Notes on the 1964

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION



Reference No. 64-46

Contents of the Volumes


Volume I    Course Lectures and Seminars

Volume II   Student Lectures


Refererence No. 64-46

The yearly programs of Geophysical Fluid Dynamics at the Woods Hole Oceanographic Institution are both interdisciplinary and international.

This year (1964) we are indebted to Dr. T. F. Malone, and through him to the Travelers' Research Center, Inc., for providing funds for the travel of Fellows from abroad.

Without this nail for the proverbial horseshoe, the valuable international aspects of the G.F.D summer programs would soon vanish.

*Participated in less than one-half of the program.

Dr. John G Pierce and Dr. Shoji Kato should have been included in the post-doctoral participants.

## Editor's Preface

Two distinctive features of large-scale geophysical flows are that they are dominated by the earth's rotation and that they are turbulent. This year's lecture program was an exploration of recent achievements in the study of, first, the simplest examples of turbulence, and second, the rotational constraint.

Progress at the frontiers of turbulence theory was described by R. Kraichnan and K. Hasselmann. The Principal Invited Lecturer, A. Robinson, gave an account, in depth, of the mathematical tools and physical ideas currently employed in theoretical oceanography.

This volume is a restatement of these lectures by the predoctoral participants in the summer program. It represents their view of the relative importance of points raised in the lectures, and their view of matters neglected. The final contributions here are seminars by senior participants and abstracts of the lectures of visitors. They are included in order to record the scope of our summer exposure to non-linear fluid dynamics.

Mrs. Mary Thayer has done all the work in assembling and reproducing the lectures. We are all indebted to her for her remarkable efforts in keeping the summer course running smoothly and to the National Science Foundation for its financial support of the program.

Willem V.R. Malkus

List of Seminars

1964

June 29 Dr. Willem V.R. Malkus "Boussinesq Equations"

June 30 Dr. Willem V.R. Malkus "Problem Discussion, and Boussinesq Energetics"

July 1 Dr. Willem V.R. Malkus "Convection: Stability, the Problems of Realizability and Uniqueness"

July 2 Dr. Frederic Bisshopp "Problem Discussions and Variational Procedures"

July 3 Dr. Willem V.R. Malkus "Thermal Convection: Experiments and Finite Amplitude Effects. 'Role' of Non-linear Processes"

Dr. Deitrich Lortz "Instability of Finite Amplitude Solutions of the Convection Problem"

Dr. George Veronis "Finite Amplitude Stability in Water Stratified by Both Salt and Heat"

July 6 Dr. Robert Kraichnan "Turbulence: Aims and Tools"

Dr. Willem V.R. Malkus "Turbulence: Modeling in Statistical Hydrodynamics"

July 7 Dr. Jackson Herring "The Role of the Mean Field in Turbulence Theory"

July 8 Dr. Robert Kraichnan "Turbulence: Foundations for a Deductive Theory"

July 9 Dr. Klaus Hasselmann "Turbulence: Lagrangian Basis for Expansion"

July 10 Dr. Robert Kraichnan "Turbulence: Recent Achievements"

July 13 Dr. Allan Robinson "Rotating Fluids and the General Oceanic Circulation" (Flow in Rotating Systems)

July 14 Dr. Melvin Stern "On the Instability of Quasigrostrophic Waves in a Baroclinic Current"

July 15 Dr. Allan Robinson "Friction Modification of Geostrophy"

List of Seminars (continued)

July 16 Dr. Nicholas Fofonoff "Oceanic Observations"

July 17 Dr. Allan Robinson "Frictional Modification of Geostrophy: Problems"

July 20 Dr. Allan Robinson "Wind-driven Flow"

July 21 Dr. Tiruvalam Krishnamurti "On the Theory of Air Flow over Mountains"

July 22 Dr. Allan Robinson "Double Boundary Layers"

Dr. Henry Stommel "Theories and Ocean Currents" (general lecture)

July 23 Dr. Pierre Welander "Some Integral Constraints for the Ocean Circulation"

July 24 Dr. Allan Robinson "Ocean Circulation"

July 27 Dr. Francis P. Bretherton "Time Dependent Motions in the Ocean"

July 28 Dr. Francis P. Bretherton "Resonant Interactions between Waves"

July 30 Dr. Shoji Kato "Response of an Unbounded Isothermal Atmosphere to Point Disturbances"

July 31 Dr. Kirk Bryan "The Ocean Circulation"

