Notes on the 1965 Summer Study Program in GEOPHYSICAL FLUID DYNAMICS at The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

1965

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Editor's Preface

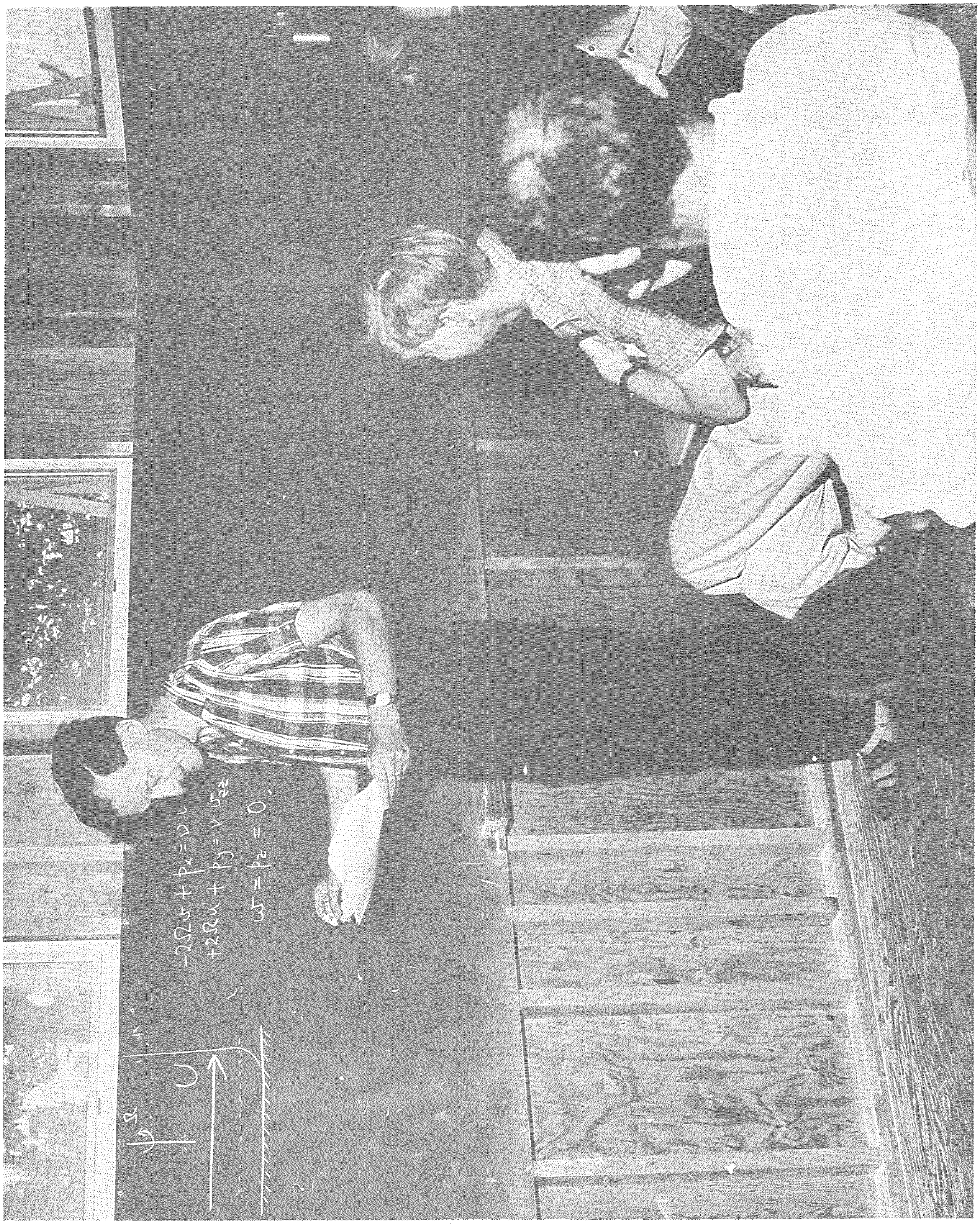
This volume contains a restatement by the pre-doctoral participants of the summer program's first lecture series. It represents their view of the relative importance of points raised in the lectures and their view of matters neglected.

Following these notes, the abstracts of a two-week symposium on recent work in turbulence theory and a two-week symposium on the frontiers of theoretical oceanography are recorded. They attest to our long summer exposure to non-linear fluid dynamics.

Some of the quick-ripening fruit of our labors has been pressed already, and appears in Volume II. However, it is hoped that more profound inquiries may emerge after reassessments in solitude of the many brash proposals concerning the turbulent world of geophysical fluid dynamics.

Mrs. Mary Thayer has done all the work in assembling and reproducing the lectures. We are all indebted to her for her remarkable efforts in keeping the summer course running smoothly and to the National Science Foundation for its financial support of the program.

Willem V.R. Malkus



Problem?

Contents of Volume I

Course Lectures

by

Dr. Francis P. Bretherton
(Principal Invited Lecturer, G.F.D., 1965)

Contents of Volume I (continued)

ABSTRACTS

Contents of Volume I (continued)

Contents of Volume I (continued)

Lecture #I

Dynamics of Rotating and Stratified Fluids

Heat sources and dissipation will be neglected in the subsequent work and our attention will be restricted to laminar, inviscid fluid flow.

Though air is a mixture, the composition of which is variable (variations in the concentration of O_3, CO_2, H_2O affect the amount of radiation absorbed by the atmosphere; condensation of water produces latent heat) it will be regarded as a perfect gas obeying the law

$$p = R\rho T$$

Now

$$dQ = C_v dT + p d\left(\frac{1}{\rho}\right)$$

$$\therefore \quad \frac{dQ}{T} = C_v\left(\frac{dp}{p} - \frac{d\rho}{\rho}\right) - \frac{R}{\rho}d\rho$$

For adiabatic changes $dQ = 0$

$$\therefore \quad p = k\rho^{\gamma} \qquad (k \text{ is some constant}),$$

where

$$\gamma = \frac{C_v + R}{C_v}$$

Hence

$$T \propto p^{\gamma - 1/\gamma}$$

For air $\gamma \sim 1.4 > 1$ and

an increase in p is accompanied by an increase in T.

Define the potential temperature θ to be the temperature of a gas when compressed adiabatically to a standard pressure p_0 (usually taken to be 1 atmosphere = 1012·4 mb

$$= 1\cdot0124 \times 10^6 \, \frac{\text{dynes}}{\text{cm}^2} \;)$$

θ is related to T and p by the relation

$$T = \theta\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

The entropy

$$S = \int \frac{dQ}{T}$$

$$= \int (C_v + R)\frac{d\theta}{\theta}$$

$$= (C_v + R) \log \theta$$

so that θ is a measure of S .

<u>Isothermal Atmospheres</u> are such that

$$T = \text{const.}$$

Consider a parcel of gas in hydrostatic equilibrium under the forces shown

$$\frac{dp}{dz} = -\rho g$$

$$= -g\,p/RT$$

$$\therefore p = p_0 e^{-gz/RT}$$

$$\rho = \rho_0 e^{-gz/RT}$$

The scale height $H_s \left(\equiv \frac{RT}{g}\right)$ is the vertical distance in which the density (of an isothermal atmosphere) falls off by a factor e .
This is usually of the order of 10 km.

<u>Adiabatic Atmospheres</u> are such that

$$\theta = \text{constant}$$

Again $$\frac{dp}{dz} = -g\rho$$

Also $$\frac{1}{T}\frac{dT}{dz} = \frac{1}{\theta}\frac{d\theta}{dz} + \frac{\gamma-1}{\gamma}\frac{1}{p}\frac{dp}{dz}$$

$$= 0 - \frac{\gamma-1}{\gamma}\frac{g}{RT}$$

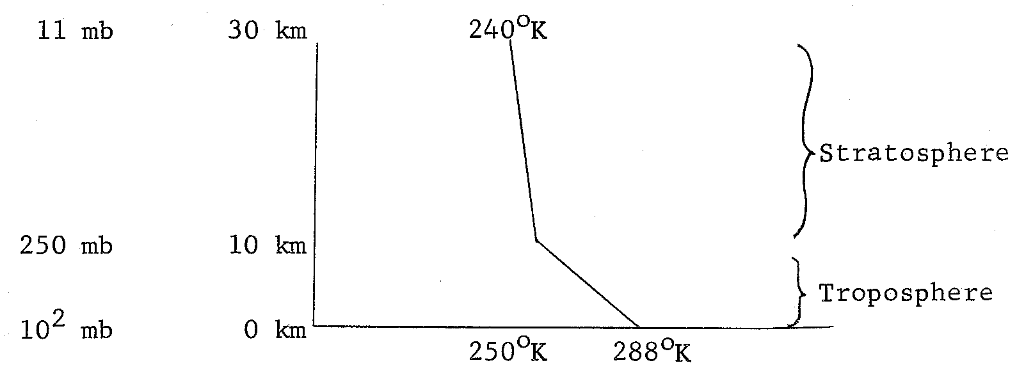
$$\therefore \frac{dT}{dz} = -\frac{\gamma-1}{\gamma}\frac{g}{R}$$

= lapse rate

$\sim 10^{\circ}C/km$ for the atmosphere,

i.e. temperature decreases linearly with height in an adiabatic atmosphere.

A typical mean temperature distribution in the atmosphere is the following:



99% of the mass of the atmosphere is in the tropo- and stratosphere. In the stratosphere the temperature is essentially constant with height, a very stable configuration (θ increasing with height).

The ocean is composed of water and salt (3% by weight or 34-37 $^o/oo$). The density is a function of pTS.

$$\rho = \rho\,(pTS)$$

Changes in ρ due to variations in p, are small . . 4% (1)

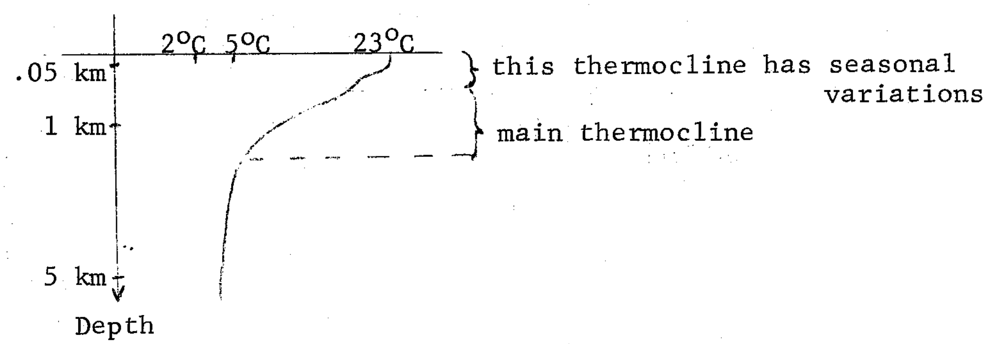
Changes in ρ due to variations in T, are smaller . .4%

Changes in ρ due to variations in S, are smaller . .2%

The potential density ρ^* is defined to be the density of a fluid when it is brought adiabatically to standard pressure (1 atmosphere).

The adiabatic temperature gradient of the ocean is approximately .1°C/km.

The normal temperature distribution of the ocean is like this:



Note that salinity decreases with depth and has a destabilizing effect.

The vertical stability of atmosphere and oceans.

Consider a parcel of air which is displaced from $z \to z + \zeta$ and let the change in density be $\rho(z) \longrightarrow \rho\,(z + \zeta)$. Let ρ' = the density at $z + \zeta$ had the displacement been an adiabatic one. Then

$$\rho' = \rho \left[\frac{p(z+\zeta)}{p(z)} \right]^{1/\gamma}$$

$$= \rho \left[1 + \frac{1}{\gamma p} \frac{dp}{dz} \zeta + \ldots \right]$$

$$\rho' - \rho(z+\zeta) = \rho(z) \left\{ \frac{1}{\gamma p} \frac{dp}{dz} - \frac{1}{\rho} \frac{d\rho}{dz} \right\} \zeta + O(\zeta^2)$$

$$\therefore \frac{\rho' - \rho(z+\zeta)}{\rho(z)} = \frac{1}{\theta} \frac{d\theta}{dz} \zeta$$

The buoyancy force per unit mass is $g(\rho' - \rho(z+\zeta))$

Hence

$$\frac{g}{\rho(z)} (\rho' - \rho(z+\zeta)) = -\ddot{\zeta}$$

and the resultant motion is simple harmonic with frequency

$$N = \sqrt{\frac{g}{\theta} \frac{d\theta}{dz}} \quad .$$

N is the Brunt-Väisälä frequency.

A similar calculation for the ocean yields

$$N = \left[- \frac{g}{\rho^*} \cdot \frac{d\rho^*}{dz} \right]^{1/2}$$

where ρ^* is the potential density, defined in a similar manner to the potential temperature so that the ocean is vertically stable provided ρ^* decreases upwards. The periods are of the order of 5 - 10 min. which is very small when compared with the periods of large-scale phenomena, e.g. rotation of the earth = 1 day.

These notes submitted by

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Lecture #II

Small Amplitude Waves in the Atmosphere

We investigate small amplitude waves in the atmosphere: Let the basic (unperturbed) state be characterised by $p_0(z), \rho_0(z)$, and zero velocity field.

$$\frac{1}{\rho_0}\frac{dp_0}{dz} + g = 0 \tag{1}$$

Consider a perturbation which is such as to cause motion in the x- z-directions only and assume

$$\underset{\sim}{u} \cdot \nabla << \frac{\partial}{\partial t} \tag{2}$$

in the operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{u} \cdot \nabla$

The relevant equations

$$\frac{D\underset{\sim}{u}}{Dt} + \frac{1}{\rho}\nabla p + g\underset{\sim}{k} = 0 \qquad \text{Momentum}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underset{\sim}{u} = 0 \qquad \text{Continuity}$$

$$\frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} = 0 \qquad \text{Eq. of state}$$

simplify to

$$u'_t + \frac{1}{\rho_0} p'_x = 0$$

$$w'_t + \frac{1}{\rho_0} p'_z + g\frac{\rho'}{\rho_0} = 0$$

$$\frac{1}{\rho_0}\rho'_t + \frac{1}{\rho_0}\frac{d\rho_0}{dz} w' + u'_x + w'_z = 0$$

$$\frac{1}{\gamma p_0} p'_t - \frac{1}{\rho_0}\rho'_t + w'\left(\frac{1}{\gamma p_0}\frac{dp_0}{dz} - \frac{1}{\rho_0}\frac{d\rho_0}{dz}\right) = 0$$

The last equation may be written $\frac{1}{\gamma p_0} p'_t - \frac{1}{\rho_0}\rho'_t + w'\frac{1}{\theta_0}\frac{d\theta_0}{dz} = 0.$

We seek solutions of the form

$$u'w'p'\rho' = \mathcal{R}e\left[\left\{\hat{u}(z)\,\hat{w}(z)\,\hat{p}(z)\,\hat{\rho}(z)\right\}e^{i(kx-\omega t)}\right]$$

Substitution yields

$$\hat{w}_z + \frac{1}{\gamma p_0}\frac{dp_0}{dz}\hat{w} - \left(\frac{i\omega}{\gamma} - RT\frac{ik^2}{\omega}\right)\frac{\hat{p}}{p_0} = 0$$

$$\frac{1}{p_0}\hat{p}_z + \frac{g}{\gamma RT}\frac{\hat{p}}{p_0} - \left(i\omega + \frac{g}{i\omega}\frac{1}{\theta_0}\frac{d\theta_0}{dz}\right)\frac{\hat{w}}{RT} = 0$$

Put

$$\hat{p} = p_0^{\frac{1}{2}}\underline{p}(z)$$

$$\hat{w} = \frac{g}{\omega}p_0^{-\frac{1}{2}}W(z)$$

Note that $c^2 = \gamma RT_0$ (speed velocity of sound)^2

$$N^2 = \frac{g}{\theta_0}\frac{d\theta_0}{dz} \quad \text{(Brunt-Väisälä frequency)}^2$$

$$H_s(z) = \frac{c^2}{\gamma g} \quad \text{scale height}$$

Then

$$\left.\begin{aligned} &\underline{P}_z + \frac{1}{H_s}\left(\frac{1}{\gamma} - \frac{1}{2}\right)\underline{P} - \frac{i}{H_s}\left(1 - \frac{N^2}{\omega^2}\right)W = 0 \\ &W_z - \frac{1}{H_s}\left(\frac{1}{\gamma} - \frac{1}{2}\right)W + iH_s\left(k^2 - \frac{\omega^2}{c^2}\right)\underline{P} = 0 \end{aligned}\right\} \quad (a)$$

<u>Isothermal atmosphere</u>:

For an isothermal atmosphere c^2, N^2, H_s are all independent of z. Consider the solution which has

$$W = 0 \quad \text{everywhere.}$$

For this case $\underline{P} = P_0 e^{-\left(\frac{1}{\gamma}-\frac{1}{2}\right) z/H_s}$

and $\omega^2 = k^2 c^2$ (a)

The magnitude of the horizontal velocity fluctuation

$$
\begin{aligned}
|\hat{u}| &= \frac{k}{\omega} \frac{|\hat{p}|}{\rho_0} \\
&= \frac{k}{\omega} \frac{\rho_0^{1/2}}{\rho_0} |\underline{P}| \\
&\rightarrow \infty \text{ as } z \rightarrow \infty
\end{aligned}
$$

However, the kinetic energy per unit volume

$$\rho |\hat{u}|^2 \rightarrow 0 \text{ as } z \rightarrow \infty$$

i.e. the energy density decreases upwards and the total energy (from $z = 0$ to ∞) is bounded. This is a Lamb wave, it is essentially a sound wave, confined to a region near the ground with the help of gravity. The level surface at $z = 0$ supports pressure fluctuations and plays an essential role in the wave mode.

The isothermal atmosphere permits another type of motion.

Assume $\underline{P}, W \propto e^{imz}$

Then the kinetic energy density $\sim \rho_0 \left[|\hat{u}|^2 + |\hat{w}|^2 \right]$

$\sim |\underline{P}|^2$ and is independent of z.

Corresponding to (a) we have

$$\left(\frac{\omega^2}{c^2} - k^2\right)\left(1 - \frac{N^2}{\omega^2}\right) = m^2 + \frac{1}{H_s^2}\left(\frac{2-\gamma}{2\gamma}\right)^2$$

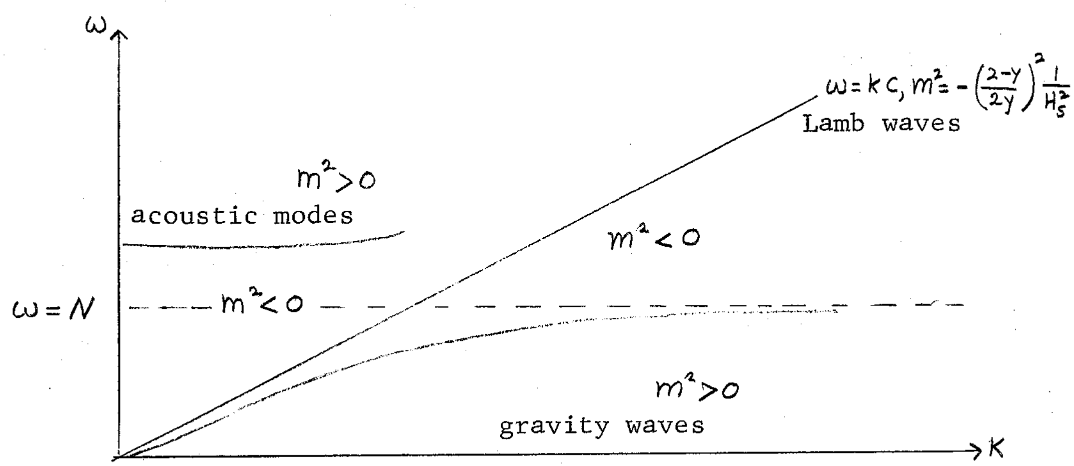
(i) if $\omega^2 > N^2$ then in fact $\omega^2 > (K^2 + m^2)C^2$ and the wave speed $>$ the speed of sound.

If in addition $(K^2+m^2)H_s^2 >> 1$ i.e. the wavelength $<<$ the scale height

$$\frac{N^2}{(K^2+m^2)C^2} << \frac{1}{(K^2+m^2)H_s^2} << 1$$

and we have acoustic waves.

(ii) If $\omega^2 < N^2$ and $(K^2+m^2)H_s^2 >> 1$ then $W^2 \sim N^2 \dfrac{K^2}{K^2+m^2}$ and the waves are internal gravity waves.



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Lecture # III

Boussinesq Approximation

Consider a motion with typical velocity, length, time, V, L, T, respectively. If

$$\frac{VT}{L} \gg 1 \quad \text{then} \quad \frac{D}{Dt} = \underline{u}\cdot\nabla \qquad \text{and we have steady motion}$$

$$\frac{VT}{L} \sim 1 \quad \text{then} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla \qquad \text{and we have non-linear motion}$$

$$\frac{VT}{L} \ll 1 \quad \text{then} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} \qquad \text{and we have linear motion.}$$

In a compressible atmosphere the significant parameters are C^2, H_s, N^2 and γ.

Denote typical values of these quantities by C^*, H_s^*, N^* and form the dimensionless parameters

$$\frac{V^2}{C^{*2}} \text{ or } \frac{L^2}{C^{*2}T^2}; \quad \frac{L}{H_s^*}; \quad \frac{1}{N_0^{*2}T^2} \text{ or } \frac{V^2}{N_0^{*2}L^2}$$

In the linear model with which we have so far been dealing, the corresponding dimensionless parameters are

$$\frac{\omega^2}{(k^2+m^2)C^{*2}}; \quad \frac{1}{\sqrt{k^2+m^2}\,H_s}; \quad \frac{\omega^2}{N^{*2}}$$

We obtained pure internal gravity waves when the first two parameters were taken to be much less than 1, while the third one was approximately equal to 1. We now assume that

1. $\frac{V^2}{C^{*2}}$ or $\frac{L^2}{C^{*2}T^2} \ll 1$
2. $L/H \ll 1$
3. $\frac{1}{N_0^{*2}T^2}$ or $\frac{V^2}{N_0^2L^2} \sim 1$

(1) $\Longrightarrow$ that the medium is essentially incompressible so that

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} << \nabla \cdot \underline{u}$$

(2) $\Longrightarrow$ $\frac{\Delta \rho_0}{\rho_0} << 1$ ($\Delta \rho$ = change of ρ_0 over the height scale of the motion)

whence $W \frac{1}{\rho_0}\frac{d\rho_0}{dz} << \frac{\partial W}{\partial z}$

These approximations enable us to write the continuity equation

$$u_x + W_z = 0 \tag{1}$$

In the momentum equation set

$$p = p_0 + p' \quad - \quad p'/p_0 << 1$$

$$\rho = \rho_0 + \rho' \quad - \quad \rho'/\rho_0 << 1$$

Then, as we have seen in Lecture II

$$\frac{DW}{Dt} + \frac{1}{\rho_0}\frac{dp'}{dz} + g\frac{\rho'}{\rho_0} = 0 \tag{2}$$

i.e. we neglect the ratio $\frac{\rho'}{\rho_0}$ except where it is coupled with the gravity term.

Similarly

$$\frac{1}{\gamma p}\frac{DP}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

becomes

$$\frac{1}{\gamma P_0}\frac{DP'}{Dt} - \frac{1}{\rho_0}\frac{D\rho'}{Dt} + \frac{1}{\theta_0}\frac{d\theta_0}{dz}W = 0$$

and since

$$\frac{\partial p'}{\partial z} \sim g\rho'$$

$$\frac{p'}{p_0} \sim g\rho' L/c_0^2 \rho_0 \sim \frac{L}{H_s} \frac{\rho'}{\rho_0}$$

we ignore $\frac{1}{\gamma p_0} \frac{Dp'}{Dt}$, getting

$$-\frac{1}{\rho_0} \frac{D\rho'}{Dt} + \frac{1}{\theta_0} \frac{d\theta_0}{dz} W = 0 . \tag{3}$$

Equations (1), (2) and (3) are the equations for a Boussinesq liquid.

We now justify these approximations by doing a scale analysis:

Let

$$\underset{\sim}{r} = z^* \hat{\underline{k}} + \varepsilon H^* \underset{\sim}{r'} \qquad \text{where } z^* \underset{\sim}{k} \text{ is a mean height} \tag{4}$$

$$\underline{r}' \text{ is dimensionless}$$

$$t = t'/N^* \tag{5}$$

$$\underline{u} = \varepsilon N^* H^* \underline{u}' \tag{6}$$

$$\rho = \rho_0(z) + \varepsilon N^{*2} H^* \rho^* \frac{1}{g} \rho'(\underline{r}', t') \tag{7}$$

$$p = p_0(z) + \varepsilon^2 \rho^* N^{*2} H^{*2} p'(\underline{r}', t') \tag{8}$$

where

$$\rho_0(z) = \rho^* + \varepsilon \frac{d\rho_0^*}{dz} H^* z' + O(\varepsilon^2)$$

$$p_0(z) = p^* + \varepsilon \frac{dp_0^*}{dz} H^* z' + \varepsilon^2 \frac{1}{2} \frac{d^2 p_0^*}{dz^2} H^* z'^2 + O(\varepsilon^3)$$

Define the basic state to satisfy the hydrostatic equation at all heights, so that

$$\frac{dp_0^*}{dz} = -g\rho_0^* , \quad \frac{d^2 p_0^*}{dz^2} = -g \frac{d\rho_0^*}{dz} ,$$

Substitute (4) ... (8) in the equations

$$\rho \frac{D\underline{u}}{Dt} + \nabla p + g\rho \underline{k} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \underline{u} = 0$$

$$\frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D}{Dt} = 0$$

Then

$$\varepsilon N^{*2} H^{*}\{\rho^{*} + O(\varepsilon)\} \frac{D\underline{u}'}{Dt} +$$

$$+ \frac{1}{\varepsilon H^{*}} \nabla' \{ p^{*} + \varepsilon \frac{dp^{*}}{dz} H^{*} z' + \varepsilon^{2} \frac{1}{2} \frac{d^{2}p^{*}}{dz^{2}} H^{*2} z'^{2} + \varepsilon^{2} \rho^{*} N^{*2} H^{*2} p' + O(\varepsilon^{3}) \} +$$

$$+ g\{\rho^{*} + \varepsilon \frac{d\rho_0}{dz} H^{*} z' + \varepsilon N^{*2} H^{*} \rho^{*} \frac{1}{g} \rho' + O(\varepsilon^{2})\} \underline{k} = 0, \text{ etc.}$$

Equating coefficients of powers of ε ,

Zero order: $\frac{dp_0^{*}}{dz} + g\rho^{*} = 0, \quad \nabla' \cdot \underline{u}' = 0.$

1st order: $\frac{D\underline{u}'}{Dt'} + \nabla p' + \rho' \underline{k} = 0, \quad \frac{D\rho'}{Dt'} - N'^{2} W = 0.$

The last three equations are known as the Boussinesq equations for an inviscid liquid.

These notes submitted by

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Lecture IIIb.

We take the Boussinesq equations derived above, in the form

$$\frac{Du}{Dt} + \frac{1}{\rho}\nabla p + \sigma \underset{\sim}{k} = 0 \qquad (1)$$

$$\nabla \cdot \underset{\sim}{u} = 0 \qquad (2)$$

$$\frac{D\sigma}{Dt} = N^2 \cdot (\underset{\sim}{u}\,\underset{\sim}{k}) \qquad (3)$$

where σ is the buoyancy force per unit mass: $\sigma = g\frac{\rho'}{\rho_0}$

and where $N^2 = \frac{g}{\theta_0} \cdot \frac{d\theta_0}{dz}$.

