+ (1-\zeta^2)(2a_2^i + 6a_3^i \zeta + \ldots)$$

This equation is satisfied only if the corresponding coefficients of the various powers of ζ are equal. Hence,

$$-a_0^i \Lambda_i = 2a_2^i$$

$$-a_1^i \Lambda_i = -2a_1^i + 6a_3^i$$

$$-a_2^i \Lambda_i = -4a_2^i - 2a_2^i + 12a_4^i$$

Looking for the lowest terminating solution, we have,

$$a_3^i = 0 \quad \therefore a_1^i = 0, \quad \Lambda_i = 6, \quad a_2^i = -3a_0^i$$

$\therefore F_1(\sin\theta) = 1 - 3\sin^2\theta$ up to a constant factor

$= 3\cos^2\theta - 2$

From the equation $\frac{d}{dr} r^2 \frac{dR_1}{dr} = 6R_1$

we now have

$$R_1 = r^2 \text{ or } r^{-3}$$

Hence

$$\psi_1 = (3\cos^2\theta - 2) \begin{cases} \left(\frac{r}{a}\right)^2 & r < a \\ \left(\frac{a}{r}\right)^3 & r > a \end{cases}$$

and $K_1(\theta) = -\frac{1}{4\pi G}\frac{5}{a}\left(3\cos^2\theta - 2\right)$

Introducing these two expressions in I:

$$\left(3\cos^2\theta - 2\right) \equiv \frac{5}{4\pi a^2\sqrt{2}} \oiint \frac{3\cos^2\theta(\sigma) - 2}{\sqrt{1-\cos\beta}}\, d\sigma$$

If we now put for the lowest solution in $\frac{\delta r}{a}$

$$\frac{\delta r}{a} = A\left(3\cos^2\theta - 2\right) + 3A \,, \quad A = \text{const.}$$

normalised so that it vanishes at the poles, we have from the integral equation:

$$\frac{\omega^2 a^2}{2} = \frac{4}{3}\pi\rho G a^2 (3A) - \frac{\rho G 4\pi a^2}{5}(3A)$$

$$\therefore 3A = \frac{15}{16}\frac{\omega^2}{\pi\rho G} \qquad \text{(ellipticity)}$$

$$= \frac{5\omega^2 a}{4g} \qquad g = \text{local gravity}$$

or, introducing numerical values, $3A \approx \frac{1}{230}$

This compares with the best present figure of $\frac{1}{297}$.

These notes submitted by

Moiz Rasiwala

EFFECTS OF ROTATION ON WAVE MOTIONS

Effects of Rotation on Wave Motions

Louis N. Howard

We shall consider three main aspects of small oscillatory motions of a rotating fluid: a) oscillations of a self-gravitating homogeneous rotating sphere, (b) shallow water theory in a rotating system, and c) tides on the surface of a sphere.

The problem of an oscillating non-rotating liquid sphere was first considered by Kelvin, and we begin by sketching his theory. Take a spherical body whose radius is given by

$$r \leq a + \zeta(\theta, \varphi)$$
$$\max \left| \frac{\zeta}{a} \right| << 1 \tag{1.1}$$

as a function of angular coordinates. The geometry is that of Figure 1, and "a" is a "mean radius" such that $\iint \zeta \, d\Omega = 0.$

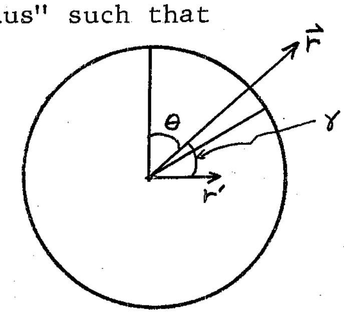


Figure 1.

It is necessary to know the gravitational potential Φ on the surface: The density ρ is assumed constant (thus prohibiting internal waves). Poisson's equation applies and the potential is found from the volume integral

$$\Phi = \rho G \int_V \frac{dV}{|\vec{r} - \vec{r}'|}$$
$$= \rho G \int d\Omega \int_0^{a+\zeta} \frac{r'^2 \, dr'}{\sqrt{r^2 + r'^2 - 2rr' \cos\gamma}} \tag{1.2}$$

where $d\Omega$ is the usual element of solid angle.

This may be conveniently rewritten as

$$\Phi = \rho G r^2 \iint d\Omega \int_0^{a+\zeta} \frac{(r'/r)^2 \, d\,(r'/r)}{\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\,\frac{r'}{r}\cos\gamma}} \tag{1.3}$$

Recall that the denominator may be expanded in terms of the Legendre polynomials, i.e.,

$$\left(1+X^2-2zX\right)^{-1/2} = \sum_0^{\infty} X^n P_n(z) \tag{1.4}$$

(This is most easily seen by directly expanding the square root and regrouping terms.)

Therefore

$$\Phi = \rho G r^2 \iint d\Omega \sum_0^{\infty} P_n(\cos\gamma) \int_0^{r'=a+\zeta} \left(\frac{r'}{r}\right)^{n+2} d\left(\frac{r'}{r}\right) \tag{1.5}$$

Performing the radial integration gives

$$\Phi = \rho G r^2 \sum_0^{\infty} \iint \frac{P_n(\cos\gamma)}{n+3} \left(\frac{a+\zeta}{r}\right)^{r+3} d\Omega \tag{1.6}$$

For $\zeta << a$ this may be expanded as

$$\Phi \approx \rho G r^2 \sum_0^{\infty} \iint \frac{P_n(\cos\gamma)}{n+3} \left(\frac{a}{r}\right)^{n+3} \left(1+(n+3)\frac{\zeta}{a}\right) d\Omega$$

$$\approx \frac{4\pi}{3}\rho G r^2 \left(\frac{a}{r}\right)^3 + \rho G r^2 \sum_0^{\infty} \iint \zeta\,(\theta',\varphi') P_n(\cos\gamma) \frac{1}{a}\left(\frac{a}{r}\right)^{n+3} d\Omega \tag{1.7}$$

On the surface the approximation $r = a$ may be used in the first-order terms of (1.7) since this produces a second-order error. So on the surface we may write

$$\Phi \approx \frac{4}{3}\pi\rho G \frac{a^3}{a+\zeta} + \rho G a \sum_0^{\infty} \iint P_n \zeta \, d\Omega$$

$$\Phi \approx \frac{4}{3}\pi\rho G a^2 \left(1 - \frac{\zeta}{a}\right) + \cdots \tag{1.8}$$

Next we will need the integral relation between surface harmonics S_n and the Legendre polynomials:

$$\iint S_n P_m(\cos\gamma)\, d\Omega = \delta_{mn} \frac{4\pi}{2n+1} S_n \tag{1.9}$$

where δ_{mn} is the Kronecker delta. Also we will decompose ζ into surface harmonics:

$$\zeta = \sum_0^\infty \zeta_n(\theta, \varphi) \tag{1.10}$$

Hence the surface potential becomes to first order

$$\Phi = \frac{4\pi}{3} \rho G a^2 - \frac{4\pi}{3} \rho G a \sum_1^\infty \zeta_n + \sum_1^\infty \rho G a \frac{4\pi}{2n+1} \zeta_n \tag{1.11}$$

or

$$\Phi = \frac{4\pi}{3} \rho G a^2 + \frac{4\pi}{3} \rho G a \sum_1^\infty \zeta_n \frac{2(1-n)}{2n+1} \tag{1.12}$$

and on the surface

$$r = a + \sum_1^\infty \zeta_n \tag{1.13}$$

With the result, we can now proceed to the oscillations of a liquid sphere. In the equation of motion we neglect non-linear terms and so use

$$\vec{U}_t + \vec{\nabla}\left(\frac{1}{\rho} P\right) = \vec{\nabla} \Phi \tag{1.14}$$

(P = pressure, ρ = density, U = velocity, subscripts denote partial derivatives. Note that $\vec{\nabla}\Phi$ is taken with positive sign, since gravity is attractive.) For irrotational motion $\vec{U} = \vec{\nabla}\varphi$ and so

$$\varphi_t + \frac{1}{\rho} P - \Phi = F_{(t)} = 0 \tag{1.15}$$

The surface boundary conditions are

$$\varphi_t - \Phi = 0$$
$$\varphi_r - \zeta_t = 0 \qquad (1.16)$$

Inside the sphere every harmonic function is expressible in terms of surface harmonics, which we call φ_n , i.e.,

$$\varphi = \sum \left(\frac{r}{a}\right)^n \varphi_n(\theta, \varphi, t) \qquad (1.17)$$

Substituting into the previous expression (1.12) for the gravitational potential and utilizing the orthogonality properties (1.9) gives

$$\varphi_{n_t} = \frac{4\pi}{3} \rho G a \, \frac{2(1-n)}{2n+1} \zeta_n \qquad (1.18)$$

$$\frac{n}{a} \varphi_n = \frac{\partial \zeta_n}{\partial t} \qquad (1.19)$$

and therefore

$$\zeta_{n_{tt}} = \frac{4\pi}{3} \rho G \, \frac{2n(1-n)}{2n+1} \zeta_n$$

or

$$\zeta_{n_{tt}} = \frac{g}{a} \, \frac{2n(1-n)}{(2n+1)} \zeta_n \qquad (1.20)$$

where

$$g = \frac{4\pi}{3} \rho G a \qquad (1.21)$$

So we find that these oscillations are harmonic with a frequency spectrum given by

$$\omega_n^2 = \frac{g}{a} \, \frac{2n(n-1)}{2n+1} \qquad (1.22)$$

Therefore there is a degenerate set of "oscillations" with $\omega = 0$ obtained from $n = 1$. These are simply displacements of the whole sphere. The lowest real oscillation is with $n = 2$ and

$$\omega_2^2 = \frac{4}{5} \, \frac{4\pi}{3} \rho G \qquad (1.23)$$

This is approximately the period (about 90 minutes for the earth) of a satellite orbit at the surface of the sphere. This type of relationship is expected on dimensional grounds for any self-gravitating spherical system.

Now add rotation. The equations of motion of a system rotating with angular velocity $\omega\vec{k}$ are

$$\vec{u}_t+\vec{u}\cdot\vec{\nabla}u+2\omega(\vec{k}\times\vec{u})+\vec{\nabla}\left(\frac{P}{\rho}-\frac{1}{2}\omega^2(x^2+y^2)\right)=\vec{\nabla}\Phi \tag{1.24}$$

for ω = constant. For small amplitude motions $\vec{u}\cdot\vec{\nabla}u$ is small and we neglect it. (This does not necessarily mean that ω is small.) The problem of the surface of a rigidly rotating body such as the earth is described by

$$\frac{P}{\rho}-\frac{1}{2}\omega^2(x^2+y^2)=\Phi+\text{const.} \tag{1.25}$$

At the surface if there is no motion, since

$$\begin{aligned} x^2+y^2 &= r^2\sin^2\theta = r^2(1-\cos^2\theta) \\ &= r^2\tfrac{2}{3}\left(1-P_2(\cos\theta)\right) \end{aligned} \tag{1.26}$$

with θ as the co-latitude, we have

$$-\frac{1}{2}\omega^2r^2\frac{2}{3}\left(1-P_2\right)=\Phi+\text{const.} \tag{1.27}$$

and for $r\approx a$

$$\frac{1}{3}\omega^2a^2P_2=-\frac{4\pi}{3}\rho G a\left(\frac{2}{5}\zeta_2\right) \tag{1.28}$$

or

$$\zeta=-\frac{5}{6}\omega^2\frac{a^2}{g}P_2 \tag{1.29}$$

At the pole $P_2(0)=1$, at the equator $P_2\left(\frac{\pi}{2}\right)=-\frac{1}{2}$, and so the difference in ζ_2 between these regions may be expressed as

$$\frac{\Delta\zeta_2}{a}=\frac{3}{2}\left(\frac{5}{6}\;\frac{\omega^2a}{g}\right)=\frac{15}{16\pi}\;\frac{\omega^2}{\rho G} \tag{1.30}$$

These notes submitted by

William C. Saslaw

Oscillations around Rigid Rotation

Louis N. Howard

From our previous results we may write the radius and surface potential as

$$r = a - \frac{5}{6}\frac{\omega^2 a^2}{g} P_2 + \sum \zeta_n \tag{2.1}$$

$$\Phi = ga + \frac{1}{3}\frac{\omega^2 a^2}{g} P_2 + g\sum \frac{2(1-n)}{2n+1}\zeta_n \tag{2.2}$$

The equation of motion is

$$\vec{u}_t + 2\omega\vec{k}\times\vec{u} + \nabla\Pi = 0 \tag{2.3}$$

where

$$\Pi = \frac{P}{\rho} - \Phi - \frac{1}{2}\omega^2(x^2+y^2) \tag{2.4}$$

and also

$$\nabla\cdot\vec{u} = 0 \tag{2.5}$$

On the surface if we consider ω to be small and neglect the displacement of the surface due to rotation we have

$$\Pi = -\Phi - \frac{1}{2}\omega^2 a^2 \sin^2\theta \tag{2.6}$$

and therefore

$$\Pi = \text{const.} + g\sum_1^\infty \frac{2(1-n)}{2n+1}\zeta_n \tag{2.7}$$

The linearized boundary condition on the surface is

$$\zeta_t = \vec{n}\cdot\vec{u} \tag{2.8}$$

where $\vec{n}$ is the normal to the surface. We can use Π instead of a velocity potential. Replace the operator $\frac{\partial}{\partial t}$ by the operator $i\sigma$.

Then

$$i\sigma\vec{u} + 2\omega\vec{k}\times\vec{u} + \nabla\Pi = 0 \qquad (2.9)$$

Taking the cross product with $\vec{k}$ and solving for $\vec{u}$ gives

$$\vec{u} = \frac{1}{\sigma^2-4\omega^2}\left\{i\sigma\nabla\Pi + \frac{4\omega^2}{i\sigma}\Pi_z\vec{k} - 2\omega\vec{k}\times\vec{\nabla}\Pi\right\} \qquad (2.10)$$

where we have assumed

$$\sigma^2 \neq 4\omega^2 \qquad (2.11)$$

Using (2.5) gives

$$\nabla^2\Pi - \frac{4\omega^2}{\sigma^2}\Pi_{zz} = 0 \qquad (2.12)$$

which is the fundamental equation for rotating oscillations.

For no rotation this reduces to Laplace's equation, and this is still true to first order in ω/σ. Since there is a first-order term (in ω/σ) in the boundary condition, we may get a first-order perturbation of the modes, even though the equation is unchanged. We consider this further, working only to first order.

Next expand

$$\Pi = \sum\left(\frac{r}{a}\right)^n\Pi_n \qquad (2.13)$$

(appropriate since Π is harmonic to this order)

with

$$\left(a\frac{\partial\Pi}{\partial r}\right)_n = n\Pi_n \qquad (2.14)$$

on the surface. Multiplying the boundary condition (2.7) by $i\sigma$

$$i\sigma\Pi = \text{const.} + g\sum_1^\infty \frac{2(n-1)}{2n+1}\left(\vec{r}_1\cdot\vec{u}\right)_n \qquad (2.15)$$

shows that the n'th component is

$$
\begin{aligned}
i\sigma \Pi_n &= g\,\frac{2(n-1)}{2n+1}\left(\vec{r}_1\cdot\vec{u}\right)_n \\
&= g\,\frac{2(n-1)}{2n+1}\,\frac{1}{(\sigma^2-4\omega^2)}\left\{ i\sigma\frac{\partial\Pi}{\partial r} + \frac{2\omega}{a}\frac{\partial\Pi}{\partial\varphi} + \right. \\
&\quad \left. + \frac{4\omega^2}{i\sigma}\cos\theta\left(\cos\theta\,\frac{\partial\Pi}{\partial r} - \frac{\sin\theta}{a}\frac{\partial\Pi}{\partial\theta}\right)\right\}_n
\end{aligned}
\tag{2.16}
$$

since

$$\vec{r}_1 \times \vec{k}\cdot\nabla\Pi = -\sin\theta\,\frac{1}{r\sin\theta}\,\frac{\partial\Pi}{\partial\theta} \tag{2.17}$$

Divide by $i\sigma$ and consider $\frac{\omega}{\sigma} << 1$. This shows that there is in fact a first order effect of rotation of the form

$$\frac{\omega}{\sigma}\;\frac{2}{ia}\;\frac{\partial\Pi}{\partial\varphi} \tag{2.18}$$

Using the surface condition (14) gives to first order

$$\sigma^2\Pi_n = \frac{g}{a}\,\frac{2(n-1)}{2n+1}\left\{ n\Pi_n + \frac{2\omega}{i\sigma}\left(\frac{\partial\Pi}{\partial\varphi}\right)_n\right\} \tag{2.19}$$

The surface hormonics Π_n may be expanded as

$$\Pi_n = \sum_{-n}^{n} A_m e^{im\varphi} P_n^m(\cos\theta) \tag{2.20}$$

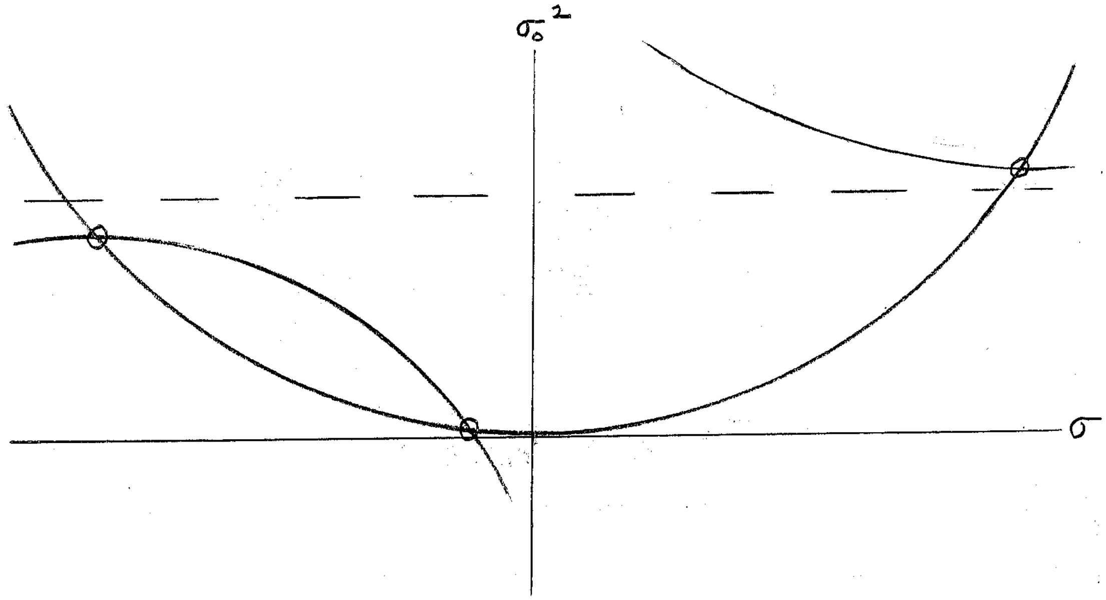
and therefore

$$\sigma^2 = \sigma_0^2\left(1 + 2\,\frac{m}{n}\,\frac{\omega}{\sigma}\right) \tag{2.21}$$

where

$$\sigma_0^2 = \frac{g}{a}\,n\,\frac{2(n-1)}{2n+1} \tag{2.22}$$

The location of the three roots of this cubic are shown graphically in the schematic figure on the following page (16). The root with small σ is such that $\frac{\sigma}{\omega} \sim 1$ and its position should be distrusted since our analysis holds only for $\frac{\sigma}{\omega} << 1$. This rotational splitting of modes is reminiscent of the Yeeman splitting of the energy levels of an atom by a magnetic field.



A procedure for investigating second-order effects is to begin with the basic equation (2.12). Denote the surface value of Π by f and solve for P in the expansion

$$\Pi = \Pi^0 + \frac{4\omega^2}{\sigma^2} P + \ldots \tag{2.23}$$

We have

$$\nabla^2 \Pi^0 = 0 \;; \quad \Pi^0 = f \text{ on surface} \tag{2.24}$$

$$\nabla^2 P = \Pi^0_{zz} , \quad P \doteq 0 \text{ on surface} \tag{2.25}$$

To solve this we use

$$\nabla^2 \left(z \Pi^0_z \right) = 2 \Pi^0_{zz} \tag{2.26}$$

A particular solution which satisfies the boundary condition is

$$P = \frac{z}{2} \Pi^0_z - H \tag{2.27}$$

where H is a suitable harmonic function.

In future discussion we will need to use the following fact: Any "reasonable" vector field $\vec{u}$ can be determined from three scalar "potentials" χ, Ψ, ω , which are essentially unique, according to

$$\vec{u} = \vec{r}_1 \times (\vec{r}_1 \times \nabla \chi) + \vec{r}_1 \times \nabla \psi + \frac{\partial \omega}{\partial r} \vec{r}. \tag{2.28}$$

This is analogous to the two-dimensional result:

$$\vec{u} = \nabla \varphi + \nabla \times (\vec{k}\, \psi) \tag{2.29}$$

Now

$$\vec{\nabla} \cdot \vec{u} = -\frac{1}{r^2} \Lambda \chi + \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \omega}{\partial r}\right) \tag{2.30}$$

where Λ is defined through

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \Lambda \tag{2.31}$$

Also the surface harmonics of degree n are eigenfunctions of Λ :

$$\Lambda Y_n + n(n+1) Y = 0 \tag{2.32}$$

and

$$\nabla \times \vec{u} = -\vec{r}_1 \times \left(\vec{r}_1 \times \nabla \left(r \frac{\partial}{\partial r}\left(\frac{\psi}{r}\right)\right)\right) + $$
$$+ \nabla\left(\frac{\partial}{\partial r}(\chi + \omega)\right) \times \vec{r}_1 + \vec{r}_1 \left(-\frac{1}{r^2} \Lambda \psi\right) \tag{2.33}$$

One representation of the general solution of $\nabla \cdot \vec{u} = 0$ can be obtained as follows:

First select ψ and χ arbitrarily. Then ω may be determined (up to an additive function of angles) from

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \omega}{\partial r}\right) = \Lambda \chi \tag{2.34}$$

with, if one wishes also $\vec{u} \cdot \vec{n} = 0$ on $r = a$ (incompressible flows in a sphere) the boundary condition

$$\frac{\partial \omega}{\partial r} = 0 \quad \text{for } r = a. \tag{2.35}$$

These notes submitted by

William C. Saslaw

Low-Frequency Oscillations in a Rotating System

Louis N. Howard

We recall the basic equations of motion

$$\vec{u}_t + 2\omega\vec{k} \times \vec{u} + \vec{\nabla}\Pi = 0 \tag{3.1}$$

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{3.2}$$

where, on the surface

$$\Pi = g\sum \frac{2(n-1)}{2n+1}\zeta_n \tag{3.3}$$

In second order the change in shape produced by rotation is given by

$$\zeta_t + \vec{u}\cdot\vec{\nabla}\zeta = \vec{r}_i\cdot\vec{u} \tag{3.4}$$

where $\vec{u}$ is the velocity normal to the surface. The term $\vec{u}\cdot\vec{\nabla}\zeta$ is of order $\frac{\omega^2 a}{g}$ and indicates the eccentricity of the rotating fluid.

For steady motions we have

$$2\omega\vec{k}\times\vec{u} + \vec{\nabla}\Pi = 0 \tag{3.5}$$

and therefore

$$\vec{k}\cdot\vec{\nabla}\Pi = 0 \tag{3.6}$$

Crossing (3.5) with $\vec{k}$ gives:

$$2\omega\left[\vec{k}(\vec{k}\cdot\vec{u}) - \vec{u}\right] + \vec{k}\times\vec{\nabla}\Pi = 0 \tag{3.7}$$

For convenience define

$$\vec{u}_2 = \frac{1}{2\omega}\vec{k}\times\vec{\nabla}\Pi \tag{3.8}$$

This is a two-dimensional (horizontal) velocity field. What we have demonstrated is the Taylor-Proudman theorem: Slow, linear, steady flow (in a rotating system) is two-dimensional. These are called geostrophic flows. Its streamlines are circles. Furthermore, since

$$\vec{u}_2 = f(r \sin\theta)\,\hat{\varphi}_1 \qquad (3.9)$$

where $\hat{\varphi}_1$ is a unit vector in the azimuthal direction, u is determined by a single function of a single variable.

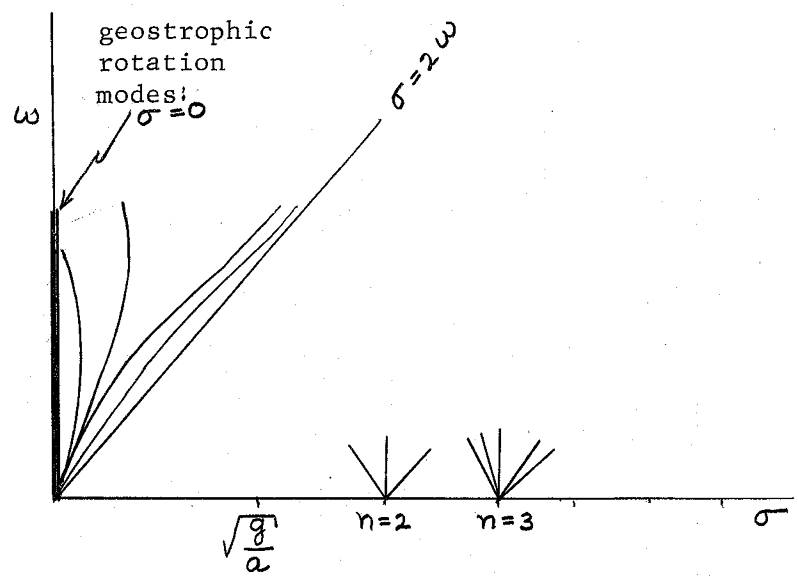
To solve for a dispersion relation, replace $\frac{\partial}{\partial t}$ by $i\sigma$ and obtain (as before) the harmonics

$$\vec{u} = \frac{1}{\sigma^2 - 4\omega^2}\left\{ i\sigma \nabla\Pi + \frac{4\omega^2}{i\sigma}\Pi_z \vec{k} - 2\omega\vec{k} \times \nabla\Pi \right\} \qquad (3.10)$$

For $\sigma^2 \approx 4\omega^2$ (i.e. of comparable magnitude)

$$(\sigma^2 - 4\omega^2)\Pi_n = \frac{g}{a}\,\frac{2(n-1)}{(2n+1)}\left\{ a\frac{\partial\Pi}{\partial r} - \frac{2\omega}{i\sigma}\frac{\partial\Pi}{\partial\varphi} - \frac{4\omega^2}{\sigma^2}\, a\cos\theta\left(\frac{\partial\Pi}{\partial z}\right)\right\}_n \qquad (3.10a)$$

Now for $\frac{\omega^2 a}{g} \ll 1$ the right-hand side of (3.10a) is much larger than the left-hand side and may therefore be equated (approximately) to zero. Doing so gives the following sort of diagram for the dispersion relation:



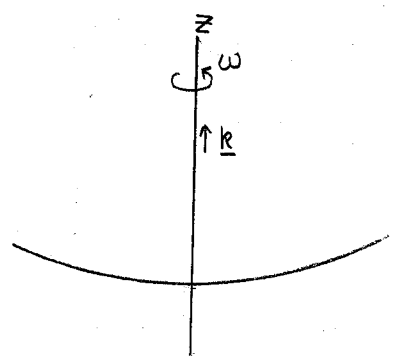
These notes submitted by

William C. Saslaw

Theory of Shallow Water Waves

Louis N. Howard

We shall consider an incompressible fluid contained in a shallow bowl rotating with angular velocity $\omega \underline{k}$. $\underline{k}$ is the unit vector in the z-direction, and gravity acts in the z-direction.



The equations of motion and of continuity are

$$\underline{u}_t + 2\omega \underline{k} \times \underline{u} + \nabla(p/\rho) + g\underline{k} = 0 \tag{3.11}$$

$$u_x + v_y + w_z = 0 \tag{3.12}$$

Let $\zeta(x, y, t)$ be the displacement of the free surface above a mean height H. Neglecting the inertial acceleration of the fluid with respect to the gravitational acceleration, the vertical component of (3.11) yields the expression

$$p = \rho g(\zeta - z).$$

The remaining components then give

$$u_t - 2\omega v + g\zeta_x = 0 \tag{3.13}$$

$$v_t + 2\omega u + g\zeta_y = 0 \tag{3.14}$$

A horizontal continuity equation may be derived by the following argument.