Aug. 4 Dr. Harvey Greenspan "Contained Rotating Fluids"

Aug. 5 Dr. Harvey Greenspan "Contained Rotating Fluids"

Aug. 6 Dr. Joseph Pedlosky "Unsteady Ocean Circulations"

Aug. 7 Dr. Harvey Greenspan "Contained Rotating Fluids"

Aug. 11 Dr. Albert Barcilon "Diffusion of a Semi-infinite Line Vortex Normal to a Stationary Plane"

Aug. 12 Dr. Frederic Bisshopp "Rapidly Rotating Convection"

Aug. 14 Dr. J. Stewart Turner "Coupled Convection of Heat and Salt"

List of Seminars (continued)

Aug. 17 Dr. Raymond Hide "Detached Shear Layers in a Rotating Fluid"

Aug. 18 Dr. George Veronis "Generation of Mean Ocean Circulation by Fluctuating Winds"

Aug. 19 Dr. Klaus Hasselmann "Non-linear Processes in Random Wave Fields"

Aug. 20 Dr. Klaus Hasselmann "Non-linear Processes in Random Wave Fields"

Aug. 21 Dr. Robert Miller "Observations on Breaking Waves" Demonstration of Experiments on Nauset Beach by Dr. John Zeigler

Aug. 27 Dr. Tiruvalam Krishnamurti "Some Transformations for the Mountain Wave Problem in Compressible and Incompressible Atmospheres"

Student Seminars

Sept. 2 Patrick A. Davis "Inertial Oscillations in Magnetohydrodynamics"

Stephen Pond "Ocean Circulation Models for Regions of High Latitude"

Uriel Frisch "Microscopic Descriptions of Long-range Correlations in a Gas"

Benjamin Halpern "Upper Bound on Heat Transport by Turbulent Convection"

Sept. 3 Kern Kenyon "Non-linear Rossby-waves"

Ruby Krishnamurti "Finite Amplitude Instability in Quasi-steady Convection"

Steven Orszag "Analytic Approach to Some Problems of Turbulence"

Sten Gösta Walin "Thermal Circulation in a Deep Rotating Annulus

Sept. 4 Michael F. Devine "A Two-layer Model of the Equatorial Undercurrent"

John G. Pierce "A Problem in Mountain Wave Theory"

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# Boussinesq Equations

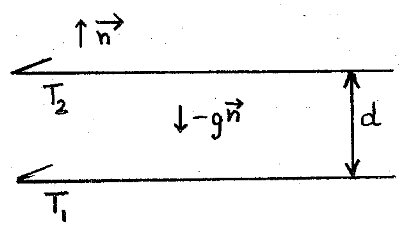
Willem V.R. Malkus

June 29, 1964

## Bénard's problem

In this first lecture we shall derive some equations which prove to be very useful in convection problems. The most typical of these is Bénard's problem:

A viscous, heat-conducting fluid occupies the space between two infinite horizontal planes ($z = 0$ and $z = d$), at different temperatures, that of the lower plane ($T_1$) being greater than that of the upper plane ($T_2$) A constant gravity $-gn$ ($n$ being a vertical unit vector) acts on the fluid.




If the temperature difference $\Delta T = T_1 - T_2$ is small, the fluid remains at rest and there is a pure thermal conduction. If $\Delta T$ exceeds a certain critical value such a state becomes unstable and convection occurs.

Let us write the Navier-Stokes equation for that problem

$$\frac{D\rho}{Dt} = -\rho \operatorname{div} \vec{V} \tag{1}$$

$$\rho \frac{D\vec{v}}{Dt} = \operatorname{div} \overline{\overline{P}} - \rho g \vec{n} \tag{2}$$

$$\rho \frac{D(C_v T)}{Dt} = \overline{\overline{P}} : \nabla \vec{V} + \mathrm{div}(K \,\mathrm{grad}\, T) \tag{3}$$

$$\overline{\overline{P}}_{ik} = -p\delta_{ik} + \gamma \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \frac{\partial v_j}{\partial x_j} \delta_{ik} \right) \tag{4}$$

The tensor $\overline{\overline{P}}$ is the sum of the scalar pressure $-p\delta_{ij}$ and of the stress tensor $\overline{\overline{\tau}}$. $K$ is the thermal conductivity. The equation of state will be

$$\rho = \frac{P}{RT} \qquad \text{(perfect gas)} \tag{5}$$

The Boussinesq approximation

As the Navier-Stokes equations are too complicated to handle, one usually simplifies them assuming:

- A constant density except in the gravity term.
- An incompressible fluid.
- The dissipation term $\overline{\overline{\tau}} : \nabla v$ is neglected compared with the heat flux term $\mathrm{div}(K \,\mathrm{grad}\, T)$.
- The equation of state is simplified.

$$\rho_0 \frac{D\vec{v}}{Dt} = -\mathrm{grad}\, P + \mu \nabla \cdot \nabla \vec{V} - \rho g \vec{n} \tag{6}$$

$$\rho_0 \frac{D(C_v T)}{Dt} = \mathrm{div}(K \,\mathrm{grad}\, T) \tag{7}$$

$$\mathrm{div}\, \vec{V} = 0 \tag{8}$$

$$\rho = (1 - \alpha (T - T_r)) \rho_r \tag{9}$$

where $\rho_o$ is some mean density and $T_r$ a reference temperature. What kind of an approximation is this? We shall show that the Boussinesq equations can be derived from the Navier-Stokes equation by non-dimensionalizing and introducing a reference field.

The reference field ( $\rho_a$, $P_a$, $T_a$ ) is a stratified, static, adiabatic solution of the Navier-Stokes equations which here reduce to

$$\frac{\partial P_a}{\partial z} = -g\rho_a \tag{10}$$

together with the adiabatic condition

$$\frac{\partial T_a}{\partial z} = -\frac{g}{C_p} \qquad \text{(a consequence of } \frac{ds}{dz} = 0 \text{ )} \tag{11}$$

and the equation of state

$$\rho_a = \frac{P_a}{R\,T_a} \tag{12}$$

The explicit solution will be written in the next paragraph.

We now introduce the departure of the actual field from the reference field by writing:

$$\tilde{\rho} = \rho - \rho_a \tag{13}$$

$$\tilde{P} = P - P_a \tag{14}$$

$$\tilde{T} = T - T_a \tag{15}$$

Before non-dimensionalizing the equations we have to find the parameters of Bénard's problem. These are

$$d,\ \Delta T,\ C_v,\ \mu,\ K,\ g,\ P_r,\ T_r,\ R$$

where $P_r$ and $T_r$ are some reference pressure and temperature and $R$ is the perfect gas constant. Using independent units

$[M]$ = unit of mass; $[L]$ = unit of length

$[T]$ = unit of time; $[\theta]$ = temperature unit

we have the following dimensional equations.

$$d = [L];\ \Delta T = [\theta];\ g = [L][T]^{-2};\ C_v = [L]^2[T]^{-2}[\theta]^{-1}$$

$$\mu = [M][L]^{-1}[T]^{-1};\ K = [M][L][T]^{-3}[\theta]^{-1}$$

$$P_r = [M][L]^{-1}[T]^{-2};\ T_r = [\theta],\ R = [L]^2[T]^{-2}[\theta]^{-1}$$

We can thus construct $9 - 4 = 5$ dimensionless independent parameters. <u>Two of them will be chosen in such a way that they vanish when $\Delta T \longrightarrow 0$ and $d \longrightarrow 0$</u>, and we shall use them as independent expansion parameters.

<u>Some algebra</u>

Solving the system (10, 11, 12) for reference field we get

$$(16)\quad \frac{T_a}{T_r} = 1 - \eta\frac{z}{d} = f,\ \frac{P_a}{P_r} = f^{\frac{1}{s}},\ \frac{\rho_a}{\rho_r} = f^{\frac{1-s}{s}}$$

$T_r$ is some reference temperature (here $T_2$ for instance).

$$\eta = \frac{gd}{C_p T_r} \tag{17}$$

is the ratio of $d$ to the hydrostatic adiabatic height.

$$S = \frac{R}{C_p} \tag{18}$$

Besides $\eta$ and $S$ we introduce two more dimensionless parameters. The Prandtl number

$$\sigma = \frac{\nu}{\chi} \tag{19}$$

where

$$\nu = \frac{\mu}{\rho} \quad \text{and} \quad \chi = \frac{K}{\rho C_p} \tag{20}$$

and the relative temperature difference

$$\varepsilon = \frac{\Delta \tilde{T}}{T_r} = \frac{\tilde{T}(d) - \tilde{T}(0)}{T_r} \tag{21}$$

We notice that $\varepsilon$ and $\eta$ vanish when $d \longrightarrow 0$ and $\Delta T \longrightarrow 0$.