We linearize them, using an x, z-plane with $N^2 = N^2(z)$. Then

$$u_t + \frac{1}{\rho} p_x = 0 \qquad (4)$$

$$w_t + \frac{1}{\rho} p_z + \sigma = 0 \qquad (5)$$

$$u_x + w_z = 0 \qquad (6)$$

$$\sigma_t - N^2 w = 0 \qquad (7)$$

By differentiating (4) with respect to z , (5) with respect to x , we get after subtracting the resulting equations

$$\frac{\partial}{\partial t}\left(u_z - w_x\right) - \sigma_x = 0 \qquad (8)$$

We note here, that σ influences the change of the horizontal component of the vorticity only.

By differentiating (8) with respect to t and (7) with respect to x and adding we get an equation between u and w . This we differentiate with respect to x and substitute u_{zx} from (6) which was differentiated with respect to z before. As a result there is an equation for w :

$$\frac{\partial^2}{\partial t^2}\left(W_{xx}-W_{zz}\right)+N^2(z)\,W_{xx}=0 \tag{9}$$

This equation can be treated further when we make certain assumptions. An easy case is when

$$\underline{N^2(z) = \text{const.}}$$

We assume $\underset{\sim}{u} = \Re\left(\hat{\underset{\sim}{u}}\, e^{i(kx+mz-\omega t)}\right)$ and get

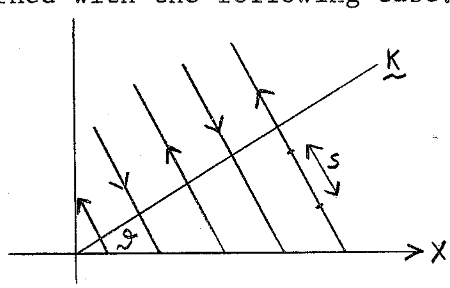
$$\omega^2 = N^2\,\frac{k^2}{k^2+m^2} \tag{10}$$

as we obtained when studying gravity waves in a compressible atmosphere.

From the continuity equation (6) we see that

$$i\,\underset{\sim}{k}\cdot\hat{\underset{\sim}{u}} = 0 \quad \text{i.e. } \hat{u}\perp\underset{\sim}{k} \tag{11}$$

we have transversal motions. From (10) we see that always $|\omega| \le |N|$. If $m=0$, then $|\omega| = |N|$, the wave number is horizontal, the motion vertical. If, on the other hand, $m \neq 0$, we are concerned with the following case:



Along the lines of the motion, there is no pressure gradient.

We have $\omega = N\cos\vartheta$.

If a parcel is moved a distance s, the buoyancy force, which only works in the vertical, is proportional to $s\cos\vartheta$. The oscillation of the parcel is due to the component along the wave front which is $\sim s \cos^2 \vartheta$ so that the parcel's frequency is $\sim \sqrt{\cos^2 \vartheta}$. Hence, in the case $\vartheta \rightarrow \frac{\pi}{2}$ we get, as it is to be expected, fully horizontal motion:

$$k^2 << m^2 \quad \omega \sim \frac{NK}{m} \rightarrow 0 .$$

But this case is the case of hydrostatic balance. We have from (6)

$$\left| \frac{\hat{W}}{\hat{U}} \right| = \frac{K}{m} \rightarrow 0 \text{ and } \frac{W_t}{\frac{1}{\rho} P_z} \sim \frac{K^2}{m^2} \rightarrow 0 \text{ from (5)}$$

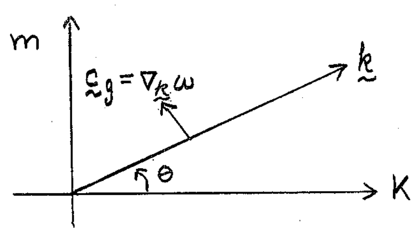
So W_t can be neglected, and (5) goes over into

$$\frac{1}{\rho} P_z + \sigma = 0, \text{ q.e.d.} \qquad (12)$$

If $\hat{u}$ varies slowly with z, x and t, we might consider also a group velocity. We have now $\underset{\sim}{u} = \mathcal{R}\left(\hat{u}(x,z,t)\, e^{i(kx+mz-\omega t)}\right)$.

The definition of group velocity $\underset{\sim}{c}_g$ is $\underset{\sim}{c}_g = \left(\frac{\partial \omega}{\partial K}, \frac{\partial \omega}{\partial m}\right) = \nabla_{\underline{K}}\, \omega$, while the phase velocity $\underset{\sim}{c}_p$ is $\underset{\sim}{c}_p = \frac{\omega}{|\underset{\sim}{k}|^2} \cdot \underset{\sim}{k}$.

Because ω depends on K/m but not on $(K^2 + m^2)$, $\underset{\sim}{c}_g$ is normal to $\underset{\sim}{K}$, as is seen in the following picture:



The magnitude of $\underset{\sim}{c}_g$ is $|\underset{\sim}{c}_g| = \frac{N}{|\underset{\sim}{k}|} \cos \vartheta$. Hence, when

$$\vartheta \rightarrow \frac{\pi}{2} \text{ then } \omega \rightarrow 0 \text{ so } \underset{\sim}{c}_p \rightarrow 0 \text{ but } \underset{\sim}{c}_g \rightarrow \frac{N}{m}$$

$$\vartheta \rightarrow 0 \qquad \omega \rightarrow N \quad \underset{\sim}{c}_p \rightarrow \frac{N}{K} \quad \underset{\sim}{c}_g \rightarrow 0$$

$\underset{\sim}{c}_g \rightarrow 0$ means that, when there are at any place slow variations of amplitude, they will stay there for all time.

$N^2(z) \neq$ const.

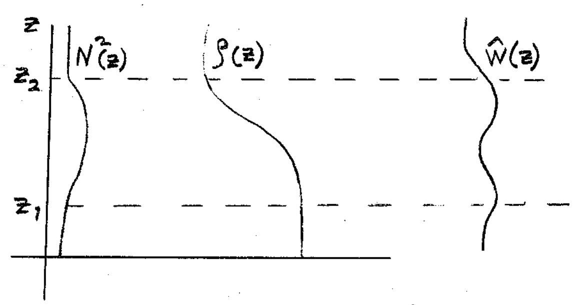
We assume $W = \Re\left(\hat{w}(z)e^{i(kx-\omega t)}\right)$ and get from (9)

$$\frac{d^2\hat{w}}{dz^2} + k^2\left(\frac{N^2(z)}{\omega^2} - 1\right)\hat{w} = 0 \tag{13}$$

If there are two boundaries at which $\hat{w}$ has to vanish, we get - for a given k - a set of eigenvalues $\omega_1, \omega_2, \omega_3 \dots$, all of which are real and smaller in absolute value than the previous ones. If we think of N^2 being constant for the moment, we have $\hat{w} = A \sin mz$ with $mH = n\pi$, where H is the distance between the boundaries and n is an integer. For ω we have $\omega^2 = \frac{k^2}{k^2+m^2} N^2$, which looks like equation (10), but here holds only for special m, namely

$$\omega = \frac{kN}{\sqrt{k^2 + n^2\frac{\pi^2}{H^2}}} \tag{14}$$

The horizontal component of $\underset{\sim}{c}_g$ becomes $\frac{d\omega}{dk} = \frac{N}{\sqrt{k^2+n^2\frac{\pi^2}{H^2}}}\cos^2\vartheta$. It is easy to see what happens with N^2 varying in the following way:



If $\omega^2 > N^2$ we obtain waves of exponential type from (13), while $\omega^2 < N^2$ gives us waves of sinusoidal shape in z.

In order to combine both types and get a smooth curve for

$\hat{W}(z)$ decaying as $z \to \pm\infty$, a restriction on ω is

$$\omega^2 < N^2_{max} \text{ and } \omega^2 > N^2_{min} \text{ at } |z| = \infty,$$

but, for given k, there is an infinite sequence of discrete values within this range, which may be ordered by the number of nodes in $\hat{W}(z)$.

These notes submitted by

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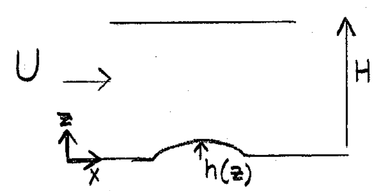
Lecture #IV

Lee Waves behind Mountains

This is a very common phenomenon, often made visible by cloud formations. References containing photographs of such formations are given at the end of this chapter.



We consider linearized, steady, two-dimensional flow with $\frac{h(z)}{H} \ll 1$. Take U independent of z. Then $\frac{D}{Dt} \cong U\frac{\partial}{\partial x}$.

The equation is: $U^2 \frac{\partial^2}{\partial x^2}\left(w_{xx} + w_{zz}\right) + N^2 w_{xx} = 0.$

The boundary conditions are: (i) $w = 0$ at $z = H$

(ii) $w = U\frac{dh}{dx}$ at $z = 0$

(iii) $w = 0$ as $x \to -\infty$

We make no assumption for w as $x \to \infty$.

The waves can be argued to appear downstream in the following way. The phase speed $C_p = -U$ (relative to the air) for stationary waves. $C_g < C_p$ for this geometry, as previously demonstrated. Thus the energy is travelling relative to the air with a velocity less than U so it is moving downstream away from the mountain.

Take $w(x,z) = \int_{-\infty}^{\infty} \hat{w}(k,z) e^{ikx} dk$.

As w does not tend to 0 as $x \to \infty$ we expect singularities in $\hat{w}$.

The boundary conditions are:

(i) $\hat{w} = 0$ at $z = H$

(ii) $\hat{w} = ikU\hat{h}(k)$ on $z = 0$

The equation becomes $\hat{w}_{zz} + \left(\frac{N^2}{U^2} - k^2\right)\hat{w} = 0$

The solution is $\hat{w} = -\frac{ikU\hat{h}(k)}{\sin(\sqrt{k_0^2 - k^2}\,H)} \sin\left(\sqrt{k_0^2 - k^2}\,[z-H]\right)$ where $k_0 = \frac{N}{U}$.

Note that for $k \to k_0^+$, or $k \to k_0^-$, $\hat{w}$ remains bounded.

The singularities are poles for $\sqrt{k_0^2 - k^2}\,H = n\pi$ $\quad n = 1, 2, 3$;

these occur for $k = k_n$ where $k_n^2 H^2 = k_0^2 H^2 - n^2\pi^2$. Thus the number of such poles occurring on the real axis is dependent on $k_0^2 H^2$. We will ignore decaying modes.

Near a pole we have $\hat{w} \cong \frac{R_n(n,z)}{k - k_n}$ + continuous function.

<u>The Riemann-Lebesque Lemma</u>: $\int_a^b f(k) e^{ikx} dk \to 0$ as

$|x| \to \infty$ if $\int_a^b |f(k)|\, dk$ exists.

We will use this lemma to study the lee waves at large distances from the mountain. The lee waves come from the neighbourhood of poles in the complex k-plane.

$$w \sim \sum_n R_n \int_{k_n-\varepsilon}^{k_n+\varepsilon} \frac{1}{k-k_n} e^{ikx}\, dk \qquad \text{as } |x| \to \infty .$$

In order to obtain a zero w for $x \leq 0$ we must close the integral in the following manner for all k_n

If $\hat{h}(k) = \hat{h}(-k)$, a symmetric mountain.

$$w(x,z) = -\sum \frac{4\pi^2 n U}{H^2} \hat{h}(|k_n|) \sin \frac{n\pi z}{H} \cos \sqrt{k_0^2 - \frac{n^2\pi^2}{H^2}}\, x$$

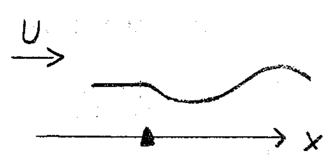
If $\hat{L}(k) = -\hat{h}(-k)$

$$w(x,z) = -\sum \frac{4\pi^2 n U}{H^2} \hat{h}(|k_n|) \sin \frac{n\pi z}{H} \sin \sqrt{k_0^2 - \frac{n^2\pi^2}{H^2}}\, x$$

<u>Note</u>: a) We have a finite number of wave trains. The number of these = integral part of $\frac{k_0 H}{\pi} = \frac{NH}{U\pi}$. b) For a point mountain $h(x) = A\,\delta(x)$. The limiting process assumed here is

$$\int h\, dx = A = \text{const}, \; \frac{h}{H} << 1, \text{and } h \to \infty .$$

Then $\hat{h} = \frac{A}{2\pi}$. The wave amplitude depends on the area, not on the shape. The vertical velocity is at a maximum directly above the mountain.



The wind velocity has its maximum in the lee of the mountain.

A physical argument for this: If the mountain has width of order L, the time for the air to pass over it is $\tau = \frac{L}{U}$. If $\tau N \ll 1$, the buoyancy forces will have a small effect and to a first approximation we have irrotational flow.

To the next approximation the buoyancy forces give a downward acceleration and the air has a resultant downward velocity after passing the mountain which can set up a long wave-length wave motion.

When $k_n L$ is either much greater than, or much less than unity, the amplitude of the lee wave will be small as it is dependent on $\hat{h}(k_n)$. For many mountain ranges $k_n L \sim 1$ for some n and this increases the importance of these waves.

References

1. Sawyer, J.S., 1960: Quart.J.Roy.Met.Soc. 86, 326.

2. Fritz, 1965: J.Atmos.Sci.

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Lecture #V

Lee Waves with Varying U

The linear analysis of standing gravity waves can be amended to include the effect of a wind speed varying with height. Suppose N^2 = const., and the incident wind speed is $U(z)$. Then the two-dimensional equations for the perturbation quantities u, w, p, σ are

$$U(z)\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} + \frac{\partial}{\partial x}\left(\frac{p}{\rho^*}\right) = 0$$

$$U(z)\frac{\partial w}{\partial x} + \sigma + \frac{\partial}{\partial z}\left(\frac{p}{\rho^*}\right) = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$U(z)\frac{\partial \sigma}{\partial x} - N^2 w = 0$$

in the Boussinesq approximation. The second term in the first equation represents the only new effect - the vertical convection of zeroth-order momentum by the perturbation flow. The perturbation quantities can be Fourier transformed in the x-direction as before, and the equation for the Fourier coefficient $\hat{w}(z;k)$ is

$$\hat{w}_{zz} + \left[\frac{N^2}{U(z)^2} - \frac{U_{zz}}{U(z)} - k^2\right]\hat{w} = 0$$

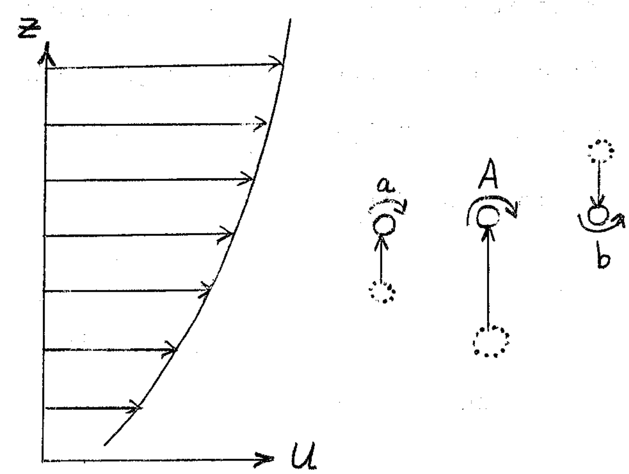
As long as $N^2 - UU_{zz}$ is positive, that equation is essentially the same as the one obtained in the last lecture. In fact $N^2 - UU_{zz}$ can be considered the square of an effective Brunt-Väisälä frequency. The $\hat{w}$ equation is usually written

$$\hat{w}_{zz} + \left[\ell^2(z) - k^2\right]\hat{w} = 0$$

where

$$\frac{N^2}{U^2} - \frac{U_{zz}}{U} = \ell^2(z) > 0$$

What is the physical meaning of the new term U_{zz}/U in the $\hat{w}$ equation? Suppose the vorticity U_z decreases upward, so U_{zz} is negative:



Suppose a parcel of fluid A is moved upward. It carries excess vorticity into its new location and induces a secondary motion. Parcels like a move upward on the left, and parcels on the right are swept downward. a carries vorticity positive with respect to its surroundings; b carries a negative vorticity. Both a and b tend to sweep A back to its original position, and the net effect is that the locus of positive vorticity tends to move upstream. The lee wave pattern, which must stand stationary with respect to the mountain exciting it, is thus helped along into the incident airstream by the displaced vorticity. If U_{zz} is positive the shear disturbance tends to propagate downstream against the gravity waves, and the effective Brunt-Väisälä frequency is driven up.

Non-Linear Theories for Mountain Flow

1. Long's Inverse Approach

By using streamline displacement as a dependent variable, Long derived an equation of motion both exact and linear for a very restricted class of flows. Consider a motion two-dimensional, steady, frictionless, and adiabatic governed by the Boussinesq equations. The continuity equation implies u and w can be found from a stream function ψ ,

$$u = \psi_z$$
$$w = -\psi_x$$

and steadiness means

$$\rho = \rho(\psi)$$

for a liquid, or

$$\theta = \theta(\psi)$$

for a gas. The vorticity equation is

$$(\underset{\sim}{u}\cdot\nabla)\nabla^2\psi = \frac{g}{\rho^*}\frac{\partial\rho}{\partial x}$$

where notation appropriate for a liquid is being used. The right-hand term can be written

$$\frac{g}{\rho^*}\frac{d\rho}{d\psi}\frac{\partial\psi}{\partial x} = -\frac{g}{\rho^*}\frac{d\rho}{d\psi}w = -\frac{g}{\rho^*}\frac{d\rho}{d\psi}(\underset{\sim}{u}\cdot\nabla)z$$

since ρ is a function of ψ only, and $(\underset{\sim}{u}\cdot\nabla)z = \underset{\sim}{u}\cdot\underset{\sim}{k} = w$ Now $\nabla\psi$ is perpendicular to the streamlines, so $(\underset{\sim}{u}\cdot\nabla)\psi = 0$. Likewise

$$(\underset{\sim}{u}\cdot\nabla)F(\psi) = \frac{dF}{d\psi}(\underset{\sim}{u}\cdot\nabla)\psi = 0$$

for any F . Hence the vorticity equation becomes

$$(\underset{\sim}{u}\cdot\nabla)\nabla^2\psi = (\underset{\sim}{u}\cdot\nabla)\left(\frac{-g}{\rho^*}\frac{d\rho}{d\psi}z\right)$$

That can be integrated along streamlines —

$$\nabla^2\psi + \frac{g}{\rho^*}\frac{d\rho}{d\psi}z = \text{const. along streamlines}$$
$$= F(\psi)$$

To get beyond this an important geometrical assumption must be made. Assume all streamlines extend to $x \to -\infty$ where they are unambiguously identified by their height z_1. Thus ψ must be a function monotonic on z_1, and we assume

$$\frac{d\psi}{dz_1} = U_1(z_1) > 0$$

where the subscript 1 means far upstream from the disturbing mountain. Then

$$\nabla^2\psi = \nabla\cdot\left(\frac{d\psi}{dz_1}\nabla z_1\right) = \frac{d\psi}{dz_1}\nabla^2 z_1 + \frac{d^2\psi}{dz_1^2}(\nabla z_1)^2$$

Since $\psi_{z_1} = U_1,\ \psi_{z_1 z_1} = U_{1\,z_1}$ and

$$-\frac{g}{\rho^*}\frac{d\rho}{d\psi} = -\frac{g}{\rho^*}\frac{d\rho}{dz_1}\frac{dz_1}{d\psi} = \frac{N_1^2}{U_1}$$

the equation of motion becomes

$$\nabla^2 z_1 - \frac{N_1^2}{U_1^2}z + \frac{1}{U_1}\frac{dU_1}{dz_1}(\nabla z_1)^2 = F(\psi) = \mathcal{F}(z_1)$$

As $x \to -\infty$, $z_1 \to z$, and $(\nabla z_1)^2 \to 1$, $\nabla^2 z_1 \to 0$. Thus, evaluating the left-hand side of the equation at $x \to -\infty$ to get $\mathcal{F}(z_1)$, we find

$$\nabla^2 z_1 - \frac{N_1^2}{U_1^2}(z - z_1) + \frac{1}{U_1}\frac{dU_1}{dz_1}\left[(\nabla z_1)^2 - 1\right] = 0$$

Let the displacement of a streamline from its original height be

$$\zeta = z - z_1$$

Then $\nabla z_1 = \nabla z - \nabla \zeta = \underset{\sim}{k} - \nabla \zeta, \nabla^2 z_1 = -\nabla^2 \zeta$, and

$$\nabla^2 \zeta + \frac{N_1^2}{U_1^2} \zeta = \frac{1}{U_1} \frac{dU_1}{dz_1} \left[(\nabla \zeta)^2 - 2 \frac{\partial \zeta}{\partial z} \right]$$

The equation for ζ becomes simple and linear if N_1/U_1 is independent of z_1 and $\frac{1}{U_1} \frac{dU_1}{dz_1} = 0$. Thus for

$$U_1 = \text{const.}$$

$$N_1 = \text{const.}$$

the equation is

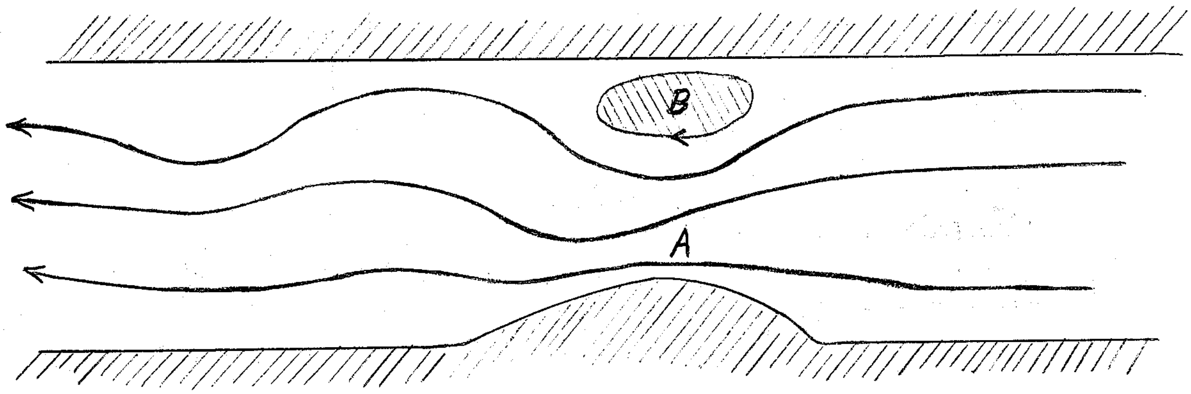
$$\nabla^2 \zeta + \frac{N_1^2}{U_1^2} \zeta = 0$$

exactly the same equation as that obtained for W_{xx} in the linear analysis. The boundary condition at the ground is still non-linear; since the lowest streamline must follow the mountain surface $h(x)$,

$$\zeta(z_1 = 0) = h(x)$$

But any solution to the linearized problem is a solution to the non-linear problem for <u>some</u> mountain shape, and Long simply tailors the mountain to the solution. He confirms his work with flow visualization experiments over models shaped to fit existing solutions to the ζ equation.

In practice, the assumption hardest to meet is that all streamlines emanate from $x \to -\infty$. As the amplitude of the mountain is increased, the ζ solution ceases to cover the entire x - z-plane, and closed streamlines appear:

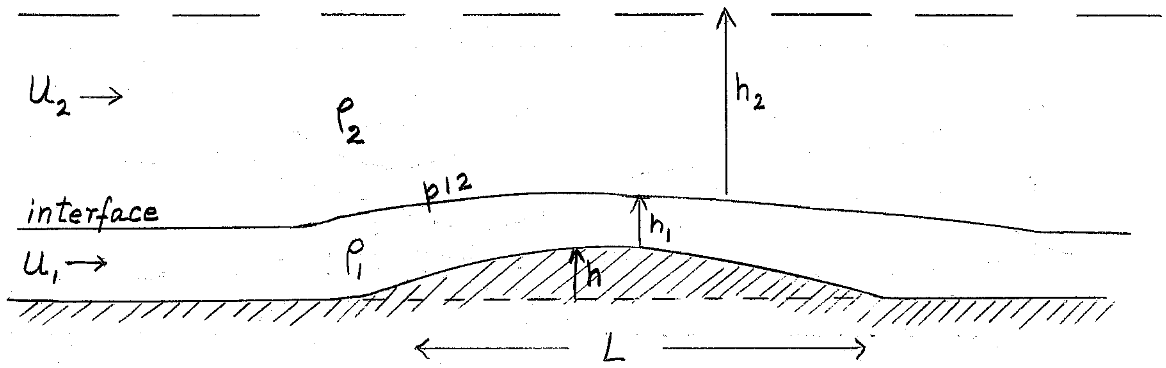


Physically, a rotor or dead air region like B is plausible. Fluid around region A must be both raised and accelerated over the mountain - its kinetic and potential energy must increase. Hence energy for region A must come from some other part of the flow, so the fluid around B both stagnates and drops down to meet the requirement.

Though Long's equation gives such solutions, they undermine the assumptions on which the equation is based. As the dead air regions become more prominent, Long's solutions diverge from the experimental flows. As the mountain height is increased further, a stagnation region of fluid which never gets over extends upstream. That phenomenon of blocking completely escapes Long's formulation, and a radically different approach is required to explain it.

2. The Hydraulic Model

If an analogy between the flow of an atmosphere over a mountain and free-surface channel flow can be constructed, the concepts of hydraulic engineering will be at our disposal. To show how this can be done without too much mathematics, we shall consider the simplest two-layer model. The physical situation is illustrated below:



Assume
$$\frac{h_1}{L} << 1$$
$$\frac{h_1}{h_2} << 1$$

and ρ_1 , ρ_2 are constant in their layers. Since the density is constant the vorticity in the layers can remain zero (remember the Boussinesq vorticity equation $\omega_t = \sigma_x$), and we can consider the case $u_1 = u_1(x)$. Bernoulli's equation just below the interface is

$$\frac{p_{12}}{\rho_1} + \frac{u_1^2}{2} + g(h+h_1) = \text{const.}$$

Just above,

$$\frac{p_{12}}{\rho_2} + g(h+h_1) \approx \text{const.}$$

since fractional changes in upper layer are very small for $h_1/h_2 << 1$. The continuity equation for the lower layer is

$$u_1 h_1 = \text{const.} = Q$$

and the three equations combine to give

$$\frac{1}{2}\frac{Q^2}{h_1^2(x)} + g' h_1(x) = B - g' h(x)$$

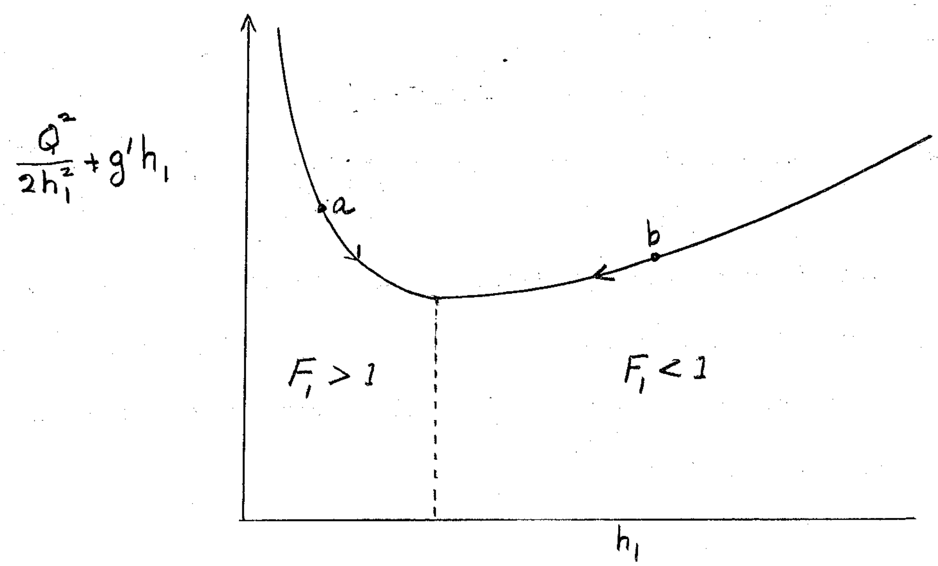
with
$$g' = g\frac{\rho_1 - \rho_2}{\rho_1}$$

and $\mathcal{B}$ a constant. That equation gives $h_1(h)$ implicitly.