According to the Taylor-Proudman theorem, the vertical velocity component may depend only linearly on z . Furthermore, it must satisfy the restrictions

$$w\Big|_{z=H} = w_{\text{bottom}}, \quad \text{say}$$

$$w\Big|_{z=\zeta} = \zeta_t .$$

It follows that, to lowest order in $\frac{\zeta}{H}$,

$$w = \zeta_t\left(1+\frac{z}{H}\right) - \frac{z}{H}\, w_{\text{bottom}} \tag{3.15}$$

Also, for a fluid particle to remain on the surface, we might have

$$\frac{D}{Dt}\left(z+H\right) = 0 \tag{3.16}$$

Now $w_{\text{bottom}} \equiv \frac{Dz}{Dt}$; hence $w_{\text{bottom}} = -u\cdot\nabla H$

Thus, using (3.12) and (3.15) we obtain

$$-\left(u_x + v_y\right) = \frac{\zeta_t}{H} + \frac{u\cdot\nabla H}{H},$$

i.e., $\nabla'\cdot(\underset{\sim}{u} H) = -\zeta_t$

where ∇' indicates the operator $\left(\frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}, 0\right)$. (3.17)

Let us now assume axial symmetry, and write

$$\zeta = \varphi(r)\, e^{im\theta}$$, using cylindrical polar coordinates.

It follows from (3.13), (3.14) and (3.17) that for a paraboloidal basin $(H = H_0(1-r^2/a^2)$, assumed fixed independent of ω)

$$\frac{d}{dr}\left(r\left(1-\frac{r^2}{a^2}\right)\frac{d\varphi}{dr}\right) + \left[-\frac{m^2}{r}\left(1-\frac{r^2}{a^2}\right) - 4m\frac{\omega r}{\sigma a^2} + r\,\frac{\sigma^2-4\omega^2}{g H_0}\right]\varphi = 0 \tag{3.18}$$

Write
$$-4\frac{m\omega}{\sigma a^2} - \frac{4\omega^2}{gH_o} + \frac{\sigma^2}{gH_o} = \frac{\sigma_o^2}{gH_o} \qquad (3.19)$$

and so
$$\frac{d}{dr}\left(r\left(1-\frac{r^2}{a^2}\right)\frac{d\varphi}{dr}\right) + \left[-\frac{m^2}{r}\left(1-\frac{r^2}{a^2}\right) + \frac{\sigma_o^2}{gH_o}r\right]\varphi = 0 \qquad (3.20)$$

This is of the same form as would be obtained in a non-rotating system. Expecting ϕ to be of the form

$$\phi = r^{\mu}\sum_{K=0}^{\infty} C_K\left(\frac{r}{a}\right)^K, \qquad (3.21)$$

regularity requirements at the center lead to $\mu = |m|$. The C_K's then obey the formula of recurrence

$$C_{K+2} = -\frac{m^2-(|m|+k)(|m|+k+2)+\frac{\sigma_o^2 a^2}{gH_o}}{(K+2)(K+2+2m)}C_K \qquad (3.22)$$

Searching for a finite series (21), we demand the numerator of (22) to vanish for some value of K, which implies

$$\left(\frac{a^2\sigma_o^2}{gH_o}\right)_{(K)}^{(m)} = \left(K+|m|\right)\left(K+|m|+2\right) - m^2 \qquad (3.23)$$

Only even values of K can contribute to finite series (21) since the sole cutoff condition, (23), involves even terms alone (C_o has to be non-zero because of the regularity requirement at the center).

The lowest non-trivial eigenvalue corresponds to $m = \pm 1$, $K = 0$:

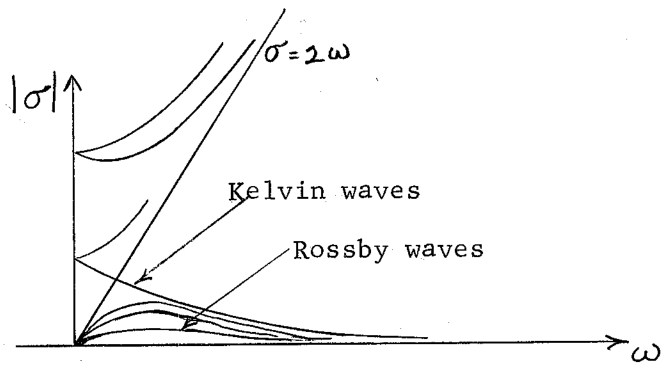
$$\left(\frac{a^2\sigma_o^2}{gH_o}\right)_{(0)}^{(\pm 1)} = 2 \longrightarrow \left(\sigma_o^2\right)_{(0)}^{(\pm 1)} = 2\,gH_o/a^2 \qquad (3.24)$$

with its two eigensolutions

$$\zeta \sim e^{\pm i\varphi}\, r \quad \text{or} \quad \begin{array}{l}\zeta_1 \sim x \\ \zeta_2 \sim y\end{array} \qquad (3.25)$$

which describe the space-dependent part of "sloshing" motions.

Solving (23) for larger values of m and k and expressing σ_0^2 again in terms of σ and ω, we find that the dependence of σ on ω may look like (see eq.(3.19)):



Rotation splits up each eigenfrequency of the liquid; the eigenfrequency $\sigma = 0$ appears to be infinitely degenerate and related to low-frequency Rossby waves belonging to non-axisymmetric modes only.

Curves crossing the line $4\omega^2 - \sigma^2 = 0$ correspond to values of k and m obeying

$$k(2|m| + k + 2) = 0 \longrightarrow k = 0 \longrightarrow m \neq 0 \qquad (3.26)$$

(see (3.19)). They originate from the lowest points on the $|\sigma|$-axis of the diagram and represent "Kelvin" wave types of motion.

These notes submitted by

Joseph I. Silk

Effects of Rotation on Shallow Water Motions

Louis N. Howard

To examine the effect of rotation on a fluid moving in a bowl of general shape, z = -H(x,y), we consider velocities of the form

$\underline{u} = e^{i\sigma t}(u(x,y), v(x,y), w)$ and the equations

$$i\sigma u = 2\omega v + g\zeta_x = 0 \tag{4.1}$$

$$i\sigma v = 2\omega u + g\zeta_y = 0 \tag{4.2}$$

$$i\sigma \zeta + (Hu)_x + (Hv)_y = 0 \tag{4.3}$$

which contain the x and y components of velocity only. Combining these into a two-dimensional vector $\underline{u}_2 = (u, v, 0)$, we have to express the usual boundary condition that the fluid velocity at the surface of the bowl be tangential to the surface, by a condition on $\underline{u}_2$ and $\underline{n}_2$, the projection of the surface normal into the (x,y) plane. We choose the formulation:

$$H\,\underline{u}_2 \cdot \underline{n}_2 = 0 \quad \text{on the surface of the container.} \tag{4.4}$$

(Thus $\vec{u}_2 \cdot \vec{n}_2$ <u>bounded</u> on a sloping beach, <u>zero</u> on a vertical wall). Insertion of (3) into (1) and (2) yields, with $\underline{k} = (0,0,1)$

$$\sigma^2 \underline{u}_2 - 2i\sigma\omega\, \underline{k} \times \underline{u}_2 + g\nabla\nabla\cdot(H\underline{u}_2) = 0 \tag{4.5}$$

Are waves possible in the "general" case? How would the frequency spectrum look?

We define an inner product of two vectors <u>a</u> and <u>b</u> by

$$(\underline{a}, \underline{b}) = \int_D H\, \underline{a}^* \cdot \underline{b}\; dx\, dy,$$

D being the domain of the (x,y) plane that is delimited by the intersection with the bowl's surface. Thus, we have

$$(\underline{a}, i\underline{k}\times\underline{b}) = \int_D H\underline{a}^*\cdot i(\underline{K}\times\underline{b})\,dx\,dy = -\int_D iH\underline{b}\cdot(\underline{k}\times\underline{a}^*)\,dx\,dy = \int_D H(i\underline{k}\times\underline{a})^*\cdot\underline{b}\,dx\,dy = (i\underline{k}\times\underline{a},\underline{b}) \tag{4.6}$$

and

$$(\underline{a}, \nabla\nabla(H\underline{b})) = \int_D H\underline{a}^*\cdot\nabla\nabla\cdot(H\underline{b})\,dx\,dy = \int_D \nabla\cdot\left[H\underline{a}^*\nabla\cdot(H\underline{b})\right]dx\,dy - \int_D \nabla\cdot(H\underline{a}^*)\nabla\cdot(H\underline{b})\,dx\,dy; \tag{4.7}$$

here the first integral on the right vanishes for all vectors $\underline{a}^*$ satisfying (4). Also, for $\underline{a} = \underline{a}_r + i\,\underline{a}_i$.

$$(\underline{a}, \underline{K}\times\underline{a}) = -2i(\underline{a}_i, \underline{K}\times\underline{a}_r). \tag{4.8}$$

Operating $\int_D dx\,dy\; H\,\underline{u}_2^*\cdot\ldots$ on eq. (5) and using (7) and (8) leads to

$$(\sigma_r+i\sigma_i)^2(\underline{u}_2,\underline{u}_2) - 4(\sigma_r+i\sigma_i)\omega\int_D H\underline{u}_{2i}(\underline{K}\times\underline{u}_{2r})\,dx\,dy - g\int_D |\nabla\cdot(H\underline{u}_2)|^2\,dx\,dy = 0 \tag{4.9}$$

the imaginary part of which reads

$$2i\sigma_r\sigma_i(\underline{u}_2,\underline{u}_2) = 4i\sigma_i\,\omega\int_D H\underline{u}_{2i}\cdot(\underline{K}\times\underline{u}_{2r})\,dx\,dy. \tag{4.10}$$

For $\sigma_i \neq 0$ (10) can be used to transform the real part of (9) into:

$$(\sigma_r^2-\sigma_i^2)(\underline{u}_2,\underline{u}_2) - 2\sigma_r^2(\underline{u}_2,\underline{u}_2) - g\int_D |\nabla\cdot(H\underline{u}_2)|^2\,dx\,dy = 0 \tag{4.11}$$

which allows for the only solution $\underline{u}_2 \equiv (0,0,0)$. To obtain non-vanishing velocities we consequently require $\sigma_i = 0$, i.e. all possible waves will be stable. Rewriting (5) in the form

$$\mathcal{H}\,\underline{u}_2 = \sigma^2\underline{u}_2 \tag{4.5a}$$

and using (4), (6), (7) and σ = real, the operator $\mathcal{H}$ is found to be self-adjoint. $\mathcal{H}$ contains a parameter $\alpha = \sigma\cdot\omega$ which, if small, suggests a perturbation calculation to describe the effects rotation induces in the oscillatory modes in shallow basins of general shape. We put

$$\mathcal{H}\underline{b} = -g\nabla\nabla\cdot(H\underline{b}) + 2i\alpha\,\underline{k}\times\underline{b} = \mathcal{H}_o\underline{b} + \alpha\,\mathcal{H}_1\underline{b}$$

and calculate the influence of rotation on a particular normal mode $\underline{u}^\circ$ of $\mathcal{H}_o$, with eigenvalue $(\sigma^\circ)^2$. Let $\underline{u}^\circ$ be non-degenerate and normalized. Expanding $\underline{u}_2$ and σ^2 in powers of α :

$$\underline{u}_2 = \underline{u}^\circ + \alpha\,\underline{u}^1 + \alpha^2\underline{u}^2 + \dots$$
$$\sigma^2 = (\sigma^\circ)^2 + \alpha\,\lambda_1 + \alpha^2\lambda_2 + \dots$$

gives the equation of first order in α :

$$\left(\mathcal{H}_o - (\sigma^\circ)^2\right)\underline{u}^1 = (\lambda_1 - \mathcal{H}_1)\,\underline{u}^\circ. \tag{4.12}$$

Since $\mathcal{H}_o$ is self-adjoint,

$$\left(\underline{u}^\circ, \left(\mathcal{H}_o - (\sigma^\circ)^2\right)\underline{u}^1\right) = 0, \tag{4.13}$$

consequently

$$\lambda_1(\underline{u}^\circ, \underline{u}^\circ) = (\underline{u}^\circ, \mathcal{H}_1\underline{u}^\circ) = \int_D H\,\underline{u}^{\circ *}\cdot 2i\,(\underline{k}\times\underline{u}^\circ)\,dx\,dy = 0 \tag{4.14}$$

as non-degeneracy of $\underline{u}^\circ$ implies $\underline{u}^{\circ *}\times\underline{u}^\circ = 0$. Thus, rotation has no effect of first order on the wave frequency. To obtain the effect on the velocity field, we represent $\underline{u}^1$ by the eigenvectors of $\mathcal{H}_o$ (which, being self-adjoint, possesses a complete set of eigenvectors). All of those vectors corresponding to non-zero eigenvalues ("$\underline{u}_n$") are irrotational as can be seen from

$$\mathcal{H}_o\underline{u}_n = -g\nabla\nabla\cdot(H\underline{u}_n) = (\sigma^n)^2\underline{u}_n\,;$$

the eigenvalue $\sigma = 0$ has eigenvectors ("$\underline{v}_n$") with $\nabla\cdot(H\underline{v}_n)$ = a constant which (4) indicates to be zero. For these $\underline{v}_n$, then,

$$\underline{v}_n = \frac{1}{H}\nabla\times\underline{a} = \frac{1}{H}(\nabla\Psi\times\underline{K})$$

in the two-dimensional case, with ψ an arbitrary function constant along the boundary.

Inserting

$$\underline{u}^1 = \sum_n C_n \underline{u}_n + \frac{1}{H}\left(\nabla \psi^1 \times \underline{K}\right) \tag{4.15}$$

into (12):

$$\sum_n \left((\sigma^n)^2 - (\sigma^0)^2\right) C_n \underline{u}_n - (\sigma^0)^2 \frac{1}{H}\left(\nabla \psi^1 \times \underline{K}\right) = -2i\, \underline{K} \times \underline{u}^0, \tag{4.16}$$

multiplying this by $H\underline{u}_n^*$ and integrating determines C_n :

$$C_n = \frac{-2i\left(\underline{u}_n, \underline{K} \times \underline{u}^0\right)}{(\sigma^n)^2 - (\sigma^0)^2} \;; \tag{4.17}$$

the component of $\underline{u}^1$ parallel to $\underline{u}^0$ is obtained by a normalization condition. Operating $\nabla \times \ldots$ on (12) leads to

$$(\sigma^0)^2 \nabla \times \left(\frac{1}{H} \nabla \psi^1 \times \underline{K}\right) = 2i\, \nabla \times \left(\underline{K} \times \underline{u}^0\right) \tag{4.18}$$

or

$$(\sigma^0)^2 \nabla \cdot \left(\frac{1}{H} \nabla \psi^1\right) = -2i\, \nabla \cdot \underline{u}^0 \tag{4.19}$$

which has to be solved subject to ψ^1 = const. along the boundary.

Neglecting centrifugal effects, one can use the resulting $\underline{u}'$ to calculate the second order perturbation on the eigenvalues σ^2. The second order term in (5a) is

$$\left(\mathcal{H} - (\sigma^0)^2\right)\underline{u}^2 = -\mathcal{H}_1 \underline{u}^1 + \lambda_2 \underline{u}^0 \tag{4.20}$$

which gives, by the above arguments,

$$\begin{aligned} \lambda_2 &= \left(\underline{u}^0, \mathcal{H}_1 \underline{u}^1\right) = \\ &= \sum_n \left(\underline{u}^0, 2i\, \underline{K} \times C_n \underline{u}_n\right) + \left(\underline{u}^0, 2i\, \underline{K} \times \left(\frac{1}{H} \nabla \psi^1 \times \underline{K}\right)\right) = \\ &= -4 \sum_n \frac{\left|\left(\underline{u}^0, \underline{K} \times \underline{u}_n\right)\right|^2}{(\sigma^n)^2 - (\sigma^0)^2} + (\sigma^0)^2 \int_D \frac{1}{H}\left(\nabla \psi^{1*}\right) \cdot \left(\nabla \psi'\right) dx\, dy \end{aligned} \tag{4.21}$$

since

$$2i\int_D \underline{u}^{o*}\cdot(\underline{K}\times(\nabla\psi'\times\underline{K}))\,dx\,dy = 2i\int_D \underline{u}^{o*}\cdot\nabla\psi'\,dx\,dy =$$
$$= \underbrace{2i\oint_{(D)}\psi'\underline{u}^{o*}\cdot d\underline{n}}_{=0} - 2i\int_D \psi'\nabla\cdot\underline{u}^{o*}\,dx\,dy =$$
$$= (\sigma^o)^2\int_D \frac{1}{H}(\nabla\psi'^*)\cdot(\nabla\psi')\,dx\,dy\,,$$

where (19) has been used.

These notes submitted by

Johannes Schmid-Burgk

<u>Supplement to "Effects of Rotation on Shallow Water Motions"</u>

Louis N. Howard

The qualitative features of the waves in a paraboloidal bowl and the effect of rotation on them would appear to be pretty much typical of more general cases. Here we consider briefly the case of "Rossby" waves at high rotation rate, which for the paraboloid have $\sigma \sim \omega^{-1}$. We shall see that similar modes can be found in general. This limit, $\frac{a^2\omega^2}{gH_0} \gg 1$ and $\sigma/\omega \ll 1$ is however physically somewhat artificial, since to achieve it in an actual "bowl", the bowl would have to be itself a very deep and slender paraboloid in order to preserve the depth parallel to the z-axis at $H_0(1-r^2/a^2)$ - under such circumstances one might wonder about the relevance of shallow water theory. But there is no particular difficulty with this limiting case regarded as a mathematical device, and similarity between the paraboloid and a more general ease in the limit suggests similarity for more moderate values of $a^2\omega^2/g H_0$.

Let $\sigma = \mu \cdot \frac{g H_0}{a^2} \cdot \frac{1}{\omega}$ and introduce dimensionless variables in the basic equation (4.5) using a as length scale and $H = H_0 h(\frac{r^2}{a})$. Then we get

$$\frac{\sigma}{\omega}\mu\vec{u} - 2i\mu\vec{k}\times\vec{u} + \nabla\nabla\cdot(h\vec{u}) = 0 \tag{4.22}$$

We are interested in modes for this equation with μ of order 1 and $\frac{\sigma}{\omega}$ small. It thus seems natural to drop the first term and consider

$$-2i\mu\vec{k}\times\vec{u} + \nabla\nabla\cdot(h\vec{u}) = 0.$$

(Warning: we shall see presently that this is in fact a singular perturbation.) Taking the curl of this equation shows that $\nabla\cdot\vec{u} = 0$ so $\vec{u} = \nabla\psi\times\vec{k}$, and $\vec{k}\times\vec{u} = \nabla\psi$. Thus $-2i\mu\psi + \nabla\cdot(h\vec{u}) = 0$, since the constant of integration can be absorbed by ψ. But since $\nabla\cdot\vec{u} = 0$, this now becomes

$$0 = -2i\mu\psi + \vec{u}\cdot\nabla h = -2i\mu\psi + (k\times\nabla h)\cdot\nabla\psi \tag{4.23}$$

If s is arc length along a line of constant ψ and $\beta = |\nabla h|$, (4.23) becomes

$$\beta\frac{\partial\psi}{\partial s} = 2i\mu\psi \tag{4.24}$$

Now in general this equation presents a problem, for we must usually expect closed contours of h in basins, and integration of the above first order equation around such a contour will not in general return one to the same value of ψ. This can be achieved for a particular contour by choice of μ, but in general will not then work for other contours. In the case of the paraboloid it happens to work for all contours at once, but this is not usually true. This would suggest perhaps that the disturbance must be localized along a particular contour, depending on μ , but the situation is in fact different. To clarify it we must reconsider the first term in (4.22). The reason that neglecting it is a singular perturbation is that this forces $\nabla \cdot \vec{u} = 0$ which then reduces the order of (4.22). This can be seen more clearly by writing the equation in terms of the surface elevation ζ , which is related to $\vec{u}$ by

$$\vec{u} = \frac{2\omega}{\sigma^2 - 4\omega^2}\left[\nabla\zeta \times k + \frac{i\sigma}{2\omega}\nabla\zeta\right] \tag{4.25}$$

One finds from this that $\mu\left(\frac{\sigma}{\omega}\vec{u} - 2i\vec{k} \times \vec{u}\right) = i\frac{\mu}{\omega}\nabla\zeta$

so (4.22) may again be integrated to

$$i\frac{\mu}{\omega}\zeta + \nabla\cdot\left(h\vec{u}\right) = 0, \quad \text{or}$$

$$\frac{1}{2}i\mu\left(4 - \frac{\sigma^2}{\omega^2}\right)\zeta - \frac{i\sigma}{\omega}\nabla\cdot\left(h\nabla\zeta\right) - \left(\vec{k} \times \nabla h\right)\cdot\nabla\zeta = 0 \tag{4.26}$$

It is clear in this equation that dropping the term $\frac{\sigma}{\omega}\nabla\cdot\left(h\nabla\zeta\right)$ reduces the order and constitutes a singular perturbation. Furthermore, if this term <u>is</u> dropped, along with the one in σ^2/ω^2 , we get exactly the equation (4.24) with ζ replacing ψ . But (4.26) now shows that a special contour along which (for given μ) the reduced-order equation is satisfied, plays the role of a <u>turning point</u>, ψ being exponentially decaying on one side and sinusoidal, with scale $\sqrt{\sigma/\omega}$ times the overall scale, on the other. The reader is invited to examine this more fully in the axisymmetric case.

Spherical Tides

Louis N. Howard

We shall now extend the results of the previous lecture to apply to shallow water tides on a solid sphere.

Motions normal to the spherical surface are neglected, and the velocity is written

$$\underline{u} = u\,\underline{\theta}_1 + v\,\underline{\varphi}_1 \,, \qquad \text{in spherical polar coordinates.}$$

For generality, we shall include a forcing term in the formulation of the problem, to represent external influences due to sun, moon, etc.; however we shall only solve the equations for the case of free oscillations.

The equations of motion are

$$\underline{u}_t + 2\omega\, \underline{k} \times \underline{u} + \frac{1}{\rho}\nabla p + g\,\underline{r}_1 = \text{forcing term} \tag{5.1}$$

where $\underline{k} = (\cos\theta, \sin\theta, 0)$.

We suppose that $\frac{\omega^2 a}{g}$ and $\frac{v}{\omega a}$ are small compared to unity, and deduce from the radial component of (5.1) that $p = \rho g(\zeta - r)$ where $\zeta(r,\theta,\varphi,t)$ is the displacement of the free surface above the mean height, and $H(\theta,\varphi)$ is the mean height of the shallow water above the sphere. The radial coordinate r is measured from the mean height level.

Representing the forcing term by $g\nabla\hat{\zeta}(\theta,\varphi)$, we find for the remaining components:

$$u_t - 2\omega\cos\theta\, V + \frac{g}{a}\frac{\partial}{\partial\theta}(\zeta - \hat{\zeta}) = 0$$

$$V_t + 2\omega\cos\theta\, u + \frac{g}{a\sin\theta}\frac{\partial}{\partial\varphi}(\zeta - \hat{\zeta}) = 0$$

Considering only axisymmetric vibrations, of period σ , we obtain

$$i\sigma u - 2\omega \cos\theta v + \frac{g}{a}\frac{\partial}{\partial\theta}(\zeta - \bar{\zeta}) = 0 \qquad (5.2)$$

$$i\sigma v + 2\omega\cos\theta u = 0 \qquad (5.3)$$

We also require to make use of an equation of continuity. This may be obtained by considering the flux of fluid through a surface element $r^2 d\theta d\varphi$. Note that the surface density per unit area is $(H+\zeta)\rho$.

Therefore we must have

$$\frac{\partial}{\partial t}\left[\rho(H+\zeta)\right] + \nabla\cdot\left[\rho\underline{u}(H+\zeta)\right] = 0$$

Assuming that $\frac{\zeta}{H} \ll 1$, we obtain (with ρ = constant and H = $H(\theta,\varphi)$)

$$i\sigma\zeta + \frac{1}{a\sin\theta}\frac{\partial}{\partial\theta}(Hu\sin\theta) = 0 \qquad (5.4)$$

From (5.2) and (5.3) it follows that

$$\sigma^2 u - 4\omega^2 u\cos^2\theta - i\sigma\frac{g}{a}\frac{\partial}{\partial\theta}(\zeta - \bar{\zeta}) = 0 \qquad (5.5)$$

These equations are the basis of Laplace's tidal theory (axisymmetric case). We now take $\bar{\zeta} = 0$, considering only the free oscillations. Then from (5.4) and (5.5), for the case of uniform mean depth, we find that

$$\sigma^2 u - 4\omega^2 u\cos^2\theta + g\frac{H}{a^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(u\sin\theta)\right] = 0$$

Now define $q = 4\frac{\omega^2 a^2}{gH}$, and $\lambda = \frac{\sigma^2 a^2}{gH}$;

and so

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}(u\sin\theta)\right) + (\lambda - q\cos^2\theta)u = 0$$

Finally, write $u = V\sin\theta$, $\mu = \cos\theta$, and then

$$\frac{d^2}{d\mu^2}\left[\left(1-\mu^2\right)v\right]+\left(\lambda-q\mu^2\right)V=0$$

Multiply by $V(1-\mu^2)$ and integrate with respect to μ from $\mu=+1$ to $\mu=-1$, whence

$$\lambda\int_{-1}^{+1}(1-\mu^2)V^2 d\mu = q\int_{-1}^{+1}\mu^2(1-\mu^2)V^2 d\mu + \int_{-1}^{+1}\left[\frac{d}{d\mu}(1-\mu^2)V\right]^2 d\mu$$

This is a variational equation for V. For a trial function we take $V=1+\alpha\mu^2+\ldots$, and we obtain for the approximate value of λ:

$$\lambda \simeq 2+\frac{q}{5}.$$

For illustration, let us calculate the frequency for the earth. We take for the mean depth of the ocean $H \sim 5000$ m, $a \sim 6.10^6$m and $\omega \sim 10^{-4}$ radians/sec. It maybe noted that rotation effects are necessarily of importance, since the time for a long wave-length disturbance to travel around the earth is approximately $\frac{a}{\sqrt{gH}}$, or 3.10^4secs., of the order of a rotation period.

Then we have that $q = \frac{4\omega^2 a^2}{gH} \simeq 7$, and so the frequency σ is given by

$$\sigma \cong \sqrt{gH/a^2}\sqrt{\lambda} \cong 224\frac{m}{sec}\cdot 1.84 = 411\, sec.^{-1}$$

Thus the lowest axisymmetric mode has a period of about 7 hours. Also, in the vicinity of the equator, we have approximately that

$$\frac{d^2V}{d\mu^2}+\left(\lambda-q\mu^2\right)V=0$$

This equation is that of the quantum mechanical simple harmonic oscillator. The solutions have appreciable amplitudes mainly inside $\mu < \sqrt{\lambda/q}$ which is indeed small compared with 1 if the frequency is small compared to the rotation rate. Thus low frequency axisymmetric free tidal oscillations have their significant motions mostly restricted to the neighborhood of the equator, and look like Hermite functions.

The surface displacement ζ may be expanded in spherical harmonics

$$\zeta(\mu) = \sum_n C_n P_n(\mu).$$

From equations (5.4) and (5.5) we obtain

$$\frac{\partial}{\partial\mu}\left[\frac{1-\mu^2}{\frac{\sigma^2}{4\omega^2}-\mu^2}\frac{\partial\zeta}{\partial\mu}\right] = -q\zeta \tag{5.6}$$

where (as before)

$$q = \frac{4\omega^2 a^2}{gH}$$

Using the identity

$$\frac{d}{d\mu}\left[(1-\mu^2)P_n'\right] = -n(n+1)P_n,$$

we obtain from (5.6)

$$\sum_n C_n P_n' = q\left(\frac{\sigma^2}{4\omega^2}-\mu^2\right)\sum\frac{C_n P_n'}{n(n+1)}$$

We now utilize the identities

$$\frac{2n+1}{n(n+1)}(\mu^2-1)P_n' = P_{n+1} - P_{n-1}$$

$$(2n+1)P_n = P_{n+1}' - P_{n-1}'$$

The result is the equation

$$\sum_n C_n P_n'\left[-1+q\left(\frac{\frac{\sigma^2}{4\omega^2}-1}{n(n+1)}\right)\right] = q\sum_n\frac{C_n}{2n+1}\left\{\frac{P_{n+2}'}{2n+3}+\frac{P_{n-2}'}{2n-1}-\frac{2(2n+1)}{(2n-1)(2n+3)}P_n'\right\}.$$

Thus the coefficients C_n are related by

$$C_n L_n = \frac{C_{n-2}}{(2n-3)(2n-1)} + \frac{C_{n+2}}{(2n+5)(2n+3)} \quad (n \geq 2);$$

We take $C_0 = 0$ so that the perturbation gives no total volume change, and this formula then holds also for $n = 1$, if we define $C_{-1} = 0$.