The Navier-Stokes equations are non-dimensionalized by setting

$$x = (d)x_1; \quad y = (d)y'; \quad z = (d)z' \tag{22}$$

$$\chi = \chi(T_r)\chi' \tag{23}$$

$$K = k(T_r)K' \tag{24}$$

$$\tilde{T} = (\varepsilon T_r)T' \tag{25}$$

$$\tilde{\rho} = (\varepsilon \rho_r)\, \rho' \tag{26}$$

$$\tilde{v} = (V)\, v' \tag{27}$$

$$\tilde{P} = (\rho_r V^2)\, P' \tag{28}$$

$$t = \left(\frac{d}{V}\right) t' \tag{29}$$

where

$$V = \left(\frac{g\,\tilde{\Delta} T d}{T_r}\right)^{\frac{1}{2}} \tag{30}$$

The following relations prove to be useful

$$V = (\varepsilon \eta)^{\frac{1}{2}} C \quad ; \quad C = (C_p T_r)^{\frac{1}{2}} \tag{31}$$

$$\frac{d}{V} = \left(\frac{d}{\varepsilon g}\right)^{\frac{1}{2}} \tag{32}$$

This non-dimensionalizing procedure is not entirely arbitrary. It is the one which gives rise to the Boussinesq equations. For instance the pressure being proportional to $\varepsilon \eta$ is in some sense of higher order than the density which is proportional to $\varepsilon$.

Expressed with the new primed variables the Navier-Stokes equations are transformed into the set

$$\rho' = \frac{\eta P' - s \int^m T'}{s(\int + \varepsilon T')} \quad ; \qquad m = \frac{1-s}{s} \tag{33}$$

$$\frac{D}{Dt'} \log\left(1 + \varepsilon \int^m \rho'\right) = -\mathrm{div}' \vec{v}' + \eta m \int^{-1} w'^{-1} \; ; \; w' = \vec{v}' \cdot \vec{n} \tag{34}$$

$$(\rho^m + \varepsilon\rho')\frac{D\vec{v}'}{Dt'} = -\rho'\vec{n} - \mathrm{grad}'p' + \left(\frac{\sigma}{R_G}\right)^{1/2}\mathrm{div}'\bar{\bar{\tau}}' \tag{35}$$

$$\varepsilon(\rho^m + \varepsilon\rho')\frac{DT'}{Dt'} = \varepsilon\left(\frac{1}{R_G\sigma}\right)^{\frac{1}{2}}\mathrm{div}'k'\mathrm{grad}'T + \eta\left(\frac{1}{R_G\sigma}\right)^{\frac{1}{2}}\frac{\partial K'}{\partial z'} + \varepsilon\eta\left[w'\rho' + \frac{DP'}{Dt'} + \left(\frac{\sigma}{R_G}\right)^{1/2}\phi'\right] \tag{36}$$

$$R_G = \frac{g d^3 \Delta\tilde{T}}{\chi\nu T_r} \quad \text{is the generalized Rayleigh number.} \tag{37}$$

$\phi'$ stands for $\bar{\bar{\tau}}' : \nabla'\vec{v}'$

We now expand any one of the independent variables with respect to $\varepsilon$ and $\eta$

$$P' = \sum_{i,j=0}^{\infty} P'_{ij}\,\varepsilon^i\eta^j \quad \text{etc.} \tag{38}$$

and get as zero, zero equations

$$\rho'_{00} = -T'_{00} \tag{39}$$

$$\frac{D'_{00}\vec{v}'_{00}}{Dt'} = -\rho'_{00}\vec{n} + \left(\frac{\sigma}{R_G}\right)^{\frac{1}{2}}\nabla'\cdot\nabla'\vec{v}'_{00} - \mathrm{grad}'P_{00} \tag{40}$$

$$\frac{D'_{00}T'_{00}}{Dt'} = \left(\frac{1}{R_G\sigma}\right)^{\frac{1}{2}}\mathrm{div}'\mathrm{grad}'T'_{00} \tag{41}$$

where

$$\frac{D'_{00}}{Dt'} = \frac{\partial}{\partial t'} + \vec{V}'_{00}\cdot\nabla' \tag{42}$$

These are the Boussinesq equations written in non-dimensional form. As an important feature of this method we notice that the heat transfer and the momentum transfer equations remain coupled even with vanishing $\varepsilon$ .

As another one we notice the impossibility of sound-wave-like solutions, the pressure being of higher order than the density.