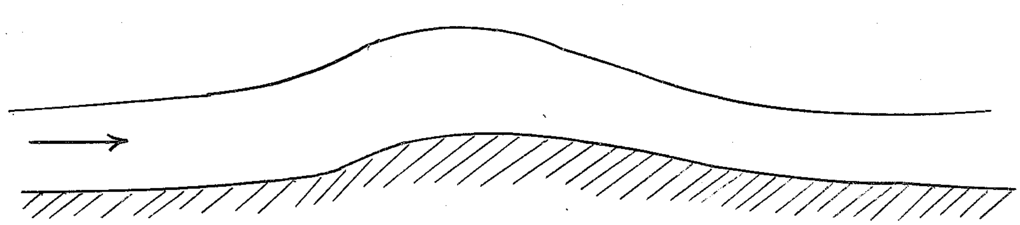
The character of the flow depends critically on the upstream value of the Froude number

$$F_1 = \frac{Q^2}{g' h_1^3}$$

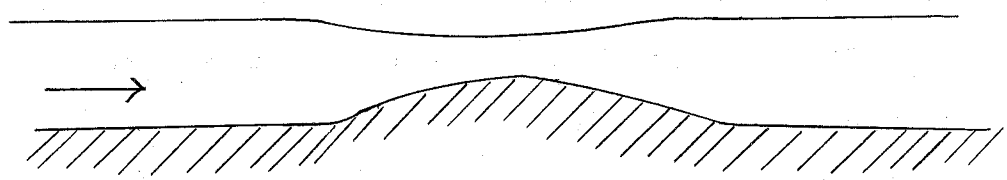
Let us draw a diagram of the left-hand side of the equation for h_1 :



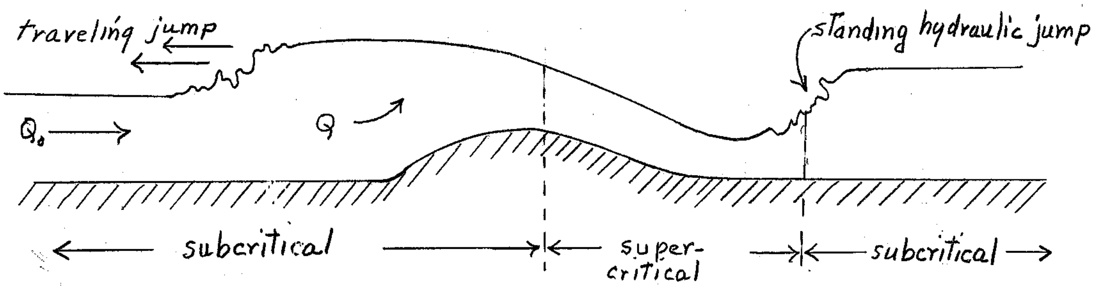
The curve has a minimum at $F_1 = 1$. If the flow far upstream has $F_1 > 1$ (situation a , a super-critical flow), then as h increases and we move down the curve, h_1 increases as well. Thus the interface streamline bows upward:



If the flow begins subcritically at b the interface drops over the mountain:



But if h is increased beyond the point where the maximum Froude number has risen to 1, the flow changes completely, and blocking begins:



The hydraulic model thus indicates what may happen when it is no longer energetically possible to move all the fluid over the mountain. Whether or not the predicted hydraulic jumps are observed is a matter of current controversy.

References

Scorer, 1949, Q.J.Roy.Meteor.Soc. 79, 41 (for lee waves with varying U)

Long, R. R., Tellus, 7 (for non-linear theory).

These notes submitted by

Steven Crow

Lecture #VI

Dynamics of a Rotating Fluid

Three questions should be considered in the future:

1. What exactly do we mean by energy density and radiation stress?

If one solves the time dependent lee wave problem, a perturbation to the initial velocity will propagate upstream. If the mode is leaky, then this upstream wave will radiate energy and the associated drag is not understood.

Understanding this upstream radiation may shed light on what happens if the velocity profile reverses sign at some height z_1 .

2. Neglecting the blocking effect and considering $U(z)$ and N^2 constant, what happens to Long's solution if you put on a small time dependent perturbation? What would happen to any of the theories considered? In fact, what is the connection between all three?

3. What happens if one considers three-dimensional flow around mountains? A suitable model to try would be a uniform ridge with a sine height perturbation.

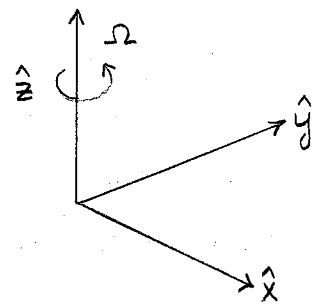
<u>Dynamics of a Rotating Fluid</u>

ref: Squire, <u>Surveys in Mechanics</u>; Eliasser and Kleinschmidt, Handbuch der Physik, <u>42</u>.

1. <u>Equations</u>:

Consider an inviscid incompressible fluid: the continuity equation becomes: $\nabla \cdot \vec{u} = 0$.

In the following rotating coordinate system:



$\hat{z}$ = unit vector in ~~z~~ direction

where Ω is assumed to be constant, the Navier-Stokes equation for the motion observed in the rotating frame is:

$$\frac{D\vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} + \nabla\left[\frac{P}{\rho} + \frac{1}{2}\Omega^2(x^2+y^2)\right] = 0$$

where

$$2\vec{\Omega} \times \vec{u} = \text{coriolis term}$$

$$\frac{1}{2}\Omega^2(x^2+y^2) = \text{centrifugal term}$$

If the pressure is not explicitly involved in the boundary conditions we can define a new effective pressure

$$P' = P + \frac{1}{2}\rho\Omega^2(x^2+y^2)$$

Let us scale the problem in terms of:

U - velocity scale

L - length scale

T - time scale

then consider the relative sizes of the first two terms in the N-S equation:

$$\frac{D\vec{U}}{Dt} \Big/ \vec{\Omega} \times \vec{U} = \frac{L}{T^2} \times \frac{T}{\Omega \cdot L}$$
$$= \frac{1}{\Omega T} = \frac{U}{\Omega L}$$
$$= \text{Rossby } \#$$

If the Rossby number is

large - rotation is a small perturbation on the motion.

small - rotation is dominant.

2. Two-dimensional flow on the x, y plane

Two-dimensional flow implies that $\partial/\partial z = 0$, and since the flow is inviscid and incompressible we may write the x and y components of $\vec{u}$ as

$$u = \psi_y$$
$$v = -\psi_x$$

where

ψ = stream function, independent of z .

Plugging these components of $\vec{u}$ into the coriolis term,

$$2\vec{\Omega} \times \vec{u} = 2\Omega \left[\psi_x \hat{x} + \psi_y \hat{y} + 0 \right]$$
$$= \nabla (2\Omega\psi)$$

Thus
$$\frac{D\vec{u}}{Dt} + \nabla \left(\frac{p'}{\rho} + 2\Omega\psi \right) = 0$$

Therefore, in the two-dimensional case we can absorb the effect of rotation into an effective pressure term provided the pressure does not appear in the boundary conditions. Hence the dynamical effects of rotation disappear.

Note: this is true for all Rossby numbers.

3. Vorticity Equation

Vorticity in the rotating frame is defined as the curl of the velocity field: $\vec{\zeta} = \nabla \times \vec{u}$

the conservation of vorticity is expressed by the equation:

$$\frac{D\vec{\zeta}}{Dt} = \left(\vec{\zeta} + 2\vec{\Omega}\right) \cdot \nabla \vec{u}$$

which says that the change of vorticity is due to stretching and twisting of the vortex lines of the <u>total</u> vorticity, $\vec{\zeta}+2\vec{\Omega}$, by the velocity field $\vec{u}$. Writing the two terms of the r.h.s. of the vorticity equation as two separate terms and remembering that we have defined $\vec{\Omega}$ as constant in the $\hat{z}$ direction, we have

$$\frac{D\vec{\zeta}}{Dt} = \left(\vec{\zeta} \cdot \nabla\right)\vec{u} + 2\Omega \frac{\partial}{\partial z} \vec{u}$$

From the definition of vorticity we see immediately that:

$$\vec{\zeta} \sim U/L$$

Therefore comparing the size of the two terms on the r.h.s. of the vorticity equation we have:

$$\frac{(\vec{\zeta} \cdot \nabla)\vec{u}}{2\Omega \frac{\partial u}{\partial z}} = \frac{\frac{U}{L} \cdot \frac{1}{L} \cdot U}{\Omega \cdot U/L} = \frac{U}{\Omega L}$$

which is the Rossby number again. Thus if the Rossby number $<< 1$, the stretching and twisting of the basic frame dominates the picture.

Note: For two-dimensional flow $\partial/\partial z = 0$, and, since all fluid motions are in x , y planes, $\vec{\zeta}$ is parallel to z . Therefore:

$$\vec{\zeta} \cdot \nabla = \zeta \frac{\partial}{\partial z} \rightarrow 0$$

hence

$$\frac{D\vec{\zeta}}{Dt} = 0$$

4. Geostrophic Balance

Considering small Rossby numbers, the N-S equation becomes:

$$2\vec{\Omega} \times \vec{u} + \nabla\left(\frac{p'}{\rho}\right)^{*} = 0$$

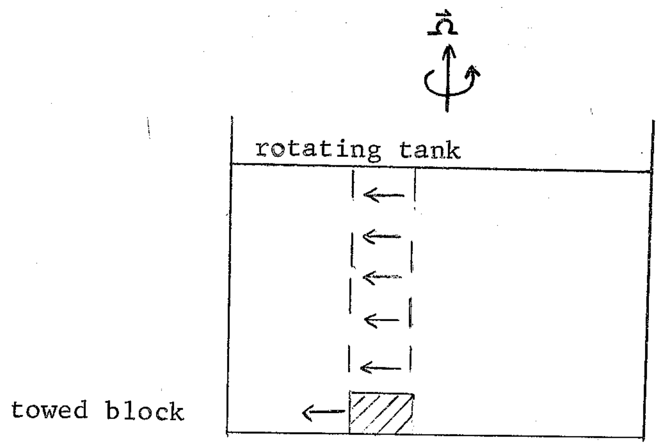
Take curl of this equation:

$$\nabla \times (2\vec{\Omega} \times \vec{u}) = 0$$

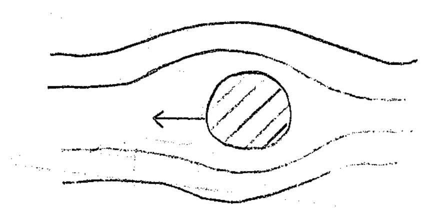
which implies that $$\frac{\partial \vec{u}}{\partial z} = 0$$

Thus the flow is independent of z and becomes two-dimensional.

G.I.Taylor did an experiment which showed this property quite well.



The column of fluid directly above the towed block moved with the block. A top view would be:



A qualitative explanation might be that if the energy required for a vortex to shorten and hop over the obstacle is greater than is required to swirl around the side, then the fluid will take the lower energy path.

Taylor did not investigate what happens if viscosity is present and/or the tank is very tall.

5. Linearized plane waves in a rotating fluid

ref: Chandrasekhar, 1961, Hydrodynamic and Hydromagnetic Stability, p. 85-86.

Consider: $\frac{1}{T} >> \frac{U}{L}$

Then: $\vec{u}\cdot\nabla << \frac{\partial}{\partial t}$

and the equations are:

$$\frac{\partial \vec{u}}{\partial t} + 2\vec{\Omega}\times\vec{u} + \nabla\left(\frac{p}{\rho}\right) = 0$$

$$\nabla\cdot\vec{u} = 0$$

Look for solutions of the form:

$$\vec{u} = Re\left\{\hat{\underline{u}}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\}$$

$$\frac{p}{\rho} = Re\left\{\hat{p}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)}\right\}$$

plugging in we get:

(i) $\quad -i\omega\hat{\underline{u}} + 2\vec{\Omega}\times\hat{\underline{u}} + ik\hat{p} = 0$

(ii) $\quad \vec{k}\cdot\hat{\underline{u}} = 0$

now dot (i) with $\vec{k}$

$$ik^2 p + \vec{k}\cdot(2\vec{\Omega}\times\hat{\underline{u}}) = 0$$

and cross (i) with $\vec{k}$

(iii) $\quad -i\omega\vec{k}\times\hat{\underline{u}} + 2\vec{\Omega} - (\vec{k}\cdot 2\vec{\Omega})\hat{\underline{u}} = 0$

dot (iii) with $\hat{\underline{u}}$

$$(k\cdot 2\Omega)\hat{\underline{u}}\cdot\hat{\underline{u}} = 0$$

hence we must have:

$$\underline{\hat{u}}_r \cdot \underline{\hat{u}}_i = 0$$

$$\underline{\hat{u}}_r \cdot \underline{\hat{u}}_r - \underline{\hat{u}}_i \cdot \underline{\hat{u}}_i = 0$$

where $\underline{\hat{u}}_r$ = real part of $\underline{\hat{u}}$

$\underline{\hat{u}}_i$ = imaginary part of $\underline{\hat{u}}$

These conditions imply that $\underline{\hat{u}}_r \perp \underline{\hat{u}}_i$ and from the continuity equation that: $\vec{k} \perp \underline{\hat{u}}_r \perp \underline{\hat{u}}_i$

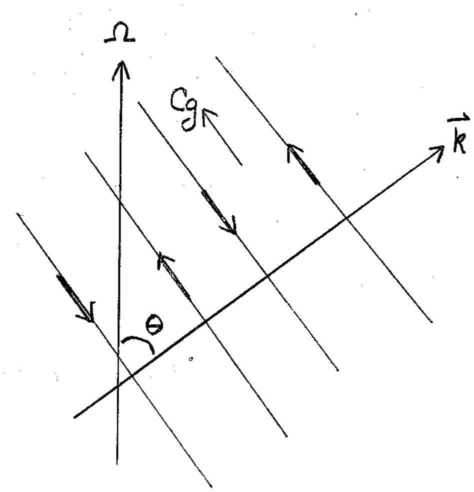
Now taking the real part of (iii),

$$\omega |k| = \pm \left(\vec{k} \cdot 2\vec{\Omega} \right)$$

Hence the dispersion relation is

$$\omega = \frac{\vec{k} \cdot 2\vec{\Omega}}{|k|}$$

ω depends only on the direction of $\vec{k}$ and not on its magnitude. This property is reminiscent of gravity waves and the reason is similar. Just as the Brunt-Väsälä frequency was the constant of the medium in the gravity wave case, so the rotation frequency is the constant in this case.



P = const. in the plane of the wave $\quad \omega = 2\Omega \cos\theta = 2\Omega_\perp$

where: $\Omega_\perp$ is conponent of $\vec{\Omega}$ perpendicular to the plane of the wave.

The motion is simple harmonic and the particles travel in circles perpendicular to the plane of the diagram.

The group velocity $\underset{\sim}{C_g} = \nabla_{\underset{\sim}{k}}\, \omega$ is:

$$C_g = \frac{2\Omega}{|k|} \sin\theta \quad , \text{ perpendicular to } \underset{\sim}{k}$$

we see

$$\begin{array}{ll} \text{if } \theta \to 0 & C_g \to 0 \\ & \omega \to \Omega \\ \text{if } \theta \to \pi/2 & C_g \parallel Oz \\ & \frac{\omega}{\Omega} << 1 \end{array}$$

This may be compared to the case of the gravity waves and one sees that in the two cases the direction of C_g is opposite and in the gravity wave case the particles execute <u>linear</u> simple harmonic motion.

These "inertial" waves can be connected with Taylor columns by considering that the column is simply an inertial wave with $\theta = 0$, for which the reflection time is much greater than the period of the wave.

These notes submitted by

John R. Booker

Lecture #VII

Viscous Boundary Layer Effects

1. The Ekman Spiral

Consider a homogeneous liquid bounded below by a plane rigid surface. Orient the coordinates axes as in Fig. 1, and assume that the system rotates with constant velocity $\underline{\Omega}$, the direction of which is not necessarily vertical.

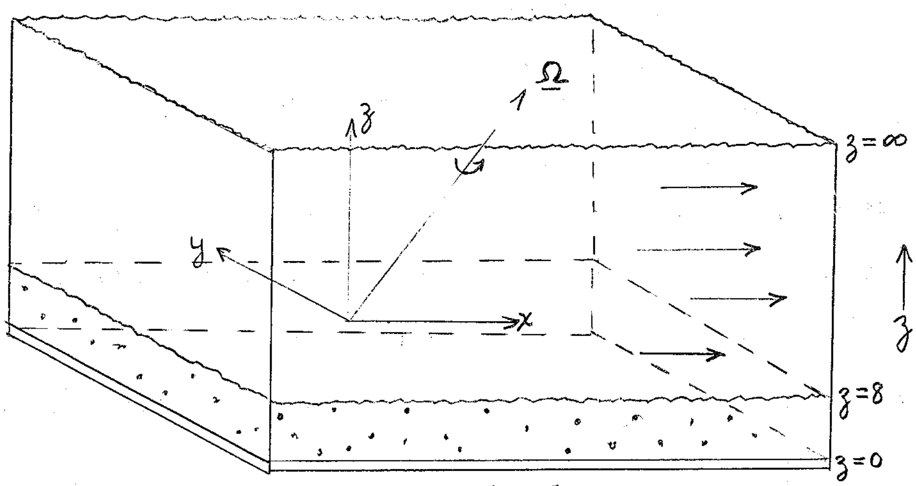


Fig. 1

Adjacent to the surface there exists a viscous boundary layer, the thickness (δ) of which is to be determined[1]. Above this layer assume the motion is uniform and in the x-direction, i.e.

$$\underline{u} = \underline{U}(U, 0, 0) ,$$

where U is constant. Within the boundary layer we shall seek

---

[1]It may be noted that since the system is rotating there exist solutions for a viscous boundary layer of <u>uniform thickness</u>, whereas without rotation, diffusion of the layer upward would cause the boundary layer thickness to <u>increase</u> downstream.

steady-state solutions of the form

$$\underline{u} = \left[u(z), v(z), 0\right],$$

while at the surface we shall adopt the no-slip condition

$$\underline{u} = 0.$$

With these restrictions on the motion, the basic set of equations becomes

$$-2\Omega_z v + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) = \nu \frac{\partial^2 u}{\partial z^2} \tag{1}$$

$$2\Omega_z u + \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) = \nu \frac{\partial^2 v}{\partial z^2} \tag{2}$$

$$-2\Omega_x v + 2\Omega_y u + \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) = 0 \tag{3}$$

where $\underline{\Omega} = (\Omega_x, \Omega_y, \Omega_z)$. The terms of the continuity equation, as well as the convective acceleration terms of the equations of motion vanish identically. Above the boundary layer, viscous forces vanish and geostrophic balance obtains. Therefore, $\underline{u} = (U, 0, 0)$ implies

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) &= 0 \\ \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) &= -2\Omega_z U, \end{aligned}\right\} \tag{4}$$

and our governing equations reduce to

$$-2\Omega_z v = \nu \frac{\partial^2 u}{\partial z^2} \tag{5}$$

$$2\Omega_z (u - U) = \nu \frac{\partial^2 v}{\partial z^2} \tag{6}$$

with boundary conditions

$$\left.\begin{aligned} \underline{u} &= 0 \quad \text{at} \quad z = 0 \\ \underline{u} &\to \underline{U} \quad \text{as} \quad z \to \infty \end{aligned}\right. \tag{7}$$

Multiplying (6) by $i = \sqrt{-1}$ and adding (5) yields

$$2\Omega_z\left(\frac{u+vi}{V}\right)i - \nu\frac{\partial^2}{\partial z^2}\left(\frac{u+vi}{V}\right) - 2\Omega_z i = 0 \tag{8}$$

Letting

$$Z(z) = \frac{u+vi}{V}$$

eqn. (8) may be rewritten

$$\left.\begin{aligned} &\nu\frac{d^2Z}{dz^2} - 2\Omega_z Zi + 2\Omega_z i = 0 \\ \text{with b.c. :}\quad &Z = 0 \quad \text{at} \quad z = 0 \\ &Z \to 1 \quad \text{as} \quad z \to \infty. \end{aligned}\right\} \tag{9}$$

The solution of (9) is

$$Z = 1 - \exp\left[-\left(\frac{2\Omega_z}{\nu}\right)^{1/2}\frac{1+i}{2^{1/2}}z\right] ,$$

with real part

$$R_e(Z) = 1 - \exp\left[-\left(\frac{\Omega_z}{\nu}\right)^{1/2}z\right] .$$

The quantity $\delta = \left(\frac{\nu}{\Omega_z}\right)^{\frac{1}{2}}$ is called the <u>boundary layer thickness</u>.

Transforming Z gives the components of motion

$$\begin{aligned} u &= V\left(1 - e^{-z/\delta}\cos\frac{z}{\delta}\right) \\ v &= V\left(e^{-z/\delta}\sin\frac{z}{\delta}\right) \end{aligned} \tag{10}$$

The hodograph for this solution is the Ekman spiral illustrated in Fig. 2. At $z = 0$, by applying l'Hôpital's rule, the slope is found to be 1, i.e. the direction of flow at the bounding surface makes an angle of 45° with the x-axis. The physical interpretation of the Ekman spiral is as follows:

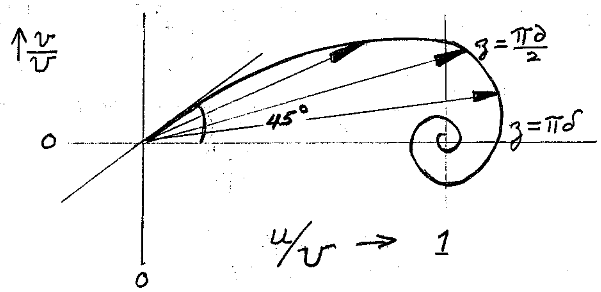


Fig. 2

Above the boundary layer the Coriolis force, $2\Omega_z V$, tends to accelerate particles to the right of the flow, i.e., along the negative y-axis; while the pressure gradient force $\frac{\partial}{\partial y}\left(\frac{p}{\rho}\right)$ acts equally, but along the positive y-axis as prescribed by geostrophic balance. Thus, the motion of particles above the boundary layer is restricted to planes parallel to the x, z-axes. As we shall see, this is not so in the boundary layer. Eqn. (3) implies that $\frac{\partial p}{\partial y}$ is independent of z. Therefore, since the Coriolis force is less in the boundary layer due to viscous stress[2] while the pressure gradient has the same value throughout the liquid, particles in the boundary layer tend to drift along the positive y-axis.

---

[2]The surface stress τ_0 is given by

$$\tau_0 = \left(\mu \frac{\partial u}{\partial z}, \mu \frac{\partial v}{\partial z}\right) = \frac{\mu V}{\delta}(1, 1),$$

and at $z = \frac{\pi \delta}{4}$, the stress reduces to

$$\tau_{\frac{\pi}{4}} = V(1, 1).$$

The Coriolis force acts in turn to the right of the particles. The exponential decrease of this effect[3] between the surface, where the crossing angle is 45°, and ∞, where the crossing angle is zero, combined with the S.H.M. due to rotation, leads to the turning of the flow in the boundary layer.

---

[3]If we define the volume flux $\underline{Q}$ as

$$\underline{Q} = \left[\int_0^\infty (u - v)\,dz, \int_0^\infty v\,dz \right] ,$$

then since the integrals are convergent (i.e. they decrease exponentially)

$$\underline{Q} = \frac{v\delta}{2} \left[-1, 1 \right] \qquad (11)$$

Because of the ageostrophic component of flow in the boundary layer, a flux deficit exists in the x-direction as can be seen from (11). There is an apparent contradiction between the second boundary condition (7) and the fact that the motion becomes approximately geostrophic at $z = \delta$. This, as well as several other ramifications of the Ekman spiral, are discussed, for example, in Hess, S. L., 1959: Introduction to Theoretical Meteorology. Holt, Rinehart and Winston, 362 pp.

These notes submitted by

W. Alan Bowman

Lecture #VIII

The Effect of Two Ekman Layers

Next let us consider the case where a top is placed on the liquid and, for convenience, the axis of rotation is parallel to the z-axis. Adjacent to each surface, there exists a viscous boundary layer. These two Ekman layers are separated by an interior region where the flow is geostrophic. We shall be interested in the flow between these two parallel plates.

Let the rate of rotation be given by Ω, the distance between plates by H, a characteristic horizontal velocity by V and a characteristic horizontal length by L. For the problem outlined above the scaling approximations are

$$R_o = \frac{V}{\Omega L} << 1 \qquad (12)$$

$$\varepsilon = \frac{\nu}{\Omega H^2} << 1 \qquad (13)$$

$$\frac{\nu}{\Omega L^2} << 1 \qquad (14)$$

where R_o is the Rossby nomber and (12) and (13) represent horizontal and vertical Taylor numbers respectively. In addition, we assume $w << u, v$ throughout the liquid and $H\frac{\partial}{\partial z} << 1$ in the interior region. Our rather heuristic approach to this problem will be to construct the flow in the interior and in the Ekman layers, and then to match the flow at the two intermediate "boundaries".

Zero-order approximation. In the <u>interior region</u> we assume geostrophic balance. The liquid then moves uniformly as Taylor columns, the equation of motion for which reduces to

$$2\underline{\Omega} \times \underline{u} + \nabla \frac{p}{\rho} = 0 , \tag{15}$$

where $\underline{u} = (u, v, 0)$ and $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$. By defining a stream function ψ_0

$$\psi_0 = -\frac{p}{2\Omega\rho}$$

eqn. (15) becomes

$$\underline{u} = -\underline{k} \times \nabla \psi_0 . \tag{16}$$

In the boundary layer, the vertical component of motion takes on small values, in contrast to the interior region (where it is zero). From the continuity equation, the vertical motion at the edge of the lower boundary layer is

$$w_{z=0+\delta\infty} = -\nabla \cdot Q = -\frac{1}{2}\delta \nabla^2 \psi_0 (x, y) , \tag{17}$$

where $\delta = \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}$, Q is the volume flux and $\nabla^2 \psi_0$ is the vertical component of relative vorticity of the basic flow. Notice, when the vorticity is zero, the volume flux and the vertical motion also become zero[4]. For the upper boundary layer, the vertical motion is similarly

$$w_{z=H-\delta\infty} = \frac{1}{2}\delta \nabla^2 \psi_0 (x, y) \tag{18}$$

which is equal and opposite that of the lower layer. If this result is true, the matching condition for the interior region requires the

[4]Indeed, the fact that w is proportional to vorticity is a consequence of so-called "Ekman layer suction", which refers to the drawing away of the boundary layer by motion normal to the boundary.

basic vortex lines in the interior to shrink, thus compressing the Taylor columns. But this implies that the motion of the interior fluid cannot be entirely geostrophic. We therefore proceed to the next approximation.