Here L_n is given by

$$L_n = \frac{\frac{\sigma^2}{4\omega^2} - 1}{n(n+1)} + \frac{2}{(2n-1)(2n+3)} - \frac{1}{g}$$

The above formula can be used effectively to calculate the normal modes by suitably truncating the infinite system of equations for the C_n. For instance, for the lowest mode we have

$$C_1 L_1 - \frac{C_{-1}}{-1} - \frac{C_3}{7.5} = 0, \left(\text{or } C_1 L_1 - \frac{C_3}{7.5} = 0 \right) - \frac{C_1}{3.5} + C_3 L_3 - \frac{C_5}{11.9} = 0$$

and so on. Truncating at the first step we find $L_1 = 0$, which is easily seen to lead to $\lambda = 2 + g/5$, as in the simple variational calculation above. The next approximation gives

$$\begin{vmatrix} L_1 & -\frac{1}{7.5} \\ -\frac{1}{3.5} & L_3 \end{vmatrix} = 0$$

i.e. $L_1 = \frac{1}{3 \cdot 5 \cdot 5 \cdot 7 \, L_3}$. Using the 1st approximation to evaluate L_3 we find

$$\lambda = 2 + g/5 + \frac{2g}{3 \cdot 5 \cdot 5 \cdot 7} \; \frac{1}{\left(\frac{2}{g} - \frac{4}{5}\right)\frac{1}{3 \cdot 4} + \frac{2}{5 \cdot 9} - \frac{1}{g}} = 2 + g/5 - \frac{12}{35} \; \frac{g^2}{2g + 75}$$

For the earth, this correction term increases the period of the lowest mode by about 3%. Thus convergence appears to be quite rapid, at least for the low modes. This approach is also useful in calculating forced motions. For classical references see Lamb.

These notes submitted by

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Hydromagnetic Planetary Waves

Willem V.R. Malkus

ABSTRACT

A study is made of hydromagnetic oscillations in a rotating fluid sphere. The basic state is chosen as a uniform current parallel to the axis of rotation. This state is stable or marginally stable to axisymmetric disturbances for all values of current, rotation, viscosity and magnetic diffusivity. It is found that the non-dissipative normal modes are described by a modified form of the Poincaré eigenvalue problem. For small rotation rates, the lowest non-axissymmetric modes are unstable. For rotation rates of geophysical interest all normal modes are stable. The introduction of ohmic dissipation produces a hydromagnetic boundary layer problem. Solutions for the boundary layer are outlined indicating its role in altering the free periods, damping the oscillations and producing external poloidal magnetic fields. Dispersion relations are derived which establish that the zonal phase velocities of both "fast" hydrodynamic and "slow" hydromagnetic waves can be of either sign. Observations of the secular variations of the earth's magnetic field indicate motion primarily towards the west. A mechanism for selective excitation of the observed motion is discussed.

ANALOGY BETWEEN ROTATING AND STRATIFIED FLUIDS

Steady, Two-dimensional Motion

George Veronis

The plan of this series of lectures is to develop the theory of a non-rotating, stratified fluid in parallel with that of a rotating, homogeneous fluid in order to emphasize the striking similarities between the two systems. Parts of the material on the stratified problem in lectures 4 and 5 follow a presentation of the same problems by R. Dickinson in a term paper which he wrote for a course on rotating fluids by H. Greenspan.

I. Basic Equations

The Navier-Stokes equation of an incompressible, viscous fluid becomes

$$\frac{\partial \underset{\sim}{v}}{\partial t} + (\underset{\sim}{v} \cdot \nabla)\underset{\sim}{v} + 2\underset{\sim}{\Omega} \times \underset{\sim}{v} = -\frac{1}{\rho}\nabla p - g\hat{k} + \nu\nabla^2 \underset{\sim}{v}$$

when the fluid is placed in a uniform gravitational field and is made to rotate at a uniform angular velocity. ν is the kinematic viscosity and all other symbols have their usual meaning.

The equation for the conservation of mass is

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underset{\sim}{v} = 0,$$

and the heat equation, with the viscous heating and compressibility neglected, is

$$\frac{\partial T}{\partial t} + \underset{\sim}{v} \cdot \nabla T = \kappa \nabla^2 T,$$

where κ is a function of the thermal conductivity k, i.e.

$$\kappa = \frac{k}{\rho c_p}.$$

Note: If the temperature is composed of a fluctuating temperature, T, plus a constant imposed temperature gradient in the z-direction, we may of course write

$$T_{total} = T + \frac{z}{L}\Delta T,$$

and

$$\frac{\partial T_{total}}{\partial t} + \underset{\sim}{v} \cdot \nabla T + \frac{\Delta T}{L} w = \kappa \nabla^2 T.$$

We shall be considering cases where the density varies primarily in the direction of the gravitational field (the z-direction). Then we write

$$\rho(x,y,z,t) = \rho_0(z) + \rho'(x,y,z,t)$$

$$P(x,y,z,t) = P_0(z) + P'(x,y,z,t)$$

where ρ' and P' are small perturbing functions and P_0 is defined such that

$$\frac{\partial P_0(z)}{\partial z} = -g\rho_0(z)$$

Under these conditions, in component form, the Navier-Stokes equations become

$$u_t + \underset{\sim}{v}\cdot\nabla u - 2\Omega v = -\frac{1}{\rho}P_x + \nu\nabla^2 u$$

$$v_t + \underset{\sim}{v}\cdot\nabla v + 2\Omega u = -\frac{1}{\rho}P_y + \nu\nabla^2 v$$

$$w_t + \underset{\sim}{v}\cdot\nabla w \qquad = -\frac{1}{\rho}P'_z + \nu\nabla^2 w - g\frac{\rho'}{\rho},$$

where u, v and w are respectively the x, y and z components of the velocity.

If now, in addition, we apply the Boussinesq approximation, ρ becomes ρ_0, leaving ρ' only in the gravity term. The equation for the conservation of mass takes the form of the continuity equation for incompressible flows.

$$\nabla\cdot\underset{\sim}{v} = 0 \quad \text{and} \quad \frac{d\rho}{dt} = 0$$

Finally if ρ is given by $\rho_0(1-\alpha T)$ we substitute $(-\alpha\rho_0 T)$ for ρ'.

II. Steady Motion

We shall be considering the small, steady motions of an incompressible

rotating, homogeneous	non-rotating, stratified

fluid which is now considered to be inviscid and non-conducting.

In two dimensions the equations describing this system are

$\rho(u u_x + \omega u_z - 2\Omega v) = -P_x$	$\rho(u u_x + \omega u_z) = -P_x$
$\rho(u \omega_x + \omega \omega_z) = -P_z - g\rho$	$\rho(u \omega_x + \omega \omega_z) = -P_z - g\rho$
$u_x + \omega_z = 0$	$u_x + \omega_z = 0$
$\rho = \text{constant}$	$u P_x + \omega P_z = 0$

Let P, ρ and $\vec{V}$ be expanded in a series in terms of a small parameter ϵ such that P and ρ have zero-order contributions while $\vec{V}$ does not; i.e. the zero order state is one of no motion.

$$(P, \rho) = \sum_{n=0}^{\infty} \epsilon^n (P, \rho)_n$$

$$\underset{\sim}{V} = \sum_{n=1}^{\infty} \epsilon^n \underset{\sim}{V}_n$$

Then the zero order system reduces to the statement of hydrostatic equilibrium.

$$P_{0x} = 0$$

$$P_{0z} = -g\rho_0$$

Using the above conditions the first order system reduces to

$2\Omega v_1 = \frac{1}{\rho} P_{1x}$	$0 = P_{1x}$
$2\Omega u_1 = 0 \Rightarrow u_1 = 0$	
$0 = P_{1z} \Rightarrow v_{1z} = 0$	$-g\rho_1 = P_{1z}$
$u_{1x} + \omega_{1z} = 0 \Rightarrow \omega_{1z} = 0$	$u_{1x} + \omega_{1z} = 0$
	$\omega_1 \rho_{0z} = 0 \Rightarrow \omega_1 = 0$

Higher orders only give the same equations with differing subscripts. Consequently we conclude that the system is

geostrophic in one dimension	hydrostatic

In <u>three dimensions</u>, expanding variables as before, we again find hydrostatic equilibrium at zeroth order

$$P_{0x} = P_{0y} = 0$$
$$P_{0z} = -g\rho_0$$

To first order

$2\Omega v_1 = \frac{1}{\rho} P_{1x} = \frac{1}{\rho_0} P_{1x}$	$0 = P_{1x}$
$2\Omega u_1 = -\frac{1}{\rho} P_{1y}$	$0 = P_{1y}$
$0 = P_{1z} \Rightarrow u_{1z} = v_{1z} = 0$	$-g\rho_1 = P_{1z}$
$u_{1x} + v_{1y} + w_{1z} = 0$	$u_{1x} + v_{1y} + w_{1z} = 0$
	$w_1 \rho_{0z} = 0 \Rightarrow w_1 = 0$

and we find

$$u_{1x} + v_{1y} = 0$$

That is, the two systems to this order are horizontally non-divergent with the flow satisfying the Taylor-Proudman theorem, i.e.,

geostrophic, not varying in the direction of rotation and moving as columns of fluid following contours of equal height. (The Taylor column was illustrated in the laboratory by Dr. A. Ibbetson and an example of this type of flow is seen in the Görtler-wave experiment of the second lecture. The column appears below the oscillating disc.)	hydrostatic and moving in a horizontal plane. (This constraining effect of stratification is sometimes called blocking. The fluid moves in horizontal layers. A laboratory demonstration was arranged to illustrate this flow.)

The equations to second order in ϵ are

$$\rho(u_1 u_{1x} + v_1 u_{1y} - 2\Omega v_2) = -P_{2x}$$
$$\rho(u_1 v_{1x} + v_1 v_{1y} + 2\Omega u_2) = -P_{2y}$$
$$\rho(u_1 w_{1x} + v w_{1y}) = -P_{2z}$$
$$u_{2x} + v_{2y} + w_{2z} = 0$$

$$\rho_0(u_1 u_{1x} + v_1 u_{1y}) = -P_{2x}$$
$$\rho_0(u_1 v_{1x} + v_1 v_{1y}) = -P_{2y}$$
$$g\rho_2 = -P_{2z}$$
$$u_{2x} + v_{2y} = 0$$
$$w_2 \rho_{0z} = 0$$

As conditions on the second order velocity components we find after some cross-differentiation and manipulation, that

$$u_{2x} + v_{2y} \neq f_n(z)$$
$$u_{2z},\ v_{2z},\ w_{2z} \neq f_n(z)$$

no work at all, that

$$u_{2x} + v_{2y} = 0$$
$$w_2 = 0$$

Up to this order then, if we consider time dependencies to be of order ϵ (and higher),

and upon taking $u_z = 0, v_z = 0$

$$u_t + uu_x + vu_y - 2\Omega v = -\frac{1}{\rho} P_x$$
$$v_t + uv_x + vv_y + 2\Omega u = -\frac{1}{\rho} P_y$$
$$w_t + uw_x + vw_y = -\frac{1}{\rho} P_z - g$$
$$u_x + v_y + w_z = 0$$

We may integrate the last equation to give:

$$u_x + v_y = -\frac{1}{h}\frac{dh}{dt}$$

where $h(x,y,t)$ = thickness of the layer of fluid. Cross-differentiating the full equations to get rid of pressure and using above equation,

$$\rho_0(u_t + \underset{\sim}{V}_{horiz} \cdot \nabla u) = -P_x$$
$$\rho_0(v_t + \underset{\sim}{V}_{horiz} \cdot \nabla v) = -P_y$$
$$g\rho = -P_z$$

$$w = 0,\ u_x + v_y = 0$$

We note simply that the flow is hydrostatic to 3rd order and horizontally non-divergent to 2nd order.

we derive an equation for the conservation of potential vorticity,

$$\frac{\xi + 2\Omega}{h} = \frac{v_x - u_y + 2\Omega}{h} ,$$

$$\frac{d}{dt}\left(\frac{\xi + 2\Omega}{h}\right) = 0 .$$

Again in three dimensions, but this time for the combined rotating and stratified case, the equations are

$$\rho \underset{\sim}{V} \cdot \nabla u - 2\Omega \rho v = -P_x$$

$$\rho \underset{\sim}{V} \cdot \nabla v + 2\Omega \rho u = -P_y$$

$$\rho \underset{\sim}{V} \cdot \nabla \omega = -P_z - g\rho$$

$$\nabla \cdot \underset{\sim}{V} = 0$$

$$\underset{\sim}{V} \cdot \nabla \rho = 0$$

To zeroth order the system is hydrostatic.

$$P_{0x} = P_{0y} = 0$$

$$P_{0z} = -g\rho_0$$

To first order it is hydrostatic, geostrophic, horizontally non-divergent, with no vertical velocity

$$P_{1z} = -g\rho_1$$

$$2\Omega\rho_0 v_1 = P_{1x} , \qquad 2\Omega \rho_0 u_1 = -P_{1y}$$

$$\omega_1 = 0 \qquad , \qquad u_{1x} + v_{1y} = 0$$

Differentiating and combining the first three equations

$$2\Omega\left(\rho_{0z} v_1 + \rho_0 v_{1z}\right) = -g\rho_{1x} ,$$

$$2\Omega\left(\rho_{0z} u_1 + \rho_0 u_{1z}\right) = g\rho_{1y} .$$

If the system is Boussinesq, we drop $\rho_{0z}\binom{u_1}{v_1}$ and get the thermal wind relations

$$2\Omega\rho_0 v_{1z} = -g\rho_{1x}$$

$$2\Omega\rho_0 u_{1z} = g\rho_{1y}$$

Once again, the second order equations are

$$\rho_0\left(u_1 u_{1x} + v_1 u_{1y}\right) - 2\Omega\rho_0 v_2 = -P_{2x}$$

$$\rho_0\left(u_1 v_{1x} + v_1 v_{1y}\right) + 2\Omega\rho_0 u_2 = -P_{2y}$$

$$g\rho_2 = -P_{2z}$$

$$u_{2x} + v_{2y} + w_{2z} = 0$$

$$u_1\rho_{1x} + v_1\rho_{1y} + w_2\rho_{0z} = 0$$

and to this order, including first order time derivatives

$$u_t + \underset{\sim}{V}_h \cdot \nabla u - 2\Omega v = -\frac{1}{\rho}P_x$$

$$v_t + \underset{\sim}{V}_h \cdot \nabla v + 2\Omega u = -\frac{1}{\rho}P_y$$

$$g\rho = -P_z$$

These notes submitted by

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Ertel's Theorem

George Veronis

In an inviscid non-diffusive stratified fluid with density ρ and velocity $\underset{\sim}{V}$, measured relative to axes rotating with uniform angular velocity $\underset{\sim}{\Omega}$, and in which the equation of state may be written

$$\frac{\partial s}{\partial t} + \underset{\sim}{V}\cdot\nabla s = 0, \quad s = s(P,\rho) \tag{1}$$

the quantity $\frac{(\nabla\times\underset{\sim}{V}+2\underset{\sim}{\Omega})}{\rho}\cdot\nabla s$ is conserved following the motion of fluid elements.

Proof. The inviscid form of the Navier-Stokes equation is

$$\frac{\partial}{\partial t}\underset{\sim}{V} + \underset{\sim}{V}\cdot\nabla\underset{\sim}{V} + 2\underset{\sim}{\Omega}\times\underset{\sim}{V} = -\frac{1}{\rho}\nabla P \tag{2}$$

The vorticity equation may be written

$$\frac{\partial}{\partial t}\underset{\sim}{\omega_a} + \nabla\times(\underset{\sim}{\omega_a}\times\underset{\sim}{V}) = \nabla P\times\nabla\left(\frac{1}{\rho}\right) \tag{3}$$

where $\underset{\sim}{\omega_a} = \nabla\times\underset{\sim}{V} + 2\underset{\sim}{\Omega}$

The scalar product of (3) with $\frac{\nabla s}{\rho}$ and use of (1) and (2) then gives

$$\left(\frac{\partial}{\partial t} + \underset{\sim}{V}\cdot\nabla\right)\left(\frac{\underset{\sim}{\omega_a}\cdot\nabla s}{\rho}\right) = \frac{1}{\rho}\nabla s\cdot\nabla P\times\nabla\left(\frac{1}{\rho}\right)$$

which is zero if $s = s(P,\rho)$ as assumed.

Modes of Oscillation in Rotating and Stratified Fluids.

These may be found by linearisation of the equations of motion. We first examine motions which are independent of space.

1) Rotating: Inertial Oscillations

By elimination of u between the two linearised space independent equations

$$\frac{\partial u}{\partial t} - 2\Omega v = 0 \quad \text{and} \quad \frac{\partial v}{\partial t} + 2\Omega u = 0,$$

we find $\frac{\partial^2 v}{\partial t^2} + (2\Omega)^2 v = 0.$ Motion is therefore oscillatory with frequency 2Ω. If we take the local value of Ω on the Earth, the period of inertial oscillations varies from half a day at the poles to infinity at the equator.

2) Stratified: Buoyancy Oscillations

From the two equations

$$\frac{\partial \omega}{\partial t} + \frac{g\rho}{\rho_0} = 0 \quad \text{and} \quad \frac{\partial \rho}{\partial t} + \frac{\partial \rho_0}{\partial z}\omega = 0$$

, we obtain

$\frac{\partial^2 \omega}{\partial t^2} - \left(\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}\right)\omega = 0$. Motion is oscillatory with a frequency $\left(-\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}\right)\omega = 0$, the Brunt Väisälä Frequency.

If we allow spatial variation of the waves in two dimensions the modes are modified as follows:

1) Rotating: Inertial Waves

The equations of motion become

$$\text{N-S equs.}\begin{cases} u_t - 2\Omega v = -P_x \\ v_t + 2\Omega u = 0 \\ \omega_t = -P_z \end{cases}$$

continuity $u_x + \omega_z = 0$, whence we define a stream function ψ so that

$$u = \psi_z , \quad \omega = -\psi_x$$

Elimination of P and v then leads to

$$\nabla^2 \psi_{tt} + 4\Omega^2 \psi_{zz} = 0$$

and if $\psi \propto e^{i(\sigma t + kx + nz)}$ the dispersion relation becomes

$$\sigma^2 = \frac{4\Omega^2 n^2}{k^2 + n^2}$$

We note that when $n \gg k$, i.e., when the motion is largely in horizontal layers, the frequency approaches that of pure inertial oscillations. When $k \gg n$, i.e., when the motion is in vertical columns, the frequency approaches zero, the flow thus becomes steady and the Taylor-

Proudman conditions are approached. This type of wave motion was illustrated by an experiment in which a disc was oscillated along the axis of a vertical cylinder of water in solid body rotation. The wave motion was made apparent by its effect on a suspension of aluminium particles in the water illuminated by a vertical sheet of light. The shearing motions in the waves rotate the aluminium particles and the waves were seen as alternate dark and bright bands aligned with the wave crests. Waves are possible if the disc frequency is less than twice the frequency of rotation of the cylinder and radiate away from the disc in a cross pattern, being reflected at the cylinder boundary.

2) Stratified: <u>Internal waves</u>

The equations of motion

$$\rho_0 u_t + P_x = 0$$
$$\rho_0 \omega_t + P_z = -g\rho$$
$$u_x + \omega_z = 0 \quad \text{(whence } u = \psi_z, \ \omega = -\psi_x\text{)}$$

and $\rho_t + \frac{\partial \rho_0}{\partial z}\omega = 0$

may be reduced using the Boussinesq approximation to $\nabla^2 \psi_{tt} = \frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}\psi_{zz}$,

and if $S = \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}$ is constant and $\psi \propto e^{i(\sigma z + kx + nz)}$ the dispersion relation is

$$\sigma^2 = \frac{-gsk^2}{k^2+n^2}.$$

(These waves were produced experimentally in a stratified brine solution with a nearly constant density gradient by slowly oscillating a block of wood at one end of a tank containing the fluid. A film of the internal wave rays produced in this way, made by Dr. Stewart Turner, was also shown.

The experiments mentioned above were set up and demonstrated by Dr. Alan Ibbetson.)

The Analogy between Rotating and Stratified Fluids

The analogy between two-dimensional flows of a homogeneous fluid relative to a rotating frame and the two-dimensional flows of a stratified fluid with a constant mean density gradient in cases when it is appropriate to make the Boussinesq approximation is now extended by a study of the equations of motion including the effects of viscosity and diffusivity. The two sets of equations are non-dimensionalised and are then as described below.

1) 2-D rotating homogeneous flows (Ω constant).

Scaling all lengths with a length scale L, time with $\frac{1}{2\Omega\delta}$, velocity components with a speed V and pressure with $2\Omega V\rho L$, yields the equations

$$\left.\begin{aligned} \delta u_t + \epsilon \underset{\sim}{V}\cdot\nabla u - v &= -P_x + E\nabla^2 u \\ \delta v_t + \epsilon \underset{\sim}{V}\cdot\nabla v + u &= E\nabla^2 v \\ \delta w_t + \epsilon \underset{\sim}{V}\cdot\nabla w &= -P_z + E\nabla^2 w \\ u_x + w_z &= 0 \end{aligned}\right\} I$$

where $\underset{\sim}{V} = (u, v, w)$, $E = \frac{\nu}{2\Omega L^2}$ is the Ekman number ($\frac{1}{E^2}$ = Taylor number), $\epsilon = \frac{V}{2\Omega L}$ is the Rossby number and δ is, as yet, at our disposal.

2) 2-D stratified fluid; Boussinesq Approximation.

Lengths are scaled with a length L. In the basic state the temperature is $T + \frac{z}{L}\Delta T$ where the temperature difference between top and bottom of the fluid is ΔT and the corresponding density difference is $-\alpha\rho_0 \Delta T$. The time is scaled by $\frac{1}{\delta}\sqrt{\frac{L}{-g\alpha\Delta T}}$, proportional to the reciprocal of the Brunt-Väisälä frequency. ΔT_c is a scale of the temperature fluctuations driving the motions from the basic state and we write $T = \Delta T_c \theta$. The pressure is scaled with $-\rho L g \alpha \Delta T_c$ (corresponding to the change in hydrostatic balance) and the velocity components with $\sqrt{-g\alpha L \Delta T}\,\frac{\Delta T_c}{\Delta T}$. The equations become

$$\left.\begin{aligned} \delta u_t + \epsilon \underset{\sim}{V}\cdot\nabla u &= -P_x + \sigma^{-1/2} E \nabla^2 u \\ \delta \omega_t + \epsilon \underset{\sim}{V}\cdot\nabla \omega &= -P_z + \sigma^{-1/2} E \nabla^2 \omega + \theta \\ \delta \theta_t + \epsilon \underset{\sim}{V}\cdot\nabla \theta &= -\omega + \sigma^{1/2} E \nabla^2 \theta \\ u_x + \omega_z &= 0 \end{aligned}\right\} \; II$$

where $E = \sqrt{\frac{\nu K}{g\alpha \Delta T L^3}}$ ($\frac{1}{E^2}$ is the Rayleigh number), $\epsilon = \frac{\Delta T_c}{\Delta T}$, and $\sigma = \frac{\nu}{K}$ the Prandtl number.

Notice the similarity between the equations of sets I and II. u, ω, x, z and v on set I correspond to ω, u, z, x and θ, respectively, in set II. The equations are then exactly analogous with ϵ and E as defined, except for the presence of the Prandtl number, σ, in set II. At unit Prandtl number or in small steady motions (when in II we may define $\sigma^{-1/2} P = \mathbb{P}$, $\sigma^{-1/2}\theta = \Theta$) the analogy between the rotating and stratified systems becomes complete, provided that the boundary conditions may similarly be matched and made to correspond. Results deduced in one system may be applied directly to the other.

These notes submitted by

Stephen A. Thorpe

$E^{\frac{1}{2}}$-Layers in Rotating and Stratified Systems

George Veronis

Consider first a stratified system with unperturbed temperature gradient $\Delta T/L$. The total temperature is written $T_{mean} + \frac{z}{L}\Delta T + (\sigma^{1/2}\Delta T_c)T$ where ΔT_c is a scale factor for temperature perturbations, σ is the Prandtl number ν/κ, and T is the dimensionless temperature deviation. We will be considering steady two-dimensional motions so that the quantities u_t, w_t, T_t are set equal to zero. Furthermore we consider $\epsilon = \Delta T_c/\Delta T$ small, enabling the non-linear terms to be dropped. The basic equations are then

$$-T = -P_z + E\nabla^2 w \tag{1}$$

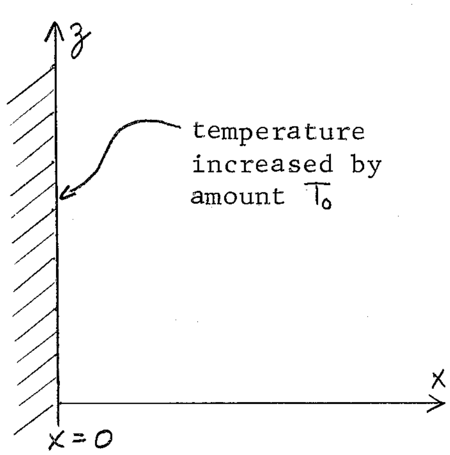
$$w = E\nabla^2 T \tag{2}$$

$$0 = -P_x + E\nabla^2 u \tag{3}$$

$$u_x + w_z = 0 \tag{4}$$

$E = \sqrt{\frac{\nu\kappa}{g\alpha L^3\Delta T}}$ is the reciprocal square root of the Rayleigh number. Velocity is measured in units of $\sqrt{g\alpha L\Delta T}(\Delta T_c/\Delta T)$ and pressure in units of $\sigma^{-1/2}\rho L g\alpha\Delta T_c$.

<u>First application</u> (<u>buoyancy layer</u>).



Consider a boundary at $x = 0$ of indefinite extent in the z-direction. The stable stratification of the system is perturbed by applying a constant temperature deviation, T_0, at the boundary. The temperature deviation vanishes at large x. The above equations are to be solved subject to the boundary conditions

$$T = T_0 \,,\; w = 0 \quad at \quad x = 0$$
$$T = 0 \,,\; w = 0 \quad at \quad x = \infty$$

Since the applied temperature is independent of z we seek a solution in which all quantities are independent of z. Hence the relevant equations are

$$E \frac{d^2 w}{dx^2} + T = 0 \tag{1'}$$

$$E \frac{d^2 T}{dx^2} - w = 0 \,. \tag{2'}$$

Multiply the first equation by i and subtract the second equation, letting $\phi = w + iT$. The result is $iE \frac{d^2\phi}{dx^2} + \phi = 0$, for which the solution is

$$\phi = C_1 \exp\left(\frac{x e^{i\pi/4}}{\sqrt{E}}\right) + C_2 \exp\left(\frac{x e^{5i\pi/4}}{\sqrt{E}}\right).$$

But since $e^{i\pi/4} = (1+i)/\sqrt{2}$ has a positive real part, the condition that w and T vanish at $x = \infty$ implies $C_1 = 0$. At $x = 0$, $\phi = iT_0 = C_2$, hence

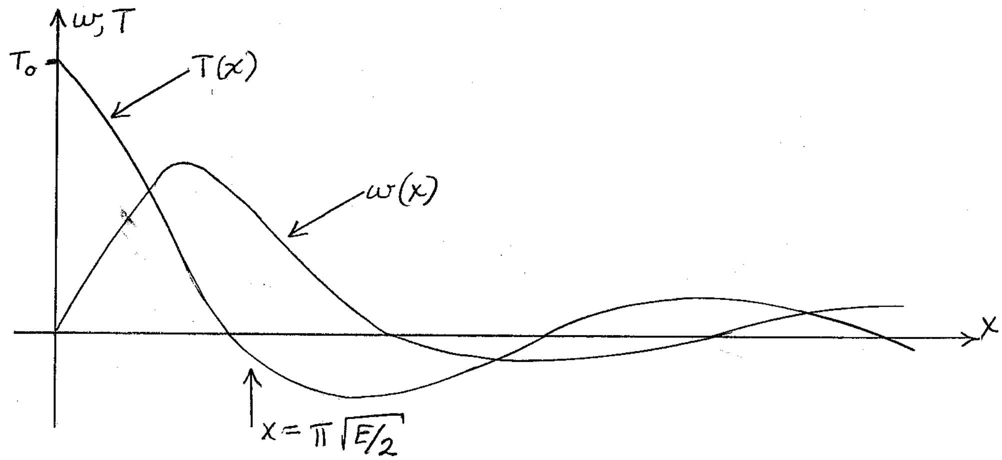
$$\phi = w + iT = iT_0 e^{-\frac{x}{\sqrt{2E}}} e^{-\frac{ix}{\sqrt{2E}}}$$

or

$$\left.\begin{aligned} T &= T_0 e^{-\frac{x}{\sqrt{2E}}} \cos\left(x/\sqrt{2E}\right) \\ w &= T_0 e^{-\frac{x}{\sqrt{2E}}} \sin\left(x/\sqrt{2E}\right) \end{aligned}\right\} \tag{5}$$

This solution shows that the temperature adjustment and corresponding flow are confined to a relatively small "buoyancy layer" where $x \lesssim \sqrt{2E}$.