Here are some numerical values of the parameters for

$d$ = 10 cm $\Delta T$ = .01°C in air, (N.T.P.)

$\mathcal{E} = 2.10^{-5}$ ; $\eta = 3.10^{-6}$ ; $R_G = 2.10^3$ ; $\sigma = 0.7$ ; $S = 0.4$

Liquids

What happens if we replace the gas by a liquid?

The equation of state is now

$$\rho = \rho_r\left[1-\alpha(T-T_r)+\beta(P-P_r)\right] ; \alpha = -\left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p\right]_r ; \beta = \left[\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T\right]_r \quad (43)$$

The adiabatic equation is now

$$\rho_a C_v \frac{\partial T_a}{\partial z} = P_a \frac{\partial \log \rho_a}{\partial z} \quad (44)$$

or

$$\frac{\partial T_a}{\partial z} = -\frac{P_a \beta \rho_r g}{\rho_a C_v + \alpha P_r} \quad (45)$$

We approximate this expression by

$$-\frac{P_r \beta \rho_r g}{\rho_r C_v + \alpha P_r} \quad (46)$$

The parameters of the problem are now

$d, \Delta T, C_v, \mu, K, g, P_r, T_r, \rho_r, \alpha, \beta$

Keeping the same value for $\varepsilon$ we now choose

$$\eta = \frac{P_r \beta \rho_r g d}{(\rho_r C_v + \alpha P_r) T_r} \tag{47}$$

and introduce $\quad C_1 = \alpha T_r , \quad C_2 = \beta P_r \qquad (48)$

The velocity scale becomes

$$V = \left( \frac{C_1 g \Delta \tilde{T} d}{T_r} \right)^{1/2} \tag{49}$$

and the Rayleigh number

$$R = C_1 R_G \tag{50}$$

Boussinesq energetics

A careless use of the Boussinesq equations leads to a paradoxial form of the energy equation. Let us try to explain this difficulty.

In the steady state an energy equation can easily be derived from the Navier-Stokes momentum equation multiplying both members by $\cdot \vec{v}$ and averaging over the whole system.

$$\langle \tfrac{1}{2} \rho \frac{D v^2}{Dt} \rangle_m = - \langle \vec{v} \cdot \text{grad}\, p \rangle_m - \langle \rho g w \rangle_m - \langle \vec{v} \cdot \nabla \bar{\bar{\tau}} \rangle_m \tag{51}$$

The left-hand side is zero because

$$\rho \frac{D v^2}{Dt} = \rho \vec{v} \cdot \text{grad}\, v^2 = \text{div}(\rho v^2 \vec{v}) - \rho v^2 \text{div}(\rho \vec{v})$$

$\text{div}(\rho \vec{v})$ is zero

The other term, when averaged, becomes a surface integral which vanishes because $\vec{v}\cdot\vec{n} = 0$ at the boundaries. $\langle \rho g \vec{v}\cdot\vec{n} \rangle_m$ vanishes too because there is no mass flux across the boundaries.

We are left with

$$\langle \vec{v}\cdot \text{grad } p \rangle_m + \langle \vec{v}\cdot \nabla \overline{\overline{\overline{\tau}}} \rangle_m = 0 \tag{52}$$

If we do the same with the Boussinesq momentum equation dropping the subscripts we get

$$\left\langle \frac{1}{2} \frac{Dv'^2}{Dt^2} \right\rangle_m = -\langle \rho' w' \rangle_m + \left\langle \left(\frac{\sigma}{R_v}\right)^{\frac{1}{2}} \vec{v}'\cdot\nabla'\cdot\nabla'\vec{v}' \right\rangle_m - \langle \vec{v}'\cdot \text{grad}' p' \rangle_m \tag{53}$$

The left-hand side again vanishes; the second vanishing term is now $\langle \vec{v}'\cdot \text{grad } p' \rangle_m$ because $\text{div}'\vec{v}' = 0$ . $\rho'$ is replaced by $-T'$. $\langle \vec{v}'\cdot\nabla'\cdot\nabla'\vec{v}' \rangle_m$ can be written $-\langle (\nabla'\vec{v}')^2 \rangle_m$.