First-order approximation. In the momentum equation

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\,\underline{u} + 2\,\underline{\Omega}\times\underline{u} + \nabla\frac{p}{\rho} = 0,$$

let $\underline{u} = \underline{u}_0 + \underline{u}_1 = -\underline{k}\times\nabla\,\psi_0 + \underline{u}_1$

and $p = p_0 + p_1$,

where the subscript zero refers to the zero-order solution discussed above and subscript 1 is a correction. Then

$$\frac{\partial \underline{u}_0}{\partial t} + \underline{u}_0\cdot\nabla\underline{u}_0 + \left[2\underline{\Omega}\times\underline{u}_0 + \nabla\frac{p_0}{\rho}\right] + 2\underline{\Omega}\times\underline{u}_1 + \nabla\frac{p_1}{\rho} = 0 \tag{18}$$

the terms in brackets being zero by (15), thus

$$\left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)\underline{u}_0 + 2\,\underline{\Omega}\times\underline{u}_1 + \nabla\frac{p_1}{\rho} = 0 \tag{19}$$

To eliminate the pressure term in (19) we take the curl

$$\left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)(\nabla\times\underline{u}_0) - 2\,\underline{\Omega}\cdot\frac{\partial \underline{u}_1}{\partial z} = 0$$

and for two-dimensional flow

$$\left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)(\nabla^2\psi_0\,\underline{k}) = -2\,\underline{\Omega}\cdot\frac{\partial \underline{u}_1}{\partial z} \tag{20}$$

which gives the individual rate of change of vorticity. The vertical component of (20) is independent of z and may be integrated to give

$$\omega_1 \Big|_{z=0+\delta\infty}^{z=H-\delta\infty} = \frac{H}{2\Omega}\frac{D_0}{Dt}\nabla^2\psi_0 = \delta\nabla^2\psi_0(x,y) \qquad (21)$$

which represents the stretching of the Taylor columns. Equation (20) then leads to the prediction equation

$$\frac{D_0}{Dt}\nabla^2\psi_0 = -2\left(\frac{\nu}{\Omega H^2}\right)^{\frac{1}{2}}\Omega\,\nabla^2\psi_0 \qquad (22)$$

which expresses the rate of change of vorticity by viscous decay. For the problem of flow in an inviscid interior region bounded on bottom and top by an Ekman layer, the vertical motion is non-zero at the intermediate "boundaries". This allows the Taylor columns to shrink as the basic relative vorticity decreases, and thus avoids the contradiction found for the zero-order approximation. On the other hand, for an entirely inviscid liquid eqn. (22) indicates that the Taylor columns conserve their vorticity, just as in the zero order case.

In order to guarantee that the analysis above is internally consistent let us consider the following. For <u>linearized motion</u>

$$\frac{UT}{L} << 1$$

where T is a characteristic time corresponding to U and L. Equation (22) is then

$$\frac{\partial}{\partial t}\nabla^2\psi_0 = -2\Omega\mathcal{E}^{\frac{1}{2}}\nabla^2\psi_0$$

with solution[5]

---

[5]The time $\tau = \frac{1}{2\Omega\mathcal{E}^{1/2}}$, given by (23) is called the <u>spin-up time</u>, and represents the effective time required for a fluid, initially rotating with uniform speed Ω, to reach a state of <u>solid rotation</u> with speed $\Omega + \Delta\Omega$, after having received an impulse $\Delta\Omega$. Here $T = \tau$.

$$\nabla^2 \psi_0 = \left(\nabla^2 \psi_0\right)_{t=0} \exp\left(-2\Omega\, \varepsilon^{\frac{1}{2}} t\right). \tag{23}$$

The vertical motion scale W is

$$W = \frac{U}{L}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} = U\left(\frac{\nu}{\Omega L^2}\right)^{\frac{1}{2}} << U,$$

as required. In the interior region where the flow is initially irrotational

$$\underline{u} = (\varphi_x, \varphi_y)$$

and

$$\nabla^2 \varphi = 0$$

and to this order of approximation the viscous effects vanish.

For <u>non-linearized motion</u>, the inviscid case, for example, requires

$$\varepsilon = 0$$
$$\Omega \frac{W}{H} << T \frac{U}{L} \sim 1$$

with vertical motion scale

$$W = 0$$

i.e., the vertical component is the only component of vorticity. To verify the consistency of the prediction equation (22) we recall (18) in the form

$$\frac{D_0}{Dt}\underline{u}_0 + 2\underline{\Omega}\times\left(\underline{u}_0 + \underline{u}_1\right) + \frac{1}{\rho}\nabla\left(p_0 + p_1\right) = 0.$$

The corresponding inhomogeneous equation for viscous flow is

$$\frac{D_0}{Dt}\underline{u}_0 + 2\underline{\Omega}\times\left(\underline{u}_0 + \underline{u}_1\right) + \frac{1}{\rho}\nabla\left(p_0 + p_1\right) = \nu\nabla^2\left(\underline{u}_0 + \underline{u}_1\right) + \left(\frac{\partial}{\partial t} + \underline{u}_0\cdot\nabla\right)\underline{u}_0 - \left[\frac{\partial}{\partial t} + \left(\underline{u}_0 + \underline{u}_1\right)\cdot\nabla\right]\left(\underline{u}_0 + \underline{u}_1\right) \tag{24}$$

Now we wish to compare the magnitudes of terms based on the approximations

$$\frac{UT}{L} \sim 1$$
$$\frac{1}{T} \sim \varepsilon^{\frac{1}{2}}\Omega.$$

For example

$$\frac{\nu \nabla^2 \underline{u}_o}{\frac{D_o \underline{u}_o}{Dt}} = \frac{\frac{V\nu}{L^2}}{\frac{V}{T}} = \frac{\nu T}{L^2} = \frac{\nu}{\Omega L^2} \quad \frac{1}{\varepsilon^{1/2}} = \frac{H^2}{L^2}\varepsilon^{\frac{1}{2}}$$

which, by (13), is very small. Therefore it is permissible to drop the first term from the right-hand side of (24). To find the magnitude of $\underline{u}_1$ we consider (19). In the <u>interior region</u>

$$\nabla \cdot \underline{u}_1 = 0$$

and

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = -\frac{\partial w_1}{\partial z} = \frac{V}{H}\left(\frac{\nu}{\Omega L^2}\right)^{\frac{1}{2}}$$

or

$$u_1 \sim \frac{VL}{H}\left(\frac{\nu}{\Omega L^2}\right) = V\varepsilon^{\frac{1}{2}} ,$$

and (13) implies

$$\underline{u}_1 < \underline{u}_o$$

which was also required. In the <u>boundary layer</u>

$$2\underline{\Omega} \times \underline{u} + \frac{1}{\rho}\nabla p - \nu\frac{\partial^2 \underline{u}}{\partial z^2} = \nu\left(\frac{\partial^2 \underline{u}}{\partial x^2} + \frac{\partial^2 \underline{u}}{\partial y^2}\right) - \frac{D\underline{u}}{Dt} . \qquad (25)$$

The terms on the left-hand side are comparable in magnitude if

$$\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} \sim \frac{\partial}{\partial z} >> \frac{1}{L}$$

while the Coriolis acceleration and individual change of $\underline{u}$ are of the same order when

$$\frac{V}{\Omega L} << 1 .$$

Therefore, the last two terms of (24) may also be neglected. That is, equation (19) as well as the prediction equation (22) are internally consistent with the zero- and first-order approximations.

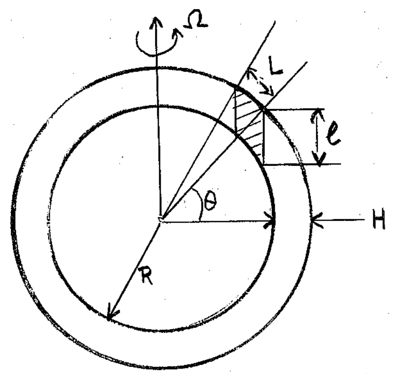
These notes submitted by

W. Alan Bowman

Lecture #IX

Spherical Case

Let us consider a thin, spherical, rotating shell. Exclude the equatorial region and consider an inviscid fluid.



let: $\frac{U}{\Omega L} << 1$

$\frac{L}{R} << 1$

$\frac{H}{L} << 1$

$\frac{H}{R} << 1$

The fluid will move in Taylor columns of length

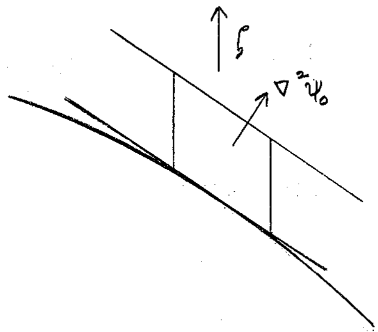
$$\ell = \frac{H}{\sin\theta} \qquad \text{(except near equator)}$$

with horizontal extent, L . As these columns move their length will change. The change in vorticity is:

$$\frac{D\vec{\zeta}}{Dt} = -\frac{2\Omega}{\ell}\frac{d\ell}{dt}$$
$$= 2\Omega \cot\theta \frac{d\theta}{dt}$$
$$= 2\Omega \cot\theta \frac{V}{R}$$
$$\therefore \zeta \cong 2\Omega \times \frac{\text{northward displacement}}{R}$$

β-plane approximation

There are several ways we may take the variation of $\vec{\zeta}$ with latitude into account. One of the most common methods is to map the spherical surface onto a plane:



ψ_0 = basic stream function

$\nabla^2\psi_0$ = basic vorticity

If we neglect stretching of vortex lines we have:

$$\frac{D_0}{Dt}\nabla^2\psi_0 = 0$$

and rotation drops out. However, suppose we allow ζ to vary linearly with θ :

$$\frac{D_0}{Dt}\nabla^2\psi_0 = \left[\frac{2\Omega\cot\theta}{R}\right] V\sin\theta$$

let $\beta = \dfrac{2\Omega\cos\theta}{R}$

$$\frac{D_0}{Dt}\nabla^2\psi_0 = -\beta\psi_x$$

<u>where</u>: x subscript denotes differentiation w.r.t. a coordinate pointing east.

Now consider a volume $\mathcal{V}$ on the sphere whose projected area is S , and whose boundary is Γ .

$$\mathcal{V} = \ell\sin\theta\, S = SH$$

define $f = 2\Omega\sin\theta$

then $\dfrac{\Gamma}{S} = \left(-\nabla^2\psi_0 + f\right)$

Both S and Γ are conserved during the fluid motion and hence both $\mathcal{V}$ and the vorticity are also

$$\therefore \frac{D}{Dt}\left[-\nabla^2\psi_0 + f\right] = 0$$

which leads to the previous result if $\frac{df}{dy} = \beta$, β taken independent of y . y points north.

See N. Phillips: "Reviews of Geophysics".

Rossby Waves:

$$\frac{D}{Dt} \nabla^2 \psi + \beta \psi_x = 0$$

linearize with $\frac{UT}{L} << 1$

$$\frac{\partial}{\partial t} \nabla^2 \psi + \beta \psi_x = 0$$

now assume plane wave solution of the form:

$$\psi = \hat{\psi} \, \mathcal{R}e \left\{ e^{i(kx + \ell y - \omega t)} \right\}$$

we get:

$$-i\omega(-k^2 + \ell^2)\hat{\psi} + \beta i k \hat{\psi} = 0$$

and hence we have the dispersion relation:

$$\omega = - \frac{\beta k}{k^2 + \ell^2}$$

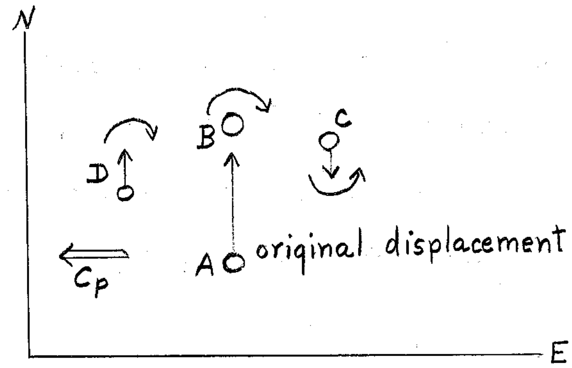
with phase velocity:

$$C_p = \frac{\omega}{k^2 + \ell^2}(k, \ell)$$

$$= - \frac{\beta}{(k^2 + \ell^2)^2}(k^2, h\ell)$$

$\therefore$ the x-component of the wave <u>always</u> propagates westward.

A simple physical argument is to consider the relative vorticities induced by a northward displacement of a fluid parcel.



The original displacement is from A to B ; this induces a clockwise circulation around B . Parcel C will be swept south and parcel D will be swept north. It is evident from the diagram that the secondary circulations induced by the displacements of C and D will tend to sweep the original parcel back to its position A . It is also evident that the northward displacement of D represents a progression of the phase to the left or west.

We get an inconsistency when $\omega \rightarrow \Omega$.

$$\beta \sim \frac{\Omega}{R}$$

$$\therefore \frac{\omega}{\Omega} \sim \frac{\beta L}{\Omega} \sim \frac{L}{R} << 1$$

so we have a further restriction on the regime of validity of this analysis.

One interesting observation has been made about these waves, that is that the energy flux is not parallel to the group velocity.

$$C_g \equiv \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial \ell}\right)$$

$$= \frac{\beta}{(k^2+\ell^2)^2}\left(k^2-\ell^2, 2k\ell\right)$$

this is at some angle θ with respect to the phase velocity. The energy flux is $(\bar{p}u, \dot{p}v)$ and is parallel to the wave fronts and hence not parallel to C_g. One finds, however, that C_g only differs from the energy flux by a non-divergent vector.

See: Longuet-Higgins, Deep Sea Research, 11, p.35, 1964.

These notes submitted by

John R. Booker

Lecture X

Rotating, Stratified, Boussinesq Flow

We shall look briefly at the linearized problem when both rotation and stratification are important. Suppose $N^2 =$ const. The linearized equations are

$$\underset{\sim}{u}_t + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} + \frac{1}{\rho^*}\nabla p + \sigma \underset{\sim}{k} = 0$$

$$\sigma_t - N^2 \underset{\sim}{u} \cdot \underset{\sim}{k} = 0$$

$$\nabla \cdot \underset{\sim}{u} = 0$$

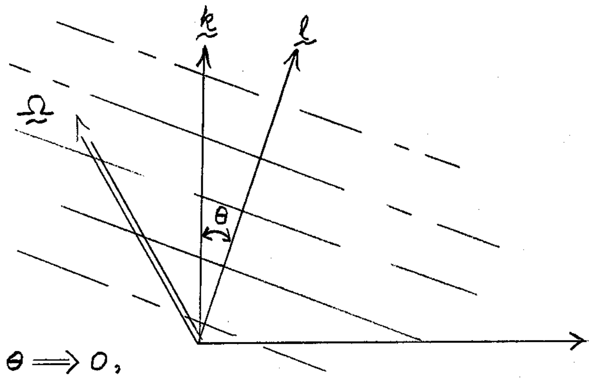
where $\underset{\sim}{\ell}$ need not be parallel to $\underset{\sim}{k}$. There exist plane wave solutions of frequency ω and wave number $\underset{\sim}{\ell}$ such that

$$\omega^2 = N^2\left[1 - \frac{(\underset{\sim}{\ell} \cdot \underset{\sim}{k})^2}{|\underset{\sim}{\ell}|^2}\right] + \frac{(\underset{\sim}{\ell} \cdot 2\underset{\sim}{\Omega})^2}{|\underset{\sim}{\ell}|^2}$$

The first term is the square of the frequency for gravity waves and the second is the square of the frequency for pure inertial waves.

Accelerations from the two kinds of restoring force simply add.

In the atmosphere $\Omega^2 << N^2$, so the effect of rotation is unimportant unless the motion is nearly horizontal. But suppose the wave propagates nearly vertically:



As $\theta \Longrightarrow 0$,

$$\omega^2 \Longrightarrow N^2\theta^2 + (2\underset{\sim}{l}\cdot\underset{\sim}{k})^2$$

and rotation begins to dominate. Only $\underset{\sim}{\Omega}\cdot\underset{\sim}{k}$, the vertical component of $\underset{\sim}{\Omega}$, can affect the vertical vorticity, - stratification cannot.

Hydrostatic Approximation

When the horizontal length of the disturbances L is great compared with the depth of the fluid H, some terms can be dropped from the compressible flow equations. Consider a compressible atmosphere at latitude (angular elevation above equator) ϕ. Then the x- and z-momentum equations will be truncated as shown:

$$\frac{Du}{Dt} + \cancel{2\Omega\cos\phi\, w} - 2\Omega\sin\phi\, v + \frac{1}{\rho_0(z)}\frac{\partial p}{\partial x} = 0$$

$$\cancel{\frac{Dw}{Dt}} - \cancel{2\Omega\cos\phi\, u} + \frac{g\rho}{\rho_0(z)} + \frac{1}{\rho_0(z)}\frac{\partial p}{\partial z} = 0$$

Two assumptions have to be made -

(i) vertical acceleration is negligible

(ii) horizontal component of $\underset{\sim}{\Omega}$ is unimportant.

The continuity equation is

$$\frac{\rho t}{\rho_0(z)} + \frac{1}{\rho_0}\frac{d\rho_0}{dz} w + u_x + v_y + w_z = 0$$

If we exclude vertical bouncing by assuming $w \sim U H/L$, where U is a typical horizontal speed, then the two assumptions can be made more precise as follows:

$$\text{(i)}\quad \frac{P_z}{P_x} \sim \frac{L}{H} \qquad \frac{Dw}{Dt} \sim \frac{H}{L}\frac{Du}{Dt}$$

$$\frac{\frac{Dw}{Dt}}{\frac{1}{\rho_0}P_z} \sim \frac{H^2}{L^2} \qquad \frac{\frac{Du}{Dt}}{\frac{1}{\rho_0}P_x} \lesssim \frac{H^2}{L^2} << 1$$

$$\text{(ii)}\quad \frac{2\Omega\cos\phi w}{2\Omega\sin\phi v} \sim \cot\phi\frac{H}{L} << 1$$

$$\frac{2\Omega\cos\phi u}{\frac{1}{\rho_0}P_z} \sim \frac{H}{L}\frac{2\Omega\cos\phi u}{\frac{1}{\rho_0}P_x} \leq \frac{H}{L}\frac{2\Omega\cos\phi u}{2\Omega\sin\phi v}$$
$$\leq \cot\phi\frac{H^2}{L^2} << 1$$

The $\lesssim$ sign comes in because $\frac{1}{\rho_0}\,\partial p/\partial x$ must be the order of the <u>larger</u> of Du/Dt and $2\Omega\sin\phi\, v$.

Near the equator $\cot\phi \to \infty$, and the second of the hydrostatic assumptions cannot be taken for granted. But if the Rossby number is small enough the argument still goes through:

$$\frac{2\Omega\cos\phi w}{\frac{Du}{Dt}} \sim \frac{\Omega T H}{L} \text{ or } \frac{\Omega H}{U} << 1$$

$$\frac{2\Omega\cos\phi u}{\frac{1}{\rho_0}P_z} \sim \frac{H}{L}\frac{2\Omega\cos\phi u}{\frac{1}{\rho_0}P_x} \sim \frac{H}{L}\frac{2\Omega\cos\phi u}{\frac{Du}{Dt}}$$
$$\sim \frac{\Omega T H}{L} \text{ or } \frac{\Omega H}{U} << 1$$

Hence near the equator the hydrostatic assumption requires

(i) $\frac{H}{L} << 1$

(ii) $R_o = \frac{\Omega T H}{L}$ or $\frac{\Omega H}{U} << 1$

For the atmosphere $\Omega H \sim 2\frac{1}{2}\, km/hr$ and for the sea $\Omega H \sim 1\, km/hr$.

Hydrostatic Waves

Let us linearize the hydrostatic equations and assume Boussinesq flow to simplify the algebra. Assume N^2 is a function of z alone and set $f = 2\Omega \sin\phi$. The equations become

$$u_t - fv + \frac{1}{\rho^*} P_x = 0$$
$$v_t + fu + \frac{1}{\rho^*} P_y = 0$$
$$\sigma + \frac{1}{\rho^*} P_z = 0$$
$$u_x + v_y + w_z = 0$$
$$\sigma_x + N^2(z) w = 0$$

The functions can be separated -

$$w = w_n(z)\, w^*(x,y,t)$$
$$\frac{P}{\rho^*} = P_n(z)\, P^*(x,y,t)$$

and so on. The equations in the starred functions become

$$u_t^* - fv^* + P_x^* = 0$$
$$v_t^* + fu^* + P_y^* = 0$$
$$u_x^* + v_y + \frac{1}{C_n^2} P_t^* = 0$$

where the separation constant C_n^2 plays the part of an eigenvalue in, say, the equation for $w_n(z)$:

$$\frac{d^2 w_n}{dz^2} + \frac{N(z)^2}{C_n^2} w_n = 0$$

If $f = 0$, the shallow water wave equation is recovered:

$$\left(\frac{\partial^2}{\partial t^2} - C_n^2 \nabla^2\right)\left(\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right) = 0$$

In the ocean, the first mode - the barotropic mode - depends on the motion of the free surface and has a characteristic speed $c_1 \sim \sqrt{gH} \sim 200 m./s.$ The second and higher modes - the baroclinic modes - are internal waves and have characteristic speeds from a few meters per second on down.

Let us use a β-plane approximation for f and position the plane directly on the equator. Then

$$f = \beta y$$

and we must accept the limitation $R_o << 1$ explained in the last section. The coefficients in the starred equations depend on y alone, so a further separation is possible:

$$v^* = Re\left\{\hat{v}(y) e^{i(kx - \omega t)}\right\}$$

and so on. The equation for $\hat{v}$ is

$$\frac{d^2\hat{v}}{dy^2} + \left\{\frac{\omega^2}{C_n^2} - \beta\frac{k}{\omega} - k^2 - \frac{\beta^2 y^2}{C_n^2}\right\}\hat{v} = 0$$

When the coefficient in brackets is greater than zero, $\hat{v}(y)$ is oscillatory. But for large y^2 the coefficient becomes negative, so solutions can be found which decay rapidly as $y \longrightarrow \pm\infty$. Thus the β-effect traps these Rossby waves in a band around the equator.

The natural length for the mode under consideration is $\sqrt{\frac{C_n}{\beta}}$ and the natural frequency is $\sqrt{\beta C_n}$. Define $\tilde{k}$ and $\tilde{\omega}$ as follows:

$$k = \tilde{k}\sqrt{\frac{\beta}{c_n}} \qquad \omega = \tilde{\omega}\sqrt{\beta c_n}$$

Then functions $\hat{v}$ decaying toward $y \Rightarrow \pm \infty$ can be found providing

$$\tilde{\omega}^2 - \frac{\tilde{k}}{\tilde{\omega}} - \tilde{k}^2 = 2m+1 \qquad m = 0, \pm 1, \pm 2, \cdots$$

They have the form

$$\hat{v}(y) = A\, H_m\left(\frac{y}{\sqrt{\frac{c_n}{\beta}}}\right) \exp\left(-\frac{1}{2}\frac{y^2\beta}{c_n}\right)$$

where H_m are the Hermite polynomials. As $m \to \infty$ and the number of oscillations in the y-direction increases, it becomes easier to see the physical significance of these solutions. That limit will be explored in the next lecture.

These notes submitted by

Steven Crow

Lecture #XI

Special Cases

We shall now consider the asymptotic case $m \to \infty$. The dispersion relation is

$$\tilde{\omega}^2 - \frac{\tilde{k}}{\tilde{\omega}} = 2m + 1 + \tilde{k}^2$$

from which it can easily be seen that in this limit either $\tilde{\omega}^2$ or $\frac{\tilde{k}}{\tilde{\omega}}$ is large. We shall consider the two cases separately.

a) $\tilde{\omega}^2$ large

The dispersion relation then takes the form

$$\tilde{\omega}^2 \approx \tilde{k}^2 + 2m + 1$$

and $\frac{\tilde{k}}{\tilde{\omega}} \lesssim 1$. In this case in the equation

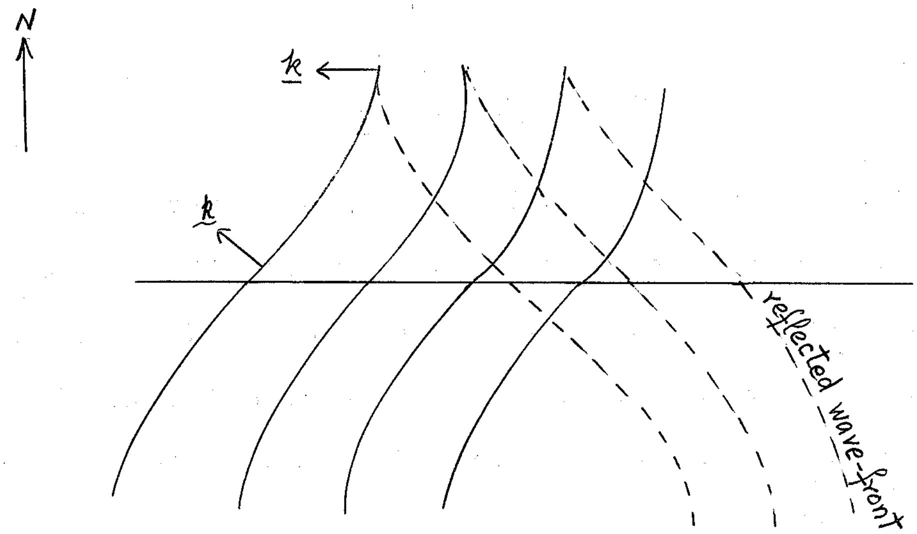
$$\frac{d^2\hat{v}}{dy^2} + \left[\frac{\omega^2}{C_n^2} - \beta\frac{k}{\omega} - k^2 - \frac{f^2}{C_n^2}\right]\hat{v} = 0$$

the term $\beta\frac{k}{\omega}$ is neglected. Because in this asymptotic limit the y-component of the wavelength vector is so small that f may be regarded as constant over a single wavelength, $\hat{v}$ becomes sinusoidal in character and the dispersion relation is

$$\omega^2 = f^2 + (k^2 + \ell^2)c_n^2$$

where ℓ is the wave number in the N-S direction. We see that locally the β-effect is not important; these waves are inertio-gravitational waves.

These waves are trapped in the equatorial regions. That this should be so can be seen by the following argument:



Consider northward-travelling waves. ω and k will both be constant but f increases with latitude. Hence ℓ^2 must decrease and if f^2 becomes sufficiently large ℓ^2 must eventually become zero which means that $\underset{\sim}{k}$ turns to the E-W direction. Since $\underset{\sim}{C}_g$ is parallel to $\underset{\sim}{k}$ the wave energy cannot propagate beyond this latitude. For barotropic motions ($n = 1$) the above considerations must be viewed with caution.