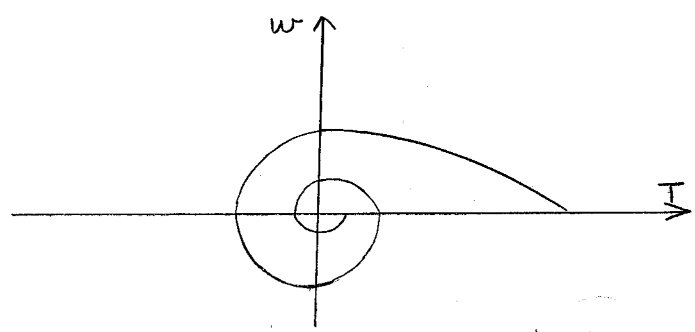
In the interior of the fluid where $x \gg \sqrt{2E}$ conditions remain the same as before the perturbation was applied. The accompanying sketches of $T(x)$ and $w(x)$ show how the temperature is raised adjacent to $x = 0$ (with relatively minor subsequent oscillations) and how the fluid rises close to $x = 0$, but maintains $w = 0$ on the boundary.



The net upward flux (for unit length in the y-direction) is

$$\int_0^\infty w(x)\,dx = T_0\sqrt{\frac{E}{2}}$$

For comparison purposes with the analog for the rotating case it is instructive to plot w vs. T with x as parameter:



Rotational analog of the buoyancy layer: The Ekman Layer

The equations for small steady two-dimensional motions in a rotating system are

$$\left.\begin{aligned} -v &= -P_x + E\nabla^2 u \\ u &= E\nabla^2 v \\ 0 &= -P_z + E\nabla^2 w \\ u_x + w_z &= 0 \end{aligned}\right\} \qquad (6)$$

Let a constant wind stress be applied at the surface $z=0$ parallel to the y-direction. The boundary conditions are then

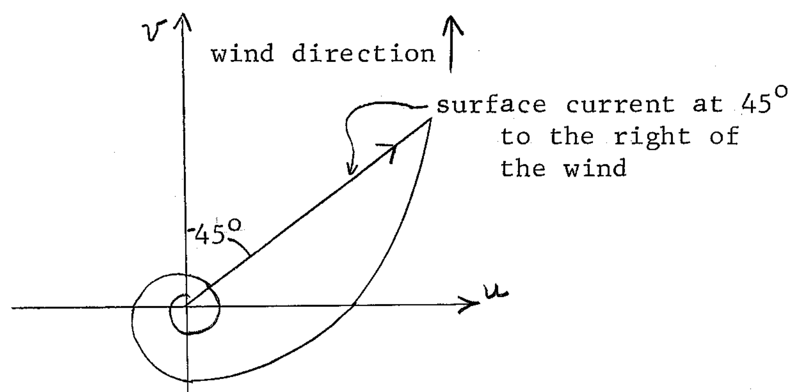
$$\frac{\partial v}{\partial z} = T_0 \qquad \frac{\partial u}{\partial z} = 0 \qquad \text{at} \quad z = 0$$
$$v = 0 \qquad u = 0 \qquad \text{at} \quad z = -\infty .$$

Again, these conditions imply a solution having no x-dependence. The solution to the first two equations in (6) satisfying the given boundary conditions is

$$\left.\begin{aligned} u &= \sqrt{\tfrac{E}{2}}\, T_0 \left\{\cos\left(z/\sqrt{2E}\right) - \sin\left(z/\sqrt{2E}\right)\right\} \exp\left(z/\sqrt{2E}\right) \\ v &= \sqrt{\tfrac{E}{2}}\, T_0 \left\{\cos\left(z/\sqrt{2E} + \sin\left(z/\sqrt{2E}\right)\right\} \exp\left(z/\sqrt{2E}\right). \end{aligned}\right\}$$

(Note: the positive z-direction is taken to be vertically upwards.)

Plotting v versus u yields the <u>Ekman spiral</u>:



To compute the net transport we write $T_y = \partial v/\partial z$ and $T_x = \partial u/\partial z$

Then the first two equations in (6) become

$$-v = \frac{\partial}{\partial z}\left(E\, T_x\right)$$
$$u = \frac{\partial}{\partial z}\left(E\, T_y\right)$$

But since T_x and T_y vanish at $z = -\infty$,

$$\int_{-\infty}^{0} u\, dz = E\, T_y\Big|_{z=0} = E\, T_0$$
$$\int_{-\infty}^{0} v\, dz = -E\, T_x\Big|_{z=0} = 0 ,$$

which shows that the net transport is at right angles to the stress. Since we have assumed a constant viscosity throughout the fluid, the inclusion of E inside the partial derivative in Eq. (8) is trivial. A more detailed analysis in which the eddy viscosity coefficient is variable, leads to equations having the same form as (8). The subsequent derivation then shows that the net transport depends only on the surface conditions and is independent of detailed structure. (cf. GFD Lecture Notes 1961, p.83).

Buoyancy layer for a non-uniform temperature perturbation

If the applied temperature at the walls is not uniform, but is a slowly varying function $T_0(z)$, we can consider to what order in E various quantities will be affected. The complete form of Eq. (1) is

$$-T = -P_z + E\left(w_{xx} + w_{zz}\right)$$

Since $w_{xx} = O(E^{-1})$ but $w_{zz} = O(1)$ we are still justified in neglecting the term w_{zz}. But what about P_z ? To learn about this we return to the continuity equation

$$u_x = -w_z = -\left(\frac{dT_0}{dz}\right) e^{-x/\sqrt{2E}} \sin\left(x/\sqrt{2E}\right) = -\frac{1}{T_0}\frac{dT_0}{dz} w .$$

Thus

$$u_\infty = -\frac{1}{T_0}\frac{dT_0}{dz}\int_0^\infty w\,dx = -\sqrt{\frac{E}{2}}\,\frac{dT_0}{dz},$$

hence implying a horizontal flow of order $E^{1/2}$.

Alternatively, $u_x = -\frac{1}{T_0}\frac{dT_0}{dz} w = O(1)$ and using the stretching transformation $x = E^{1/2}\xi$ this shows $\frac{\partial u}{\partial \xi} = \frac{\partial x}{\partial \xi} u_x = O(E^{1/2})$, and therefore $u = O(E^{1/2})$.

Now going back to Eq. (3):

$$P_x = E\,\frac{\partial^2 u}{\partial x^2} = E\frac{\partial}{\partial x}\left(-w_z\right)$$

But $\omega_z = O(1)$, hence $P = O(E)$. It is therefore justified to neglect the term P_z in Eq.(1) in comparison with the other terms, which are of order 1. In summary, we have found

$$\begin{aligned} \omega &= O(E^0) \\ T &= O(E^0) \\ P &= O(E) \\ u &= O(E^{1/2}) \\ \psi &= O(E^{1/2}) \end{aligned}$$

where ψ is the stream function associated with the two-dimensional flow. (i.e. $u = \partial\psi/\partial z$ implies ψ and u are of the same order.)

<u>Temperature perturbation applied throughout the fluid</u>.

A converse problem to the one just considered is the following: For $t < 0$ the system is in the base state of stable stratification with a linear temperature gradient. At $t = 0$ the fluid is heated up by a uniform amount T_0 , but the boundary is maintained at the original temperature distribution. Describe the resulting steady flow.

From Eq.(1) we see that in the interior $P_z = T_0$. The pair of equations to be satisfied throughout is therefore

$$\begin{aligned} -T &= -T_0 + E\frac{d^2\omega}{dx^2} \\ \omega &= \qquad\quad E\frac{d^2T}{dx^2} \end{aligned}$$

The boundary conditions are

$$\begin{aligned} T = 0,\ \omega = 0 \quad &at \quad x = 0 \\ T = T_0,\ \omega = 0 \quad &at \quad x = \infty \end{aligned}$$

It can readily be checked that the solution is

$$\begin{aligned} T &= T_0\left[1 - \cos(x/\sqrt{2E})\,e^{-x/\sqrt{2E}}\right] \\ \omega &= -T_0 \sin(x/\sqrt{2E})\,e^{-x/\sqrt{2E}} \quad . \end{aligned}$$

The time-dependent Ekman layer.

When linearized acceleration terms are added to the equations governing the Ekman layer we obtain

$$u_t - v = E u_{zz},$$
$$v_t - u = E v_{zz}.$$

The initial conditions are $u = v = 0$ at $t = 0$, and the boundary conditions for $t \geq 0$ are the same as stated following Eq. (6). It can be shown that the solution to this problem is

$$\left.\begin{aligned} u &= T_0\sqrt{\frac{E}{\pi}}\int_0^t \frac{\sin\tau}{\sqrt{\tau}}\, e^{-z^2/4E\tau}\, d\tau \\ v &= T_0\sqrt{\frac{E}{\pi}}\int_0^t \frac{\cos\tau}{\sqrt{\tau}}\, e^{-z^2/4E\tau}\, d\tau. \end{aligned}\right\} \qquad (9)$$

(For derivation via Laplace transform techniques see GFD Lecture Notes 1961, p.85.)

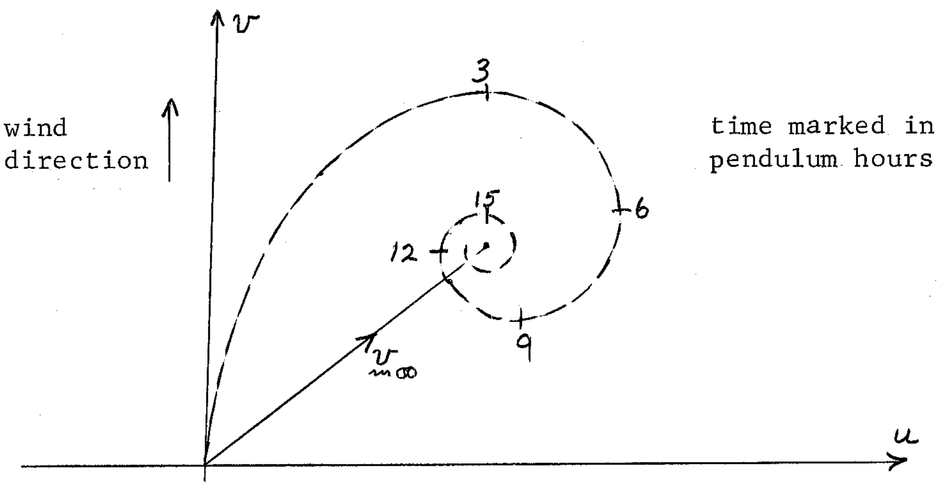
The evolution in time of the current at any particular depth can be studied by plotting v versus u from Eq.(9) using t as parameter. For example, at the surface $z = 0$:

$$u = T_0\sqrt{\frac{E}{\pi}}\int_0^t \frac{\sin\tau}{\sqrt{\tau}}\, d\tau = T_0\sqrt{2E}\, S\left(\sqrt{2t/\pi}\right)$$
$$v = T_0\sqrt{2E}\, C\left(\sqrt{2t/\pi}\right)$$

where C and S are the Fresnel integrals (Abramowitz and Stegun, Handbook of Mathematical Functions, Section 7.3). The plot of S versus C is the familiar Cornu spiral. Since

$$dv/du = (dv/dt)/(du/dt) = \cot t,$$

vertical and horizontal tangents occur where $t = \text{even}\ \pi/2$, $t = \text{odd}\ \pi/2$ respectively. The dimensionless time $t = \pi/2$ corresponds to $\pi/4\Omega = \frac{1}{4}$ of a pendulum day. At the end of one pendulum day the velocity at any depth is within 20% of its final value. Hence the time scale for the development of the Ekman layer is $t = O(1)$.



Time development of the surface current.

The limit as $t \to \infty$ is $\underset{\sim}{V}_\infty = (u_\infty, v_\infty) = \left(T_0\sqrt{\frac{E}{2}}, T_0\sqrt{\frac{E}{2}}\right)$.

Since it was previously shown that the horizontal velocity $u = O(E^{1/2})$ when the temperature perturbation T_0 is a slowly varying function of z, it can be seen that portions of the fluid which lie at distances of order 1 from the boundary will not be affected until $t = O(E^{-1/2})$, i.e. an extremely long time after the perturbation is applied. For a full treatment of the corresponding problem in a rotating system (the spin-up problem) see Howard and Greenspan, Journal of Fluid Mechanics, 1963.

These notes submitted by

Dudley H. Towne

The Role of $E^{1/3}$ and $E^{1/4}$ Boundary Layers

George Veronis

Before going on to discuss other possible boundary layers let us put the equations describing small, steady, stratified fluid motion into a more convenient form. Previously, in two dimensions, we found

$$E\nabla^2\psi_z = P_x$$
$$E\nabla^2\psi_x = -P_z + T$$
$$E\nabla^2 T = -\psi_x$$
$$u_x + w_z = 0$$

Eliminating the pressure

$$E\nabla^4\psi = T_x$$
$$E\nabla^2 T = -\psi_x$$

Finally, combining these two equations we get

$$E^2\nabla^6\phi = -\phi_{xx} \quad \text{where } \phi \text{ is } T, \psi \text{ or } P.$$

When variations with respect to the coordinates are $O(1)$ and $E \ll 1$, the last equation reduces (to $O(E^2)$) to

$$\phi_{xx} = 0.$$

From this equation we derive the interior system

$$P_x = 0, \; P_z = T, \; w = 0, \; u_x = 0.$$

With variations with respect to x are $O(E^{-1/2})$, the term $E\partial^6_{6x}\phi$ balances the term $\partial^2_{2x}\phi$ and the system of equations for the buoyancy layer results $\left(\partial^n_{nx} \equiv \frac{\partial^n}{\partial x^n}\right)$.

The interior plus the buoyancy layer are sufficient to describe flows which have infinite vertical extent and which do not require horizontal boundary layers in which to complete the circulation. Most problems are bounded in the vertical and it is then necessary to consider the possible occurrence of boundary layers adjacent to horizontal boundaries or in the vicinity of abrupt variations in the vertical structure.

One obvious boundary layer to look for occurs when $\partial_x = O(1)$ and the z-coordinate is stretched so that the principal balance is

$$E^2 \partial_{6z}^6 \phi + \phi_{xx} = 0$$

Letting $\partial_z = E^b \partial_\zeta$ where $\partial_\zeta = O(1)$, we conclude that

$$b = -1/3$$

To complete the descritpion of the boundary layer, we note that, if this boundary layer is to connect to the side boundaries, it must be capable of transporting the same amount of fluid as the buoyancy layer. It is then possible to consider flows where fluid is transported along the buoyancy layer, turns a corner (we cannot describe the flow in this region as yet), and then flows along the $E^{1/3}$ layer. (This requirement is plausible but is also somewhat arbitrary. It turns out that the results are useful and for this reason we pursue the argument.) Thus, summarizing the above two points we require that $\partial_z = O(E^{-1/3})$ and that $\psi = O(E^{1/2})$.

The heat equation is

$$E \nabla^2 \psi \approx E^{1/3} \partial_{\zeta\zeta}^2 \psi = T_x .$$

Since $\partial_x = O(1)$ and $\psi = O(E^{1/2})$ we deduce that

$$T = O(E^{1/6})$$

From similar arguments using the remaining equations we derive the following magnitudes and equations:

$$
\begin{aligned}
\partial_z &= O(E^{-1/3}) & \\
u &= O(E^{1/6}) & E^{1/3}\partial^2_{\zeta\zeta}u &= -\partial_x P \\
\omega &= O(E^{1/2}) & E^{1/3}T &= \partial_\zeta P \\
T &= O(E^{1/6}) & E^{1/3}\partial^2_{\zeta\zeta}T &= \omega \\
P &= O(E^{1/2}) & E^{1/3}\partial_x u + \partial_\zeta \omega &= 0 \\
\psi &= O(E^{1/2}) & &
\end{aligned}
$$

At this point it is useful to recall an additional result derived from the analysis of the buoyancy boundary layer. When the imposed temperature at the boundary varied in the z-direction, we found that

$$u_I = \sqrt{\frac{E}{2}}\,\frac{\partial T_0}{\partial z} \quad \text{or} \quad \psi_I = \sqrt{\frac{E}{2}}\,T_0$$

where the subscript I refers to the interior (away from the buoyancy boundary layer). Now this type of balance must exist at the outside edge of the buoyancy layer if the flow is to be connected to the buoyancy layer. It is clear from the balances derived for the $E^{1/3}$ layer that if the layer is to be capable of transporting the necessary amount of fluid from the buoyancy layer $(\psi = O(E^{1/2}))$, then $T = O(E^{1/6})$ and the above requirement cannot be satisfied. The question is: does a boundary layer exist which can adjust the temperature as required by the above relation?

We consider the heat equation

$$\psi_x = E\nabla^2 T$$

If we stretch z by

$$\partial_z = E^c \partial_\eta, \quad c > 0$$

we derive

$$\psi_x = -E^{1-2c}\partial^2_{\eta\eta}T$$

and we note that with $c = 1/4$ we have

$$\psi_x = -E^{1/2}\partial^2_{\eta\eta} T$$

Thus if $\psi = O(E^{1/2})$, T is $O(E^0)$, i.e., we can adjust T as required.

To complete the system as before we note that from the sixth-order equation we obtain to lowest order

$$\phi_{xx} = 0,$$

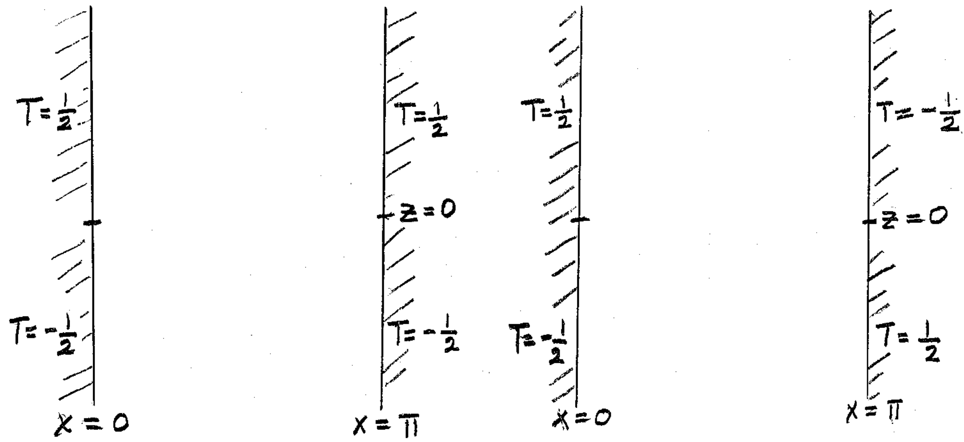
and the remaining balances of the $E^{1/4}$ layer are

$$\begin{array}{ll}
\partial_z = O(E^{-1/4}) & \\
u = O(E^{1/4}) & 0 = \partial_x P \\
\omega = O(E^{1/2}) & E^{1/4} T = \partial_\eta P \\
T = O(E^0) & E^{1/4}\partial^2_{\eta\eta} T = \omega \\
P = O(E^{1/4}) & E^{1/4}\partial_x u + \partial_\eta \omega = 0 \\
\psi = O(E^{1/2}) &
\end{array}$$

It should be noted that the $E^{1/4}$ layer involves only a limited amount of variation in z. It can adjust the temperature but cannot satisfy all of the dynamical requirements. For the latter it is necessary that an $E^{1/3}$ layer be used.

To exhibit the ideas presented above we treat two simple flows which are forced by temperatures imposed at the lateral boundaries. The mathematical problems are identical to those which were treated by Stewartson for the exactly analogous situation in rotating fluids. However, the method which we shall use is much simpler mathematically and makes use of the foregoing results. The procedure is essentially that used by Dickinson in a term paper at M.I.T.

We wish to find the characteristics of the steady flow between vertical boundaries which may occur when the boundaries are maintained at the temperatures indicated in the diagrams



(a) Symmetric Problem (b) Antisymmetric Problem

There is a discontinuity in the wall temperature at $z = 0$. For $z \neq 0$ the wall temperature is constant. In the earlier analysis it was shown that the interior horizontal velocity, $u_I = -\sqrt{\frac{E}{2}}\frac{\partial T_0}{\partial z}$. Proceeding formally to the limit in which the temperature is a step function we find here that

$$u_I = -\sqrt{\frac{E}{2}}\,\delta(z) \quad \text{near } x = 0$$

and $$u_I = +\sqrt{\frac{E}{2}}\,\delta(x) \quad \text{near } x = \pi \qquad \text{(problem (a))}$$

or $$u_I = -\sqrt{\frac{E}{2}}\,\delta(x) \quad \text{near } x = \pi \qquad \text{(problem (b))}.$$

We now wish to derive the horizontal $E^{1/3}$ boundary layers which will conduct the flow in the region $z = 0$, taking the values of u_I found above to represent the horizontal components of the flow at the edge of the boundary layers.

From the equation $E^2\nabla^6 u = -u_{xx}$, we deduce as before that in the horizontal boundary layer

$$\frac{\partial^6}{\partial \zeta^6}u + \frac{\partial^2}{\partial x^2}u = 0 \quad \text{where } \frac{\partial}{\partial \zeta} = E^{1/3}\frac{\partial}{\partial z}.$$

Taking the Fourier transform of u, $\tilde{u}(x,s) = \int_{-\infty}^{\infty} u\, e^{\iota s \zeta}\, d\zeta$, we find that the equation for $\tilde{u}$ is

$$-s^6 \tilde{u} + \tilde{u}_{xx} = 0$$

with boundary conditions (deduced from $u = u_I$ at $x = 0, \pi$)

$$\tilde{u} = -\sqrt{\frac{E}{2}} \text{ at } x = 0$$

and $\tilde{u} = \pm\sqrt{\frac{E}{2}}$ at $x = \pi$ for problems (a) / (b).

The solutions for the stream functions may then be written (using $u = \frac{\partial \psi}{\partial \zeta}$) as

(problem a) $$\psi_{sym} = \frac{1}{2\pi}\sqrt{\frac{E}{2}} \int_{-\infty}^{\infty} \frac{\sinh[k^3(x - \pi/2)]}{\sinh(k^3 \pi/2)} \frac{\sin k\zeta}{k}\, dk$$

(problem b) $$\psi_{anti} = \frac{1}{2\pi}\sqrt{\frac{E}{2}} \int_{-\infty}^{\infty} \frac{\cosh[k^3(x - \pi/2)]}{\cosh(k^3 \pi/2)} \frac{\sin k\zeta}{k}\, dk$$

(This is the solution valid in the interior region near $z = 0$.) Making the substitution $k\zeta = \ell$, we evaluate the solution at the edge of the boundary layer when $\zeta \to \infty$.

Since

$$\lim_{\zeta \to \infty} \int_{-\infty}^{\infty} \frac{\sinh[(\frac{\ell}{\zeta})^3 (x - \pi/2)]}{\sinh[(\frac{\ell}{\zeta})^3 \pi/2]} \frac{\sin \ell}{\ell}\, d\ell = \frac{x - \pi/2}{\pi/2} \int_{-\infty}^{\infty} \frac{\sin \ell}{\ell}\, d\ell$$
$$= 2(x - \pi/2)$$

and $$\lim_{\zeta \to \infty} \int_{-\infty}^{\infty} \frac{\cosh[(\frac{\ell}{\zeta})^3 (x - \pi/2)]}{\cosh[(\frac{\ell}{\zeta})^3 \pi/2]} \frac{\sin \ell}{\ell}\, d\ell = \pi$$

we find that, since $\frac{\partial}{\partial x} \lim_{\zeta \to \pm\infty} = \lim_{\zeta \to \pm\infty} \frac{\partial}{\partial x}$,

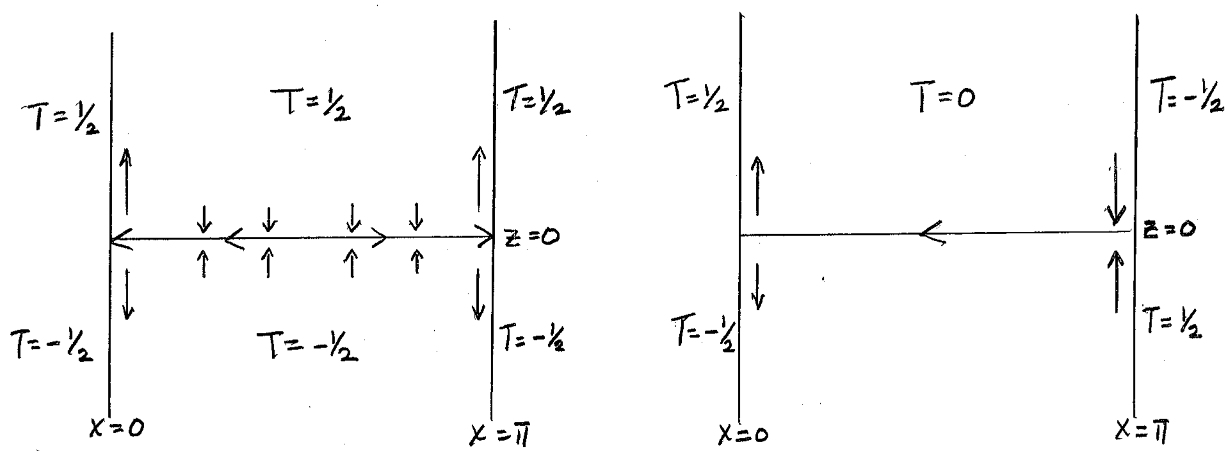
$$\lim_{\zeta \to \pm\infty} \frac{\partial \psi_{sym}}{\partial x} = -\omega_I = \pm \frac{1}{\pi}\sqrt{\frac{E}{2}} \qquad \text{in problem (a)}$$

and $$\lim_{\zeta \to \pm\infty} \frac{\partial \psi_{anti}}{\partial x} = -\omega_I = 0 \qquad \text{in problem (b)}$$

where ω_I is the vertical velocity component at the edge of the $\frac{1}{3}$ layer.

Thus we note that in the anti-symmetric problem all of the fluid ejected from the right-hand buoyancy layer (because of the discontinuous boundary temperature) is transported by the $E^{1/3}$ layer across to the left-hand buoyancy layer. In the symmetric problem fluid is ejected to both buoyancy layers. Hence in the latter situation the $E^{1/3}$ layers must receive fluid from the interior.

The combined flow patterns of the $E^{1/2}$ and $E^{1/3}$ layers is therefore as follows:



(a) symmetric (b) antisymmetric

As things now stand in the symmetric case warmer fluid is being drawn from above and cooler from below, which would imply a temperature discontinuity of order E^0 across $z=0$. Since this cannot occur there must be a compensating flow in an $E^{1/4}$ layer on both sides of $z=0$. No such requirement need be satisfied in the antisymmetric case, since the fluid in the $E^{1/3}$ layer is drawn from the $E^{1/2}$ layers at $x=\pi$ and is at the mean temperature.