We are thus left with

$$\langle T' w' \rangle_m = \left(\frac{\sigma}{R_G}\right)^{\frac{1}{2}} \langle (\nabla'\vec{v}')^2 \rangle_m \tag{54}$$

This equation seems to involve no work term; actually <u>$\langle T' w' \rangle_m$ is the work term</u>: expanding the work term of the Navier-Stokes energy equation (52)

$$-\langle \vec{v}\cdot \text{grad } p \rangle_m = -\left\langle w \frac{\partial p_a}{\partial z} \right\rangle_m - \langle \vec{v}\cdot \text{grad } \tilde{p} \rangle_m$$

The second term is zero as we already know. Using equation (10) and

$$\langle \rho w \rangle_m = \langle \rho_a w \rangle_m + \langle \tilde{\rho} w \rangle_m = 0 \text{ (vertical momentum)}$$

we get

$$-\langle \vec{v} \cdot \text{grad}\, p \rangle_m = -g \langle w \tilde{\rho} \rangle m$$

After non-dimensionalizing and using $\rho' = -T'$ the work term becomes just

$$\langle T' w' \rangle_m$$

<u>Some useful relations</u> (steady state)

Expanding the heat equation of Boussinesq horizontally averaged ( $\bar{A}$ is the horizontal mean value of $A$ )

$$\frac{\partial}{\partial z}(\overline{WT}) = \left(\frac{1}{R_{G\sigma}}\right)^{\frac{1}{2}} \frac{\partial^2 \overline{T}}{\partial z^2} \tag{55}$$

which can be integrated yielding

$$\overline{WT} + \left(\frac{1}{R_{G\sigma}}\right)^{\frac{1}{2}} \beta = H = \text{constant}; \quad \beta = -\frac{\partial \overline{T}}{\partial z} \tag{56}$$

Multiplying the heat equation by $T$ and averaging over the whole space

$$\left\langle \frac{1}{2} \frac{D}{Dt} T^2 \right\rangle_m = \left\langle \left(\frac{1}{R_{G\sigma}}\right)^{\frac{1}{2}} T \,\text{div grad}\, T \right\rangle_m \tag{57}$$

The left-hand side is zero as

$$\left\langle \frac{1}{2} \text{div} \left(\vec{V} T^2\right) \right\rangle_m$$

We are left with

$$\langle \operatorname{div}(T \operatorname{grad} T) \rangle_m = \langle (\operatorname{grad} T)^2 \rangle_m \tag{58}$$

The first term, expressed as a surface integral gives rise to

$$(R_{G\sigma})^{\frac{1}{2}} H (T_2 - T_1)$$

where $H$ is the heat flux through any one of the limiting planes. planes.

$$\left(\frac{1}{R_{G\sigma}}\right)^{\frac{1}{2}} \langle (\operatorname{grad} T) \rangle^2 = H \Delta T \tag{59}$$

Introducing the horizontal fluctuation of the temperature

$$T = \theta + \overline{T} \qquad (\overline{T} = \text{mean horizontal temperature}) \tag{60}$$

we obtain

$$\left(\frac{1}{R_{G\sigma}}\right)^{\frac{1}{2}} \langle (\operatorname{grad} \theta)^2 \rangle_m = \langle \beta W T \rangle_m \, ; \; \beta = -\frac{\partial \overline{T}}{\partial z} \tag{61}$$

using $\overline{W} = 0$.

The shear flow analogue of the two preceding equations were first used by Landau to study finite amplitude problems.

Notes submitted by

Uriel Frisch.

## Convection:
## Stability, the Problems of Realizability and Uniqueness

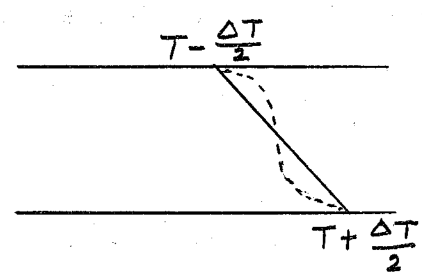
Willem V.R. Malkus

When a temperature gradient is maintained in a horizontal layer of fluid by heating from below and cooling from above, the arrangement is top-heavy and will become unstable to perturbations. We can anticipate the convection problem as follows:

Since the heat flux

$$H = \kappa \beta_m + (WT)_m$$

is independent of the vertical coordinate $z$, and since there is no vertical motion $W$ near the boundaries, the temperature gradient which is constant in conduction (solid line) must increase at the boundaries as indicated by the dashed line




Although the situation is top-heavy and motions will tend to rearrange the fluid, these motions will be inhibited by the viscosity of the fluid. We may thus ask under what conditions there will be instability. If we consider only infinitesimal amplitude perturbations and solve the resulting linearized perturbation equations, we will be able to state only that below the critical conditions, the infinitesimal perturbation will decay. However it is not necessary that motions of infinitesimal amplitude will be the first to become unstable. The following analysis will allow us to ask under what conditions is the infinitesimal amplitude stability theory meaningful.