If $f^2 << C_n^2(k^2 + \ell^2)$ the modes are pure gravity waves. The case $f^2 >> C_n^2(k^2 + \ell^2)$ describes "inertial oscillations" whose frequency is independent of wavelength. These motions have been detected in moored buoy experiments but it is not understood why they should be excited to such large amplitudes in the ocean. The transition between these two extremes occurs when $L \approx \dfrac{C_n}{f}$.

b) $\dfrac{\tilde{k}}{\varpi}$ large

In this case $\dfrac{\omega^2}{C_n^2}$ will be neglected in the equation for and we obtain locally

$$\omega = \frac{-\beta k}{k^2 + \ell^2 + f^2/c_n^2}$$

These are "Divergent Rossby Waves". It is worth remarking that this derivation does not assume geostrophy, $(\omega << f)$ and is therefore valid for motions near the equator. It is, however, illuminating to rederive this relation in a manner which clearly illucidates the way in which the dynamics affects the problem. The vorticity equation is

$$\frac{\partial}{\partial t}(u_y - v_x) - \beta v - f(u_x + v_y) = 0$$

If we suppose that $\omega << f$ then $\left|\frac{\partial}{\partial t}(u_y - v_x)\right|$ is no bigger than $\left|f(u_x + v_y)\right|$ and hence

$$|u_x + v_y| << |u_y - v_x| \ .$$

Therefore, to a first approximation, we can introduce a stream function to describe the horizontal motion:

$$u = \psi_y \ , \quad v = -\psi_x \ .$$

Kinematically the motion is approximately non-divergent, but the horizontal divergence can, in some cases, play an important role in the dynamics of the flow through the stretching of the vortex lines. Consistent with the assumption $\omega << f$ we can neglect the time derivatives in the first two tidal equations which can then be integrated to yield

$$f\psi = -P \ ,$$

to within a constant of integration, which has no physical significance. Substituting this into the third tidal equation gives

$$u_x + v_y = \frac{f}{c_n^2}\psi_t \ .$$

The vorticity equation now becomes

$$\frac{\partial}{\partial t}\nabla_h^2\psi+\beta\psi_x-\frac{f^2}{C_n^2}\psi_t=0$$

This is the equation for divergent Rossby waves. If we Fourier transform the equation we recover the dispersion relation

$$\omega=\frac{-\beta k}{k^2+\ell^2+f^2/C_n^2}$$

We shall now show that even in the equatorial regions where f is small and $\omega \gtrsim f$ the dispersion relation derived above tends to the correct limit as $f \longrightarrow 0$.

We are considering the case $\frac{\tilde{k}}{\tilde{\omega}}=\frac{k}{\omega}C_n \gg 1$. That is, the frequency ω is very much smaller than that of gravitational waves of the same wave number. Furthermore, from the tidal equations, if f is small

$$|u_x+v_y| \sim \frac{\omega P}{\rho C_n^2},$$

$$|u_x| \sim \frac{k}{\omega}k\frac{P}{\rho}.$$

Hence $|u_x+v_y| \sim \frac{\omega^2}{k^2C_n^2}|u_x| \ll |u_x|$.

Similarly $|u_x+v_y| \ll |v_y|$, which means that in this case the horizontal divergence is dynamically unimportant too.

The vorticity equation is now

$$\frac{\partial}{\partial t}(u_y-v_x)-\beta v=0,$$

which describes non-divergent Rossby waves. Introduction of a stream function and Fourier transforming leads to the dispersion relation

$$\omega=\frac{-\beta k}{k^2+\ell^2}.$$

It is not the gravitational restoring force which is providing the essential dynamics. When $u_x + v_y \approx 0$ we have area-preserving motions and the vertical component of vorticity is conserved. On the equator although $f = 0$, β is non-zero and a fluid parcel still appears to have a relative vorticity as it moves north or south, and the Rossby wave mechanism still exists. The divergence will be unimportant dynamically provided $\frac{f^2}{c_n^2} << k^2 + \ell^2$

$$i.e.\ L << \frac{c_n}{f}.$$

If the horizontal divergence is dynamically significant, it can be seen from the dispersion relation that its effect is to decrease the frequency. Qualitatively this result can be understood in the following way: When a fluid parcel moves north it will exhibit a relative anticyclonic spin which means that the pressure will be highest at its centre. This must have resulted from a horizontally convergent motion which causes a stretching in the vertical direction of the vortex lines. The absolute vorticity is increased. Then the relative anticyclonic spin must decrease which decreases the Rossby restoring force, and hence the frequency.

For long wavelengths $\left(\omega \sim \sqrt{\beta c_n}\right)$ we find by comparing the time scales, that our hydrostatic approximation is still good if $\sqrt{\frac{\Omega R}{c_n}}\, \frac{H}{L} << 1$, where R is the radius of the earth.

For very short length scales the approximation is good for inertia waves but for Rossby waves it breaks down near the equator. We can no longer set $p_z + \sigma = 0$ but have

$$p_z + \sigma - 2\Omega \cos\varphi\, u = 0$$

Fortunately, however, the last term does not play any role in our treatment of the problem, because vertical displacements drop out, producing a decoupling between the horizontal layers. The horizontal wave motions are then independent of the vertical structure.

These notes submitted by

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Lecture #XII

Non-Linear Motions

Measurements at the Meteorological Office in England have shown that it is not a good approximation to regard atmospheric motions as a sum of linearised Rossby waves. We are therefore forced to consider non-linear motions.

We shall now consider geostrophic motions with the Brunt-Väisälä frequency a function of z only and, for convenience, assume that the Boussinesq approximation can be made. The equations of motion then take the form:

$$2\underset{\sim}{\Omega} \times \underset{\sim}{u} + \frac{1}{\rho^*} \nabla P + \sigma \hat{\underset{\sim}{z}} = 0$$

where $\hat{\underset{\sim}{z}}$ is a vertical unit vector. Eliminating the pressure by taking the curl we obtain the vorticity equation:

$$2(\underset{\sim}{\Omega} \cdot \nabla)\underset{\sim}{u} = -\hat{\underset{\sim}{z}} \times \nabla\sigma .$$

From the equation for the vertical component, $\underset{\sim}{\Omega} \cdot \nabla\, w = 0$, we see that w is independent of position along a line parallel to the rotation axis. The equations for the horizontal components are:

$$2\underset{\sim}{\Omega}\cdot\nabla u = \sigma_y ,$$
$$2\underset{\sim}{\Omega}\cdot\nabla v = -\sigma_x .$$

If $\frac{H}{L}\cot\varphi << 1$, these reduce to

$$2\Omega\sin\varphi\, u_z = \sigma_y ,$$
$$2\Omega\sin\varphi\, v_z = -\sigma_x .$$

These are the "Thermal Wind Equations". By means of these equations it can be found that for a typical value for the temperature gradient in the atmosphere $\left(\frac{dT}{dy} \approx -10^{\circ}C/1000km\right)$ we obtain a velocity gradient $\frac{\partial u}{\partial z} \approx 30 m/sec$ per $10 Km$. This gives a difference in westerly wind from ground to troposphere of 60 knots, which is large compared with the ground wind speed. Note that this has come from a vorticity equation. Gravitational forces are feeding vorticity into the system and this can be balanced only by variations in the Coriolis force. The thermal wind equations are a good approximation when $R_0 << 1$. However, they are not predictive equations; we cannot determine from them the time development of the system. In order to do this we must take higher order equations into account.

For simplicity, we shall consider an inviscid, Boussinesq liquid with $f = 2\Omega\sin\varphi$, constant. β-effects and viscosity could easily be considered, however, provided they are not so large as to destroy the basic balance assumed below, but the essential physics is contained in this simple model. We assume that the Rossby number based on the vertical component of the Coriolis force, $R_o = \frac{U}{fL} \ll 1$, and that $\frac{H}{L}\cot\varphi << 1$, but that $\frac{L}{UT} \sim 1$.

The vertical vorticity relative to the rotating frame is of order $\frac{U}{L}$. Hence the order of its rate of change is $\frac{U}{LT} \sim \frac{U^2}{L^2}$, which is assumed to be comparable with the stretching of the vortex lines $\sim f\frac{\partial w}{\partial z} \sim f\frac{W}{H}$. Thus $\frac{U^2}{L^2} \sim f\frac{W}{H}$ and so

$$\frac{W}{H} \sim \frac{U}{L} R_o .$$

This means that to lowest order in R_o, $u_x + v_y = 0$. Motions are horizontally non-divergent. We now introduce dimensionless variables and let primed variables be the dimensional quantities. We set

$$x', y' = L(x, y);$$
$$z' = Hz \qquad\qquad x, y \text{ are of order unity}$$
$$t' = \frac{L}{U}t.$$

The horizontal velocity

$$\underset{\sim}{u}'_h = U\left[\underset{\sim}{u}_0 + R_o \underset{\sim}{u}_1 + O(R_o^2)\right],$$
$$w' = \frac{UH}{L}\left[R_o w_1 + O(R_o^2)\right],$$
$$\frac{p'}{\rho^*} = fUL\left[p_0 + R_o p_1 + O(R_o^2)\right],$$
$$\sigma' = \frac{fUL}{H}\left[\sigma_0 + R_o\sigma_1 + O(R_o^2)\right],$$
$$N^2(z) = \frac{N'^2(z)H^2}{L^2 f^2}$$

The non-dimensionalisation of $\frac{p'}{\rho^*}$ is suggested by a balance of the pressure scale with the Coriolis forces. We are also interested in having gravitational forces and Coriolis restoring forces approximately in balance. In the linear case this occurred when $\frac{C_n}{fL} \sim 1$. But $C_n \sim NH$ and so we would expect that $\frac{NH}{fL} \sim 1$. This consideration motivates our choice for the non-dimensionalisation of the Brunt-Väisälä frequency. By substituting the dimensionless quantities into the equations of motion, the continuity equation and the adiabatic equation and equating terms in the same power of the Rossby number we obtain two systems of equations, one of zero order and one of first order in R_0.

a) Zero-Order Equations

We have already shown that to zero order in R_0

$$u_{0x} + v_{0y} = 0$$

Hence we may introduce a stream function to describe the horizontal motion and write

$$\underset{\sim}{u}_0 = \hat{\underset{\sim}{z}} \times \nabla \psi_0 (x, y, z, t).$$

The horizontal momentum equation is

$$-\nabla_h \psi_0 + \nabla_h P_0 = 0,$$

which implies that

$\psi_0 = P_0 + F(z, t)$, where F is an arbitrary function.

But $\hat{\underset{\sim}{z}} \times \nabla \psi_0$ depends only on $\nabla_h \psi_0$ and so we may choose ψ_0 such that

$$\psi_0 = P_0 .$$

In the equation for vertical momentum we can neglect the term $\frac{H}{L} \cot \varphi \, \psi_y$ and obtain an equation for hydrostatic balance:

$$P_{0z} + \sigma_0 = 0.$$

The lowest order adiabatic equation is

$$\frac{D_0 \sigma_0}{Dt} - N^2(z)\, w_1 = 0,$$

where $\frac{D_0}{Dt} \equiv \frac{\partial}{\partial t} + \underset{\sim}{u}_0 \cdot \nabla$. This introduces the first order quantity w_1 , and so the zero-order equations cannot be closed without appealing to the first-order system.

b) First-Order Equations

The continuity and horizontal momentum equations are:

$$w_{1_z} + \operatorname{div} \underset{\sim}{u}_1 = 0$$

$$\frac{D_0 \underset{\sim}{u}_0}{Dt} + \frac{H}{L} \cot\varphi\, w_1 \hat{\underset{\sim}{X}} + \hat{\underset{\sim}{z}} \times \underset{\sim}{u}_1 + \nabla_h P_1 = 0$$

where $\hat{\underset{\sim}{X}}$ is a unit vector pointing towards the east. Note that there is no vertical convection of momentum. We must neglect the second term in the momentum equation because we have assumed that $\frac{H}{L} \cot\varphi \ll 1$. Taking the vertical component of the curl of this equation we obtain

$$\frac{D_0}{Dt} \nabla_h^2 \psi_0 + \operatorname{div} \underset{\sim}{u}_1 = 0 .$$

Hence
$$\begin{aligned}\frac{D_0}{Dt} \nabla_h^2 \psi_0 &= w_{1_z} \\ &= \frac{\partial}{\partial z}\left[\frac{1}{N^2} \frac{D_0 \sigma_0}{Dt}\right] \\ &= -\frac{\partial}{\partial z}\left[\frac{1}{N^2(z)} \frac{D_0}{Dt} \psi_{0z}\right],\end{aligned}$$

using the potential vorticity equation.

We have

$$\frac{\partial}{\partial z}\left[\frac{1}{N^2} \frac{D_0}{Dt} \psi_{0z}\right] = \frac{D_0}{Dt}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2} \psi_{0z}\right)\right] - \left(\frac{\partial \underset{\sim}{u}_0}{\partial z}\right) \cdot \nabla_h \left(\frac{1}{N^2} \psi_{0z}\right) .$$

The last term vanishes because

$$\frac{\partial \underset{\sim}{u}_0}{\partial z} = \hat{\underset{\sim}{z}} \times \nabla \psi_{0z}$$

which is perpendicular to the horizontal gradient of $\frac{1}{N^2}\psi_{0z}$.

Hence

$$\underline{\frac{D_0}{Dt}\left[\nabla_h^2 \psi_0 \frac{\partial}{\partial z}\left(\frac{1}{N^2}\psi_{0z}\right)\right] = 0}$$

This, together with the boundary conditions on the level surfaces:

$$w_1 = 0 ,$$

$$\text{i.e. } \frac{1}{N^2}\frac{D_0}{Dt}\psi_{0z} = 0$$

$$\Longrightarrow \quad \frac{D_0}{Dt}\psi_{0z} = 0 ,$$

are the "Quasi-geostrophic Equations".

The quantity $\nabla^2\psi_0 + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\psi_{0z}\right)$ is called the potential vorticity. The first term is the vertical component of relative vorticity and the second describes the change in length of vortex lines due to gradients of potential temperature. The quasi-geostrophic equations express the conservation of potential vorticity as a fluid particle moves. Note that if we had an Ekman layer the boundary condition should be modified to take account of the Ekman layer suction.

viz: $\frac{D_0}{Dt}\psi_{0z} = \mathcal{K}\nabla_h^2\psi_0$,

where $\mathcal{K} = \pm 2\left(\frac{\nu\Omega}{H^2}\right)^{\frac{1}{2}}$

Comments

1) The horizontal motion is non-divergent; $\underset{\sim}{u}_0 = \hat{\underset{\sim}{z}} \times \nabla_h \psi_0$ and the vertical motion is an order smaller in the Rossby Number. So

$\left|w\frac{\partial}{\partial z}\right| \ll |\underset{\sim}{u}_o \cdot \nabla|$. There is no vertical transfer of momentum.

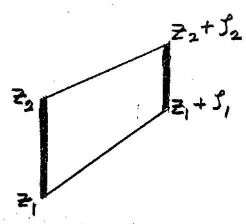
$$2)\ \frac{\partial}{\partial z}\left(\underset{\sim}{u}_o \cdot \nabla\sigma\right) = \frac{\partial \underset{\sim}{u}_o}{\partial z}\cdot\nabla\sigma + \underset{\sim}{u}_o\cdot\nabla\frac{\partial\sigma}{\partial z}$$
$$= -\left[\frac{\partial}{\partial z}\left(\hat{\underset{\sim}{z}}\times\nabla_h\psi_o\right)\right]\cdot\nabla_h\psi_{oz} + \underset{\sim}{u}_o\cdot\nabla\frac{\partial\sigma}{\partial z}$$
$$= -\left(\hat{\underset{\sim}{z}}\times\nabla_h\psi_{oz}\right)\cdot\nabla_h\psi_{oz} + \underset{\sim}{u}_o\cdot\nabla\frac{\partial\sigma}{\partial z}$$
$$= \underset{\sim}{u}_o\cdot\nabla\sigma_z\ .$$

This shows that the variation of $\underset{\sim}{u}_o$ with height associated with the presence of horizontal temperature gradients does not contribute to the term $\frac{\partial}{\partial z}\left(\underset{\sim}{u}_o\cdot\nabla\sigma\right)$. This allows us to invert the operators $\frac{\partial}{\partial z}\ \frac{1}{N^2}$ and $\frac{D_o}{Dt}$.

3) Let ζ be a small vertical displacement of a fluid element. The buoyancy force, $\sigma = N^2\zeta$. Using the hydrostatic approximation

$$\sigma = -\frac{\partial p}{\partial z} = -\psi_z ,$$

we obtain $\zeta = -\frac{\psi_z}{N^2}$.



Consider now the motion of a vortex line element. The fractional change in vertical length as line element moves is

$$\frac{z_2+\zeta_2-(z_1+\zeta_1)}{z_2-z_1} = 1 + \frac{\zeta_2-\zeta_1}{z_2-z_1}$$
$$= 1 - \frac{\partial}{\partial z}\left(\frac{\psi_z}{N^2}\right).$$

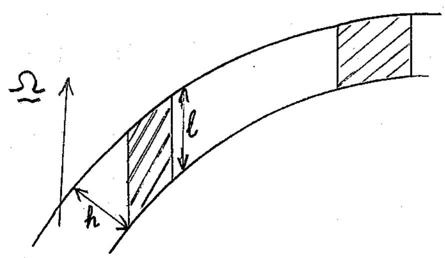
Calling the vorticity before the displacement Q , the vorticity after the displacement will be $Q + \nabla_h^2\psi$. On the other hand, this must be equal to $Q\left[1-\frac{\partial}{\partial z}\left(\frac{\psi_z}{N^2}\right)\right]$. If in the basic state

$Q = f = 2\Omega_{vertical}$ (no relative vorticity) we then obtain the equation

$$\nabla_h^2 \psi + f \frac{\partial}{\partial z}\left(\frac{1}{N^2}\psi_z\right) = 0$$

Note that $f = 1$ in our non-dimensional units.

4) In oceanography the potential vorticity for a homogeneous fluid is defined in another way. We consider the translation of a Taylor column.



$l = h \operatorname{cosec} \varphi$ where φ is the latitude.

curl $\underset{\sim}{u}$ is parallel to $\underset{\sim}{\Omega}$.

curl $\underset{\sim}{u} + 2\underset{\sim}{\Omega}$ is the absolute vorticity.

The quantity $\dfrac{|\operatorname{curl} \underset{\sim}{u} + 2\underset{\sim}{\Omega}|}{h \operatorname{cosec} \varphi} = \dfrac{\hat{\underset{\sim}{z}} \cdot \operatorname{curl} \underset{\sim}{u} + f}{h}$ is conserved as the column moves, and is called potential vorticity.

In the β-plane we can write the potential vorticity in the atmosphere as
$$f + \nabla_h^2 \psi + f \frac{\partial}{\partial z}\left(\frac{1}{N^2}\psi_z\right).$$

5) For <u>any</u> motions of an inviscid, even compressible fluid for which there exists some locally conserved quantity θ , which is a function of pressure and density alone, and which in our case is the potential temperature, we can derive the following equation:
$$\frac{D}{Dt}\left[\left(\operatorname{curl} \underset{\sim}{u} + 2\underset{\sim}{\Omega}\right) \cdot \nabla \log \theta\right] = 0,$$
which expresses the conservation of potential vorticity in general. It must be emphasized that no geostrophic assumption is involved.

The Balance Equations

The quasi-geostrophic equations were first derived by Charney for the purpose of numerical weather forecasting. He formulated them to remove gravity waves from the problem since these can lead to severe numerical instabilities unless a prohibitively small grid-spacing is employed. Gravity waves can be eliminated by assuming that $\frac{L}{TNH}, \frac{U}{NH} \ll 1$, but it is not necessary to assume geostrophy. Near the equator non-divergent Rossby waves can exist which are not geostrophic, and cannot be found from the quasi-geostrophic equations. A geostrophic motion implies that

$$f u + p_x = 0,$$
$$-f v + p_y = 0$$

and if we can regard f as constant, then $f(u_x + v_y) = 0$. A geostrophic motion, therefore, implies non-divergence on an f-plane; but the converse is not necessarily true. It is perfectly consistent to consider $\frac{W}{H} \ll \frac{U}{L}$ but $\frac{U}{fL} \not\ll 1$. This situation produces the "balance equations", a generalisation of the quasi-geostrophic equations. Let us, therefore, assume that $\frac{UT}{L} \sim 1$ and $\frac{U}{NH} \ll 1$ and that the hydrostatic approximation is valid. Take $\frac{W}{H} \ll \frac{U}{L}$, in fact $\frac{W}{H} \sim \left(\frac{U}{NH}\right)\frac{U}{L}$. As a consequence of this $|w_z| \ll |u_x|, |v_y|$ separately and so we may introduce a stream function ψ for the horizontal motion: $\underset{\sim}{u}_0 = \hat{\underset{\sim}{z}} \times \nabla \psi$. Also $\left|w\frac{\partial}{\partial z}\right| \ll \left|u\frac{\partial}{\partial x}\right|$ which implies that there is no vertical transport of momentum.

The vertical component of the vorticity equation is

$$\underset{\dfrac{U^2}{L^2}}{\frac{D}{Dt}\left(\nabla_h^2\psi+f\right)} - \underset{\dfrac{U^2}{L^2}\dfrac{U}{NH}}{\left(f+\nabla_h^2\psi\right)\frac{\partial w}{\partial z}} - \underset{\dfrac{U^2}{L^2}\dfrac{U}{NH}}{\left(2\underset{\sim}{\Omega}_h+\frac{\partial \underset{\sim}{u}}{\partial z}\right)\cdot\nabla_h w} = 0$$

where the orders of magnitude of the indicated terms are written below. The term involving $\underset{\sim}{\Omega}_h$ disappears, as previously, in view of the hydrostatic approximation. Although $f\,\dfrac{\partial w}{\partial z}$ may be small we retain it at this stage in order to include the possibility of non-geostrophic motions. Having neglected all other small terms we obtain

$$\underline{\frac{D}{Dt}\left(\nabla_h^2\psi+f\right)=f\,\frac{\partial w}{\partial z}}$$

Also $w=-\dfrac{1}{N^2}\dfrac{D}{Dt}\,p_z$ and the pressure no longer provides a stream-function for the horizontal motions. In order to complete the set we must go back to the horizontal momentum equation

$$\frac{D\underset{\sim}{u}_h}{Dt}+f\,\hat{\underset{\sim}{z}}\times\underset{\sim}{u}_h+\frac{1}{\rho^*}\nabla_h p=0\,,$$

with

$$\underset{\sim}{u}_h=\hat{\underset{\sim}{z}}\times\nabla_h\psi\,.$$

Taking the divergence of the momentum equation leads to

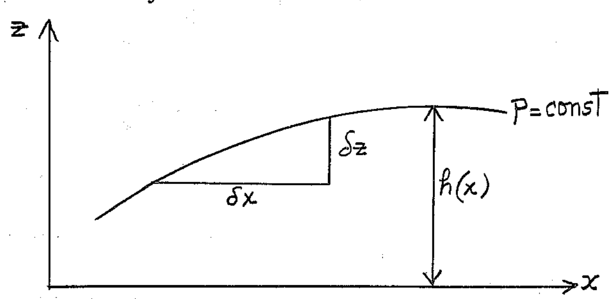
$$\nabla\cdot\left[\left(\frac{\partial}{\partial t}+\hat{\underset{\sim}{z}}\cdot\nabla\psi\right)\cdot\left(\hat{\underset{\sim}{z}}\times\nabla\psi\right)\right]-f\,\nabla_h^2\psi+\beta\,\psi_x+\frac{1}{\rho^*}\nabla_h^2 p=0\,.$$

If the scale of the motion is very much smaller than the radius of the earth we may neglect $\beta\,\psi_x$ in this equation. As a reminder, ψ is the stream-function describing the horizontal motion, p the perturbation in pressure, ρ^* a standard constant density,

f is the Coriolis parameter $2\Omega \sin\varphi$, where φ is the latitude, $\beta = \frac{\partial f}{\partial y}$ which is assumed constant and the operator ∇_h is $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. $\hat{\underset{\sim}{z}}$ is a vertical unit vector.

The only difference between the balance equations and the quasi-geostrophic equations is the inclusion of the acceleration terms in the former. Charney believes that the balance equations may have some validity in equatorial regions to describe motions from which gravity waves have been filtered. But we have ignored latent heat of condensation of moisture, heat transport and turbulent momentum transfer, all of which may be significant.

We have derived these equations for a Boussinesq liquid, but this approximation is valid only when the thickness of the layer considered is very much smaller than the pressure scale height. When H is comparable with the scale height the Boussinesq equations can still be used if pressure is used as the vertical coordinate. We briefly illustrate how this is so.



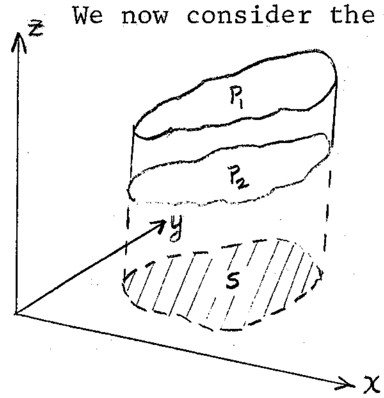
We now take h to be a function of p.

On the surface of constant p, $\left.\frac{\partial p}{\partial x}\right|_{z=\text{const.}} \delta x + \left.\frac{\partial p}{\partial z}\right|_{x=\text{const.}} \delta z = 0.$

$$\therefore \frac{\delta z}{\delta x} = \frac{-\left(\frac{\partial p}{\partial x}\right)}{\left(\frac{\partial p}{\partial z}\right)},$$

$$\text{i.e. } \left(\frac{\partial h}{\partial x}\right)_p = \frac{1}{g\rho(z)} \frac{\partial p}{\partial x}.$$

The R.H.S. is a term which occurs in the momentum equation.



We now consider the continuity equation. It is assumed that the slope of a constant pressure surface is small. $H << L$. The mass contained in the element between the two constant pressure surfaces is $\frac{1}{g}(P_2-P_1)S$, where S is the projection of the element onto a horizontal plane. This is also the "volume" in pressure coordinates. Conservation of mass is equivalent to conservation of "volume", using this pressure coordinate, which implies that the continuity equation in this system will be the same as for a Boussinesq liquid.

We assume:

1) Hydrostatic approximation,

2) $\dfrac{\rho'(x,y,z,t)}{\rho_0(z)} , \dfrac{p'}{p_0} << 1$

With an analogue pressure defined as $-\rho^* g\left[h(p)-h_0(p)\right]$ and an analogue Brunt-Väisälä frequency $N_a^2 = \dfrac{\rho^{*2} N_{true}^2}{\rho_0^2(z)}$ the equations of motion are exactly those for a Boussinesq liquid. Using this definition, N^2 varies by a factor of 25 between the ground and the troposphere and not merely a factor 2 which is usually assumed to be the case in liquid modelling of the atmosphere. The analogue ω which is required to complete the equations is proportional to the analogue $\dfrac{Dp}{Dt}$. The boundary condition at the ground is not, therefore, $\omega = 0$. Physically the ground behaves as though it were elastic in the analogue system. The atmosphere behaves like a liquid of finite height on a horizontal membrane.

There is a singularity at the top of the layer. $N_a^2 \sim \frac{1}{(H-z_a)^2}$ as $z_a \rightarrow H$, where H is the analogue height of the fluid layer.

In the quasi-geostrophic approximation the elasticity of the ground can be ignored and we may assume it to be rigid, but for the balance equations this simplification cannot be made.