To determine the $E^{1/4}$ layer required in the symmetric problem, observe that the equations $\partial_x P = 0$ and $\partial_\eta p = E^{1/4} T$ imply $T_x = 0$, i.e. that T (the temperature associated with the $E^{1/4}$ layer) is independent of x. Let T_b be the temperature at the boundary $(x = 0 \text{ or } \pi)$ attributable to the $E^{1/2}$ layer. Thus for $\eta > 0$ the net temperature at the boundaries must be the imposed temperature $1/2$, or $T_b + T = \frac{1}{2}$. Now from $u = \left(\frac{\partial \psi}{\partial z}\right)_{x=0} = -\sqrt{\frac{E}{2}}\, \frac{\partial T_b}{\partial z}$ we obtain $\psi_{x=0} = -\sqrt{\frac{E}{2}}\, T_b = -\sqrt{\frac{E}{2}}\left(\frac{1}{2} - T\right)$ and by similar reasoning

$$\psi_{x=\pi} = \sqrt{\frac{E}{2}}\left(\frac{1}{2} - T\right).$$

But in a $1/4$ layer $\psi_{xx} = 0$, and we can see that the solution satisfying the conditions at $x = 0, \pi$, is

$$\psi = \left(-1 + \frac{2x}{\pi}\right)\left(\frac{1}{2} - T\right)\sqrt{\frac{E}{2}}\,.$$

Then appealing to $E^{1/2} T_{\eta\eta} = \omega = -\psi_x$,

we obtain $T_{\eta\eta} = \frac{\sqrt{2}}{\pi}\left(T - \frac{1}{2}\right)$,

and the solution which behaves properly as $\eta \to \infty$ and satisfies $T = 0$ at $\eta = 0$ is

$$T = \frac{1}{2}\left[1 - \exp\left(-2^{1/4}\eta/\sqrt{\pi}\right)\right] \quad (\eta \geq 0)$$

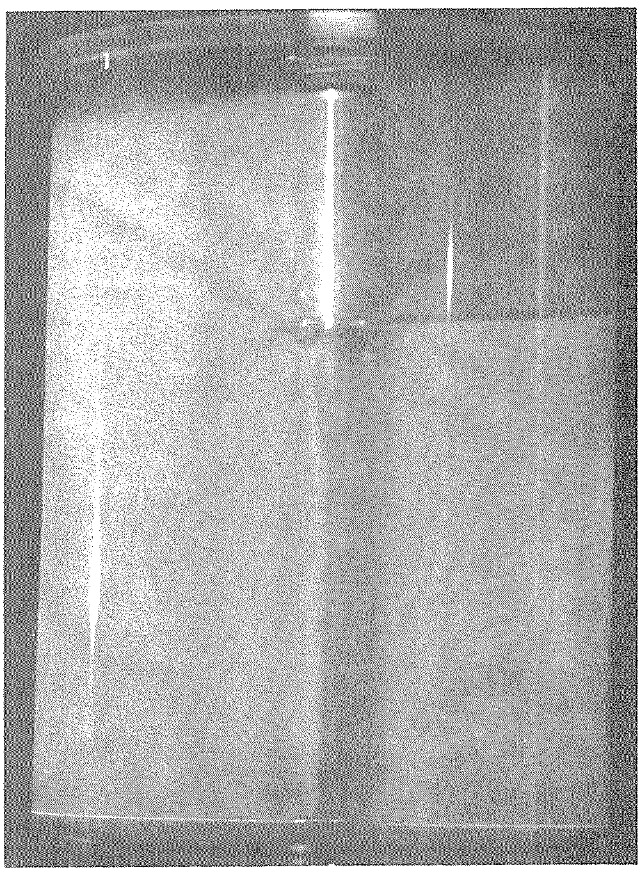
Likewise starting from $T + T_b = -\frac{1}{2}$ for $\eta < 0$ we obtain

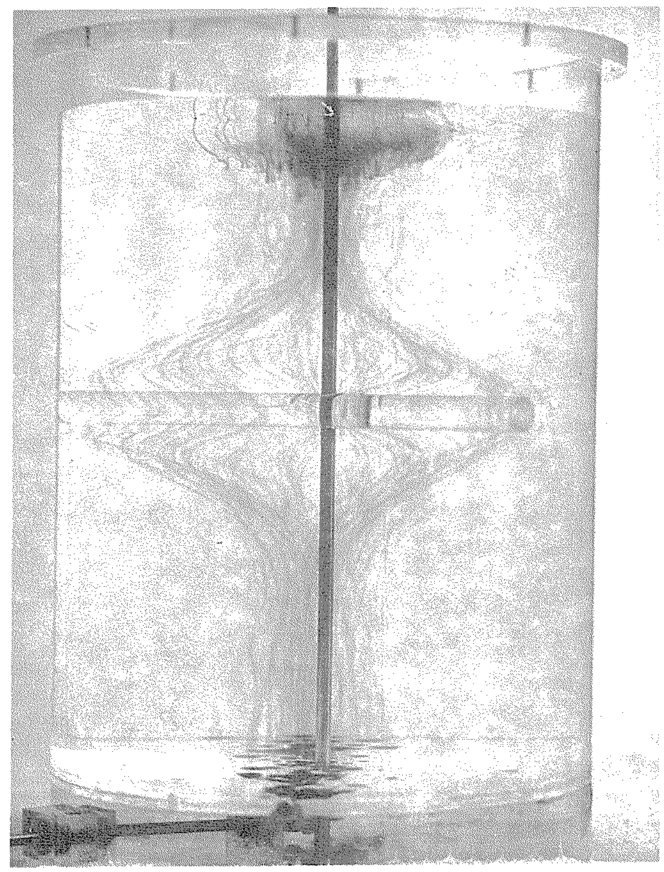
$$T = -\frac{1}{2}\left[1 - \exp\left(2^{1/4}\eta/\sqrt{\pi}\right)\right] \quad (\eta \leq 0).$$

It is to be noted that the $E^{1/4}$ layer succeeds in producing a continuous matching of the regions $\eta = \pm\infty$ between which there exists a temperature difference of order one.

Substituting these expressions for T back into the previous equation for the stream function yields

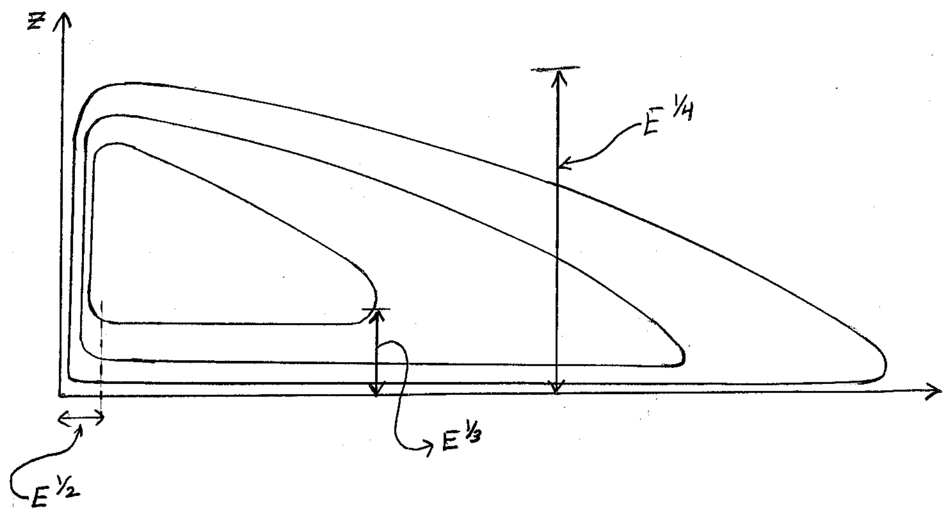
$$\psi = \begin{cases} \left(-1 + \frac{2x}{\pi}\right)\sqrt{\frac{E}{2}}\, \exp\left(-2^{1/4}\eta/\sqrt{\pi}\right) & \eta \geq 0 \\ \left(-1 - \frac{2x}{\pi}\right)\sqrt{\frac{E}{2}}\, \exp\left(-2^{1/4}\eta/\sqrt{\pi}\right) & \eta \leq 0\,. \end{cases}$$





After ½ revolution $\frac{g}{\rho} \frac{\partial \rho}{\partial z} = 2$

The streamlines for the net flow look something like the following:



These notes submitted by

Stephen A. Thorpe
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An Application of the Foregoing to a Stratified Flow Problem

George Veronis

We end this series of lectures on stratified non-rotating flows with a sketch of an analysis of the flow generated by the situation shown below in Fig. (a). The flow is caused by an inflow along the left half and an outflow along the right half of the bottom boundary. The fluid is injected and withdrawn at the temperature of the bottom boundary and must travel over the barrier from the left to the right. Shown in Fig. (b) is the analogous situation for a rotating homogeneous fluid.

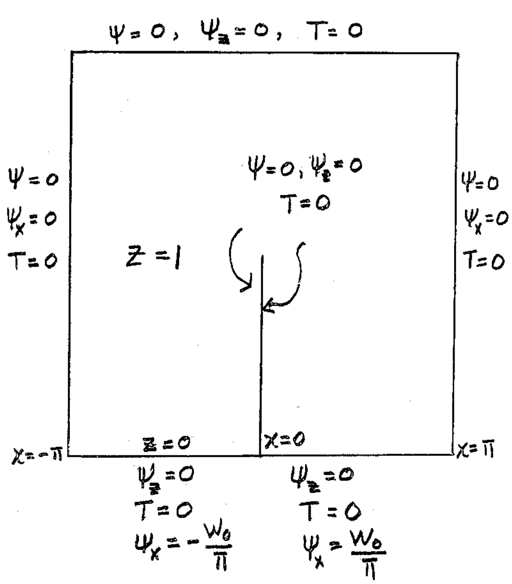


Fig. (a)

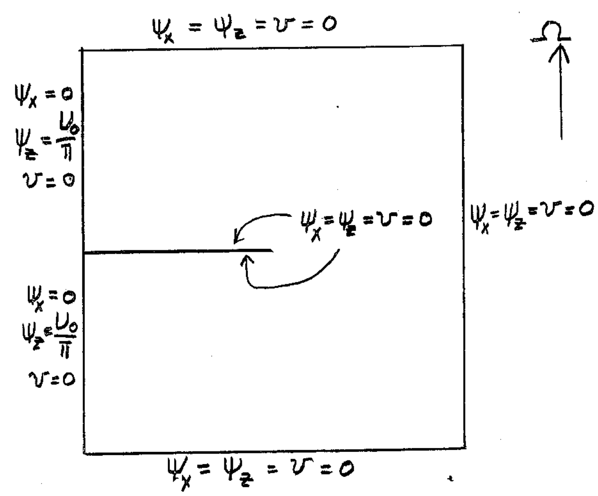


Fig. (b)

We shall use an obvious notation for the different regions. Interior regions are denoted by I, $E^{1/2}$ boundary layers by II, $E^{1/3}$ layers by III and $E^{1/4}$ layers by IV. Also subscripts L, R, T, B correspond to regions on the left, right, top and bottom. The different regions are shown in the accompanying Fig. (c).

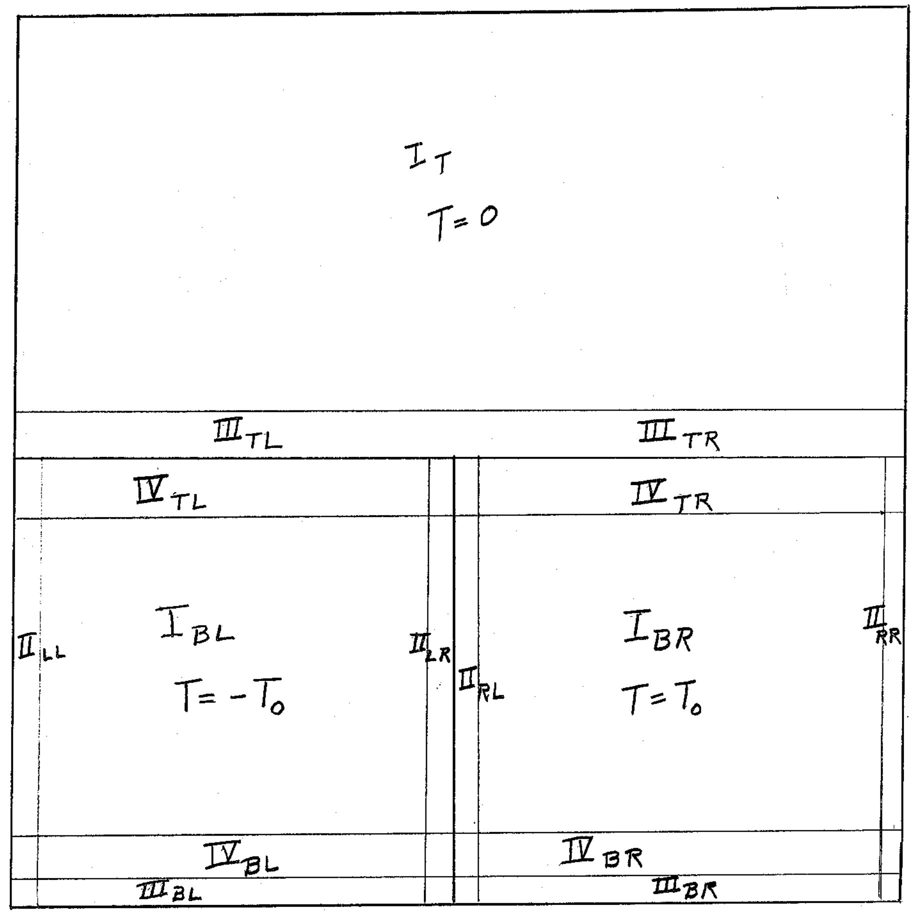


Fig.(c)

A brief description of the expected flow (and a schematic diagram of it, Fig.(d)) follows: The fluid enters the left half of the container and the $E^{1/3}$ layer there takes up the fluid and redistributes the flow so that the fluid essentially flows out of the $E^{1/3}$ layer uniformly. The $E^{1/4}$ layer receives the flow from the $E^{1/3}$ layer and diverts the fluid into the buoyancy $(E^{1/2})$ layers II_{LL} and II_{LR}. The fluid flows up along the left boundary and the left side of the dividing barrier until it reaches the $E^{1/4}$ just below the top of the barrier. This $E^{1/4}$ receives the flow from the $E^{1/2}$ layers and feeds it to the $E^{1/3}$ layer which exists at a level just above the top of the barrier. The $E^{1/3}$ layer receives the fluid from the left half and transports it to the right half of the tank at the level $z = 1$ and ejects it into the $E^{1/4}$ layer. In the right half of the tank the picture is the reverse of that of the left half. (The only possible difference may be in the details of the inflow and outflow in the $E^{1/3}$ layers at the bottom. For the steady flow that we have in mind the only requirement is that the net inflow at $z = 0, -\pi \leq x \leq 0,$ be equal to the outflow at $z = 0, 0 \leq x \leq \pi$. Superimposed on this there could be inflows and outflows (no net flow) at each boundary which could alter the details.)

The temperature of the fluid in the left interior is $-T_0$, and that of the right is T_0. The difference in temperature is to be expected because of the configuration of the system. The value of T_0 must be related to W_0, the amplitude of the net inflow and outflow at the left and right bottom boundaries respectively.

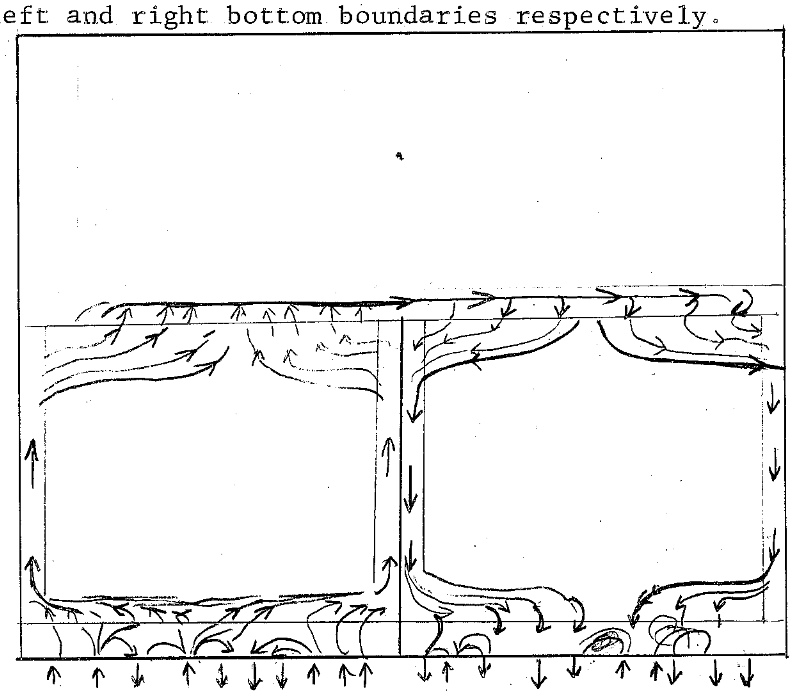


Fig. (d)

The procedure for analysis is based on the following steps:

a) Assume an interior solution involving the unknown constant temperature ϵ.

b) $E^{1/2}$ layers along the sides $0 \le z \le 1$ and along the barrier are computed connecting the interior solution with the boundaries.

c) $E^{1/4}$ layers at $z = 0$ deflect the inflow into the $E^{1/2}$ layers. The analysis for the $E^{1/4}$ layer relates the arbitrary T_o to the mass inflow.

d) $E^{1/4}$ layers at $z = 1$ suck fluid out of the $E^{1/2}$ layers on the left and eject fluid into $E^{1/2}$ layers on the right.

e) An $E^{1/3}$ layer at $z = 1$ takes the fluid from the left $E^{1/4}$ layer carries it over the barrier and pumps it into the right $E^{1/4}$ layer.

f) $E^{1/3}$ layers at the bottom satisfy the bottom boundary conditions. In the special case which we consider, viz., $W = W_o/\pi$, $-\pi \le x \le 0$; $w = -W_o/\pi$, $0 \le x \le \pi$ these $E^{1/3}$ layers at the bottom are not necessary in lowest order.

Using the results of the previous lectures we have the following results for steps a) to f):

a) $u_I = w_I = 0$, $T = 0$ in I_T, $T = -T_o$ in I_{BL}, $T = T_o$ in I_{BR}.

b) $w = \delta T_o \sin \xi e^{\xi}$, $T = T_o(1 - \cos \xi e^{\xi})$

where

$$\xi = -\frac{x+\pi}{\sqrt{2E}}, \quad \delta = -1 \quad \text{in } II_{LL}$$

$$\xi = \frac{x}{\sqrt{2E}}, \quad \delta = -1 \quad \text{in } II_{LR}$$

$$\xi = -\frac{x}{\sqrt{2E}}, \quad \delta = +1 \quad \text{in } II_{RL}$$

$$\xi = \frac{x-\pi}{\sqrt{2E}}, \quad \delta = +1 \quad \text{in } II_{RR}$$

c) $T = \delta T_0 \left(1 - e^{-\frac{2^{1/4}}{\sqrt{\pi}} \frac{z}{E^{1/4}}}\right)$; $\psi = \delta T_0 \left(1 - \frac{2|x|}{\pi}\right)\sqrt{\frac{E}{2}}\, e^{-\frac{2^{1/4}}{\sqrt{\pi}} \frac{z}{E^{1/4}}}$

$$W = -\delta \frac{2T_0}{\pi}\sqrt{\frac{E}{2}}\, e^{-\frac{2^{1/4}}{\sqrt{\pi}} \frac{z}{E^{1/4}}}$$

where $\delta = -1$ in IV_{BL}

$\quad +1$ in IV_{BR} .

From these results we conclude $\displaystyle\int_{-\pi}^{0} W_{\mathrm{IV}_{BL}}\Big|_{z=0} dx = 2T_0\sqrt{\frac{E}{2}} = W_0$

Hence $T_0 = \frac{W_0}{\sqrt{2E}}$.

Thus the influx must be of order $E^{1/2}$ to keep the problem linear (flows of 0(1)).

d) $T = \delta H(1-z)T_0\left(1 - e^{-\frac{2^{1/4}}{\sqrt{\pi}} \frac{(1-z)}{E^{1/4}}}\right)$; $\psi = \delta H(1-z)T_0\sqrt{\frac{E}{2}}\left(1 - \frac{2|x|}{\pi}\right)e^{-\frac{2^{1/4}}{\sqrt{\pi}} \frac{(1-z)}{E^{1/4}}}$

where $\delta = -1$ in IV_{TL}

$\quad = +1$ in IV_{TR} .

e) The details of the flow for the $E^{1/3}$ layer just above $z = 1$ may be derived from the following boundary value problem:

$$\partial^6_{6\zeta}\psi + \psi_{xx} = 0$$

$$\left.\begin{array}{l} \psi = \frac{T_0}{\pi}\sqrt{2E}\left(1 - \frac{2|x|}{\pi}\right) \\ \psi_\zeta = 0 \\ \psi_{\zeta\zeta\zeta\zeta} = 0 \end{array}\right\} \text{at } \zeta = 0$$

ψ and all derivatives $\longrightarrow 0$ as $\zeta \longrightarrow \infty$.

The condition $\psi_\zeta = 0$ follows from the fact that $u = \psi_z = 0(E^{1/4})$ from the solution of the $E^{1/4}$ layer (see (d) above), and in the $E^{1/3}$ layer we look for solutions with $u = 0(E^{1/6})$. To the latter order, $u = 0$. The solution may be derived by expanding ψ in a Fourier cosine series in x and solving the problem in ζ by means of a Laplace transform.

f) For the special case of $\psi = \frac{W_0}{\pi}|x|$ at $z = 0$, no $E^{1/3}$ boundary layer is needed at the bottom.

Problems in Turbulence Theory

Robert H. Kraichnan

ABSTRACT

Turbulent flows typically display two related features. First, an heirarchy of instabilities whereby the laminar flow breaks down into large-size eddies which in turn break down into a chain of smaller eddies. Second, a strong enhancement of the transport of a property like heat or momentum. Many of the gross properties of turbulent shear and Boussinesq flows can be successfully described by mixing-length analyses based on the observation that the characteristic break-down time of an eddy is the order of its circulation time. These approaches are illustrated by deriving Kolmogorov's spectrum law using mixing-length arguments.

A mathematical description of turbulence can be based on the infinite sequence of moments that arise from considering a statistical ensemble of flows. Such a description is natural because, as a consequence of the heirarchy of instabilities, it is not possible to predict the detailed evolution of a single turbulent flow no matter how much effort is lavished on controlling the initial and boundary conditions. Ordinarily, only a few of the moments are of interest in a given application. However, because the flow equations are nonlinear, the dynamical equations of all the moments are coupled. It is unclear to what extent accurate values for the moments of interest can be extracted by finite and well-defined mathematical operations, short of finite-difference integration of the flow equations for each member of the ensemble.

Outlook for Turbulence Theory

Robert H. Kraichnan

ABSTRACT

An attempt is made in this talk to assess the promise of some of the schemes that have been proposed in recent years for systematic evaluation of the statistical moments of turbulent flows. The underlying trouble in turbulence theory is the lack of a general method of handling nonlinear differential equations. The only generally applicable systematic technique now available seems to be expansion by perturbation techniques about a soluble linear problem. In incompressible Navier-Stokes turbulence this leads to expansions in powers of a turbulence Reynolds number. These expansions appear to have zero radius of convergence, but to provide valid asymptotic approximations for very small Reynolds numbers or very short times of evolution of the flow. The case of interest is large Reynolds number, and here direct calculation from the perturbation expansions is hopeless. Approximations that appear to be successful for isotropic turbulence at high Reynolds numbers have been constructed by performing (implicitly, through integrodifferential equations) certain partial summations of terms from all orders of the Reynolds-number expansion. This is *a priori* an unlikely event: It is to be expected that attempts at partial summations are almost certain to lead to disaster, and most attempts for isotropic turbulence have, in fact, not worked. The existence of successful and accessible approximations for incompressible Navier-Stokes turbulence appears to be intimately connected with the simplicity of the inviscid constants of motion. In

particular, the energy is a simple sum of squares of the amplitudes of all the degrees of freedom. It is possible, as a direct consequence of simple constant of motion, to construct partial summations for which energy is exactly conserved and the mean energy predicted for each degree of freedom is positve-definite. These consistency properties are what keep the errors bounded and prevent disaster.

Detailed results so far obtained for isotropic, incompressible turbulence suggest that the consistent approximations now available can provide rather accurate predictions of mean velocities, velocity covariances, and wavenumber spectra for general incompressible Navier-Stokes shear flows and for Boussinesq flows. However, there is no evidence that a sequence of successively more accurate consistent approximations can be constructed, and consequently errors cannot be determined quantitatively. Moreover, a higher approximation, if it does exist, would very likely be prohibitively complicated.

The applicability of similar approximations to more general turbulent flows, such as high-Mach-number turbulence, is substantially in doubt. This is because the constants of motion do not have the simple algebraic form they do in incompressible turbulence, and it is not clear that the consistency properties cited previously can be assured.

Turbulent Magnetic Fields

Robert H. Kraichnan

ABSTRACT

The evolution of a weak, random initial magnetic field in a highly conducting, isotropically turbulent fluid is discussed with the aid of the exact expression for initial growth of the magnetic energy spectrum. Equipartition arguments, the vorticity analogy, and the known turbulence approximations all are found inadequate for predicting whether the magnetic energy eventually dies away or grows exponentially. This is true for any ratio of magnetic diffusivity to kinematic viscosity. Equipartition arguments fail because they shed no light on a crucial balance between two competing processes of comparable magnitude: enhancement of local magnetic spectrum level by interactions local in wavenumber, and sweeping-out of the magnetic energy to high wavenumbers. The vorticity analogy fails even if diffusivity and kinematic viscosity are equal because magnetic energy transfer suffers no constraint analogous to that imposed on vorticity transfer by kinetic-energy conservation.

If the possibilities of eventual growth and eventual decay are both admitted, then, for each, it is possible to estimate the form of the magnetic energy spectrum by simple dynamical arguments. The results for large ratio of magnetic diffusivity to kinematic viscosity are as follows. If there is growth, the magnetic spectrum below the ohmic

cut-off eventually reaches equipartition with the kinetic energy spectrum roughly in the fashion predicted by Biermann and Schlüter, with the principal exceptions that the spectrum of kinetic energy in the equipartition inertial range evolves to the form $k^{-3/2}$ and that equipartition is maintained, with rapidly falling spectrum, through part of the ohmic dissipation range. The $k^{-3/2}$ spectrum occurs, rather than the Kolmogorov $k^{-5/3}$ spectrum of hydrodynamic turbulence, because, when the energy in the large-scale magnetic turbulence is sufficiently large, the small-scale turbulence takes the form of weakly scattering Alfvèn waves propagating along the lines of force of the large-scale field.

If the weak magnetic field decays instead of growing, then a steady spectrum can be maintained by supplying magnetic energy at low wavenumbers. In the hydrodynamic inertial range, the magnetic energy spectrum has the form k^n, where n has a value between 1/3 and 4. A more precise determination of n is not possible without detailed dynamical calculations.

The Wind-driven Ocean Circulation (a)

Melvin E. Stern

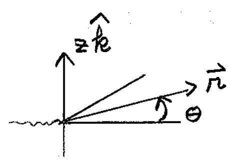
The purpose of this first lecture was to develop a general theory for the Ekman layer and then use it to build a model of the wind-driven ocean circulation. Because of the variety of physical processes which can naturally occur in the ocean and, in fact, may distort the effect of a wind stress, it is necessary to make several assumptions. The first is to consider only a homogeneous liquid of constant density. This effectively eliminates thermodynamic processes associated with a baroclinic system and may be justified by the observation that the average density gradients are quite small in the mixed surface layer (top 100-200 meters) where the wind-induced viscous stresses are large. The second is to argue that the major flux of momentum from the impressed wind stress at the surface occurs in the vertical direction. This is more difficult to rationalize since the generation of surface waves by even a homogeneous wind field leads to turbulence with both local horizontal and vertical "eddy" stresses. However, a detailed description is not wanted or needed and with the vertical scales of motion typically smaller than the horizontal, the predominant diffusive mechanism for horizontal momentum is the vertical transport of turbulent momentum. The problem is thus reduced to finding an average picture in some sense for the response to a possibly turbulent diffusive mechanism.

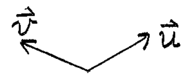
<u>The Ekman Relationship</u>: In 1905, V. W. Ekman showed with a simplified version of the N. S. equations

$$\left(\rho = const., \ \frac{du}{dt} = \frac{dv}{dt} = w = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0\right)$$

that the wind-driven current in deep water decreased in velocity and changed direction at regular intervals of depth. The wind also drove a net-horizontal transport which is only a function of the magnitude of the surface stress and the local coriolis parameter and which is directed to the right of the stress in the northern hemisphere.

Stern derived this last result with a different argument. Consider an infinite ocean rotating with a basic angular velocity of Ω. An axisymmetric surface stress is applied in the azimuthal direction.





In order to conserve mass in a circular ring element,

$$\frac{\partial}{\partial r}\left\{\int_{-\infty}^{0} \rho u (2\pi r)\, dz\right\} = 0$$

Since Stern was not interested in the details of the momentum exchange in the liquid away from the surface, he considered that all the local processes could be represented in a turbulent eddy stress function $\theta(z)$. The local steady torque balance on a thin ring of width δr and thickness δz is then

$$2\pi r\, \delta r\, \delta z \frac{\partial}{\partial z}\left(r \cdot \rho \theta\right) =$$

= (torque exerted by azimuthal eddy stress) = (radial flux of angular momentum) = $\delta z\, \delta r \frac{\partial}{\partial r}\left(u(2\pi r)\left\{r \cdot \rho\left(v + \Omega r\right)\right\}\right)$.