The Boussinesq equations are as follows:

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = -\underset{\sim}{v} \cdot \nabla T \tag{1}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \underset{\sim}{v} = -\nabla \tilde{P}/\rho_0 - \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} + \gamma (T - T_0) \underset{\sim}{k} \tag{2}$$

$$\nabla \cdot \underset{\sim}{v} = 0 \tag{3}$$

where $\kappa = \dfrac{k}{\rho c_p}$ , $k$ = thermal conductivity

$\gamma = g\alpha$ , $\alpha$ = coefficient of thermal expansion

$\tilde{P} = \rho - gz$

From these we isolated the mean equations as follows: The horizontal average (denoted by a bar) of equation (1) is

$$\frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial z}(\kappa \beta) = -\frac{\partial}{\partial z}\left(\overline{WT}\right) \tag{4}$$

where $\beta = -\dfrac{\partial}{\partial z} \bar{T}$

$T = T - \bar{T}$

Equations (1) and (4) give

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)T = \beta W - h \tag{5}$$

where

$$h = \underset{\sim}{v} \cdot \nabla T - \frac{\partial}{\partial z}\overline{WT}$$

$$\overline{h} = 0$$

Thus the non-linear term has been separated into a part which represents the interaction of the fluctuations with the mean field (and hence with each other through the mean field), and a part which represents the interaction of the fluctuations with each other.

The equations (2), (3), (5) constitute five scalar equations in the five quantities $\underset{\sim}{v}$ , $T$ , $P$ . Taking the z-component of the curl curl of equation (2) we eliminate the pressure and obtain

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 W = \gamma \nabla_1^2 T - L(\underset{\sim}{M}) \tag{6}$$

where
$$L(\underset{\sim}{M}) = \frac{\partial^2 M_x}{\partial x \partial z} + \frac{\partial^2 M_y}{\partial y \partial z} - \nabla_1^2 M_z$$

$$M_i = (\underset{\sim}{v} \cdot \nabla) v_i$$

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Since we pivot our analysis about the linear stability problem governed by a sixth order differential equation (see Pellew and Southwell 1940), we eliminate the temperature to a certain extent; taking $\nabla_1^2$ of equation (5) and $\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)$ of equation (6), we obtain

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)\left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\nabla^2 W - \gamma\beta\nabla_1^2 W$$
$$= -\gamma \nabla_1^2 h + \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) L \tag{7}$$

If $\beta$ is a constant, the left-hand side is a linear differential operator with constant coefficients on $W$. The boundary conditions at $z = 0, d$ are

$W = 0$

$T = 0$ (assuming that the boundaries are perfect conductors).

Further, if the boundaries are rigid (or more precisely if there is no slip) then $u = v = W = 0$. Then, since $\nabla \cdot \underset{\sim}{v} = 0$ we must also have

$$\frac{\partial W}{\partial z} = 0 \quad \text{at} \quad z = 0, d$$

This is a realistic boundary condition. However, a simplified condition is the stress free, or free boundary condition, i.e. $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial z} = 0$. Then it again follows from (3) that

$$\frac{\partial^2 W}{\partial z^2} = 0 \quad \text{at} \quad z = 0, d.$$

The equations are non-dimensionalized by letting

$$t = \left(\frac{d^2}{\kappa}\right) t'$$

$$\underset{\sim}{x} = (d)\, \underset{\sim}{x}'$$

$$\underset{\sim}{v} = \left(\frac{\kappa}{d}\right) \underset{\sim}{v}'$$

$$T = \left(\frac{\kappa \nu}{\gamma d^3}\right) T'$$

The equations then become

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W = \nabla_1^2 T - \frac{1}{\sigma} L \tag{8}$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) T = \frac{\beta}{\beta_m} R W - h \tag{9}$$

and

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W - \frac{\beta}{\beta_m} R \nabla_1^2 W$$
$$= -\nabla_1^2 h + \frac{1}{\sigma}\left(\frac{\partial}{\partial t} - \nabla^2\right) L \tag{10}$$

where

$$R = \frac{\gamma \Delta T d^3}{\kappa \nu}$$

$$\beta_m = \frac{\Delta T}{d}$$

The fields $W$ and $T$ are expanded in powers of $\varepsilon$

$$W = \varepsilon W_0 + \varepsilon^2 W_1 + \varepsilon^3 W_2 + \dots$$

$$T = \varepsilon T_0 + \varepsilon^2 T_1 + \varepsilon^3 T_2 + \dots$$

where $\varepsilon$ is to be identified with the amplitude of the velocity field $(W W_0)_m = \varepsilon$.