These notes submitted by

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Lecture #XIII

The Baroclinic Instability

We wish to examine the stability characteristics of a westerly air stream. The wind velocity U is assumed to be a function only of the northward coordinate y and the height z, and it will also be assumed that the quasi-geostrophic equations apply. N^2 is a function of z. The stream-function ψ may be decomposed into the basic flow, $\Psi(y,z)$ and a perturbation $\varphi(x,y,z,t)$:

$$\psi = \Psi + \varphi \, , \quad \Psi_y = -U \, .$$

The basic potential vorticity is

$$Q = \Psi_{yy} + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\Psi_z\right) + f_0 + \beta y$$

where f_0 is the value of f at $y=0$. Hence

$$\frac{\partial Q}{\partial y} = -U_{yy} - \frac{\partial}{\partial z}\left(\frac{1}{N^2} U_z\right) + \beta .$$

The first term is the gradient of relative vorticity of the basic flow, the second is that part of the potential vorticity gradient which is directly associated with temperature gradients and the third is the gradient of vorticity of the earth's rotation. On linearisation, the conservation of potential vorticity is expressed by

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left[\nabla_h^2 \varphi + \frac{\partial}{\partial z}\left(\frac{1}{N^2}\varphi_z\right)\right] + \varphi_x \frac{\partial Q}{\partial y} = 0 .$$

Although the flow has no definite boundaries expressed by y = const., in order to obtain a well-defined problem we shall postulate the existence of such boundaries across which we shall assume no fluid flows, i.e. $\varphi_x = 0$ on y = const. The condition that there is no flow through the ground at $z = -1$ is expressed by $\frac{D}{Dt}\psi_z = 0$. The linearised form is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\varphi_z - U_z \varphi_x = 0 \quad \text{on} \quad z = -1 .$$

As we approach the top of the atmosphere, in pressure coordinates $N^2 \to \infty$ it is necessary to impose the top boundary condition at a singularity. The difficulty can be overcome by means of a radiation condition. For simplicity, however, we shall regard N^2 as finite and bound the atmosphere with a rigid lid. The results are qualitatively the same as those obtained by doing the problem properly and exhibit the essential physics. We therefore take

$$\frac{D}{Dt}\psi_z = 0 \quad \text{on} \quad z = +1 .$$

We look for normal mode solutions. In order to study the stability of an arbitrary disturbance it is necessary to obtain a complete set of normal modes, but usually the problem does not yield a complete set and we cannot determine the complete stability characteristics. For the special cases $N^2 =$ const. and $N^2 \propto z$ the set has been completed and it was found that the stability characteristics were unchanged. We thus set

$$\varphi = \hat{\varphi}(y,z)e^{ik(x-ct)} .$$

The mode will be unstable if $\mathcal{Im}(c) = c_i > 0$.

Substituting this form into the potential vorticity equation leads to

$$\frac{\partial}{\partial z}\left(\frac{1}{N^2}\hat{\varphi}_z\right) + \hat{\varphi}_{yy} + \left\{\frac{1}{U-c}\frac{\partial Q}{\partial y} - k^2\right\}\hat{\varphi} = 0,$$

with $\quad (U-c)\hat{\varphi}_z - U_z\hat{\varphi} = 0 \quad \text{on} \quad z = \pm 1$

and $\quad \hat{\varphi} = 0 \quad \text{on} \quad y = 0, \pi/\ell$.

This equation has been solved only under restrictive conditions.

<u>If $U = U(y)$ only</u> ,

$\frac{dQ}{dy} = \beta - U_{yy}$ is independent of z and separable solutions of the form $\hat{\varphi} = f(z)g(y)$ can be found. The function $f(z)$ turns out to be the vertical structure for internal gravitational waves in a stratified atmosphere. $g(y)$ is essentially the same function that arises in Rayleigh's inviscid analysis for the stability of a homogeneous non-rotating liquid with $U = U(y)$. The flow is stable unless $\beta - U_{yy}$ changes sign somewhere in the flow, i.e. if $\frac{dQ}{dy}$ is single-signed everywhere, the flow is stable. Since the only effect of gravity is through N^2, which does not appear in this case, there can be no potential energy available for the disturbance. The energy of the disturbance is drawn entirely from the kinetic energy of the basic flow.

When $U = U(z)$ only,

if we furthermore assume that N^2 = const. and $\beta = 0$, then $\frac{\partial Q}{\partial y}$ is independent of y. We can take $\hat{\varphi} = \hat{\hat{\varphi}}(z) \sin \ell n y$ and obtain

$$\hat{\hat{\varphi}}_{zz} + \left[\frac{U_{zz}}{U-C} - (k^2 + n^2 \ell^2)\right] \hat{\hat{\varphi}} = 0 .$$

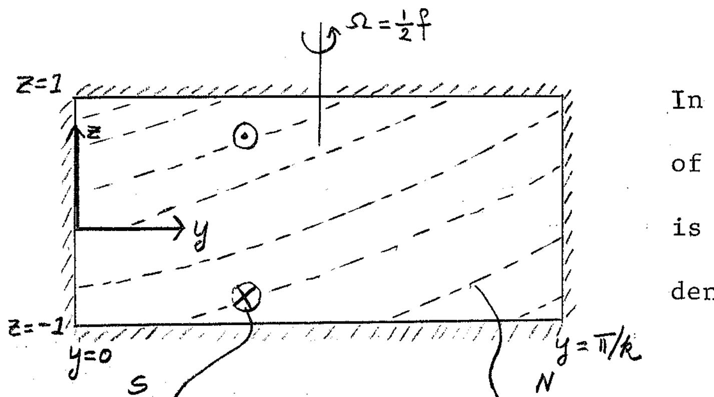
This is again the equation which arises in Rayleigh's problem but now the boundary conditions are no longer the same. In our problem the boundary conditions are $\hat{\hat{\varphi}}_z - \frac{U_z}{U-C} \hat{\hat{\varphi}} = 0$ at $z = \pm 1$ whereas in the Rayleigh problem $\phi_x = 0$ on $z = \pm 1$. Our boundary condition corresponds to a free surface in the Rayleigh problem, that is, one on which there are no pressure perturbations. In the Rayleigh problem $U_{zz} = 0$ is necessary for instability with rigid boundaries, but for free boundaries this is no longer true.

It is thought that our major weather systems are driven by the baroclinic instability.

The Eady Problem

This is a simplified model which can be solved analytically and is designed to exhibit the important features of the baroclinic instability. We consider the stability of the flow in the x-direction,

$U = z$, of a Boussinesq liquid in a rectangular channel which is rotating with an angular velocity $\Omega = \frac{1}{2}f$ about the z-direction. Furthermore $N^2 = 1$, $\beta = 0$ in our dimensionless coordinates, and we take $n = 1$ too.



In physical space the slope of the lines θ = const. is $\frac{f'U_z'}{N'^2} \ll 1$ where a prime denotes dimensional quantities.

The basic potential vorticity Q is constant because

$$Q = \underbrace{-U_y}_{=0} + \underbrace{\frac{\partial}{\partial z}\left(\frac{1}{N^2}\Psi_z\right)}_{=0} + f_0 \ .$$

If we introduce a sinusoidal variation in the x-direction the equation for the perturbed stream-function becomes

$$\hat{\hat{\varphi}}_{zz} - (k^2 + \ell^2)\hat{\hat{\varphi}} = 0 \ ,$$

which has the solution

$$\hat{\hat{\varphi}} = A \cosh \varkappa(z - \alpha)$$

where A and α are constants and $\varkappa^2 = k^2 + \ell^2$.

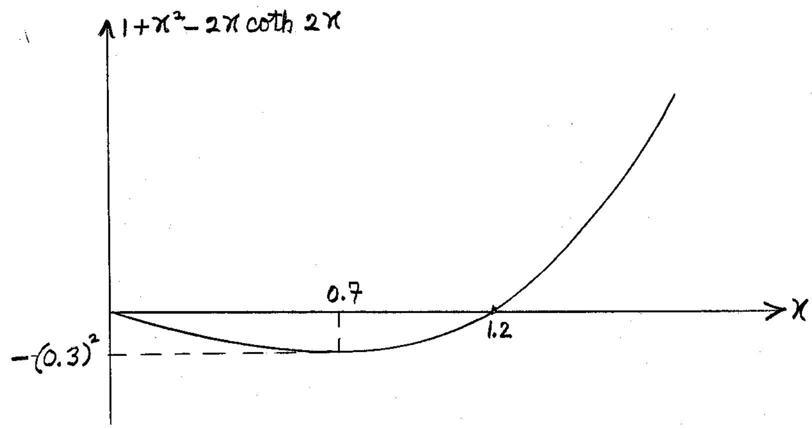
The constants are determined from the boundary conditions

$$\frac{\hat{\hat{\varphi}}_z}{\hat{\hat{\varphi}}} = \varkappa \tanh \varkappa(z-\alpha) = \frac{U'}{U-c} = \frac{1}{1-c} \quad \text{on} \quad z = 1 \ ,$$

$$= \frac{1}{-1-c} \quad \text{on} \quad z = -1 \ .$$

Eliminating α yields an equation for c,

$$\varkappa^2 c^2 = 1 + \varkappa^2 - 2\varkappa \coth 2\varkappa .$$



For $\varkappa < 1.2$, $c^2 < 0$ and the mode is unstable. This is a case of a "short-wave cut-off". Long waves are unstable. The mode is unstable if

$$\varkappa = \frac{\sqrt{k^2+\ell^2}\, H' N'}{f'} < 1.2 .$$

If the rotation rate is increased, f' is increased and the flow is unstable to smaller and smaller wavelength disturbances. The "cut-off wavelength" is independent of U_z, and hence the horizontal potential temperature gradient, although this is the cause of the instability, but the horizontal potential temperature gradient does determine the true growth rate which is $\sim \frac{f'}{N'} \frac{dU'}{dz} \sqrt{-\varkappa^2 c^2}$.

<u>The Neutral Solution</u>

It is of interest to examine the flow pattern for the neutral solution:

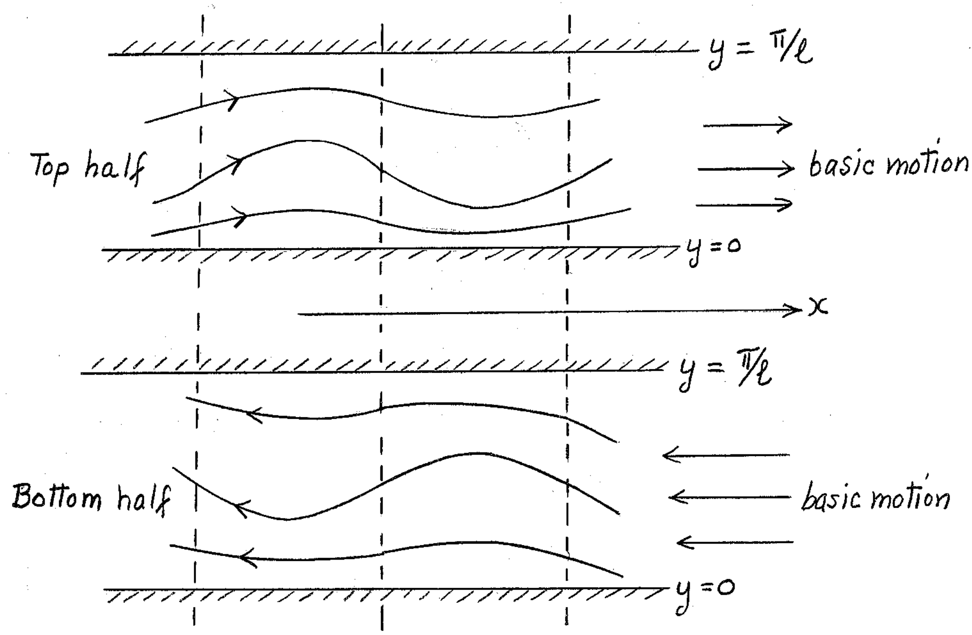
$$\varkappa = 1.2 \quad \text{and} \quad \alpha = 0$$

$$\varphi = A \cosh \varkappa z \cos kx \sin \ell y .$$

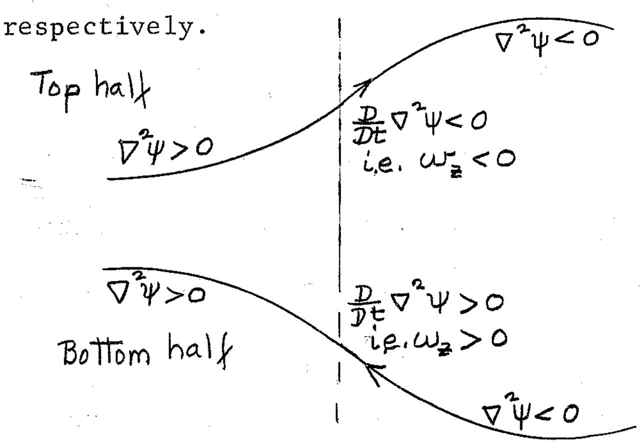
Note that normally there is a basic westerly wind superposed on this solution which does not change the stability but carries the solution to the east with a constant velocity.

φ is a pressure perturbation which is an even function of z .
$\sigma = -\varphi_z$ is odd in z and ω is even.

The following flow patterns are as seen looking down onto the disturbance.

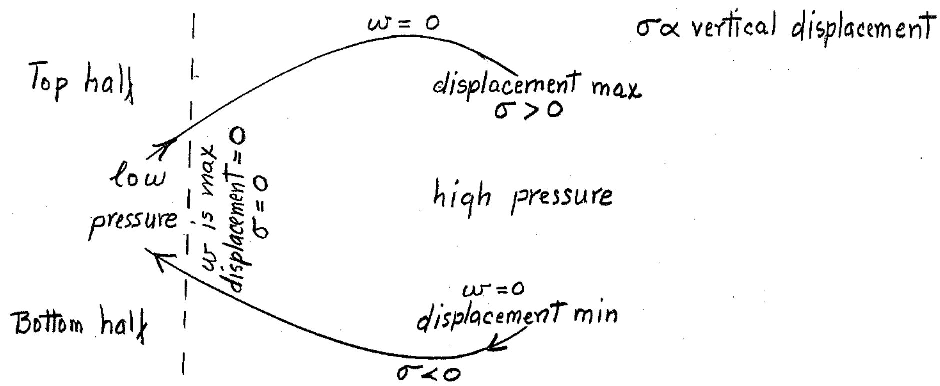


For clarity we isolate a single mode in each of the top and bottom halves respectively.



Taking the boundary conditions $\omega = 0$ at $z = \pm 1$ in consideration, the values of ω_z tell us that at the dotted line in the diagram above, there is rising motion; $\omega > 0$ in both halves.

Let us now consider the density field.



The rate of working of the buoyancy forces $\overline{\sigma\omega}$ is zero for these steady motions since σ and ω are $\frac{\pi}{2}$ out of phase.

Growing Disturbances

φ is no longer an even function of z. We then have the following expressions for σ and ω:

$$\sigma = -A\varkappa \sinh \varkappa(z-\alpha) \cos kx \sin ly\, e^{-ikct}$$
$$\omega = ikc\sigma + \sigma_x \varphi_y - \sigma_y \varphi_x .$$

Hence $\overline{\omega\sigma} = ikc\overline{\sigma^2}$ and since $ikc < 0$ for unstable disturbances $\overline{\omega\sigma} < 0$. Rising motions are, on average, associated with negative σ which means that potential energy is being converted into kinetic energy of the disturbance. For small growth rates, the flow pattern looks qualitatively like the neutral solution only with the top pattern displaced slightly to the left (west) relative to the bottom. Although this model is very idealised, the solution qualitatively agrees with observations of the atmosphere. This is the only justification for bothering the reader with the analysis.

If we have a small β-effect, then $\frac{\partial Q}{\partial y} \neq 0$. This introduces a term $\frac{\beta}{U-c}$ into the equation for $\hat{\hat{\varphi}}$ which causes a singularity when c is real. This drastically changes the form of the motion; now all modes are unstable, but the growth rate for a given k is very much less than that due to the normal baroclinic instability where we set $\beta = 0$.

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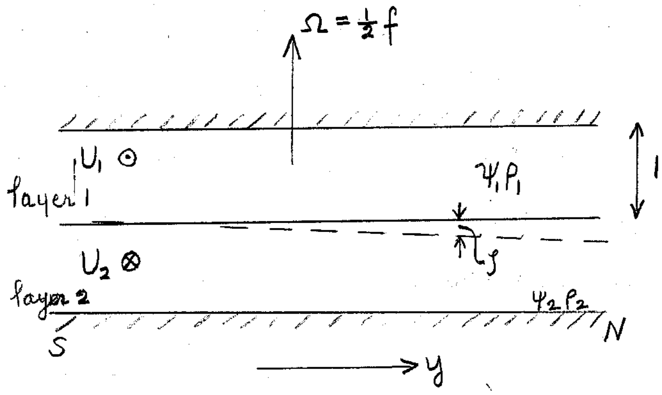
Lecture #XIV

A Physical Explanation of the Baroclinic Instability

Because of its simpler physics we shall now consider a two-layer model to explore the baroclinic instability and later compare it with the Eady model.



We assume U is a function of z only, N^2 is constant, $\rho_1 < \rho_2$ with $\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} << 1$, and that each layer is homogeneous.

The slope of the interface is assumed to be small. All the gravitational restoring force is at the interface between the two fluids. If $R_c << 1$, each layer of fluid moves in Taylor columns which do not have constant length because $\zeta = \zeta(y)$. The equation for conservation of potential vorticity in the upper layer is

$$\frac{D}{Dt}\left\{\nabla_h^2 \psi_1 + 1 . \zeta\right\} = 0, \qquad \text{ignoring } \beta\text{-effects.}$$

expresses change of vorticity due to slope of interface.

The stream functions ψ_1, ψ_2 in the upper and lower layers are equal to the pressure perturbations from the basic state divided by density; but this pressure is due only to the height of the interface. Hence, $\psi_2 - \psi_1 = \zeta$.

Thus $$\frac{D}{Dt}\left(\nabla_h^2 \psi_1 + \psi_2 - \psi_1\right) = 0 .$$

Similarly $\frac{D}{Dt}\left(\nabla_h^2\psi_2 - \psi_2+\psi_1\right) = 0$ in the lower layer. Choose axes moving such that $U_1 = -U_2$. The gradient of potential vorticity of the basic state is then

$$\frac{dQ_1}{dy} = -U_{yy} - (U_2-U_1)$$
$$= 2U, \quad \text{because } U \text{ is constant.}$$

Also $\frac{dQ_2}{dy} = -2U.$

We shall now examine the stability of the system by considering disturbances of the form

$$\psi_1 = \Psi_1 + \mathcal{R}e\, \hat{\varphi}_1 e^{i(kx+ly-kct)},$$
$$\psi_2 = \Psi_2 + \mathcal{R}e\, \hat{\varphi}_2 e^{i(kx+ly-kct)}$$

where Ψ_1, Ψ_2 are stream functions for the basic state. By substituting these expressions in the equations and matching the solutions at the interface, application of the usual boundary conditions yields the dispersion relation

$$c^2 = U^2\frac{\kappa^2-2}{\kappa^2+2}$$

where $\kappa^2 = k^2+l^2$. c is real only if $\kappa^2 > 2$. Once again we have a "short-wave cut-off" - the motion is unstable for long waves. It turns out that

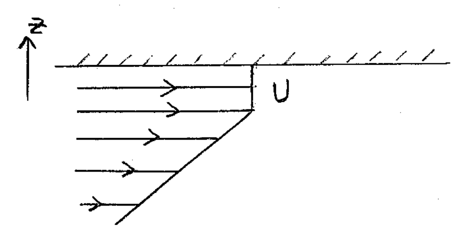
$$\frac{\hat{\varphi}_1}{\hat{\varphi}_2} = \frac{U+c}{U-c}$$

For stable modes, c is real and φ_1 and φ_2 are in phase, but have different amplitudes. For unstable modes, however, $|\varphi_1| = |\varphi_2|$ but they are out of phase.

The similarity between the results of this and the Eady problem is very marked. In the two-layer model the instability arises from variations in the potential vorticity which are absent in the Eady model. The Eady model, however, contains variations in potential temperature which are absent in the two-layer model. On considerable reflection the distinguished lecturer thought of these temperature variations as regions of extremely concentrated vorticity.

The natural boundary conditions to apply at the upper and lower boundaries is ψ_z = const, independent of x and y, which is achieved when there are no horizontal potential temperature gradients on the boundary. $\frac{D}{Dt}(\psi_z) = 0$ implies that ψ_z will continue to be the same constant for the perturbed flow. Hence $\frac{dU}{dz} = 0$ on the boundary.

However, the basic flow $U = z$ of the Eady problem does not satisfy this condition, but it could be achieved by flattening the velocity profile such that U is constant in an infinitesimally small layer near the boundary. In doing



this we have removed the horizontal potential temperature gradients at the boundary but introduced an infinitesimally small region of highly concentrated potential vorticity. Although we have modified the precise statement of the problem by changing the basic equilibrium flow, it is believed that the important physics determining the instability is the same. This change in the problem has introduced delta-functions in z into the basic potential vorticity in the neighbourhood of the boundaries. Mathematically, we have thereby removed that part of the cause of the instability arising from the boundary conditions and transferred it to the interior flow field.

Since N^2 is constant the stream-function is related to the potential vorticity by

$$q = \psi_{xx} + \psi_{yy} + \psi_{zz}$$

This may be solved by obtaining a Green's function, G which is a solution of

$$\nabla^2 G = \delta(x, y, z) ,$$

and satisfies the boundary conditions: $G_z = 0$ on $z = \pm 1$. As in electrostatics this Poisson equation may be solved by the method of images - the image system required is an array of equally-spaced images on a straight line parallel to the z-axis. This may be represented by Fourier cosine series in z.

We now do the same with the two-layer model. Take

$$q_1 = \nabla_h^2 \psi_1 + \psi_2 - \psi_1 ,$$

$$q_2 = \nabla_h^2 \psi_2 + \psi_1 - \psi_2 .$$

Set

$$q_i = \hat{q}_i \cos kx \cos ly , \quad i = 1, 2 .$$

$$\nabla^2(\psi_1 + \psi_2) = q_1 + q_2 ,$$

$$(\nabla^2 - 2)(\psi_1 - \psi_2) = q_1 - q_2 .$$

Hence $$\psi_1 + \psi_2 = \frac{\hat{q}_1 + \hat{q}_2}{\kappa^2}$$

and $$\psi_1 - \psi_2 = \frac{\hat{q}_1 - \hat{q}_2}{\kappa^2 + 2}$$ where $\kappa^2 = k^2 + \ell^2$.

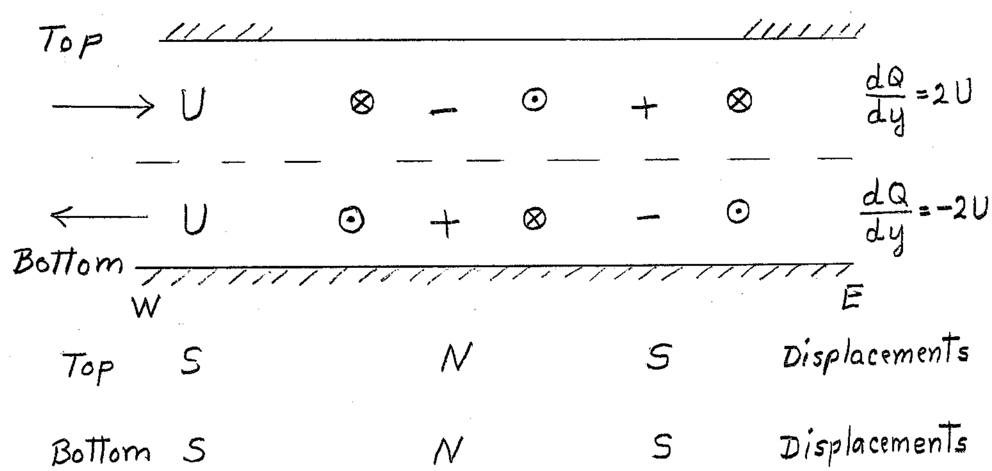
For short wavelength disturbances $\kappa^2 + 2 \approx \kappa^2$, and if $\hat{q}_2 = 0$,

$$\psi_1 \approx \frac{\hat{q}_1}{\kappa^2}$$

and $$\psi_2 \approx 0 .$$

From this it is clear that a disturbance of small wavelength originating in the upper layer is almost entirely confined to that layer. But this is not true of large wavelengths. Because the boundary conditions are $\psi_z = 0$ at $z = \pm 1$, the image system for a single vortex element is an array of vortices oriented in the same direction. Hence, if $q_1 = -q_2$, we have a line of dipoles which induces velocities whose long wavelength components largely cancel. If $q_1 = q_2$ we simply have a line vortex and $\psi_1 \sim \psi_2 \propto \frac{1}{\kappa^2}$.

We will now attempt to follow in detail the dynamics of a horizontal disturbance in the two-layer model.



This diagram qualitatively represents the excess of vorticity created in the two layers by an initially sinusoidal distribution of horizontal displacements. A northward displacement in the upper layer gives rise to a negative excess of vorticity which causes an anticyclonic rotation. In the bottom layer, however, the opposite is the case. The plus and minus signs represent the sign of the excess potential vorticity and the arrows $\odot$, $\otimes$ the direction of the corresponding flow. The potential vorticities are equal and opposite in the top and bottom layers.

$$\varphi_1 = -\varphi_2 = \varphi .$$

$$\therefore \varphi = \tfrac{1}{2}(\varphi_1 - \varphi_2) = \frac{q}{2+\chi^2} \quad \text{where} \quad q = q_1 = -q_2 .$$

The time development of this pattern is governed by the equation

$$\frac{\partial q_1}{\partial t} = \left[-U + \frac{\partial Q_1}{\partial y}\,\frac{1}{2+\chi^2}\right]\frac{\partial q_1}{\partial x} ,$$

$$= \left[-U + \frac{2U}{2+\chi^2}\right]\frac{\partial q_1}{\partial x} .$$

Since $\frac{\partial q_1}{\partial t} \propto \frac{\partial q_1}{\partial x}$ and q_1 is sinusoidal in x, the distribution of $\frac{\partial q_1}{\partial t}$ is $\pi/2$ out of phase with q_1. The sign is determined by the sign of the term in parentheses; we note that the first term is greater than the second and the situation is dominated by advection of potential vorticity by the basic flow. The effect is most easily seen by returning to the diagram.

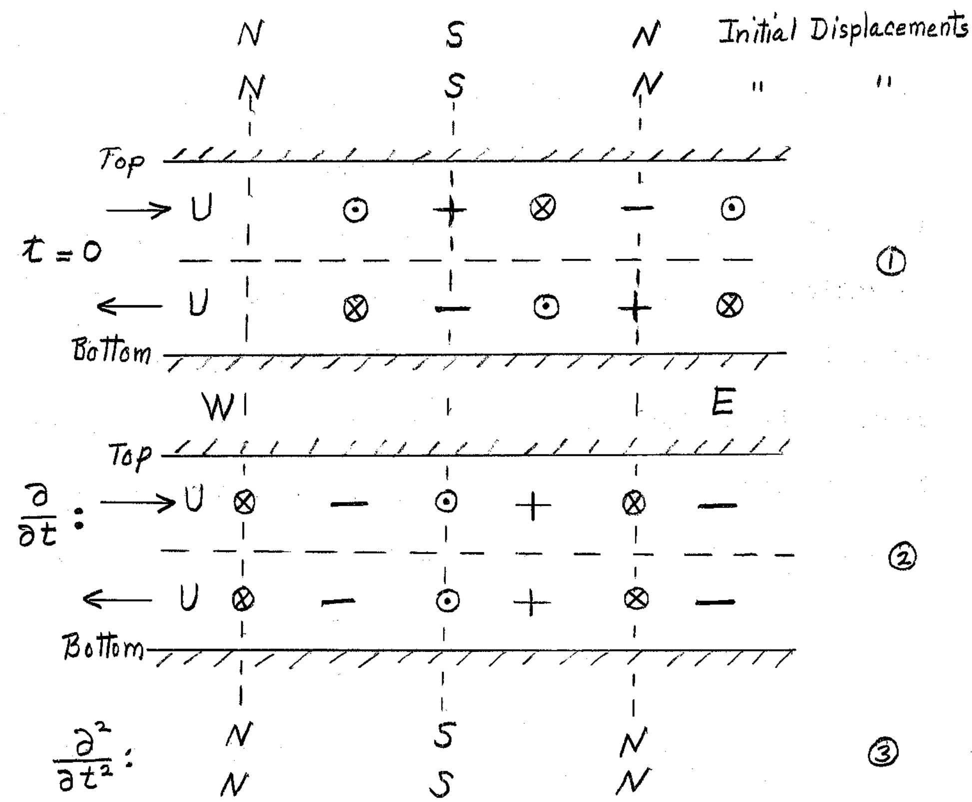


Diagram (1) represents the initial situation as previously illustrated.