The angular momentum flux arises from two sources: the first being the direct transport of local angular momentum and the second, transport of "planetary" angular momentum. Even though a particle is at rest in the rotating coordinate system, the angular rotation of the reference system gives each particle an intrinsic angular momentum of

$$\rho \Omega r^2 \hat{k}$$

in inertial space which must be conserved in the absence of any torques.

With the surface wind the only source of momentum, the eddy stress must vanish as the depth is increased. Neglecting any free motion, the horizontal velocities must also vanish as $z \to -\infty$. This requires the addition of a mass source at the origin to match any net radial flux.

An expression for the surface stress is obtained from the vertically integrated torque balance:

$$\Theta(z)\Big|_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\int_{-\infty}^{0} uv\,dr + r^2\Omega\int_{-\infty}^{0} ur\,dz\right\}$$

The radial velocity u must be $\frac{f(z)}{r}$ to satisfy continuity so that

$$\Theta(z)\Big|_0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r\int_{-\infty}^{0} f(z)v\,dz\right\} + \frac{2r}{r^2}\Omega\int_{-\infty}^{0} ur\,dz + \Omega\int\underbrace{\frac{\partial(ur)d}{\partial r}}_{0 \text{ by continuity}}$$

$$\therefore \Theta(z)\Big|_0 = \frac{1}{r^2}\int_{-\infty}^{0} f(z)\frac{\partial}{\partial r}(rv)\,dz + 2\Omega\int_{-\infty}^{0} u\,dz$$

The first term behaves as $O\left(\frac{1}{r}\right)$ so that for $r \to \infty, \Theta(z)\Big|_0 \to 2\Omega\int_{-\infty}^{0} u\,dr$.

For large r, the radial transport of "planetary" angular momentum just balances the applied wind torque with 2Ω as the coriolis parameter f. The important Ekman result is derived that a surface stress drives a net mass flux of magnitude $\frac{\Theta(0)}{f}$ and is directed towards the right of the stress.

The same result can be generalized for the case of an arbitrary smooth stress on a large deep non-homogeneous ocean. Decoupling the velocity field into its geostrophic and surface wind driven components, $\vec{u}_g + \vec{u}_s$, the local horizontally averaged Navier-Stokes expressions for the horizontal momentum balance reduce to

$$f\hat{k} \times \langle \rho \vec{u}_g \rangle_h = \langle \nabla_h p \rangle_h$$

and $f\hat{k} \times \langle \rho \vec{u}_s \rangle_h = \frac{\partial}{\partial z} \langle \vec{\theta} \rangle_h$

for

a) aquasi-steady wind stress $\left(\frac{\theta t}{\Omega \theta} << 1 \right)$

b) small motions so that $\left\langle \frac{d\vec{u}}{dt} \right\rangle = 0$

c) $\vec{u}_s$ vanishing with depth.

Then $\therefore f\hat{k} \times \int_{-\infty}^{0} \langle \rho \vec{u}_s \rangle_h dz = \langle \vec{\theta} \rangle_h \Big|_{z=0} = \vec{\tau}_0$

Identifying $\int_{-\infty}^{0} \langle \rho \vec{u}_s \rangle_h$ as M_s, the net wind-driven horizontal mass transport, then the Ekman result is

$$\vec{M}_s = \frac{\vec{\tau}_0 \times \hat{k}}{f}$$

The geostrophic and surface velocity's fields are implicitly coupled in the determination of $\vec{\tau}_0$.

The Sverdrup relation relating the integrated mass transport to the change in the coriolis acceleration is derived by taking the curl of $fk \times \vec{M}_s = \vec{\tau}_0$ and utilizing the continuity condition $\nabla \cdot \vec{M}_s = 0$.

$$\cdot k \cdot \left[\nabla \times \left(f\hat{k} \times \vec{M}_s \right) \right] = f \nabla \cdot \vec{M}_s + \frac{\partial f}{\partial y} M_s^{(y)} = (\text{curl}\, \tau_0) \cdot \hat{k}.$$

$$\therefore \frac{\partial f}{\partial y} M_s^{(y)} = \text{curl}\, \tau_0 .$$

The quantity $\frac{\partial f}{\partial y}$ is called Beta (β) in oceanographic literature. The equilibrium state represents a balance of applied torque with the northward motion of a fluid column in the surface layer up the "planetary" angular momentum or vorticity gradient.

A simple model for the wind-driven ocean circulation can be built using the Ekman theory. The Atlantic basin is roughly triangular in shape with the trade winds blowing westward in the southern zone and the westerlies blowing eastward in the northern zone. These winds drive an Ekman surface flow toward the mid-latitudes where a surface convergence is formed in the mixed layer. Continuity requires downwelling and a horizontal divergence under the surface convergence. The spreading out of a fluid column underneath the thermocline generates negative vorticity which can only be balanced by a southerly motion down the planetary vorticity gradient. A western or eastern boundary current with its higher order dynamics will be necessary to close the circulation. Although we know that the Gulf Stream has no eastern counterpart this simple Ekman-Sverdrup theory can only predict the north-south interior flow and not the position and nature of the return current.

However, we can use the Ekman-Sverdrup result and continuity to predict the net flux of the return current. The Sverdrup equation can be integrated along a latitude circle through the interior region to give the net interior meridional flux. The return boundary current must match this flux to complete the circulation and conserve mass. Using a value of $\tau = 1\,dyne/cm^2$ for the average wind stress in the Atlantic, the

Gulf Stream transport should be

$$L \cdot \frac{(\text{curl}\,\tau)\cdot\hat{k}}{\beta} = 30\cdot 10^6 m^3 sec.$$

The most recent measurements place the actual transport through the Florida Straits at $35.5 \pm 1.2 \times 10^6 m^3/sec.$[1] and that off George's Bank at between 80 to $150 \cdot 10^6 m^3/sec.$[2] While there is some uncertainty in the wind stress value, the measured transports do seem to be significantly larger than predicted and suggest that some other mechanism is complementing the wind-driven circulation.

Recent observations of neutrally buoyant Swallow floats have demonstrated the existence of large-scale eddies with kinetic energy densities of 10^2 to 10^4 above the mean wind-driven field.[3,4] These eddies appear to be more energetic in the western half of the Atlantic basin although as yet, the data is insufficient. Their characteristic time scales are in the order of a few weeks so they should be in quasi-geostrophic and hydrostatic balance.

Not much else is known about these eddies. Their special extent, their propagation characteristics, and most important, their source of energy and dissipation mechanism are all unanswered questions. There are several rather suggestive theories, one viewing these eddies as a

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[1] Direct measurement. Richardson, W.S. and W. J. Schmitz, Jr. 1965. "A technique for the direct measurement of transport with application to the Straits of Florida." J.Mar.Res., 23: 172-185.

[2] Geostrophic measurement. Fuglister, F. C. 1963. "Gulf Stream '60." Progress in Oceanography, 1: 263. MacMillan Company, N.Y.

[3] Swallow, J. C. and B. V. Hamon. 1960. "Some measurements of the deep current in the eastern North Atlantic." Deep-Sea Res., 6: 155-168.

[4] Crease, J. 1962. "Velocity measurements in the deep water of the western North Atlantic. J.Geophys.Res. 67: 3173-3176.

Rossby wave resonance excited by very low frequency wind stress variations, and another considering them as Rossby waves radiating from a somewhat unstable Gulf Stream. However, their identity as actual Rossby waves has not been confirmed, only inferred by the time and length scales, so the nature of these eddies is still quite unknown.

The formation and decay of these large-scale eddies may exert a profound effect on the overall Atlantic circulation. If the inertial terms tend to rectify the eddy motion, a secondary flow could be generated which may significantly enlarge the overall Gulf Stream transport. The next lecture will consider the energetics of moderate scale hydrostatic eddies.

These notes submitted by
Robert C. Beardsley

The Wind-driven Ocean Circulation (b)

On the Interaction of Wind Stress with Hydrostatic Eddies

Melvin E. Stern

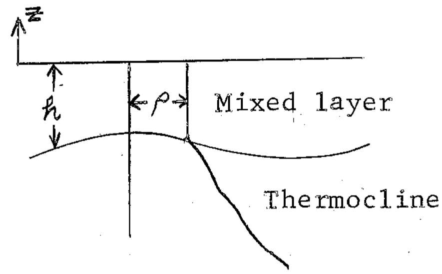
In this lecture we shall consider the effects on the "hydrostatic" (large scale) eddies in the deeper water of the wind stress applied on the surface and transmitted through the mixed layer by "small scale" turbulence.

We start with the equations of motion subject to the hydrostatic and Boussinesq approximations.

$$\left(\frac{\partial}{\partial t}+\underline{u}\cdot\nabla\right)\underline{v}+f\,\underline{k}\times\underline{v}=-\frac{1}{\rho_0}\nabla p-\frac{g\rho'}{\rho_0}\underline{k}+\frac{\partial\underline{\theta}}{\partial z} \qquad (1)^*$$

$$\nabla\cdot\underline{u}=0 \qquad (2)$$

Here $\underline{u}=\underline{v}+\underline{k}w$, i.e. $\underline{v}$ is the horizontal and w the vertical component of the total velocity $\underline{u}$.



ρ' is the departure of the density from its basic state (which is ρ = constant in the mixed layer, and stably stratified below), ρ_0 is the mean density. $\rho\underline{\theta}(x,y,z,t)$ is the lateral turbulent stress with $\underline{\theta}(x,y,0,t)=\underline{T}$. We assume that $\underline{\theta}$ decreases to zero by $y=-h(x,y,t)$; i.e. that an Ekman-type layer does exist.

In the absence of wind stress we may have an inertial motion $\underline{u}_0$ which satisfies the equations

$$\left(\frac{\partial}{\partial t}+\underline{u}_0\cdot\nabla\right)\underline{v}_0+f\,\underline{k}\times\underline{v}_0=-\frac{1}{\rho_0}\nabla p-\frac{g\rho'}{\rho_0}\underline{k} \qquad (3)$$

$$\nabla\cdot\underline{u}_0=0 \qquad (4)$$

We see from the vertical component of (3) that

$$\frac{\partial\underline{v}_0}{\partial z}=0 \text{ in } z>-h \qquad (5)$$

and hence from (4) w_0 is a linear function of z in the mixed layer. The boundary condition at the surface will be $w_0(x,y,0,t)=0$.

We now introduce a wind stress and assume that for a weak

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*In practice the inertial motions we shall consider will have a scale of O(10 km), and (1) will be the result of aveaging over a scale O(1 km).

interaction between this and the inertial motion we may write the total velocity $\underline{u} = \underline{u}_0 + \underline{u}_b$, where $\underline{u}_0$ satisfies (3) and (4) and the frictional component $\underline{u}_b$ vanishes for $z < -h$. $\underline{u}_b = \underline{v}_b + \underline{k}\, w_b$ satisfies

$$\frac{\partial \underline{v}_b}{\partial t} + \nabla \cdot \left\{ \underline{u}_0\, \underline{v}_b + \underline{u}_b\, \underline{v}_0 + \underline{u}_b\, \underline{v}_b \right\} + f\, \underline{k} \times \underline{v}_b = \frac{\partial \underline{\theta}}{\partial z} \tag{6}$$

$$\nabla \cdot \underline{u}_b = 0 \tag{7}$$

where we neglect pressure on the hydrostatic assumption, and denote

$$\left[\nabla \cdot (\underline{a}\, \underline{b}) \right]_i = \nabla \cdot (\underline{a}\, b_i)$$

The boundary condition on the vertical velocity at $z = 0$ is

$$w_0(x, y, 0, t) + w_b(x, y, 0, b) = 0 \tag{8}$$

i.e. the inertial motion satisfying (3) and (4) is driven by the "suction velocity" $w_b(x, y, 0, t)$.

The second term in (6) may be written as

$$\nabla_2 \cdot \left\{ \underline{v}_0\, \underline{v}_b + \underline{v}_b\, \underline{v}_0 + \underline{v}_b\, \underline{v}_b \right\} + \frac{\partial}{\partial z} \left\{ w_0\, \underline{v}_b + w_b\, \underline{v}_0 + w_b\, \underline{v}_b \right\}$$

where $\nabla_2 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right)$

Thus on integration of (6) from $z = -h$ to $z = 0$ and writing

$$\underline{M} = \int_{-h(x,y,t)}^{0} \underline{v}_b\, dz$$

$$\frac{\partial \underline{M}}{\partial t} + \nabla_2 \cdot \left\{ \underline{v}_0\, \underline{M} + \underline{M}\, \underline{v}_0 + \int_{-h}^{0} \underline{v}_b\, \underline{v}_b\, dz \right\} + w_b(0)\, \underline{v}_0 + f\, \underline{k} \times \underline{M} = \underline{\tau} \tag{9}$$

using (5) and (8).

Integration of (7) gives

$$\nabla_2 \cdot \underline{M} + w_b(0) = 0 \tag{10}$$

and so (9) may be written

$$\frac{\partial \underline{M}}{\partial t} + f\, \underline{k} \times \underline{M} + \nabla_2 \cdot \left\{ \underline{v}_0\, \underline{M} + \underline{M}\, \underline{v}_0 + \int_{-h}^{0} \underline{v}_b\, \underline{v}_b\, dz \right\} = \underline{\tau} + \underline{v}_0\, \nabla \cdot \underline{M} \tag{11}$$

The Ekman-Sverdrup theory is obtained on neglect of every term in (11) except $f\underline{k}\times\underline{M}$ and $\underline{\tau}$.

(11) is a momentum equation in which

$\underline{V}_o\underline{M}$ represents transport of frictional momentum by inertial velocity

$\underline{M}\,\underline{V}_o$ represents transport of inertial momentum by frictional velocity

$\int_{-h}^{0}\underline{V}_b\underline{V}_b\,dz$ represents transport of frictional momentum by frictional velocity

$\nabla\cdot\left\{\underline{V}_o\underline{M}+\underline{M}\,\underline{V}_o\right\}$ is termed the bilinear interaction term

$\nabla\cdot\left\{\int_{-h}^{0}\underline{V}_b\underline{V}_b\,dz\right\}$ is termed the self-interaction term.

$\underline{V}_o\nabla\cdot\underline{M}$ represents transport of inertial momentum by the suction velocity, i.e. momentum is redistributed in the vertical between the frictional and inertial motions.

If we assume horizontal homogeneity and take the space average of (11) we obtain

$$\frac{\partial\langle\underline{M}\rangle}{\partial t}+f\underline{k}\times\langle\underline{M}\rangle=\underline{\tau}+\langle\underline{V}_o\nabla\cdot\underline{M}\rangle \qquad (12)$$

If there is a correlation between $\underline{V}_o$ and $\nabla\cdot\underline{M}$ we may estimate the relative orders of magnitude of $\underline{\tau}$ and $\langle\underline{V}_o\nabla\cdot\underline{M}\rangle$

Take $h\sim 100$ m., $|\underline{V}_b|\sim 10$ cm/sec. $\therefore |\underline{M}|\sim 10^5$ cm^2/sec.

With a horizontal scale $\sim$10 km $\nabla\cdot\underline{M}\sim 10^{-1}$ cm/sec.,

and so if $|\underline{V}_o|\sim 10$ cm/sec $\quad \tau\sim 1$ cm^2/sec^2.

$$\therefore\quad \frac{|\underline{V}_o\nabla\cdot\underline{M}|}{|\tau|}\sim 1$$

This may be regarded as a rough upper bound, and shows that the term $\langle\underline{V}_o\nabla\cdot\underline{M}\rangle$ may be of importance in considering the Ekman transport.

Small Wind Stress Approximation

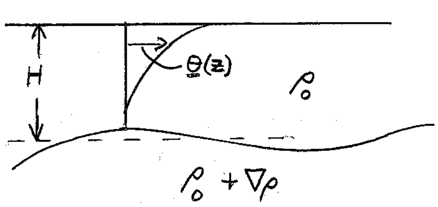
For small values of τ we assume that the self-interaction term in (11) is negligible compared with the bilinear interaction, and that this assumption is valid uniformly in time. Then we may obtain a complete set of equations by taking (11) without the self-interaction term (valid in $0 > z > -h$), the thermocline equations, the continuity equation for $\underline{V}_0$ in the mixed layer, which is $\frac{\partial h}{\partial t} + h \nabla_2 \cdot \underline{V}_0 = -\nabla \cdot \underline{M}$ (obtained from integration of $\nabla \cdot \underline{u}_0 = 0$ through the mixed layer) and the boundary condition $w_0(x, y, 0, t) = \nabla \cdot \underline{M}$.

From these one would hope to investigate the slow transfer of energy to the inertial component.

This small wind stress assumption will certainly not be valid in homogeneous non-rotating systems, as the shear will in the course of time produce a $\underline{V}_b$ of the same order of magnitude as $\underline{V}_0$. However, in the case we consider the constraints of rotation, and buoyancy beneath the mixed layer, impose a kind of rigidity on the inertial motion.

Part II

To give further weight to the ideas that a) wind stress can feed energy into hydrostatic eddies and b) neglect of the self-interaction terms in (11) is justified, we consider the following less physically relevant but more mathematically sound model.



In the inviscid two-layer system shown a horizontally uniform body force $\underline{\theta}(z)$ acts in the upper layer to produce an Ekman-like shear flow with velocity

$(U(z), V(z))$. In the upper layer we impose a small long-wave (hydrostatic) disturbance on this with $u' = e^{i(kx+ly+\omega t)}u(z)$ etc.

Then if we linearize about the basic state, the equations

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)u' + w'\frac{dU}{dz} - fv' = -\frac{\partial p'}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}\right)v' + w'\frac{dV}{dz} + fu' = -\frac{\partial p'}{\partial y}$$

$$\frac{\partial p'}{\partial z} = 0$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0$$

give

$$i\Omega u - fv = ikp - wU'$$

$$i\Omega v + fu = -ilp - wV'$$

$$\frac{dw}{dz} + iku + ilv = 0$$

$$\frac{dp}{dz} = 0$$

where $U', V' = \frac{dU}{dz}, \frac{dV}{dz}$ and $\Omega(z) = \omega + kU + lV$

Hence, eliminating u, v

$$(\Omega^2 - f^2)\frac{dw}{dz} - w\left[U'(k\Omega + ilf) + V'(l\Omega - ikf)\right] = ip\Omega(k^2 + l^2) \tag{13}$$

$$\therefore \frac{d}{dz}\left\{w(\Omega^2 - f^2)^{-1/2}\exp\int_{-H}^{z}\frac{if[kV'(\zeta) - lU'(\zeta)]}{\Omega^2 - f^2}\,d\zeta\right\}$$

$$= \frac{i\Omega p(k^2+l^2)}{(\Omega^2 - f^2)^{3/2}}\exp\int_{-H}^{z}\frac{if(kV' - lU')}{\Omega^2 - f^2}\,d\zeta \tag{14}$$

The boundary conditions at $z = 0, -H$ are $u(0) = 0$ and $w(-H) = -\frac{\partial h'}{\partial t}$, where h' is the increase in depth of the upper layer. Then

$$p = \text{const.} = g'h' \quad \text{where} \quad g' = \frac{g\Delta\rho}{\rho_0}$$

$$\therefore p = w(-H)\frac{ig'}{\omega}$$

Then on integration of (13) from $z = -H$ to $z = 0$, using the above conditions and $U(-H) = V(-H) = 0$ we have

$$(\omega^2 - f^2)^{-\frac{1}{2}} = \int_{-H}^{0} \frac{g'\Omega(k^2+\ell^2)}{\omega(\Omega^2-f^2)^{3/2}} \exp \int_{-H}^{z} \frac{if(kV'-\ell U')}{\Omega^2 - f^2} \, d\zeta \, dz$$

We note that if $U = V = 0$ we have the dispersion relation of ordinary inerto-gravity waves

$$\omega^2 = \omega_0^2 = f^2 + g'H(k^2 + \ell^2)$$

We now take $\ell = 0$ (without loss of generality) and calculate the way in which a small shear will modify this by writing $\delta U, \delta V$ for U, V and expanding in powers of $\delta U, \delta V$. The frequency $\omega_0 + \delta\omega$ is then given by

$$\delta\omega = \frac{ifk}{2H\omega_0} \int_{-H}^{0} \delta V \, dz + \text{real coefficient} \times \int_{-H}^{0} \delta U \, dz .$$

Thus waves propagating with a component of velocity upwind may grow with time, the growth rate being given by

$$\mathrm{Im}(-\delta\omega) = -\frac{fk}{2H\omega_0} \int_{-H}^{0} \delta V \, dz . \qquad (15)$$

The term in $\int_{-H}^{0} \delta U \, dz$ merely gives a change in phase speed. (13) is a two-point eigenvalue equation for $w(z)$. From a solution of this we can obtain $u(z), v(z)$ which may be split up into inertial components u_0, v_0 independent of z and frictional components $u_b(z), v_b(z)$ such that $u_b(-H) = v_b(-H) = 0$.

The frictional and inertial components u_0, v_0 both grow at the same rate (at least until the linearization breaks down) and so the ratio between them will remain small if it is so initially. This would appear to support the small stress approximation. In what sense the stress must be small can be found by estimating the relative magnitude of u_o, u_b in an eigensolution. Identifying $\int_{-H}^{0} \delta V dz = M \sim \frac{\tau}{f}$ it turns out that the relevant non-dimensional parameter for inerto-gravity waves with $k^2 \sim \frac{f^2}{g'H}$ is $\mathcal{E} = \frac{\tau}{\sqrt{g'H}\, f H}$, and the small stress approximation requires $\mathcal{E} << 1$.

In the oceans $\tau \sim 1\ cm^2 sec^{-2}$, $f \sim 10^{-4} sec^{-1}$, $H \sim 10^{-4} cm$ and $g' \sim \frac{1}{10}\ cm\ sec^{-2}$, and so $\mathcal{E} \sim \frac{1}{30}$ which is small enough.

However, some observations indicate a large contribution to the spectrum of hydrostatic eddies from waves with frequency very close to f. These are precisely the waves which grow least quickly according to (15) and moreover for these very long waves the small stress approximation breaks down (as can be seen from (13); as $\omega \to f$ it is no longer a valid first approximation to neglect the second term for small $\delta U, \delta V$, hence $\frac{\frac{dw_b}{dz}}{\frac{dw_o}{dz}}$ will not be small and nor will $\frac{|u_b|}{|u_o|}$).

Nevertheless, it is possible that the presence of a density gradient in the lower layer might lead to long wavelength motions in the upper layer growing more rapidly than shorter wavelength motions.

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These notes submitted by

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The Closure Problem

Steven A. Orszag

ABSTRACT

A theory of turbulence is described which is based on preserving all the known invariances, conservation laws and symmetries of the exact problem. For homogeneous, isotropic turbulence in an incompressible fluid these requirements include conservation of momentum and conservation of energy (when $\nu = 0$); Galilean invariance; invariance under rotations, translations and reflections in space; preservation of a complete set of "realizability" inequalities (the simplest of which is positivity of the energy spectral density); and existence of inviscid, equipartition solutions (that is, a solution (when $\nu = 0$) corresponding to equipartition of energy amongst the various modes).

At the level of fourth-order cumulants these requirements appear to uniquely define a closure of the hierarchy but at higher order, there are many possible closures consistent with the above conditions. Therefore, a more systematic technique of closure was sought. This technique involves finding the "most probable" closure consistent with the known information about the turbulence. The more precise formulation of this idea is shown to lead to a problem of evaluating certain functional integrals. These functional integrals are evaluated and the resulting closures are shown to possess all the consistency properties described above. A discussion is then made of the problem of

irreversibility in the theory of turbulence. Irreversibility enters the exact problem in at least three ways: (1) through the obvious action of viscosity on small modes; (2) through phase mixing processes; and (3) through overall energy flow in Fourier space. It is shown that the third process is probably dominant over those regions of the spectra uninfluenced by viscosity. This is because the modes of ordinary turbulence are known to be essentially critically damped.

Approximate Techniques in Turbulence Convection at High Prandtl Number

Jackson Herring

ABSTRACT

The problem of thermal convection at infinite Prandtl number is considered from the point of view of the statistical theory turbulence. Several approximate techniques available to treat this problem are compared in a set of numerical experiments. The analysis is performed for free boundaries and for a Rayleigh number of 4,000. The methods investigated are:

(1) Direct numerical integration of the equation of motion.

(2) The quasi-normal approximation.

(3) The mean field or quasi-linear approximation.

(4) The direct interaction approximation.

First of all some results of a direct integration of the Navier-Stokes equation are presented. The goal here is to establish a basis for assessing of the results of the statistical study. The static solutions for squares, hexagons, and rolls were presented for Rayleigh number up to 10^4. The plan-form which transported the most heat-flux was the roll, and that which transported the least, the square. The hexagon transported a little less heat than the roll, but appeared to be catching the roll at the larger Rayleigh numbers.

The next (3) methods may be best discussed simultaneously since they represent various prescriptions for treating the random aspects of the turbulent process. Also, as it turns out, once one is able to solve the direct interaction equations, the other methods may be obtained by repressing one or another of the terms in the equations.

These methods (i.e. 2, 3 and 4) may be briefly characterized by the way in which they treat the eddy diffusion and eddy dissipation effects associated with the fluctuating self-interactions. The mean field system deletes both effects, while the direct interaction and quasi-normal approximation give approximate treatments of both effects. The quasi-normal approximation uses a simpler approximation to represent the effect of the eddy dissipation. All these methods treat the mean-fluctuating interactions exactly.

The numerical integrations (which at present are preliminary) indicate that the direct interaction gives satisfactory results. The relaxation of the system from initial conditions predicted by this method is satisfactory. The Nusselt number predicted by the direct interaction method is about 5% larger than that found by method (1), and about 10% larger than that predicted by the mean field model. The relaxation of the mean field system is satisfactory, but not as plausible in its detailed prediction as the direct interaction.

The quasi-normal approximation does not appear to be a satisfactory statistical procedure for this system. The numerical results indicate, for example, that the response functions for the temperature fields fail to remain properly bounded.

PROBLEMS IN GALAXY FORMATION

Review of Past Work and Present Status, the Results of the Linear Analysis, and Types of Instabilities

Richard W. Michie

ABSTRACT

When the types of old stellar systems are ranked in decreasing mass, there occur the clusters of galaxies, "normal" galaxies, dwarf galaxies (with prototypes exemplified by NGC 205 and Fornax), and the globular clusters. (The latter are not necessarily all dynamically bound to a larger system.) The available observational evidence does not provide strong support for or against the assumption that these types of stellar systems are the maxima in a mass spectrum from around 10^{14} to 10^{4} suns. In this sense it will be questionable whether a theory of galaxy formation can be checked by its predictions concerning a mass distribution. Regardless of frequency of mass, it is important

to account for the relatively few irregularly shaped systems (a much harder task?), quasars, strong radio galaxies, large mass ellipticals, etc. (It is also noted that the globular clusters in our galaxy are segregated into two groups. Beyond approximately 10 kpc, these clusters all seem to possess stars whose atmospheres are deficient in "metals". Within this distance, there is a wide range in chemical composition.)

To list just a few broad requirements, a theory of galaxy formation must account for the positional and velocity distribution of stars, the rotation curve, the nucleus of the systems, the type and spatial distribution of old clusters within the systems, and should make definite predictions concerning such things as the era of galaxy formation, age of galaxies, etc.

The observational data available for construction of a theory are very sparse. At the present epoch, the density of luminous matter has been estimated by Oort to be around $7 \times 10^{-31} gm/cm^3$, while the non-luminous intergalactic matter has an upper bound placed observationally at around $10^{-29} gm/cm^3$ -- but this matter may even be absent. The temperature of the intergalactic gas has been estimated to be around 10^5 degrees (Sciama; Ginzburg and Ozernoi; G. Field), but this is uncertain also. The temperature of what appears to be cosmic black body radiation is presently around 3 degrees Kelvin. The chemical composition of the gas is largely hydrogen. The amount of helium may be 10 to 20 per cent by weight if the theoretical estimate by Peebles is correct, but the current observations (e.g. W. Sargent) favor a smaller percentage.

One of the Friedman model universes may give an adequate representation for the smoothed-out matter, but it has not been possible yet to determine the type. The curvature is not known, nor is the value of the cosmological constant. Lastly, the inverse Hubble constant, H_o^{-1}, is around 1 to 2 times 10^{10} years, and this forms a time limit to any process of galaxy formation.

The Initial Development: The Non-linear Growth of Perturbations in an Expanding Universe

Richard W. Michie

ABSTRACT

Instabilities in an expanding universe have been studied by Bonner using linearized Newtonian equations, and by Lifshitz using the linearized field equations of General Relativity. The latter author found three types of waves. There was the classical wave with a perturbed gravitational field due to a perturbed density. This mode reproduced Bonner's results, and served to strengthen the validity of the Newtonian approximation for the development of a small perturbation.* The other two modes are peculiar to General Relativity: One has a perturbed gravitational field due to a perturbed velocity field and the

*If the radius of the patch is small compared to the radius of the universe, then the metric will be locally Minkowskian. If in addition $GM/Rc^2 < 1$, the field is weak, and with $(v/c) << 1$ it is changing slowly in time. Finally, with a non-relativistic gas, we can expect to be able to use Newtonian equations for the patch.

other gives gravitational radiation due to (at least) a quadrapole mass distribution. The results indicate that adiabatic growth rate is so slow, that this mechanism cannot account for the development of a small $(\Delta\rho/\rho \sim 10^{-2})$ amplitude perturbation which may later develop into a stellar system.

The study presented during this lecture eliminated two approximations, and thus includes a non-linear dissipative flow. The radiative dissipation occurs by free-free, free-bound, and bound-bound radiation. Heat conduction also is included. The mathematical technique employs moments over the whole system giving equations for the change with time of total energy and mechanical energy. The virial equation completes the set. Comparison with full numerical solutions of the fluid equations for spherical flows indicates the accuracy of the approximate technique. The error between the two is probably not more than 5% generally, but at some phases it can be larger. There can be a difference as much as 20 - 30% for very short times (during phases of maximum compression), but the solutions always recover and then continue within a few per cent of each other.

The results indicate that small amplitude perturbations of ρ and T around 0.1% or more are unstable, and will rapidly grow to large amplitudes in cosmological time scales. At cosmic densities around 10^{-27} gm/cm^3 or less the opacity plays a minor role since cooling by expansion is so important. At higher densities, if the temperature is around 10^7 degrees, again opacity (and to some extent chemical composition) is not important, since the system radiates almost entirely due to free-free transitions for most of the total energy loss. The chemical composition of the cosmic gas becomes much more important (because of the radiation rate) if at these relatively early epochs the cosmic gas is at a low temperature. For example, at a cosmic density of 10^{-25} gm/cm^3, and temperature of 60,000 degrees or less, if the gas is pure hydrogen then at most a growth of $\Delta\rho/\rho \sim 20$ can be achieved. But this is not nearly enough to obtain fragmentation, and this is because the initial outward motion of the cloud is not reversed.

Fragmentation: The General Problem of the Fragmentation of Gas Clouds

Richard W. Michie

ABSTRACT

Viewed within the entirety of the whole problem of galaxy formation, the collapse of a small amplitude perturbation in an expanding universe is a small achievement. One hopes to account for the origin of the initial large scale and small amplitude perturbations, as well as the processes by which the cloud will at some time break up into subunits. If this latter process can occur fast enough, then there is the hope it will lead to the formation of stars. The process is often called fragmentation, and was first seriously considered by Hoyle.

Hoyle seemed to feel that a cloud would inevitably form subunits, and his argument was simple: when the cloud is collapsing, its development will be isothermal and the condition for gravitational binding (physically equivalent to the Jeans criterion) will hold for smaller masses as the cloud becomes more compact. Also, since a greater rate of energy may be released when the cloud is composed of a group of smaller systems themselves contracting, Hoyle felt it necessary that fragmentation will occur.

Hunter, and together Savedof and Vila, studied the fragmentation of a cloud using the linearized fluid equations, a $P(\rho)$ relation, and a freely falling parent cloud. (Hunter's cloud was uniform, while Savedof and Vila used a polytrope.) Both studies yielded similar results: the amplitude of the perturbation $(\Delta\rho/\rho)$ became nearly equal to one after the cloud density increased by around 2×10^3, if initially $\Delta\rho/\rho \sim 10^{-2}$. This is a slow rate of growth, but the results are important for they indicate that if the initial perturbations are present, the initial phase of growth is quite easy to obtain. Hunter found that a second iteration indicated (to this order) that the non-linear development is not any slower.

To study the effects of the radiative energy dissipation, heat conduction and non-linear flow, the moment equation technique is again used and this time the shape may vary between prolate and oblate. The tidal force exerted on the isolated fragment by the cloud is included, for it often is comparable in magnitude to the self-gravitational force.

A study of the effect of initial conditions indicates the following play a relatively minor role, within reasonable limits: the trajectory of the fragment in the cloud, the time of origin, the place of origin, the initial amplitude, the initial shape (and therefore the tidal force). For example, there is not much difference in the solutions for initial amplitudes varying from (say) 0.1% to 10%. What is found are two important things. First, following the development in time, the amplitudes of the perturbations are very unsymmetrical about the line $\Delta\rho/\rho \sim 0$, the perturbations spending about twice as much time at larger positive amplitudes as compared to the smaller negative swings. This type of oscillation, (expected of a non-linear flow), allows an enhanced rate of energy loss through radiation and keeps the fragments distinct in the cloud until it has reversed its expansion and starts to fall inwards. Second, at this later stage, the cloud is rapidly releasing gravitational energy and the conditions are favorable for the perturbation to rapidly grow. Rather figuratively, at this later stage there can be a transfer from the global (cloud) instability to local instabilities. In an example given, it was an easy matter to obtain growth of a $10^6 M_\odot$ fragment in a $10^{11} M_\odot$ cloud, with the perturbation starting its development very soon after the cloud started its development in an expanding universe.

Because of these results the usual hierarchael picture of fragmentation in the sense of large fragments breaking up into smaller ones may be incorrect. It may be more realistic to consider the cloud as being in an initial turbulent state, and through enhanced energy dissipation most of the larger turbulent elements remain distinct and then later grow. A larger element, containing a group of smaller ones (present from the start) will definitely grow first; but still the growth of the smaller ones will not lag appreciably behind.

The Era of Galaxy Formation. The Chemical Composition and Temperature of the Cosmic Gas

Richard W. Michie

ABSTRACT

While it was demonstrated during the second lecture that it is quite easy to get growth of small amplitude perturbations in an expanding cosmic gas when the cosmic density was around 10^{-27}gm/cm^3, one cannot conclude that galaxy formation occurred around this epoch. The reason is that it may require quite a bit of time for the fragmentation process to finally produce stars. The calculations discussed during the third lecture demonstrated that while it is easy to maintain perturbations in the expanding parent cloud through enhanced radiative energy loss, the perturbations will not grow rapidly until the parent cloud is itself highly unstable. This can only occur when the cloud is no longer expanding but has entered into a state of collapse -- and this may take a lot of time. The results indicate that one cannot expect galaxy formation to have occurred past that epoch when the cosmic density was around 10^{-25}gm/cm^3. (However, the growth of large scale perturbations can easily have occurred since this time.)

Since the cosmic matter and radiation field became uncoupled when the matter density was around 10^{-20}gm/cm^3, the era of galaxy formation must have been when the density was between 10^{-20} and 10^{-25}gm/cm^3. This is an outside limit, and may be more narrow if the cosmic gas during this time was at a low temperature. If, when ρ was $\sim 10^{-23}$ to 10^{-25}gm/cm^3 the temperature of the hydrogen were less than around 60,000 degrees

(which allows some range in possible heating), then the gas would have been weakly radiating and perturbations would have grown by an order of magnitude but they would not have entered into collapse. However, starting at a density of 10^{-21}gm/cm^3, growth to large amplitudes within $\sim 10^9$ years can be easily obtained for sufficiently large masses ($\sim 10^9 M_\odot$ or more) even for pure hydrogen so cool it is essentially not radiating. Such a growth is essentially adiabatic, and simply requires a perturbation in the density or velocity fields to allow a lessened rate of expansion causing later a reversal of the motion and hence collapse. The gas in the cloud is then heated and ionized. Thus perturbations can grow in a cool hydrogen gas starting when the cosmic gas was around 10^{-21}gm/cm^3. But when the background density falls to 10^{-23} to 10^{-25}gm/cm^3, in order to get collapse within a short enough period of time the gas must be either hot if "pure" hydrogen, or otherwise must contain sufficient enrichment of other elements so as to allow radiation at low temperatures. This also will narrow the era of galaxy formation depending on the amount of helium relative to hydrogen. Another important point is that the determination of the era of galaxy formation also requires a theory of fragmentation, and this cannot be accurately formulated without some knowledge of the temperature and chemical composition of the cosmic gas during these "early" stages of development of the universe. Finally, the process of initial condensation and fragmentation to stellar systems is not sensitive to the particular type of Friedmann model with zero cosmological constant.

Galactic Bores

Edward A. Spiegel

ABSTRACT

In recent attempts at explaining the spiral arms of galaxy, Lin and Shu have made plausible the idea that the arms are density waves. The question dealt with here is that of the nature of such waves as they occur in the interstellar gas when they are of sufficient amplitude to be quite nonlinear. The work reported was done in collaboration with D. W. Moore, NRC Senior Postdoctoral Fellow, N.A.S.A. Institute for Space Studies, and, for my part, received financial support from the Air Force Office of Scientific Research.

The ideas of shallow water theory are used to describe flow of the interstellar gas, in treating the gas as an isothermal ideal fluid. This means, that in general, the fluid is hydrostatic equilibrium in the direction normal to the galactic plane. (This direction will be called here, the vertical or z-direction.) Even in this approximation, a number of sorts of waves can propagate through such a medium in linear theory; in particular sound waves (p-modes) and gravity waves (g-modes) with the appropriate modifications due to rotation can propagate. We may ask what happens to such waves as they steepen and reach finite amplitude.

If a sound wave traveling in the horizontal direction, steepens into a shock, it is possible to discuss in the usual way the jump conditions across it. If we apply the jump conditions, we find, in general, that behind the jump the conditions of hydrostatic equilibrium are

violated. In such regions the ideas of shallow water theory break down and large vertical velocities arise, so that a phenomenon much like a hydraulic jump occurs.

On the other hand, if the wave which steepens is a gravity wave, it also gives rise to a hydraulic jump when it steepens into the non-linear regime. If the flow ahead of the jump is shooting, generally speaking it is also supersonic, since the gravity wave speed in an isothermal gas is $\sim \sqrt{gH}$ where $H \sim C^2/g$. Thus, the jump is also a transition from supersonic to subsonic flow as in a shock. Therefore we see that both sound waves and gravity waves, in the interstellar gas steepen into compressible hydraulic jumps, for which, perhaps, the term bores is a more appropriate name.

In this work the jump conditions for galactic bores are developed for an ionizing gas of pure hydrogen. The bore provides a source of turbulence for the gas behind the bore, and causes ionization and heating. A tendency for the gas to achieve a maximum density jump is noted and this occurs when the gas flows into the bore at speed of $v \sim \sqrt{2I/m_H} = 51\ \mathrm{km/sec}$, where I is the ionization potential.

The 3-kpc arm detected in 21cm observations provides a possible qualitative check of some of the qualitative results of the bore picture. In observing this arm in certain galactic longitudes one is able to observe tangent to the arm and thus see a cross section of the wave in a plane which is probably nearly parallel to the flow. One sees in particular that outside the arm there is ionized gas while inside the arm the matter is neutral. The ionized matter has a large vertical extent, and if the picture is that of radial outflow, the observed phenomena seem to resemble qualitatively the situation in a strong compressible bore.

This suggests as a possible model of the 3-kpc arm, that we may consider it crudely as an axisymmetric ring which represents a density wave (bore) in a steady radial flow. The source of this flow is not specified here, though we may note that motions outward from the galactic center are observed interior to the 3-kpc arm. The basic flow interior to the arm, if taken to be axisymmetric, is described by equations much like the solar wind equations with the inclusion of angular momentum. The flow, initially supersonic, begins to decelerate in its outward motion, and in order for it to decelerate through the Mach line, a discontinuity must occur, if the usual aerodynamic ideas are to be accepted. This discontinuity, according to the foregoing discussion, must be a bore, for which the jump conditions have been worked out. Hence the matching across the jump may be carried out. A complete solution for the flow follows, giving temperature, density, velocity. The general picture obtained is in qualitative agreement with many aspects of the observations. But there is a striking exception: the present model assumes that angular momentum is conserved by each parcel of fluid while the observations indicate that this is not the case. Magnetic configurations exist which appear to redistribute angular momentum in the required way, but this requires field strengths rather in excess of those observed.

DYNAMICS OF DISK GALAXIES

Alar Toomre

This series of four lectures on the stability and dynamics of thin, rotating systems of self-gravitating material was divided into two nearly independent halves. The first two lectures, reviewing i) the criteria for the avoidance of outright gravitational (or Jeans) instabilities, ii) some free and forced density waves in such systems, and iii) the galactic spiral problem, were in essence concerned with displacements only within the planes of these disks. The last two lectures, on the other hand, dealt with i) various bending oscillations, and ii) certain new buckling instabilities of such model galaxies, and thus involved significant motions at right angles as well.

Lectures I and II: Gravitational Stability of Thin Disks; Forced Responses; The Spiral Problem

ABSTRACT

This review began by noting that the elementary gravitational stability of any disklike galaxy involves two distinct length scales. One of these is the familiar Jeans wavelength (or adaptation thereof), below which the clumping tendency of the self-gravitation may be successfully resisted already by gas-kinetic or magnetic forces, or by the random motion of the material. The other is an approximate upper bound stemming from the fact that, even in the absence of such pressure-like forces, just the rotation of the thin disk tends to secure it from gravitational instabilities at large enough length scales.

It was remarked that these two critical lengths together provide a readily understood criterion for the stability of a disk of gas or stars against simple gravitational clumping: In essence, that clumping will be avoided provided the Jeans length -- which itself is of order 2π times the thickness in the case of a gas disk (e.g., Ledoux 1951) -- exceeds a certain fraction of the rotational length scale. The relatively "local" analyses of Toomre (1964, for a star disk) and of Goldreich and Lynden-Bell (1965a, for certain gas sheets) were both reviewed in this light, and it was noted from them that the critical fraction appears to be about one-quarter.

It was cautioned, however, that as yet there exists not a single stability analysis either proving or denying that a smooth, self-consistent disk of stars or a comparable gas disk could be fully saved from instabilities of all scales by sufficient random velocities or gas pressure. In this sense, the outstanding problems include whether a bar-like density disturbance could develop in an otherwise stable, axisymmetric disk of material, and also whether that system might not be prone to mild instabilities, or at least waves, of a distinctly spiral form.

The latter possibility, initially advanced by B. Lindblad, has recently been seriously reopened by Lin and Shu (1964, 1966), who indeed have found some asymptotic indications of just such an instability. Lacking the aforesaid complete analyses, however, probably the fairest comment on Lin and Shu's spiral wave hypothesis is that it is as yet unproven, and also that it probably won't be applicable to the very chaotic-looking spirals in any case. On the other hand, in its favor it must definitely be said that it appears almost impossible to attribute the most orderly large-scale spiral structures to any forces other than (primarily) the stellar gravitation.

An alternative approach to the problem of the spiral structures of the more chaotic sort was also reviewed in these lectures. Whereas one can be almost certain that the typical stars in present-day galactic disks would by now be protected from the most obvious (or Jeans) instabilities by their more or less automatically acquired random motions, it was pointed out that even that assurance is lacking as regards the interstellar material. This is due both to our ignorance of the actual gas pressures as well as to the various dissipative processes that are to be expected in a gas.

It was therefore suggested that in possibly quite a few of the galaxies -- and regardless of the existence of any really large-scale waves in the combined star and gas disks -- the gas might have been forced to adopt a distinctly uneven or lumpy distribution already by its own gravity. In such situations, it was emphasized that the approximate <u>spacings</u> between major gas fragments would not necessarily equal any Jeans length, but would rather be determined by the appropriate rotation-governed critical length scale mentioned above. Several elementary examples were offered supporting this contention. An important consequence is that, since the length scale in question happens to be proportional to the projected density of the gas, the typical fragment spacings should themselves be roughly proportional to the fractional gas content of a galaxy. That correlation, of course, is in qualitative agreement with observations.

Finally, mention was also made of some related calculations of Julian and Toomre (1966) concerning the gravitational effect of any single orbiting mass concentration (such as one of the aforementioned gas "lumps") upon a differentially rotating disk of stars possessing a velocity dispersion more than sufficient for local stability.

These calculations, to some extent foreshadowed by Goldreich and Lynden-Bell's (1965b) analyses of shearing wavelets in a gas disk, showed even such a stable star system to be remarkably responsive in a spiral-like manner to localized forcing. These forced spiral waves are not to be confused, of course, with Lin and Shu's fully self-consistent density wave proposals; however, together with the latter, they certainly contribute to the impression that the spiral phenomenon - even on the intermediate scale -- is probably explained mainly by gravitation.

Lecture III: Large-scale Bending Modes

Alar Toomre

ABSTRACT

It has been known for some time (Burke 1957, Kerr 1957) that the layer of interstellar atomic hydrogen in this Galaxy is curiously warped. Although remarkably flat over its inner half, it becomes distorted upwards by as much as 700 pc (or about 5 per cent of the radius) near one sector of its rim, and is turned down by a like amount at the opposite longitudes.

In cross section, the hydrogen disk thus resembles a shallow integral sign.

It was first thought that this might simply be a tidal distortion of our entire galactic disk (gas and stars alike) due to our presumed satellite galaxies, the Magellanic Clouds. However, already Burke and Kerr felt obliged to dismiss that possibility as numerically implausible by one or two orders of magnitude. Later workers (e.g., Lozinskaya and Kardashev 1963, Mrs. Avner 1965, Elwert and Hablick 1965) have either concurred outright, or else have had to invoke some special resonances in their attempts to explain the observed displacements as a direct tidal result of the Clouds at their present distance.

An ingenious alternative explanation was advanced by Kahn and Woltjer (1959): They noted that a not implausible motion of our Galaxy with respect to any intergalactic material would for fairly reasonable densities result in pressure forces adequate for displacing our gas layer from a plane by about the right order of magnitude. However, Kahn and Woltjer's suggestion was not worked out in very convincing detail (for instance, all gyroscopic effects were neglected, and solutions from incompressible hydrodynamics were used to describe the exterior gas motion) and it is perhaps fair to say that their idea enjoyed considerable vogue chiefly because nothing else seemed to work any better.

A third possibility, suggested by Lynden-Bell (1965), is that the observed deformation might represent a travelling wave analogous to the free or Eulerian nutation of, say, a coin thrown spinning into the air. Lynden-Bell noted that such a rapidly forward-travelling mode in a non-rigid, self-gravitating disk of constant angular speed of revolution would at any instant also be exactly plane; however, in more general disks, the corresponding mode would necessarily appear somewhat warped. He conjectured that one such disturbance might indeed have persisted since the formation of this Galaxy.

It was mainly this last suggestion which recently prompted Christopher Hunter and myself to undertake a more comprehensive study of the bending oscillations of thin, self-gravitating disks of negligible pressure. Our mathematical technique consisted largely of adapting certain Legendre polynomial expansions used previously by Hunter (1965) to study disturbances within the planes of certain model galaxies, and of numerical evaluations of the eigen-values and -vectors of (strictly speaking) infinite non-symmetric matrices.

The main conclusion of this work has been that the "Eulerian" mode is by no means the only -- nor even the most likely -- candidate: We have found many disks with relatively sharp edges to exhibit a variety of other bending modes, including one relatively slow, retrograde-travelling mode of the desired integral-sign cross section. Because of its comparative slowness of precession, the latter mode in our Galaxy appears to have been especially susceptible to excitation by external tidal forces such as those from the Magellanic Clouds during some past close passage at, say, one-half their present distance. Indeed, by assuming their mass to be one-tenth of that of the Galaxy, we have satisfied ourselves that almost the full observed amplitude and shape of bending could have been established during a single such passage, and would subsequently have persisted at least for several galactic "years". That amplitude of distortion, by the way, is of the same order as the angle of precession of a comparable rigid disk under the same circumstances.

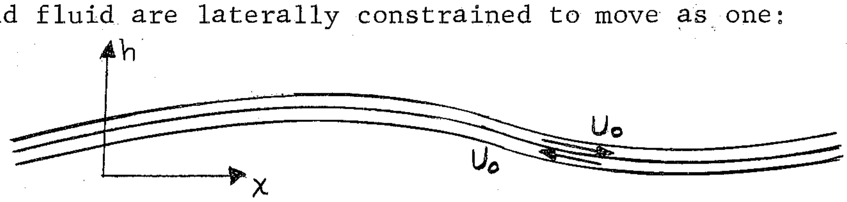
Lecture IV: A Kelvin-Helmholtz Instability

Alar Toomre

ABSTRACT

The Kelvin-Helmholtz instability in ordinary hydrodynamics refers typically to a situation where two adjacent, semi-infinite, inviscid fluids of the same density slide relative to one another with an unperturbed speed of, say, $2\,U_o$. It is then well-known (e.g., Chandrasekhar 1961, p. 481) that the vortex sheet that is their contact surface is prone to a lateral instability with an exponential growth rate $\sigma = U_o \alpha$ where α is the relevant wave number of the spacially sinusoidal perturbation.

Closely related to this instability is one which arises when two very thin, counter-streaming laminae (or extremely flexible hoses) of inviscid fluid are laterally constrained to move as one:



Then, in terms of the supposedly infinitesimal displacement (h(x,t), it is clear that a typical element of the moving to the right experiences a vertical acceleration (and hence a force per unit mass) given as

$$\left(\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}\right)^2 h = h_{tt}+2U_0 h_{tx}+U_0^2 h_{xx},$$

and that the leftward stream feels a similar acceleration with only the sign of U_0 reversed. If one further assumes the two streams to be of equal intensity, and to possess no bending stiffness or other external restoring forces, it follows that their accelerations must everywhere sum to zero, or $$h_{tt}+U_0^2 h_{xx}=0.$$

Thus, even the growth rate of sinusoidal disturbances turns out to be the same in this example as before. Note that this example also generalizes to the case of many parallel, thin streams merely by the above U_0^2 becoming replaced by the mean-square speed of all the laminae. Something like this in plasma dynamics is aptly named a "hose" instability.

Our star disk itself may be thought of as consisting of a multitude of similar -- but in this instance interpenetrating -- streams of material. We wish now to show that it would likewise be vulnerable to a centrifugal instability if the root-mean square stellar velocities, say σ_z, at right angles to the disk were very small compared to the typical random velocities in the horizontal.

The essential thing to recognize is that the necessary lateral cohesion would in this case be automatically provided by the gravity of the stars. For instance, if any group of stars were laterally displaced an increment Δh in excess of the local mean, they would be returned with an acceleration of $O(4\pi G\rho_0\Delta h)$, where ρ_0 is a typical volume density of the star disk. It is obvious that, if the corresponding "rattling" frequency $\omega_z=(4\pi G\rho_0)^{1/2}$ were large compared to others such as $U_0\alpha$, the different types of stars comprising the disk would be locally constrained to move very nearly together in any z-displacements. But such would be precisely the case if the vertical dispersion were very small, since for any prescribed surface or projected density, the volume density ρ_0 and hence ω_z would become arbitrarily large as $\sigma_z \rightarrow 0$.

This strong cohesion, of course, would not be the only effect of the stellar self-gravitation. Any extremely thin sheet of supposedly constant surface density μ would, when laterally displaced like

$$h(x,t) = H(t)\cos\alpha x,$$

itself cause a vertically-averaged restoring force per unit mass,

$$F_z' = -2\pi G\mu\alpha H(t)\cos\alpha x,$$

that would tend to restore it to a plane. With this milder restoring force included, and assuming the displacements at any given location again to be described by a single function h(x,t) for the reasons mentioned above, we obtain

$$H_{tt} - \sigma_x^2\alpha^2 H = -2\pi G\mu\alpha H,$$

where σ_x denotes the r.m.s. random velocity in a given horizontal direction.

From this last equation it emerges that the gravity of any extremely thin (or low σ_z) star disk alters the earlier result only by stabilizing sufficiently long wavelength disturbances, while the shorter bending disturbances still remain unstable. (Curiously, in this context the critical wavelength, $\sigma_x^2/G\mu$, proves to be identical with the Jeans wavelength for instability with respect to horizontal disturbances in a non-rotating star sheet!)

Of course, the above analysis -- and hence the instability conclusion -- become inapplicable at wavelengths so short that even a very small thickness of the sheet can no longer be neglected. Since that thickness is itself of $O(\sigma_z^2/G\mu)$, it must be suspected that all instabilities of the buckling kind may be avoided provided the ratio σ_z/σ_x is large enough.

The latter conjecture seems indeed borne out by a detailed analysis involving the collisionless Boltzmann equation: The critical r.m.s. velocity ratio for a stationary star sheet with Gaussian distributions of velocities in both the vertical and horizontal directions has been provisionally calculated as 0.30. Not very surprisingly, it has also been estimated from related but more complicated numerical calculations that a reasonable rate of rotation of the star disk should not alter this minimum required ratio by more than about ten per cent.

By contrast, the observed ratio of the velocity dispersions for many classes of stars is roughly 0.5 - 0.6, and thus is apparently well clear of this stability boundary.

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Topology of Magnetized Fluids

David Finkelstein

ABSTRACT

Suppose, for instance, that all of space is filled with perfectly conducting fluid, at first uniformly magnetized, $\underset{\sim}{B} = \underset{\sim}{B}_o$. Subsequent fluid motions can twist this uniform magnetic field into very complex configurations, but there is an integer K, a topological invariant we call the number of kinks, defined below, that is conserved so long as (a) the magnetic field is a continuous function of space and time (no shocks, please) and (b) the fluid can be regarded as perfectly conducting. Initially K = 0. If after some time, however, there are seen localized distortions, call them "galaxies", between which the magnetic field is approximately equal to its asymptotic value $\underset{\sim}{B}_o$, but within which there may be large departures from $\underset{\sim}{B}_o$, then each "galaxy" can be assigned a conserved kink number K_i, and

$$\sum K_i = 0.$$

A specimen of a single kink (K = +1) is the magnetic field

$$\underset{\sim}{B}(\underset{\sim}{r}) = \left[\exp\,(\alpha\,\underset{\sim}{r}\,\times)\right]\,\underset{\sim}{B}_o,$$

where the exponential of the linear operator $\alpha\,\underset{\sim}{r}\,\times$ (cross-product) is

simply a rotation about a radial axis, through the angle of rotation $\theta = \alpha r$, $r = |\underset{\sim}{r}|$. We demand that $\alpha = \alpha(r)$ be such that

$$\begin{array}{ll} \theta \rightarrow 0 & r \rightarrow \infty \\ \rightarrow 2\pi & r \rightarrow 0 \end{array}$$

in order that B satisfy the boundary conditions at ∞ and be continuous at 0. $\alpha \longrightarrow -\alpha$ yields the anti-kink (K = -1).

The kink number K is defined as follows in general.[1] Because $\underset{\sim}{B}(\underset{\sim}{r})$ is carried with the fluid, $\underset{\sim}{B}$ is never zero, and defines a mapping of $\underset{\sim}{r}$-space into "punctured" $\underset{\sim}{B}$-space. The boundary condition at $r = \infty$ makes it possible to regard $\underset{\sim}{r}$-space as a three-sphere S^3, and the number of kinks K is the number of times the field point $\underset{\sim}{B}$ surrounds the origin $\underset{\sim}{B} = 0$ as $\underset{\sim}{r}$ ranges over its S^3.

The concept of kink applies for other boundary conditions as well. For example, a magnetized fluid in a conducting toroidal chamber with tangential boundary field (Stellerator) can be assigned conserved kink-numbers.

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[1] D. Finkelstein, Kinks, J.Math.Phys., July, 1966.

Spin-Orbit Coupling in the Solar System

Peter Goldreich (and Stanton Peale)

ABSTRACT

Several theoretical discussions of planetary and satellite rotation rates have followed the recent radar determination of Mercury's rotation period (Pettengill and Dyce 1965). (Hereafter we only refer to planets, although our results may be applied to satellites as well.) In the first, Peale and Gold (1965) showed that in an eccentric orbit tidal friction could bring an axially symmetric planet to an asymptotic rotation rate which is somewhat faster than its orbital mean motion. In the asymptotic spin state the tidal torque averaged over an orbit period vanishes. For a broad class of tidal torques the maximum torque occurs at perihelion, and the final spin velocity will lie between the mean motion and the instantaneous orbital angular velocity at perihelion. The precise value of the final spin is determined by the amplitude and frequency dependence of the planet's Q, where 1/Q is the specific dissipation function (MacDonald 1964).