Diagram (2) is the difference between the situation at time Δt and the initial state; it represents the first time derivative of the flow which is dominated by the advection of potential vorticity by the basic flow.

Diagram (3) shows the displacements induced by this time development of the initial displacements.

It is clear that the displacements have the same sign in the two layers and are such as to reinforce the initial disturbance. The flow is therefore unstable.

But we have not explained the "short-wave cut-off". The velocities arising from the initial displacements compete with the basic velocity in advecting vorticity. In obtaining (3) it has been assumed that it is the new induced vorticities which dominate over the basic advection in determining the displacements which ensue. This must be justified.

The difference in advection between time $t = 0$ and $t = \Delta t$ is estimated by

$$\frac{\partial q_1}{\partial t} = \left[-U + \frac{\partial Q_1}{\partial y} \frac{1}{\kappa^2} \right] q_{1x} \qquad \left(\frac{\partial Q_1}{\partial y} = 2U \right)$$

because now $\varphi_1 = \varphi_2$ and $q_1 = q_2$ and so $\varphi_1 = \frac{q_1}{\kappa^2}$. The equation is dominated by advection due to the perturbation if $\kappa^2 < 2$ in which case situation (3) ensues. The flow is thus unstable. If $\kappa^2 > 2$, this physical mechanism is removed. Advection is dominated by the basic flow and this instability is no longer evident.

Thus we see how the two-layer model exhibits a qualitatively correct physical explanation for the instability mechanism arising in the Eady problem.

These notes submitted by

Douglas Gough
Hans C. G. True

ABSTRACTS

SOLAR CONVECTION

Edward A. Spiegel

A short review of observations of solar convection was presented and physical interpretations of several of the phenomena were suggested.

LARGE-AMPLITUDE CONVECTION

George Veronis

The stream function and the temperature fluctuation in the two-dimensional system of equations for Bénard convection are expanded in a series of eigenfunctions. The series is then truncated and the resulting set of ordinary non-linear differential equations is integrated numerically to derive the steady-state solutions. Comparison is then made of the heat flow for a given Rayleigh number for different numbers of terms in the expansion. Convergent results are derived for values of the Rayleigh number which are 30 times the critical.

It is shown that the most severely truncated system corresponds to ordinary second-order theory in the Malkus-Veronis expansion scheme and that more complete representations (more eigenfunctions) yield heat transports which are more than twice the value derived from the second-order system. Different values of the Prandtl number, σ , yield different heat fluxes with the maximum (the change is very small) values occurring for small σ .

MINIMUM AND MAXIMUM PRINCIPLES FOR VISCOUS FLOW

Joseph B. Keller

Helmholtz considered the rate of energy dissipation in a slow steady flow of an incompressible viscous fluid acted upon by forces derivable from a single valued potential. He asserted that this rate is smaller than that of any other incompressible flow satisfying the same boundary conditions, but he proved only that it is stationary. Korteweg proved the statement when the velocity is prescribed at the boundary. In this lecture the theorem is proved for other boundary conditions and generalized to include flows containing moving solid objects, liquid drops or gas bubbles. From these results it follows that the flow in question is unique. It also follows that the Stokes flow yields a lower bound for the drag on an object, that laminar flow in a pipe has a lower resistance coefficient than turbulent flow and various other facts. It is shown how these results can be used to obtain upper bounds on the effective viscosity and sedimentation velocity of a suspension and a lower bound on the velocity of rise of a gas bubble. The results are also used to clarify some aspects of the principle of the minimum rate of entropy production. Finally it is shown how corresponding maximum principles can be proved and used to obtain opposite bounds on various quantities.

NONLINEAR WAVES

Frederic E. Bisshopp

The equation,

$$\frac{\partial^2\varphi}{\partial t^2} - \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial V}{\partial \varphi} = 0,$$

which becomes the Klein-Gordon equation when $V = \frac{1}{2}\varphi^2$, was considered for the purpose of illustration of some properties of nonlinear waves. When V has a relative minimum at $\varphi = 0$ (say), there are 'plane wave' solutions

$$\varphi = f(kx - \omega t),\ \omega^2 - k^2 = 1,\ \tfrac{1}{2}f'^2 + V(f) = E;$$

where prime denotes differentiation with respect to argument, and E is a constant. The problem treated here was that of 'almost plane waves' where the quantities k, ω, and E exist and are slowly varying functions of position and time. A quantitative measure of slowness of variation of the appropriate quantities can be introduced by defining two scales of variation according to the ansatz:

$$\varphi = \varphi(X, T, \vartheta, \epsilon),\ X = \epsilon x,\ T = \epsilon t,\ \vartheta = P(X, T, \epsilon)/\epsilon .$$

'Almost plane' waves are then ones for which

$$\epsilon \to 0,\ k \equiv P_X = O(1),\ \omega \equiv -P_T = O(1).$$

The key point in the determination of asymptotic solutions in the above limit is the requirement that for any fixed X and T, the approximation to φ should be a bounded function of ϑ. It follows that

$$\varphi \sim f(\vartheta, E) + \epsilon f_\vartheta(\vartheta, E)\, g(\vartheta, E)$$

where

$$\tfrac{1}{2} f_\vartheta^2 + V(f) = E,\ \vartheta = P/\epsilon .$$

The slowly varying quantities, P and E, are determined by the relation

$$E = E(J) \text{ where } J(E) = \oint f_\vartheta\, df .$$

J is the 'action' and its dependence on X and T is determined by $(\omega J)_T + (kJ)_X = 0,\ \omega_X + k_T = 0,\ \omega^2 - k^2 = 1$.

The periodicity of the oscillation is $\Lambda = \frac{1}{2\pi} J'(E)$ and this determines the local wave number and frequency, which are k/Λ and ω/Λ.

The phase is given by
$$P \sim kX - \omega T - \epsilon \vartheta_0(X,T)$$
where
$$(\omega \vartheta_{0T} + k\vartheta_{0X}) + \frac{1}{\pi\Lambda}\int_0^{\pi\Lambda} \frac{(\omega \mathcal{J})_T + (k\mathcal{J})_X}{f_\xi^2}\, d\xi = 0,$$
$$\mathcal{J} = \int_0^\xi f_{\bar\xi}^2\, d\bar\xi,$$
and $f(\xi, E) = 0$, $f_\xi(\xi, E) > 0$ at $\xi = 0$.

Finally $g(\vartheta, E) = g(0,E) + (\omega\vartheta_{0T} + k\vartheta_{0X})\vartheta + \int_0^\vartheta \frac{(\omega\mathcal{J})_T + (k\mathcal{J})_X}{f_\xi^2}\, d\xi$

where $g(0,E)$ can be obtained, with difficulty, from initial data at $T = 0$ (say).

The exact form of the ansatz introduced to describe 'almost plane' waves was suggested to me by Martin Kruskal. The results obtained here display one or two intriguing points of similarity with his detailed results for Hamiltonian systems of ordinary differential equations. Indeed G.B.Whitham has pointed out on several occasions that the equations governing k, ω and J can be derived from an averaged Lagrangian density of the original problem, but his formulation does not produce equations for the first order quantities, ϑ_0 and g.

References

Kruskal, Martin, 1962: J.Math.Phys. 3, p.806.
Whitham, G.B., 1965: J.F.M. 22, p.273.

A VARIABLE DENSITY MODEL OF THE GULF STREAM

Stanley Jacobs

The equations governing a variable-density boundary current on a meridional boundary are transformed so that Y, y, and T are the independent variables, where Y is a stream-like function, y is distance measured northwards, and T is the temperature above the mean temperature. It is found that when a certain non-dimensional parameter δ is small, corresponding to strong stable stratification, solutions can be obtained by making an ordinary perturbation expansion in δ in a region away from the upper and lower boundaries of the system. Since the vertical velocity in this region is of order δ, the flow in the main body of the fluid to lowest order is planar.

It is found that the vertical velocity as computed from the ordinary perturbation expansion does not vanish at a horizontal boundary surface unless this surface is isothermal. As the temperature of the ocean surface is non-constant, it is inferred that the ordinary perturbation expansion is invalid near the surface, and that the region near the surface is a boundary layer in which flow varies sharply with changes in T. This boundary layer is identified as the main thermocline of the ocean. The thermocline equations are formulated but have not as yet been solved.

FLUCTUATING OCEAN CIRCULATION

Joseph Pedlosky

The unsteady motions of a homogeneous bounded ocean on the β -plane are studied. Both the free normal modes and the forced solutions for the linearized problem are computed. The non-linear response is computed by a perturbation analysis. Of particular interest is the steady (time-dependent) circulations produced by a fluctuating wind stress with zero time-mean due to the non-linearities of the dynamics. It is shown that the structure of the resulting circulations, their strength, and their sense are strong functions of the frequency of the forcing stress. Depending on the magnitude of the frequency the resulting circulations may have: 1) only a western boundary layer (low frequency), 2) no boundary layers (frequencies less than a typical Rossby wave frequency for the basin), 3) boundary layers on both eastern and western walls (very high frequency).

HYDRODYNAMIC STABILITY OF THE EKMAN BOUNDARY LAYER

Louis N. Howard

This lecture reports some results of a numerical study of the stability of the simplest Ekman layer, the non-divergent one. In addition to its interest in geophysical fluid dynamics this problem is of basic interest in hydrodynamic stability theory since the non-divergent Ekman flow is in fact an exact solution of the Navier-Stokes equations, and thus its stability problem appears as fundamental as that of the Couette flow between rotating cylinders, the Poiseuille pipe flow, and the two-dimensional channel flows. Indeed, it appears to be the only steady exact solution of boundary layer flow for which the parallel flow assumption is rigorous.

The problem is however not very easy, and the present study was undertaken in part because of the availability of a fairly convenient and efficient program developed originally for the study of boundary layer stability problems based on the Orr-Sommerfeld equation by R. Kaplan, and adapted for use on the M.I.T. time-sharing system by M. Landahl and L. N. Howard. The use of this program, or family of programs, is however also the principal limitation of this study, because the actual stability equations for the Ekman-layer problem form a sixth-order system which while similar to the Orr-Sommerfeld problem is not identical with it. However, V. Barcilon, in his study of the problem by the asymptotic method, showed that the Coriolis terms, while essential for the basic flow, are of relatively small importance in the stability problem when the Reynolds number is fairly large, and this appears to be the case of main interest. When these terms are neglected, the sixth-order system splits into the fourth-order Orr-Sommerfeld equation with a basic velocity profile which is the projection of the Ekman flow onto the plane orthogonal to the wave crests, and a second-order equation. The stability characteristics can then be obtained by studying the Orr-Sommerfeld equation alone. However, a precise estimate of the errors implied by this neglect of the Coriolis terms is not yet available, and the results must be taken with this in mind.

Calculations were made for wave angles β between 60^o and 105^o in steps of 7.5^o; here $\beta = 90^o$ corresponds to a wave whose crests are at right angles to the direction of the geostrophic flow above the Ekman layer, and with this definition Faller's experiments gave for the average β of the observed waves a value of about 75^o.

The lowest critical Reynolds number found by the calculation was about 84.8, corresponding to $\beta = 89^o$ and a wave number $\alpha_c = 0.485$. This value of the wave number agrees pretty well with Faller's observations. His critical Reynolds number was somewhat larger, about 125, and the wave angle somewhat smaller. It is probable that the observed waves would correspond to slightly amplified rather than neutral waves in the theory, and in fact the wave angle for the most unstable wave does decrease somewhat from 89^o as the Reynolds number is raised.

OVERSTABILITY IN A COMPRESSIBLE ATMOSPHERE

Edward A. Spiegel

It was demonstrated that in a compressible, convectively unstable atmosphere, thermal dissipation may destabilize the acoustic modes.

THE MOTION OF A SPHERE THROUGH A ROTATING, VISCOUS FLUID

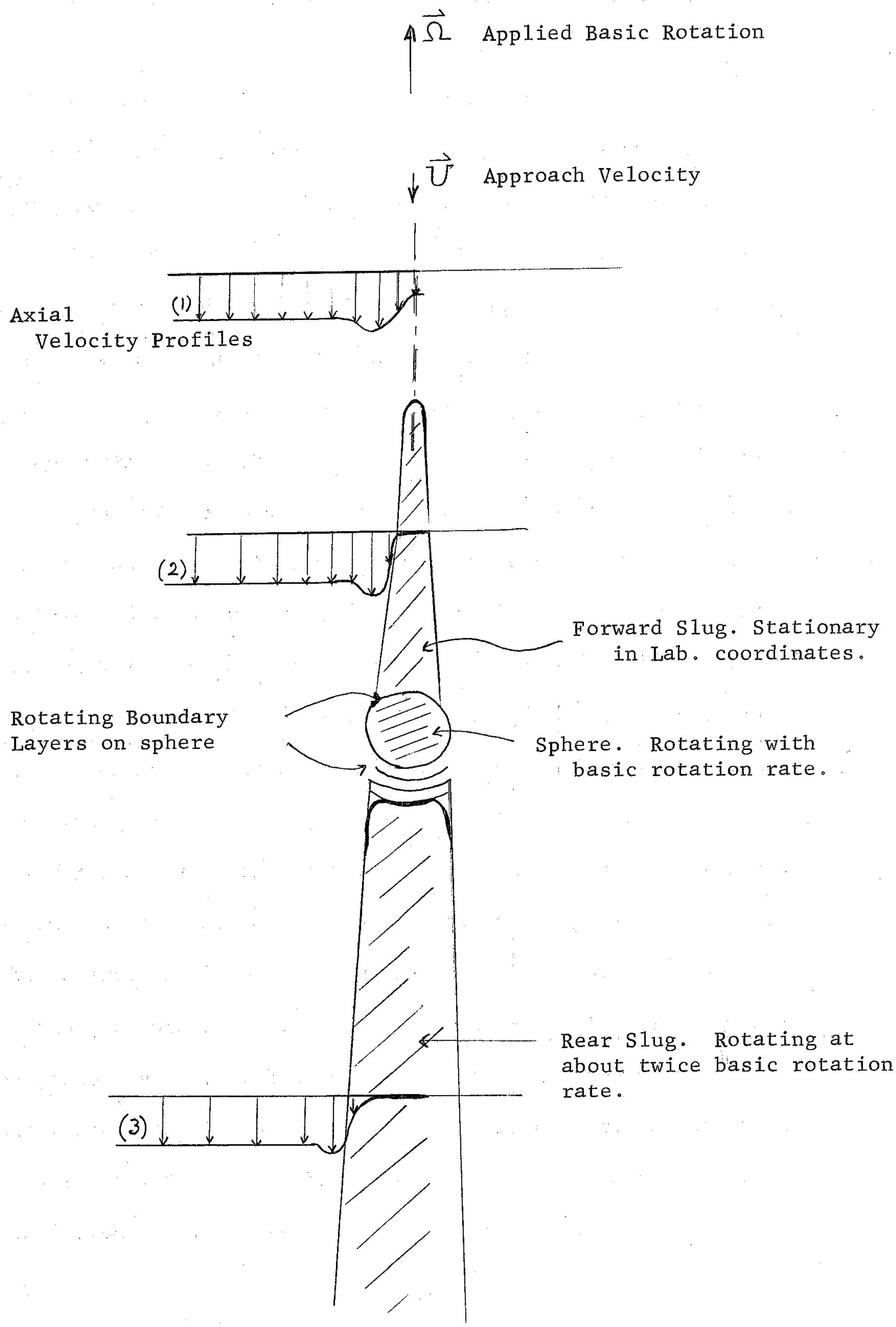
Tony Maxworthy

We are studying the effect of adding a Coriolis force to the forces normally acting on the fluid particles in a viscous flow field. Two parameters are important: a Taylor number (T), representing the relative magnitudes of Coriolis and viscous forces and a Rossby number (R_o), a measure of the relative magnitudes of inertia to Coriolis forces.

In the present work two approaches have been followed: to study very viscous flow in which rotation merely causes a small perturbation from Stokes' flow and an extension of the previous observations on the motions in a fluid of small viscosity, where the work discussed by Derek Moore, in the notes of the 1963 Geophysical Fluid Dynamics Programme, is centered. The former is described in detail in J.F.M. Vol. 22, so that only the latter, unpublished work, will be abstracted here.

Measurements of sphere drag at large T and varying R_o show that at small R_o (~ 0.1) the drag is only a function of the inviscid parameter R_o. At large values of R_o (> 1) the drag is less than the drag with no rotation. For all values of $R_o > 0.1$, the drag is a complicated function of R_o and T. Dye studies of the wake structure of a sphere at small R_o indicate a growing stagnant slug ahead of the body as R_o is decreased. At $R_o < 0.2$ the slug reaches a limiting length which is then only a linear function of T, for the small range of T available. The slug formed behind the body shows no such tendency to a limiting length and at small R_o is probably as long as the maximum length of test section ($\sim$ 5 feet). Both wakes are rotating with respect to the sphere, which under most circumstances is itself rotating at the same speed as the basic rotation. Thus complicated rotating boundary layers are formed fore and aft of the sphere in order to satisfy the non-slip boundary conditions (see sketch).

The "hydrogen-bubble technique" has been used to further study the flow field produced by the sphere. A thin platinum wire and the sphere are towed along together through the rotating fluid. At a given instant of time a short pulse of D.C. power is applied to the wire. The water in contact with the wire is hydrolysed and a thin line of H_2 bubbles formed; these are swept off and distorted by the flow and when photographed a short time later indicate the velocity field created by the sphere. These show a wake-like structure extending beyond the slugs mentioned above. At low R_o the velocity profile in the forward disturbance has many features in common with the theoretical calculations. However the rearward wake has a rather different structure which requires more analysis before it is completely understood. At moderate R_o it is rotating very rapidly ($\sim$6 times the basic rotation rate), a state which persists until R_o becomes quite small in which limit it approaches more closely to the predicted form. Further work is to be carried out on these latter aspects of the problem.



$\vec{\Omega}$ Applied Basic Rotation
$\vec{U}$ Approach Velocity
Axial Velocity Profiles
(1)
(2)
(3)
Forward Slug. Stationary in Lab. coordinates.
Rotating Boundary Layers on sphere
Sphere. Rotating with basic rotation rate.
Rear Slug. Rotating at about twice basic rotation rate.

HYDROMAGNETIC INSTABILITIES OF THE SUB-ALFVÉN EQUATIONS

Willem V.R. Malkus

A class of hydromagnetic instabilities is found which may be related to the observed secular variations of the earth's magnetic field. This study was suggested by recent experiments on flow in precessing spheroids. It is observed that steady precession induces zonal (toroidal) flows with sharp changes in slope (jets). Within a part of the laboratory range of parameters, quasi two-dimensional wave-like instabilities occur on these zonal jets. These "planetary" waves move only to the "east", drawing on the energy of the toroidal flow to strengthen a poloidal circulation. In this study, an experimental situation is visualized in which a toroidal magnetic field is imposed on the existing precession-induced flow. Appropriate equations are derived for velocities and frequencies small compared to those of Alfvén waves. The conditions for the growth of hydromagnetic instabilities is explored. The problem proves to be non-singular in those terms responsible for electrical dissipation. The characteristic equation for marginal stability sets bounds on the basic zonal fields and determines a dispersion relation for the disturbances. The significant bound set on the magnetic field is that instability is possible only when the Lorentz force is less than the Coriolis force. The growing poloidal disturbance tends to stabilize itself by non-linear interactions which increase the underlying toroidal field. All disturbances move to the "west" relative to the jet which produces them. A concluding thought is that interaction of the poloidal fields with the boundary may increase the boundary stress sufficiently to reestablish the destabilizing jet flow, hence providing a closed description of a geodynamo.

A THEORY OF THE EQUATORIAL UNDERCURRENT

Allan R. Robinson

A uniform wind blows across a horizontal infinite ocean of finite and constant depth. All horizontal variation is due to the variation of Coriolis parameter (β -effect). Various natural scales and regions occur. Far from the equator the Ekman-Sverdrup flow determines the zonal pressure gradients and the transports which are latitudinally independent. At the equator the flow is studied, firstly for a zonal wind, whence the vertical and zonal velocities and the meridional velocity gradient at the equator form a separable problem. The constant eddy viscosity result yields qualitatively a Cromwell current. To inquire more precisely into the validity of this result, a turbulent model is developed free of independent parameters. The model is calibrated by Von Karman's constant and the proportionality constants for the variation of Austausch with Richardson number for low stability. This model indicates that the wind can account for the undercurrent phenomena. Further qualitative results are obtained for constant eddy viscosity but more general wind and pressure gradient forcing.

PROBLEMS OF GALACTIC DYNAMICS - NOTABLY SPIRAL ARMS

C. C. Lin

I. General Background

1. Galaxies of stars are usually classified in terms of their appearance into elliptical galaxies, normal spirals, barred spirals, and irregular galaxies. Most galaxies (about 70 per cent) are normal spirals whose side view is a disk with a central bulge.

2. The main contents of a galaxy are the stars, the gas, and the associated magnetic field. Components which are less important from a dynamical point of view are cosmic ray particles, (including high energy photons), other electromagnetic radiation of various wave lengths, dust, etc. The basic equations governing these main components are

(a) the equations of stellar dynamics, which consists of the collisionless Boltzmann equation, and Poisson's equation, with mass density contributed both by the stars and the gas, and

(b) the equations of hydromagnetics, including both the gas dynamical equations and the Maxwell equations.

The gas is "infinitely conducting" because of the large scales involved. Thus, the equations governing the magnetic field essentially state that the magnetic flux is frozen into the gas.

3. The dimensions of a normal spiral galaxy are of the order of 10^4 parsecs in radius and 600 parsecs in thickness outside of the central bulge and excluding the hato region. Thus, to a first approx-

imation, it may be regarded as an infinitesimally thin disk. (One parsec, (1 pc.) is approximately 3.24 light years or the distance covered in one million years at the speed of one kilometer per second.)

4. The stars are in differential rotation about the center of the galaxy. Indeed, for a major part of our own galaxy, the linear speed is nearly constant at 250 km/sec. The period of revolution about the galactic center is about 250 million years for our vicinity.

5. Besides the circular motion mentioned above, the individual stars have peculiar velocities, like the molecules of a gas. These velocities are however only of the order of 10 per cent of the circular velocity.

6. It is known that the contrast of stellar density between the spiral arms and the inter-arm regions is small. The contrast in gas density may, however, be as large as 3 or 4. The brilliant young stars are mostly associated with the gas.

7. It can be shown, by an estimation of the orders of magnitude of the various forces, that the magnetic field might be important for the scale of a spiral arm, but is definitely not important for the scale of a whole disk.

TURBULENCE MICRO-SYMPOSIUM

THE DIRECT-INTERACTION AND LAGRANGIAN-HISTORY DIRECT-INTERACTION CLOSURE APPROXIMATIONS FOR TURBULENCE

Robert H. Kraichnan

The expansion of the velocity covariance as a power series in the turbulent Reynolds number is discussed in the context of isotropic turbulence. The direct-interaction approximation for the covariance is obtained by a modification of the lowest non-trivial truncation of the power series. At very low Reynolds numbers, this approximation is presumed to be asymptotically exact. At all Reynolds numbers, the direct-interaction equations have certain invariance and boundedness properties in common with the exact dynamics: Conservation of energy by the nonlinear interaction, the existence of formal inviscid equipartition solutions, and non-negativity of the turbulence energy spectrum.

The analytical and numerical consequences of the approximation are discussed and compared with experiment. There seems to be fairly good quantitative agreement at modest Reynolds numbers (those of laboratory wind-tunnel experiments), but the approximation gives qualitatively incorrect predictions for the inertial range at high Reynolds numbers. This trouble is traced to failure of the approximation to preserve a further property of the exact equations: invariance of the dynamics under a uniform translation of the flow system which changes randomly from realization to realization.

In order to incorporate this invariance property, it is necessary to expand the formalism so that Eulerian and Lagrangian statistical quantities can be treated simultaneously. This is done, and the direct-interaction approximation then is altered in such a way that the history of the energy-transfer process is traced along the particle paths (Lagrangian-history) instead of at fixed stations in space. The resulting equations incorporate the desired invariance property and give inertial-range predictions in quantitative agreement with experiment.

The significance of these studies seems to be the following: No convergent expansion schemes for high Reynolds number turbulence are known, and consequently it is not possible to construct turbulence approximations whose errors are assuredly small. In this situation, it is very important that the approximations which are used preserve as far as possible the fundamental invariance and boundedness properties of the dynamics. This gives a hope of satisfactory qualitative behavior, and the latter, together with quantitative accuracy in some limit (low Reynolds number), gives the hope that errors will stay within reasonable bounds at all Reynolds numbers.

<u>References</u>

Kraichnan, R. H., 1964, <u>Phys.Fluids</u> <u>7</u>, 1030.

Kraichnan, R. H., 1965, <u>Phys.Fluids</u> <u>8</u>, 995.

THE SELF-CONSISTENT FIELD APPROACH TO TURBULENCE

Jackson Herring

A self-consistent field type perturbation theory is developed to treat the dynamics of stationary and homogeneous turbulence. The method consists in expanding the full probability distribution function about the product of exact single-mode distributions. The theory is used in second order to find expressions for the turbulent energy spectrum and associated response frequencies. The results for the energy spectrum are identical to a simplified form of the direct interaction approximation of Kraichnan, and closely resemble the results of the generalized random phase approximation of Edwards. The relation of the present method to both the above approaches is discussed.

OPTIMAL PROPERTIES OF THE MEAN FIELD EQUATIONS

Willem V.R. Malkus

Korteweg established that solutions of the Stokes equations, for any body force derivable from a potential, had a smaller mechanical dissipation rate than any other velocity field satisfying the divergence conditions and the boundary conditions. Keller considered a general body force, and established that the dissipation rate minus twice the total rate of work done by this force was a minimum for solutions of the Stokes equations. In this study the class of velocity fields considered is limited further by the general steady state integral condition that the mechanical dissipation rate must equal the

total rate at which work is done. Hence, solutions of the Stokes equations set an upper bound on the rate of work done by the imposed force. If the imposed force is the buoyancy in the Boussinesq form of the Navier-Stokes equations, then the Stokes solutions set an upper bound on the convection of heat. A similar study is made of the solutions of the Boussinesq form of the heat equation. It is found that, for a given velocity field, the rate of entropy production is an extremum for solutions of the thermal mean field equation. The constraints on the comparison temperature fields are that they satisfy the boundary conditions and the general steady state integral condition that the thermal dissipation rate equal the total production of the thermal fluctuations by the advection. For boundary conditions leading to either maximum or minimum entropy production, the convection of heat is a maximum for solutions of the thermal mean field equation, the velocity being given.

Despite the suggestive overlap of these two theorems, Howard has established by example that a joint solution of both the mechanical (Stokes) and thermal mean field equations does not upper bound heat flux under the imposed boundary and integral constraints.