Colombo (1965) has suggested that Mercury may be spinning with an angular velocity of precisely 1.5 n, where n is its orbital mean motion. Subsequent investigations have shown that a sufficient deviation from axial symmetry would stabilize this resonant spin (Liu and O'Keefe 1965; Colombo and Shapiro 1965; Goldreich and Peale 1966a). The existence of other resonant spin states at rotation rates of pn, where p is any half-integer (negative or positive), has been suggested by the latter two sets of authors. We have shown that the exceedingly small

value of $(B-A)/C \approx 10^{-8}$ would suffice to stabilize many of these resonances against the disruptive influence of the solar tidal torque. Here, A, B and C are the principal moments of inertia in order of increasing magnitude. For the moon in its present orbit with the known value of $(B-A)/C \simeq 2\text{x}10^{-4}$, stable resonant spin states exist at rotation rates of $0.5n$, n, $1.5n$, $2n$, $2.5n$ and perhaps at several others. Thus for Mercury and the moon, stable spin states appear to exist with rotation rates which are both faster and slower than the observed values. Almost certainly Mercury and the moon must have bypassed some of these stable resonances on the way to attaining their present spins. Similar strong indications that satellites may pass through stable resonances are provided by the observed synchronous rotations of several satellites of the major planets. Of special interest is Iapetus, because of its weak tidal torque, relatively high orbital eccentricity, and well-confirmed synchronous rotation (Widorn 1950).

What conditions must be satisfied if a planet is to be captured at one of the resonant states? The present investigation is devoted to answering this question. In the following, approximate equations of motion for a spinning planet are derived by averaging the complete equation of motion over an orbit period. From the averaged equations criteria for the stability of resonant spin states are established. Details of the capture into stable spin states are emphasized, and techniques for calculating capture probabilities are developed. Capture probabilities are calculated for both the synchronous and the $1.5n$ resonance and are applied to Mercury and the moon. Similar techniques are used in a discussion of the Venusian spin, which may be commensurate with its synodic mean motion. A stability criterion is determined, and the possibility of capture into such a resonant state is considered. As a check on the validity of the averaged equations of motion the complete equations of motion were directly integrated in several cases, and the results were shown to agree with those derived from the averaged equations.

Relativistic Cosmology

James L. Anderson

ABSTRACT

The problem of formulating a consistent description of the gravitational interaction of matter is essential for all cosmological considerations since, as far as we know, gravity is the only force that is operative on the cosmological scale. Historically, the first description of the gravitational interaction was given by Newton in terms of an instantaneous action-at-a-distance between gravitating masses. Later, after the experience gained with the electromagnetic interaction, the gravitational interaction was re-expressed in terms of a gravitational field. This state of affairs was satisfactory until attempts were made to describe the gravitational interaction within the framework of special relativity. The simplest possibility appeared to be to describe the gravitational interaction by means of a scalar field. However, such a description predicted a precession of the planetary perihelia in the

opposite direction to that observed. Before other attempts were made to describe the gravitational interaction within the framework of special relativity Einstein proposed a description that we now call the general theory of relativity. Basing his development on Mach's principle, the principle of equivalence and the principle of general covariance he was lead to associate the gravitational field with the underlying geometrical structure of physical space-time. Much later it was found that all known gravitational effects could be described within the framework of special relativity by using a second rank symmetric tensor to describe the gravitational field.

In constructing relativistic cosmologies then, one has two alternatives: one can use the full mechanism of general relativity or one can work with the special relativistic theory described above. It is important to recognize that the general theory involves two universal constants, one the gravitational constant, G, and the other the velocity of light, c. In the limit $G \rightarrow 0$ one obtains from the general theory the special relativistic theory. On the other hand, in the limit $1/c \rightarrow 0$ one obtains the Newtonian theory. Thus a cosmology based on the Newtonian theory neglects both the non-linear effects of the general theory and the velocity dependent effects which are essentially special relativistic in origin. One sees then that Bonner's considerations concerning the stability of an expanding universe, based as they are on a Newtonian theory are included in the treatment of Lifschitz which is based on the general theory. It would be interesting to study a special relativistic cosmology to see if the non-linearity of the general theory plays any essential role in such stability considerations.

One other feature of the gravitational interaction in both the special and the general theory is worthy of mention. In any field theory one must impose suitable boundary conditions. In most treatments one imposes Sommerfeld radiation type boundary conditions which lead to the usual retarded solutions of the wave equation. However it is not necessary to do so and in particular the statistical arguments of Wheeler and Feynman for the elimination of the advanced solution in the electromagnetic case involving, as they do, the concept of a complete absorber and not applicable in the gravitational case since there are no gravitational absorbers. The possibility exists therefore that the gravitational interaction between two bodies might involve both advanced and retarded solutions. It is proposed that such an interaction could be observed by careful measurements of tidal forces of the moon on a resonant system of the type employed by Weber for the detection of gravitational radiation.

Problems of Relativistic Hydrodynamics

James L. Anderson

ABSTRACT

The relativistic analogue of the Eulerian equations of motion for an ideal gas have been long known.[1] Two extensions of these equations are of considerable interest, especially for some astrophysical applications: the inclusion of gravitation effects and the inclusion of transport processes.

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[1] see L. D. Landau and E. M. Lifschitz, Fluid Mechanics, Chap. XV (Pergamon Press, London, 1959).

In this lecture we discussed the problems that arise when one attempts to make these extensions.

In our first lecture we showed how gravitational interactions could be taken into account in special relativity by a symmetric tensor field $h_{\mu\nu}$. By a standard method of the Noether theorem one can construct a stress tensor, $T_G^{\mu\nu}$, for this field and it, together with the matter tensor $T_M^{\mu\nu}$, must be conserved; i.e. $(T_M^{\mu\nu} + T_G^{\mu\nu})_{,\nu} = 0$. If $T_M^{\mu\nu}$ is the matter tensor of an ideal fluid, given by $T_M^{\mu\nu} = (G+p)u^\mu u^\nu - p\eta^{\mu\nu}$ where G is the internal energy, p the pressure and u^μ the four-velocity of a fluid element, we obtain, from these continuity equations, the relativistic analogue of the Eulerian equations for an ideal fluid including self-gravitating forces. These equations, together with the number continuity equation $(n u^\mu)_{,\mu} = 0$, the equations of motion $\Box(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\rho\sigma} h^{\rho\sigma}) = \chi T_M^{\mu\nu}$ for the gravitational field, and an equation of state $\epsilon = \epsilon(n, p)$ constitute a complete set of equations for ϵ, n, p, u^μ and $h^{\mu\nu}$.

An alternate, and more satisfactory, approach to the problem of including gravitational effects is to make use of the mechanism of general relativity. The gravitational field, now denoted by $g_{\mu\nu}$, is assumed to satisfy the Einstein equations $R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \chi T_M^{\mu\nu}$ where, because of the contracted Bianchi identities, $T_M^{\mu\nu}$ must satisfy $T_M^{\mu\nu}{}_{;\nu} = 0$ (semi-colon denotes covariant derivative). With $T_M^{\mu\nu} = (\epsilon + p)u^\mu u^\nu - p g^{\mu\nu}$ these latter equations give the "Euler" equations of motion for the fluid. If we make a weak-field approximation (χ small) so that $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, we recover the special relativistic equations discussed above. On the other hand, if one makes a "slow-motion" approximation, i.e., makes an expansion in powers of 1/c where c is the light velocity, one appears to obtain an essentially different set of equations of motion, first given by Chandrasekhar.[2] The first set of equations allows for arbitrary motions of the fluid but requires that the gravitational fields are weak. The second set restricts one to motions with small velocities but takes account of non-linear gravitational effects. It appears that the physics of the situation will determine which equations one is to employ.

The problem of including transport processes in a relativistic framework is still, to a large extent, an unsolved problem, due chiefly to the fact that one does not yet possess a satisfactory kinetic theory of a relativistic system of particles. To date two approaches have been proposed, one due to Eckart[3] and the other to Landau and Lifschitz.[1] In the Eckart approach one maintains the number continuity equation $(nu^\mu)_{,\mu} = 0$ and modifies the stress-energy tensor by adding to it, in the case of heat conduction, a term $u^\mu q^\nu + u^\nu q^\mu$ where q^μ is a heat flux vector satisfying $u_\mu q^\mu = 0$. In the Landau-Lifschitz approach one retains the form of the stress-energy tensor but modifies the number continuity equation to $(\rho u^\mu + \nu^\mu)_{,\mu} = 0$ where $\nu^\mu u_\mu = 0$ and ρ is the proper-mass density of the fluid. Only in the limit where $q/(\epsilon + p)$ is small do the two approaches agree. Further progress on the problem will require a relativistic Boltzmann equation.

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[2] S. Chandrasekhar, Ap.J. 142, 1488 (1965).

[3] C. Eckart, Phys.Rev., 58, 919 (1940).

Galaxy Formation

George B. Field

ABSTRACT

The problem of growth of density fluctuations in a Friedmann universe containing radiation can be discussed conveniently by dividing the history of the universe into three distinct phases. During Phase I, extending from a few hours to about 10^3 years, radiation dominates matter both in terms of pressure and in terms of density. In Phase II, lasting up to about 10^5 years, pressure is due to radiation, and density is due to matter. Phase III occurs after the radiation and matter decouple at 10^5 years; the density is due to matter, and pressure which affects hydrodynamic motions is due mostly to the matter also.

Phase III can be discussed with Newtonian mechanics according to Bonnor. Density perturbations of the form $\delta\rho/\rho = h(t)(\sin kr)/kr$ have the solution

$$h(t) = t^{m_\pm}$$

where

$$m_\pm = -\frac{1}{6} \pm \frac{5}{6}\left(1 - \frac{24}{25} k_0^2 C_{s0}^2 t_0^2\right)^{1/2}$$

if the gas pressure obeys $p = \rho^{4/3}$, where C_s is sound speed and "0" refers to the moment t_0 (=10^3 years) when radiation and gas density are equal. Hence there is growth like $t^{2/3}$ for wavelengths much larger than the Jeans length

$$\lambda_J = C_{s0}\left(\frac{2\pi}{3G\rho_0}\right)^{1/2}$$

corresponding to a mass of about $10^5 M_\odot$ for $T_0 = 80{,}000^\circ K$ and $\rho_0 = 10^{-16}$ g cm^{-3}, numbers which are consistent with observation. It is shown that such a growth starting from a perturbation h at time t can be projected approximately into the non-linear regime, $\delta\rho$ itself (as against $\delta\rho/\rho$) starting

to increase at $t_A = 1.2\,t h^{-3/2}$ and ultimately rising sharply as $t \rightarrow 2t_A$ presumably to form a galaxy. Hence if galaxies are to form at $\sim 10^8$ years, a 1% perturbation is necessary at the beginning of Phase III.

Phase II is not yet well understood. Approximate Newtonian solutions by Peebles indicate that perturbations present at the start of Phase II are "frozen in" by radiative drag if their wavelength, and hence optical depth is small. General relativity is necessary for long wavelengths, but the only treatment available (Lifschitz) has not been worked out in detail for the equation of state valid for adiabatic changes in Phase II:

$$\frac{dp}{d\rho} = \frac{p_{RAD}}{\rho_{RAD} \frac{3}{4} \rho_{MAT}}$$

Sufficiently large objects may grow secularly in density during Phase II, but objects of intermediate size may oscillate.

Phase I has been treated by Lifschitz for sufficiently large objects $(>1 M_{\odot})$ where $dp/d\rho \rightarrow \frac{1}{3} c^2$. He finds an early period of growth, $h = c_1 t^{1/2} + c_2 t$, followed by oscillation in time with constant amplitude, corresponding to acoustic waves with speed, $(c/\sqrt{3})$, where c is the speed of light.

In summary, the theory so far developed suggests that galaxies can be formed by the growth of density perturbations. However, a finite perturbation $(\sim 1\%)$ is needed at 10^5 years, and since there is no exponential growth at previous times, the source of such a perturbation is a problem. Since the order of magnitude of h does not change much during Phases I or II, growth to $\sim 1\%$ is required in the very earliest phases. This can occur only if $\frac{dh}{dt}(t \rightarrow 0)$ is finite, which seems to mean that the perturbation must be put in as initial conditions.

Similarity Techniques applied to Taylor Columns.

Source Flows in a Stratified Fluid.

Bruce R. Morton

ABSTRACT

The purpose of these two seminars was to explore some of the more important physical characteristics of steady flows produced in extensive regions of rotating or stratified fluid by sources (or sinks) of mass or momentum. The order-of-magnitude and similarity arguments used serve to identify properties of special regimes of flow with a minimum of mathematical detail, and also bring out the significance of mass flux and flow force (defined as longitudinal momentum flux plus pressure perturbation) in the various flows. More complicated systems may then be interpreted in terms of the elementary source flows; for example, the Taylor wakes produced in a rotating environment by slow motion of a body along an axis of rotation may be related asymptotically to the flow due to a doublet formed as the limit of a flow-force source and sink combination.

The flow due to a point source of mass in an environment otherwise at rest is spherically symmetrical with mass flux constant and flow force decreasing with increasing radial distance, while that due to a point source of momentum has increasing mass flux and constant flow force with increasing axial distance. Flows due to mass sources in rotating or stably stratified environments suffer an additional lateral constraint which gives rise to strongly preferred asymptotic directions of outflow parallel to the rotation axis or normal to the gravitational field: such flows tend to be narrow, and viscous forces are generally significant far from the source.

The background rotation or stratification also impose a constraint on inflow from large radial distances producing a modified pattern of entrainment. Most previous solutions (particularly those for mass sources in a rotating fluid) have neglected viscosity.

An order-of-magnitude analysis of the Navier-Stokes equations for source-like flow in a rotating or stratified environment may be based on the single assumption that the flow subtends a small angle at some virtual source. In terms of local scales for the disturbance pressure and for each coordinate distance and velocity component, a flow may be characterised locally by a local Rossby number (the ratio of scale inertia to scale Coriolis forces) and Reynolds number (based on flow along the column), and as a whole by overall Rossby and Reynolds numbers based on the source strength. (In stratified fluids the Rossby number is replaced by a Richardson or internal Froude number.) The Reynolds rather than Ekman number is chosen to represent physical effects due principally to the source and not the boundaries of the flow region, though it is necessary to determine subsequently the degree to which these may be separated in actual cases. If the equations are now taken (for the case of ambient rotation) in the order: continuity, azimuthal momentum, longitudinal momentum, and radial momentum, special flow regimes are obtained representing laminar regions of inertial-viscous force balance, inertial-Coriolis balance, and Coriolis-viscous balance; in the first two cases there are two types of regime according as the longitudinal and azimuthal equations are or are not pressure coupled, while in the third there must always be pressure coupling. The one additional relationship needed fully to determine the similarity structure in each of these regimes is provided by a 'source strength relation' representing constancy of mass flux for the mass source or of flow force for the generalised momentum source. Existing solutions for swirling jets in still and vortical environments fit into this classification, and additional solutions have been obtained by Herbert (PhD thesis, University of Manchester, 1965) including especially the Coriolis-viscous jet produced at small Rossby numbers from a source of flow force directed along a rotation axis of the environment (J. Fluid Mechanics, 1965). This Herbert solution has a number of interesting properties: while the flow force is independent of axial distance as in a simple jet, both the entrainment flux at large radial distance and the axial mass flux in the jet are zero as a consequence of the lateral constraint of ambient rotation; and the flow force is transmitted predominantly by the pressure field since the velocity disturbance is weak.

There are two types of Taylor column: longitudinal Taylor columns produced by slow motion of a body <u>along</u> an axis of ambient rotation, and lateral Taylor columns due to motion of a body normal to the rotation axis. The latter type is most naturally treated as a wave phenomenon (c.f. ship wave patterns), but the former may usefully be treated as a <u>wake</u> problem in the normal aerodynamic sense (and in this case a better term might be Taylor wakes). An order-of-magnitude analysis may again be carried out for flow in the Taylor wake under the assumptions that the spread angle of the wake is small and that all disturbance velocity components are small relative to the speed of translation of the body (although the Reynolds number for motion of the body need not be small). These restrictions exclude a region of flow in some neighbourhood of the body but a good deal of physical insight may still be gained from consideration of the more distant flow. Inertial-viscous, inertial-Coriolis and Coriolis-viscous regimes of flow may again be identified in both upstream and downstream wakes, generally of unequal strength. The similarity structures are in each case determined finally by the drag contribution of the particular wake (and it may be noted that Taylor wakes serve the purpose of transmitting the force exerted by the body on the fluid via the flow-force field towards the boundaries). The most interesting regime is again the Coriolis-viscous flow, where the solution proves to be identical with the Herbert vortex jet. Thus at low overall Rossby numbers the upstream Taylor wake has the asymptotic structure of a viscous vortex jet corresponding with a source of positive flow force and hence has anticyclonic relative vorticity, while the downstream Taylor wake is a cyclonic vortex jet corresponding with a negative source or sink of flow force. The drag experienced by the body is transmitted by a positive upstream pressure perturbation and a negative downstream pressure.

The distant flow field of a source in a rotating or stratified environment depends strongly on viscous forces, has an essential pressure coupling of axial and azimuthal velocity fields and has axial and azimuthal velocities of comparable magnitude. These features appear to be incompatible with the assumptions on which Stewartson, Barua and Squire have previously based solutions for a mass source in a rotating environment. Moreover, the asymptotic solutions indicate that when the flow force is constant the mass flux is zero, while if the condition of constant mass flux is imposed then either the flow force must necessarily be an increasing function of axial distance, or there must be axial pressure gradients in the outer field. A simple experiment to test some of these ideas has been carried out in the WHOI Hydrodynamics Laboratory during the summer with the help of Mr. Robert Frazel, in which a time-dependent, two-dimensional source flow was produced in a stratified environment. A long narrow tank was stratified stably with a linear density gradient using salt and water, or water and ethyl alcohol, and dyed source fluid was run in slowly through a porous pipe fastened horizontally across the tank at middle depth. The density of the source fluid was equal to that of tank fluid at the level of the source axis, and flow visualisation was provided using electrical release of dye lines from vertical wires in a tank coloured with thymol blue. Although this is an unsteady flow, it exhibited many of the anticipated features. When the source was turned on and maintained at steady mass flow, a velocity field was generated which extended horizontally (and vertically) some distance beyond the lens-shaped contact surface enclosing the fluid released from the source; the thin layer of outflow, both within and beyond the contact surface, was sandwiched between upper and lower layers of reversed flow; the whole velocity field, in both outward and reversed layers, decreased in magnitude away from the source, and for the period of useful observation was very small near the end walls; no visible effect had been produced in a dye sheet extending over the end walls of the tank up to times at which the source-fluid lens extended through about half the tank length; the pressure head required to maintain constant mass flux increased steadily as the lateral extent of the flow increased. The sink flow produced by extraction of fluid at constant rate through the pressure pipe was in most respects very similar to the source flow with reversal of all flow directions; and, in particular, the pressure head necessary to produce a constant outflow increased steadily with time after the flow was started. These and other experiments with related theory will be described in greater detail elsewhere.

Stochastic Equations and the Theory of Turbulence

Joseph B. Keller

ABSTRACT

In the analysis of the propagation of sound waves, electromagnetic waves or other waves in turbulent media, linear partial differential equations with random coefficients are encountered. The statistical properties of the coefficients are determined by the turbulent motion of the medium. For the purpose of analyzing wave propagation, these statistical properties are assumed to be known. Then the statistics of the propagating waves are sought in terms of them. The theory of turbulence, on the other hand, seeks to determine the statistical properties of the turbulent flow itself. Since this flow is governed by nonlinear equations, it is not susceptible to the same type of analysis as the wave motion, which satisfies linear equations. In this lecture we present a theory of linear stochastic equations, i.e. equations with random coefficients, and

show how it can be applied to the Navier-Stokes equations to yield a theory of turbulence.

Let $L(\alpha)$ be a linear operator depending upon a random variable α with probability distribution $p(\alpha)$. Then we write a linear stochastic equation as

$$Lu = g \tag{1}$$

Here g is a given non-random vector and u is the vector to be found. If L is invertible, as we assume it to be, the solution of (1) is

$$u = L^{-1} g \tag{2}$$

From (2) we see that u also depends upon α since L does, so u is also random. The mean value of any function of α, say $F(\alpha)$, is denoted by $\langle F \rangle$ and defined by

$$\langle F \rangle = \int F(\alpha) p(\alpha) d\alpha \tag{3}$$

Thus from (2) we obtain

$$\langle u \rangle = \langle L^{-1} \rangle g \tag{4}$$

Multiplying (4) on the left by $\langle L^{-1} \rangle^{-1}$ yields the following equation for $\langle u \rangle$:

$$\langle L^{-1} \rangle^{-1} \langle u \rangle = g \tag{5}$$

Suppose u is a vector function of the vector x and let $u_i(x)$ denote a component of u. Then the two-point correlation matrix of u, denoted by $C(x, x_1)$, has components $C_{ij}(x, x_1)$ defined by

$$C_{ij}(x, x_1) = \langle u_i(x) u_j(x_1) \rangle \tag{6}$$

In matrix form (6) becomes

$$C(x, x_1) = \langle u(x)\, u(x_1) \rangle \tag{7}$$

By using (2) in (7) we obtain

$$C(x, x_1) = \langle L^{-1}(x)\, g(x)\, L^{-1}(x_1)\, g(x_1) \rangle \tag{8}$$

Here $L^{-1}(x)$ denotes the inverse operator evaluated at x. If we introduce the convention that an operator acts only on a vector with the same argument as the operator, we can rewrite (8) as

$$C(x, x_1) = \langle L^{-1}(x)\, L^{-1}(x_1) \rangle\, g(x)\, g(x_1) \tag{9}$$

Multiplying (9) on the left by $\langle L^{-1}(x) L^{-1}(x_1) \rangle^{-1}$ yields for C the equation

$$\langle L^{-1}(x)\, L^{-1}(x_1) \rangle^{-1} C(x, x_1) = g(x)\, g(x_1) \tag{10}$$

Similar equations can be obtained for higher order correlations of u.

To make the equations (5) and (10) practically useful, we assume that L is the sum of a non-random invertible operator M and a smaller random operator V,

$$L = M + V \tag{11}$$

Then by using the binomial theorem, we can write

$$L^{-1} = M(1 + M^{-1}V) = (1 + M^{-1}V)^{-1} M^{-1} = \sum_{n=0}^{\infty} (-M^{-1}V)^n M^{-1} \tag{12}$$

Taking the mean of (12) yields

$$\langle L^{-1} \rangle = \sum_{n=0}^{\infty} \langle (-M^{-1}V)^n \rangle M^{-1} \tag{13}$$

Inverting both sides of (13), using the binomial theorem again, leads to

$$\langle L^{-1} \rangle^{-1} = M \sum_{q=0}^{\infty} \left(-\sum_{n=1}^{\infty} \langle (-M^{-1}V)^n \rangle \right)^q \tag{14}$$

The first few terms in (14) are

$$\langle L^{-1}\rangle^{-1} = M + \langle V\rangle + \langle V\rangle M^{-1}\langle V\rangle - \langle VM^{-1}V\rangle + O\left[(M^{-1}V)^3\right] \tag{15}$$

From (13) we have

$$M^{-1} = \langle L^{-1}\rangle - \sum_{n=1}^{\infty}\langle(-M^{-1}V)^n\rangle M^{-1} \tag{16}$$

This equation can be solved by iterations for M^{-1} in terms of $\langle L^{-1}\rangle$. However to the order shown in (15) it suffices to use (16) as it stands, on the right side of (15). This yields

$$\langle L^{-1}\rangle^{-1} = M + \langle V\rangle + \langle V\rangle\langle L^{-1}\rangle\langle V\rangle - \langle V\langle L^{-1}\rangle V\rangle + O\left[(M^{-1}V)^3\right] \tag{17}$$

By using (17) in (5) and omitting terms which are $O\left[(M^{-1}V)^3\right]$ we obtain

$$\left[M + \langle V\rangle + \langle V\rangle\langle L^{-1}\rangle\langle V\rangle - \langle V\langle L^{-1}\rangle V\rangle\right]\langle u\rangle = g \tag{18}$$

In a similar way, we can evaluate $\langle L^{-1}(x)\, L^{-1}(x_1)\rangle^{-1}$ and rewrite (10). When $\langle V\rangle = 0$, the result simplifies to the following

$$\left[M_1 M - \langle V_1\langle L_1^{-1}\rangle V_1\rangle M - M_1\langle V\langle L^{-1}\rangle V\rangle - M_1\langle V\langle L^{-1}\rangle\langle L_1^{-1}\rangle V_1\rangle M\right]C(x,x_1) = g g_1 \tag{19}$$

Here a subscript "one" indicates the argument x_1, while no subscript indicates the argument x.

To determine $\langle L^{-1}\rangle$, which occurs in the final equations (18) and (19), we multiply (17) on the right by $\langle L^{-1}\rangle$ and drop the $O\left[\langle(M^{-1}V)^3\right]$ term to obtain

$$\left[M + \langle V\rangle + \langle V\rangle\langle L^{-1}\rangle\langle V\rangle - \langle V\langle L^{-1}\rangle V\rangle\right]\langle L^{-1}\rangle = I \tag{20}$$

Here I denotes the identity operator. This equation, (20), is a non-linear equation for $\langle L^{-1}\rangle$. Its solution can be used in (18) and (19), which are equations for $\langle u\rangle$ and C. When $\langle V\rangle = 0$, which is the condition already used in simplifying (19), both (18) and (20) simplify to

$$[M - \langle V \langle L^{-1} \rangle V \rangle] \langle u \rangle = g \tag{21}$$

$$[M - \langle V \langle L^{-1} \rangle V \rangle] \langle L^{-1} \rangle = I \tag{22}$$

It is convenient to introduce the Green's matrix $G(x, x')$ associated with L. It is defined by

$$L\,G(x, x') = I\delta(x - x') \tag{23}$$

In terms of G we can write L^{-1} as an integral operator

$$L^{-1} h(x) = \int G(x, x') h(x') \, dx' \tag{24}$$

From (24) it follows that $\langle L^{-1} \rangle$ is an integral operator with kernel $\langle G \rangle$,

$$\langle L^{-1} \rangle h(x) = \int \langle G(x, x') \rangle h(x') \, dx' \tag{25}$$

Then (22) becomes an equation for $\langle G \rangle$, which we can write as follows

$$M(x) \langle G(x, x') \rangle - \int \langle V(x) \langle G(x, x'') \rangle V(x'') \rangle \langle G(x'', x') \rangle \, dx'' = I\delta(x - x') \tag{26}$$

Once $\langle G \rangle$ is found from (26), then (21) and (19) are equations for $\langle u \rangle$ and C.

To apply these results to turbulence, let us consider the velocity v and pressure ρp of an incompressible fluid of density ρ with kinematic viscosity ν and external force per unit mass f in a domain D bounded by a surface S. The Navier-Stokes equations for v and p can be written in the form

$$\begin{bmatrix} \partial_t - \nu \Delta + v \cdot \nabla & \nabla \\ \nabla \cdot & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix} \text{ in } D \tag{27}$$

As initial and boundary conditions, we assume that v is given throughout

D at $t=0$ and on S for $t \geq 0$. These conditions determine the solution of (27) up to an additive constant in p. This problem is of the form (1) if we take u to be the four vector v, p and L to be the matrix operator in (27), which itself depends upon v. We now define M to be the same as L with v replaced by $\langle v \rangle$. Then V is given by

$$V = \begin{bmatrix} (v - \langle v \rangle) \cdot \nabla & 0 \\ 0 & 0 \end{bmatrix} \tag{28}$$

When M and V are used in (26), it becomes

$$\left[\partial_t - \nu\Delta + \langle v \rangle \cdot \nabla\right] G_{i\ell}(x,x') + \partial_{x_i} G_{4\ell}(x,x') - \int \left[C_{ik}(x,x'') - \langle v_i(x) \rangle \langle v_k(x'') \rangle\right] \partial_{x_i} \langle G_{ij}(x,x'') \rangle \cdot \tag{29}$$

$$\cdot \partial_{x_k''} \langle G_{d\ell}(x'',x') \rangle \, dx'' = \delta_{i\ell}\, \delta(x-x'), \qquad \begin{array}{l} i = 1, 2, 3 \\ \ell = 1, 2, 3, 4 \end{array}$$

$$\partial_{x_i} G_{i\ell}(x,x') = \delta_{4\ell}\, \delta(x-x'), \qquad \ell = 1, 2, 3, 4 \tag{30}$$

In these equations x denotes both space and time coordinates and the summation convention applies over the range 1, 2, 3. The initial and boundary conditions on G follow from those in the original problem above. The equations for $\langle u \rangle$ are similar to (29) and (30) while those for C are somewhat different. This set of equations is similar to that obtained by using Kraichnan's "direct interaction" approximation.