ON THE LARGE SCALE DYNAMICS OF THE OCEANIC MIXED LAYER

Melvin E. Stern

We consider the weak non-linear interaction between hydrostatic eddies and the turbulent Ekman flow which is produced by a uniform wind stress (τ) acting at the top of the mixed layer. The theory describes the space-time fluctuation in the local Ekman transport due to interactions of a barotropic component ($\underline{V}_o$) of the total velocity with τ , and the effect of those fluctuations in amplifying the $\underline{V}_o$.

It has been applied to the problem of the generation of inertia-gravity waves in a two-layer density model driven by a constant τ . We show that plane waves oriented at right angles to τ , and propagating upwind, are unstable. The possibility of an experimental test of the theory and its controversial hypothesis is briefly discussed.

A WAVE-GUIDE MODEL FOR PRESSURE FLUCTUATIONS IN A TURBULENT SHEAR FLOW

Marten T. Landahl

A theory is presented in which the pressure fluctuations in a turbulent boundary layer, or other almost parallel shear flow, are expressed in terms of integrals involving squares of the fluctuating velocity components. It is shown that the resulting fluctuations, to a good approximation, may be represented by a superposition of vorticity waves of random phases and directions, i.e., the mean shear flow acts as a wave guide for the disturbances created by the non-linear turbulent interaction terms. The wave propagation constants are determined from the solution of a modified Orr-Sommerfeld stability problem for the turbulent mean velocity profile. Numerical calculations have been carried out for a flat-plate boundary layer. In this case it was found possible, without any additional assumptions, to predict the rate of decay of the turbulent eddies with downstream distance, and their propagation speed, which both are in excellent agreement with measured values.

OBSERVATIONS ON OBSERVATIONS

Erik Mollø-Christensen

A discussion of experiments on transition and turbulence, selected to illustrate the facts that:

(a) Transition, under ideal conditions, and with excitation to lock in the phase, often consists of a sequence of quite orderly processes.

(b) Even in fully turbulent flows, the flow may be instantaneously quite regular spatially, and, it is often possible, using suitable methods of observation, to discern large scale regular flow fields.

(c) A harmonic oscillation with randomized phase and amplitude, may look very much like turbulence as far as the lowest order statistical measures are concerned.

The examples chosen were:

Klebanoff's observation of boundary layer transition, Browand's observation of a free shear layer, Kresa's measurements of the large scale structure of turbulent cylinder wakes, Kholman's pictures of Couette Flow turbulence and Coles' observations of turbulent Taylor cells.

In the second lecture, methods of observation of turbulence and data processing were briefly covered, and results of jet turbulence measurements discussed. In particular, the covariance of pressure fluctuations within frequency bands were shown, showing how certain frequencies are unstable for limited ranges of correlation distance.

UNIFORMLY SELF-SIMILAR SPORADIC TURBULENCE

Benoit Mandelbrot

At very high Reynolds numbers (oceans, atmosphere), turbulence presents features that make it very doubtful whether it can be represented as a stationary stochastic process, and therefore mean that great care must be exerted in interpreting measured spectra. Turbulent flows are indeed an alternation of laminar and turbulent regions; however, any region that seems turbulent when examined with slow instruments turns out, when inspected at a finer scale, to contain a number of laminar inserts, so that the distinction between laminar and turbulent is at best hard to establish empirically. Moreover, whichever the unit of time, successive time units contain widely varying amounts of total turbulent energy.

The "uniformly self-similar sporadic random processes" are a new family of (generalized) random functions, specifically constructed to account for those properties (and some others). It has already proved to have predictive value as well. It is described in (1), and it may be noted that it was originally inspired by certain fluctuation phenomena in electronics (2), that appear extremely close in structure to turbulence in fluids.

It may be noted that the overwhelming bulk of the models of turbulence is a continuation of statistical mechanics, and attempts to construct explicit mechanical mechanisms, thanks to which the results of macroscopic experiments may turn out to be predictable. On the contrary, the self-similar sporadic processes provide a "model" of turbulence analogous to the model of matter provided by thermodynamics (3). The latter is known to deduce a substantial body of experimental results from very few "principles", that draw their value from their predictive and organizing power rather than from any close relation with the core of physics that is constituted by mechanics.

Among the "principles" of the present approach to turbulence, the main one is that of "self-similarity". Its roots go back to the Kolmogoroff theory (whose dimensionality considerations are also hardly at all "physical"). This idea has been reinterpreted and strengthened, and it has been required that it hold "uniformly", that is, over the whole frequency range. This was shown to imply that turbulence must by "sporadic", a concept that expresses an extreme form of "intermittent" (on-and-off) character. Not only such a chance phenomenon cannot be stationary in the usual sense, but its theory turns out to require a special (but very natural) generalization of the concept of random function.

One of the basic properties of self-similar sporadic processes is the Kolmogoroff law, that the measured spectral density (when properly interpreted) is proportional to an inverse power of the frequency. But the factor that multiples $k^{-5/3}$ is no longer a constant (designated by $\varepsilon^{2/3}$ in Kolmogoroff's theory); however long one's sample of turbulence may be, this factor is the product of a random variable and of a term that depends upon the method by which the spectrum was defined and measured. It is possible to speak of the spectrum of ε and experiments appear to confirm the predictions of the self-similar theory on this account.

Notes:

(1) B. Mandelbrot, "Sporadic random functions: a generalization of spectral analysis and conditional self-similarity" (or some closely approaching title), Publication expected in spring 1966 in Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability, held in 1965, University of California Press.

(2) B. Mandelbrot, "Self-Similar Error Clusters in Communications Systems, and the Concept of Conditional Probability", Institute of Electrical and Electronics Engineers, IEEE Transactions on Communications Technology, Vol. 1 COM-13 (1965), pp. 71-90. Concerning the spectra of such phenomena in electronics, see B. Mandelbrot, "Noises

with an $1/f^{B}$ Spectrum as Bridge between Direct Current and White Noise", a privately circulated memorandum to appear, and B. Mandelbrot, "Time varying channels, $1/f$ noises and the infrared catastrophe", Conference Record of the First IEEE Communications Convention, Boulder 1965.

(3) By speaking of "thermodynamics", I refer to the new variety, statistical but not mechanical, that I have expounded in B. Mandelbrot, "Derivation of Statistical Thermodynamics from Purely Phenomenological Principles". Journal of Mathematical Physics, Vol. 5 (1964), pp. 164-171.

STATISTICAL MECHANICS OF FIELDS

Joseph B. Keller

According to classical statistical mechanics, all the properties of a Hamiltonian mechanical system in thermal equilibrium are determined by its Gibbs distribution. In this lecture this principle is applied to fields satisfying linear equations of motion in bounded domains. First infinitely many pairs of canonical variables are introduced to represent the field. Then the Hamiltonian is found and seen to be a quadratic function of the variables. Therefore the Gibbs distribution is Gaussian. In terms of it the generating functional of the field is defined, from which all moments of the field can be found by functional differentiation. For linear fields this functional is evaluated explicitly in terms of the two point, two time correlation function of the field. Then an expression for this correlation function is obtained. To exemplify the result this expression is evaluated explicitly for fields obeying the one-dimensional wave equation in a finite, semi-infinite or infinite interval. These explicit results pertain to the displacement of a vibrating elastic string, the vector potential of an electromagnetic field, or the velocity potential of a compressible fluid.

The correlation function is found to be piecewise linear with the discontinuities in slope occurring on certain characteristics of the wave equation. Therefore the correlation function of two derivatives of the field is zero except on these characteristics where it is infinite. Consequently the energy density is infinite. This infinite energy density is a consequence of the excessive excitation of the high frequency modes, which is manifested by the Rayleigh-Jeans law of spectral distribution of black body radiation. When classical mechanics is replaced by quantum mechanics this law is replaced by the Planck distribution law which leads to finite energy density. This suggests that in attempting to apply statistical mechanics to a fluid in order to describe turbulence, we should introduce a modification of the Gibbs distribution to obtain a finite energy density. In analogy with Planck's law, it should involve a constant like Planck's constant h. The product μL^3 of viscosity μ and the cube of a typical length L has the same dimensions as h.

TURBULENT THERMAL CONVECTION WITH A SMALL MEAN SHEAR

Andrew P. Ingersoll

We consider thermal convection between rigid horizontal plates which move relative to each other with constant horizontal velocity. We have been able to approximate this situation in the laboratory and to measure the vertical fluxes of heat and horizontal momentum at Rayleigh numbers up to 3×10^7. The experiment was only feasible for small rates of shear, and therefore it was not possible to model the turbulent shear layer of the atmosphere near the ground. Rather, convective turbulence dominated the flow in this experiment, as in the atmosphere above the shear layer during unstable conditions.

At high values of the Rayleigh number, R , the data are consistent with the behavior $$Nu \sim \sigma^{1/2} Mo \sim R^{1/3},$$

where σ is the Prandtl number, Nu is the Nusselt number, and Mo is the dimensionless momentum flux analogous to the Nusselt number. Thus the dimensional fluxes are independent of the plate separation, d , for d large, and it is reasonable to apply similarity arguments in analyzing the flow near the boundaries. Kraichnan's mixing length theory of turbulent thermal convection* was extended to the present experiment with shear, and the observed asymptotic dependence, given above, was obtained. The success of this method gives considerable support to Kraichnan's concise treatment of convection, and suggests that it is applicable to a variety of other flows as well.

*R. H. Kraichnan. 1962. Phys.Fluids 5, 1374.

A BILINEAR APPROXIMATION FOR TURBULENT CONVECTION

Edward A. Spiegel

An approximate general form for the fluctuating interactions in turbulent convection was presented. The resulting equations have not yet been completely solved, but an approximate solution yields

$$N = .1\left(\frac{\sigma R^2}{\sigma R + R_c}\right)^{1/3}$$

where N is Nusselt number, σ is Prandtl number, R is Rayleigh number, and R_c is critical Rayleigh number.

BOUNDARY LAYER TREATMENT OF THE MEAN FIELD EQUATIONS

Louis N. Howard

By a change of scales the mean field equations are written in the form:

$$P^{-1}(D^2-a^2)\Theta + (1-W\Theta)W = 0$$

$$(D^2-a^2)^2 W = \Theta$$

with $P = Ra^2N$ and $N^{-1} = \int_0^1 (1-W\Theta)\,dz$.

The solution of these equations is then explored, in the case of free boundary conditions, asymptotically for $P \to \infty$ using the methods of singular perturbation theory. The "inner solution", valid off the boundaries is found to satisfy $(D^2-a^2)^2 W = \Theta$, $W\Theta = 1$, with $W = D^2W = 0$ on $z = 0, 1$. It is thus necessary to solve $f(D^2-a^2)^2 f = 1$ on $(0,1)$ with $f = D^2 f = 0$ on the boundaries. This can be done effectively with good accuracy using a sine-series representation of f, suitable care being used because of the somewhat singular behavior of f near the boundaries.

This solution is then matched to the "boundary layer solution", which is expressed in terms of the variable $\zeta = P^{\frac{1}{4}} z$, and is of the form:

$$W = P^{-\frac{1}{4}} W_1(\zeta), \ \Theta = P^{\frac{1}{4}} \Theta_1(\zeta)$$

The principal term in W_1 is simply $k\zeta$, where $k = f'(0)$, and Θ_1 satisfies:
$$\frac{d^2\Theta_1}{d\zeta^2} - k^2\zeta^2\Theta_1 + k\zeta = 0$$

The relevant solution of this equation is $\Theta_1 = k^{-\frac{1}{2}} g(k^{\frac{1}{2}}\zeta)$, where
$$g(x) = \frac{1}{2} x \int_0^1 e^{-\frac{1}{2}x^2 t} (1-t^2)^{-\frac{1}{4}} dt .$$

Using this one finds for the Nusselt number: $N = (2K)^{-\frac{4}{3}} (ka)^{\frac{2}{3}} R^{\frac{1}{3}}$ where $K = \frac{1}{\sqrt{2}} \Gamma(3/4)^2$.

Fairly accurate calculations of k as a function of a, followed by maximization of N with respect to a give for the maximum heat transport:

$$N = 0.325 R^{\frac{1}{3}}$$

with a maximizing a of about 1.8 π. Recent accurate numerical calculations of Herring for $R = 10^6$, 10^7 and 10^8, when extrapolated to $R^{-\frac{1}{3}} = 0$ are in almost perfect agreement with these numbers.

The present treatment has similarities with the analytic approximation study of Herring and with the investigation of Orszag reported in the 1964 G.F.D. notes, but differs from them in that their treatments make use of the boundary layer character of the solutions to get a fairly good numerical approximation while the present strict boundary layer approach gives the exact asymptotic result.

The case of rigid boundaries appears to introduce certain essential differences.

OCEANOGRAPHIC MICRO-SYMPOSIUM

LINEAR HINDCAST OF WIND-GENERATED WAVES

Stanley Jacobs

Wave heights at the Lake Michigan Research Tower off Muskegon, Michigan, are hindcast using the Neumann energy spectrum. The procedure consists of: (1) obtaining representative values of the wind through analysis of weather maps or ship reports, (2) estimating the fetch and duration of the wind field, (3) using empirical and theoretical filters to find the frequency band of the waves, and (4) calculating the wave heights using standard statistical methods.

Hindcasts were made for periods of two weeks in August and two weeks in September of 1964, and the results compared with measurements. For strong winds and large amplitude waves the hindcast wave heights are in good agreement with observed wave heights. For light winds the agreement is poor.

NONLINEAR RANDOM WAVES

David J. Benney

Given the initial statistical properties of a weakly nonlinear system of dispersive waves one can derive a set of closed integro-differential equations governing the asymptotic behavior of the spectral functions.

GENERALIZED EKMAN MODELLING OF THE OCEANIC CIRCULATION

Pierre Welander

The problem of the wind-driven oceanic circulation is considered in the following simplified form. A rectangular basin in the β -plane contains a two-layer fluid (no mixing of mass and heat across the interface, but momentum can be transferred by pressure and interface stresses). At the top a steady wind-stress acts, the curl of which vanishes at the northern and southern boundaries. The solution to this problem is derived in the case where the dynamic equations are of the Ekman type (balance of Coriolis force, pressure gradient and friction forces due to vertical shear), and the lower layer is much deeper than the upper (in the ocean a depth ratio of about 1:10 seems realistic). The boundary conditions are vanishing normal transport at the vertical boundaries for each of the two layers, vanishing velocity at the bottom, and vanishing vertical velocity at the top. The tangential stress at the top is further given by the wind stress.

The solution has the following general characteristics. The circulation in the upper layer resembles the one-layer solution derived by Stommel (1947), apart from a geometric distortion due to the interface changes. In the lower layer motion occurs only at the western boundary and takes the form of a narrow boundary gyre with the same sense of rotation as the upper gyre. This lower gyre intensifies the shore-side transport and creates a counter-transport at the edge. This is in general agreement with the Gulf Stream picture recently suggested by Worthington, but the model is certainly too simplified to allow any closer comparison.

It is of interest to note that a solution exists only in the case of a non-vanishing interface stress. From the potential vorticity equation one can show that interface stresses must occur also if non-linear acceleration terms are included (the argument is analogous to the proof that bottom stresses are required in a one-layer model, one has only to replace the bottom by the interface). The argument is, of course, only valid in the absence of lateral friction. If one includes lateral friction the solution degenerates to a one-layer solution: no motion exists anywhere in the lower layer. The solution thus becomes identical with the one given by Munk (1950).

References:

1. Stommel, H. (1948). The westward intensification of wind-driven ocean currents. Trans.Amer.Geophys.Union 29: 202-206.

2. Munk, W. H. (1950). On the wind-driven ocean circulation. J.Met. 7: 79-93.

3. Welander, P. (1965). A two-layer frictional model of the oceanic circulation. I. Wind-driven motion in a rectangular basin. Tech.Rep. WHOI, Ref.65-23.

(The works in Refs. 1 and 2 are also summarized in Stommel's book "The Gulf Stream". Ref. 3 covers the material presented in the seminar.)

THE THEORY OF FREE INERTIAL JETS: PATH AND STRUCTURE

Allan Robinson and Pearn P. Niiler

The general problem of three-dimensional jets in a rotating fluid of variable density is developed. Transforming to temperature as an independent coordinate, a first integral is obtained. This is the potential vorticity, which is a function of the temperature and a stream-like function which gives the horizontal velocities weighted by the inverse stability. This functional may be expanded about its mean value, downstream in a coordinate system following the path of the jet. This is the structure problem. The path of the free jet is controlled by bottom topography, β-effect, and the exchange of mass with the geostrophic environment. The vertical vorticity equation integrated across the cross section of the jet provides an equation for the jet axis. Simple examples are studied. A formal two-scale expansion is made separating the meander scale from the divergence scale, and providing a closed problem for both structure and path. A result of particular interest is that a meandering baroclinic jet in equilibrium with its environment imposes a geostrophic divergence with the scale of the meander. Numerical calculations are in progress employing real Gulf Stream topography.

EXISTENCE AND STRUCTURE OF INERTIAL BOUNDARY CURRENTS

Stanley L. Spiegel

An investigation is made of the properties of inertial boundary currents in a stably stratified, inviscid, non-diffusive ocean. The Boussinesq and β-plane approximations are adopted. The equations are transformed so that density replaces the vertical coordinate as an independent variable, and after a suitable non-dimensionalization of variables, the various fields are expanded as power series in the downstream coordinate η. The motion is shown to conserve potential vorticity. The equations and boundary conditions are obtained to order η^2 and are solved in the region of formation of the coastal jet (i.e. the case of no mass flux through the plane $\eta = 0$) for several simple forms of the potential vorticity function. It is found that for a constant depth ocean, a boundary current can exist only if the geostrophic drift at the boundary layer edge is westward at all depths. This constraint, which holds for any potential vorticity consistent with stable stratification, is relaxed if the depth increases rapidly enough in the downstream (northward) direction. For slopes just in excess of this critical value, a deep onshore countercurrent is predicted. Solutions of the first order problem using realistic values of the various parameters have been computed, and are found to be in qualitative agreement with observed features of the Florida Current.

AN INERTIAL THEORY OF THE FLUCTUATING OCEAN CIRCULATION

Joseph Pedlosky

The response of a simple bounded ocean model to a surface wind stress which oscillates at or near one of the resonance frequencies of contained Rossby waves is investigated. Both the fluctuating and resulting steady circulations are found under the assumption that the amplitude of the response at resonance is limited primarily by non-linear, finite-amplitude effects.

The resulting amplitude-frequency response curve shows many interesting features in common with the characteristics of simple non-linear mechanical oscillators.

THE DEGRADATION OF INTERNAL WAVES

Owen M. Phillips

This paper is concerned with a mechanism that may account for the appearance of isolated turbulent patches in the stably stratified fluid below the thermocline. It is suggested that these may be the result of sporadic local instabilities in the large scale, low mode internal waves that are capable of propagating in this layer. It is shown that the maximum rate of shear and the minimum local Richardson number occur in such motions at the point where the density gradient is strongest. In these motions, the wave frequency n is much less than the maximum value N_m of the Brunt-Väisälä frequency in the layer; the stability criteria then approximate those in <u>steady</u>

stratified flow. For a single wave component, the motion is potentially unstable at the wave crests and troughs when the wave slope

$$ka = 2n/N_m \, .$$

This mechanism places an upper limit on the spectral density of lowest mode internal waves in a way analogous to that in which breaking imposes an equilibrium range limit for surface waves. In its saturated state, the two-dimensional spectrum of the vertical displacement is shown to be proportional to k^{-2} when the depth d of the thermocline is such that $kd \ll 1$ and to k^{-3} when $kd \gg 1$. The corresponding frequency spectra are proportional to n^{-1} and n^{-3}.

The patches of turbulence so formed flatten out in the stable ambient density gradient to form pancakes or 'blini'. The process seems also to provide a mechanism for the vertical mixing in a stable layer, below the direct influence of the surface stirring.

NUMERICAL EXPERIMENTS WITH LARGE-SCALE SEASONAL FORCING

Eric B. Kraus

Experiments with six different heating fields in a numerical general circulation model are described. Three different vertical heating gradients are each used once with and once without variations on the continental/oceanic scale along parallel circles. The zonal and the meridional heating fields are forced to vary seasonally. Integration has been carried out over a simulated period of one century for one particular configuration, and over periods of five years for each of the five other configurations.

Results - which may be represented by an electrical analogue - are rather similar to actual general circulation observations. They also show stronger summer westerlies and North-South temperature gradients in the model without schematic oceans and continents. Dynamic lag effects cause differences between the "climates" of spring and fall. In all experiments there was a breakdown in fall of a predominantly zonal circulation, accompanied by the development of "equinoctial storms".

Lag correlations computed for the mean zonal thermal wind in the 100-year experiment show persistence in summer between successive ten-day means and significant negative values over longer lag periods. No significant lag correlations were found during the winter months.

THERMAL INSTABILITY OF A WIND-DRIVEN OCEAN CIRCULATION

Elliott E. Schulman

The basic state is driven by a constant east-west wind and a longitudinal temperature gradient known to be necessary for upwelling in the mid-ocean thermocline. The effect of wind is to force a vertical velocity at the bottom of the Ekman layer, and the resulting flow consists of a meridional and vertical circulation. Instabilities with respect to thermal advection, i.e. potential energy release, are investigated.

Low wave number disturbances are stabilized by vertical heat diffusion and only for $P_e H/L m > 120$ is amplification possible, where

H = height of ocean, L = width of ocean, m = north-south wave number, and P_e = Péclet Number. Wind towards the west is found to be destabilizing. The growth rates of these disturbances however, are very slow (an e-folding time of a few years) and an analysis pivoted around the high wave number limit is required.

For $m \gg 1$ but $B \ll 1$, e.g. $\ell \approx$ 100 km, disturbances having a growth rate <u>and</u> frequency of the order of three months are found, where

B = Burger number = $R_o m^2 = (D_R/\ell)^2$

R_o = Rossby number of basic state

D_R = Deformation radius $\approx$ 50 km for ocean

ℓ = Wave length of disturbances

Instabilities in the lowest vertical mode are possible only when $g = |\theta n/m| < 6.6$ and have a maximum growth rate for $g = 4.1$, where θ = ratio of east-west to vertical temperature differences, and n = east-west wave number.

These waves appear to correspond to the Swallow eddies observed in the North Atlantic. For $B \ll 1$, the growth rates are proportional to m, but we anticipate a high wave number cut-off as ℓ approaches to deformation radius.

MIXING NEAR THE SURFACE OF THE OCEAN

Eric B. Kraus and J. Stewart Turner

Many theoretical models have been proposed to explain the structure of the upper mixed layer in the ocean. These have usually invoked horizontal advections, vertical mixing due to shear and the earth's rotation in various combinations, and have dealt with the steady state rather than the time dependent behaviour. We have developed a one-dimensional model based on much simpler but previously neglected processes, which can however be used to predict the seasonal changes of the temperature and depth of such a layer. One of us approached the problem through a laboratory experiment and the other using a more general theoretical argument, but in essence the results are the same.

It is supposed that all the heat and mechanical energy affecting the water column are put in near the surface, and propagated downwards, with no advection effects, horizontal velocities or rotation. In the experiments the process is an intermittent one, with buoyant fluid being added in discrete amounts to simulate the storage of heat, and then mixed downwards by stirring mechanically with an oscillating grid near the surface which simulates the effect of the wind. In the theory heating can be continuous, and additional stirring can be provided by convective processes near the surface. For a fixed stirring rate the behaviour is determined completely by the variations of heat input. During periods of increasing rate of heating, the depth of the well-mixed layer is decreasing. When heating is continued but at a decreasing rate, the depth of the layer increases slowly but its temperature can continue to increase. During periods of surface cooling the layer depth increases and it also cools.

Many features observed in the ocean are thus reproduced well in these experiments. The depth and temperature dependence of the upper mixed layer, and especially the phase relationships to the heating and cooling cycle are in good qualitative agreement. An important difference from previous approaches, which can easily be tested, is the prediction that mixing should only be significant in the layer directly affected by surface processes. Thus features of the density structure below the topmost density interface could be laid down early in the heating season, and persist until the well-mixed layer reaches them again late in the winter.

PENETRATIVE CONVECTION IN THE SOLAR ATMOSPHERE

Derek W. Moore

Penetration of motions in the solar convection zone into the stable layers above is of interest in explaining the observed solar photosphere. The purpose of the present work is to examine what meteorologists have learned about penetrative convection by direct observation of the earth's atmosphere and by laboratory experiments.

In particular the buoyant vortex ring model of a penetrating convective element is discussed.

STRATIFIED FLUID FLOW OVER AN OBSTACLE

Kathleen Trustrum

The problem is to determine the steady two-dimensional flow of a Boussinesq liquid between parallel horizontal planes over a vertical strip. This problem has been solved recently by Moore and Drazin for an incompressible stabley-stratified fluid. They assume that the density gradient far upstream is constant and that the horizontal velocity U satisfies the condition ρU^2 = constant, where ρ is the density. Under these conditions the non-linear equations reduce to a linear equation. Their solutions for the Froude number F smaller than a certain constant λ, show that behind the obstacle there is a jet, which winds its way through rotors and whose intensity increases with the height of the strip. They also show that it is possible to find a solution for a strip of any height, which contradicts some earlier work by Long.

However, the assumption of uniform upstream conditions for $F < \lambda$ is unlikely to be realistic for theoretical reasons and from experimental observations. If the hydrostatic approximation is made in the equations of motion for a Boussinesq liquid, it can be shown that non-linear, non-dispersive long waves can be propagated upstream for $F < \lambda$. This suggests that initial uniform upstream conditions will be disturbed in the limit of infinite time by such long waves.

A solution is obtained to the above problem by solving the initial value problem with an Oseen approximation to the non-linear inertia terms. Unfortunately there are insufficient conditions to determine all the arbitrary constants in the solution for infinite time. This difficulty is resolved by assuming that for flows for which $F > \lambda$, the solution is the same as that obtained by Moore and Drazin. This is justified as disturbances cannot propagate upstream for $F > \lambda$. Assuming that the constants are 'continuous' functions of F , the solution for $F < \lambda$ can be obtained. The solution for a line source on the bottom of the channel satisfies the above assumptions. The solution has the properties that it reduces to the irrotational solution in the limit of infinite Froude number and to the 'Taylor column' solution for zero Froude number. Calculations have still to be done to determine how the flows are modified for $F < \lambda$, from those obtained by Moore and Drazin.

BJERKNES FORCE

G. Brian Trustrum

For a considerable time it has been known that a small sphere pulsating in an oscillating pressure field experiences a Bjerknes force, tending to make it migrate in a particular direction. According to an oversimplified theory, the instantaneous translational force equals the local pressure gradient multiplied by the volume of the sphere. The Bjerknes force is the mean of this product. Such a simple calculation neglects the interaction between the motion of the sphere and the external pressure field.

When the radial acoustic mode of vibration of water in a spherical container is excited, any small air bubble in the water will pulsate. There will usually be positions at which the Bjerknes and gravitational forces on the bubble balance each other. A calculation is described for finding these positions. The water is assumed to be inviscid and irrotational and using the Proudman-Pearson technique, inner and outer expansions are made about the bubble. A different length scale is used for the two expansions and the expansion parameter is a Mach number for the water based on the bubble radius. To first order the results are identical with the over-simplified theory.

Trapping of bubbles near the centre of radial sound fields has been observed experimentally, but it has also been noticed by M. Strasberg and others that the bubble becomes unstable when the amplitude exceeds a critical level depending on bubble radius. These instabilities are discussed but are not yet properly understood.

The speaker acknowledges helpful conversations with Dr. T. Brooke Benjamin, who is also interested in these problems.