Clearly the amplitude will be a function of the Rayleigh number:

$$R = \sum_{i=0}^{\infty} R_i \varepsilon^i$$

with $R_o$ = critical Rayleigh number.

Substituting these expansions into (10) generates the following sequence of linear equations:

$$\mathcal{L}(W_o) \equiv \left(\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 W_o - R_o \nabla_1^2 W_o = 0. \tag{11}$$

$$\mathcal{L}(W_1) = R_1 \nabla_1^2 W_o - \nabla_1^2 h_{oo} + \sigma^{-1}\left(\frac{\partial}{\partial t} - \nabla^2\right) L_{oo}. \tag{12}$$

$$\begin{aligned}\mathcal{L}(W_2) = R_1 \nabla_1^2 W_1 + R_2 \nabla_1^2 W_o + \left\{(W_o T_o)_m - \overline{W_o T_o} + G_{oo}\right\} \nabla_1^2 W_o - \\ - \nabla_1^2 (h_{o1} + h_{1o}) + \sigma^{-1}\left(\frac{\partial}{\partial t} - \nabla^2\right)(L_{o1} + L_{1o}).\end{aligned} \tag{13}$$

$$\begin{aligned}\mathcal{L}(W_i) = \sum_{i=0}^{n-1} R_{i-n} \nabla_1^2 W_n + \sum_{n+\ell=0}^{i-2} \left\{(W_n T_\ell)_m + \right. \\ \left. + G_{n\ell} - \overline{W_n T_\ell}\right\} \nabla_1^2 W_{i-(n+\ell+2)} \\ + \sum_{n=0}^{i-1} \left\{-\nabla_1^2 h_{n,i-(n+1)} + \sigma^{-1}\left[\frac{\partial}{\partial t} - \nabla^2\right] L_{n,i-(n+1)}\right\}.\end{aligned} \tag{14}$$

where $G$ is that part of $\beta$ which vanishes when time variations of $\overline{WT}$ vanish:

$$\frac{\beta}{\beta_m} = 1 + \frac{1}{K\beta_m}\left\{(\overline{WT})_m - \overline{WT} + G\left(\frac{\partial \overline{WT}}{\partial t}\right)\right\}$$

Equation (11) is a linear characteristic value problem. The linear constant coefficient operator $\mathcal{L}$ is called the Rayleigh operator.

Its solution will be discussed below. Equation (12) is a linear inhomogeneous equation, the left-hand side consisting of the known solutions of (11) and the unknown $R_1$. The solubility condition for such an inhomogeneous equation is

$$\left(\widetilde{W}_0 \, \mathcal{L}(W_1)\right)_m = 0$$

For, if an operator $\mathcal{L}$ has a linear homogeneous eigenvalue problem

$$\mathcal{L}(W_0) = 0$$

and also

$$\mathcal{L}(W_1) = F$$

where $F$ is an inhomogeneous term, then, multiplying the second equation by $\widetilde{W}_0$ and integrating over the entire volume gives

$$\int \widetilde{W}_0 \, \mathcal{L}(W_1) \, d\tau = \int \left(\widetilde{\mathcal{L}}(\widetilde{W}_0)\right) W_1 \, d\tau$$

But $\widetilde{\mathcal{L}}(\widetilde{W}_0) = 0$. Thus the solubility condition follows, stating that the inhomogeneous term must be orthogonal to $\widetilde{W}_0$.

This condition gives us

$$R_1 = \frac{\left(\widetilde{W}_0 \left(\nabla_1^2 h_{00} + \sigma^{-1}\left(\frac{\partial}{\partial t} - \nabla^2\right) L_{00}\right)\right)_m}{\left(\widetilde{W}_0 \nabla_1^2 W_0\right)_m}$$

It has recently been shown that $R_1 = 0$ for all boundary conditions. The solubility condition applied to (13) gives

$$R_2(\widetilde{W}_0 \nabla_1^2 W_0)_m = -\left(\widetilde{W}_0\left[(W_0 T_0)_m - \overline{W_0 T_0}\right]\nabla_1^2 W_0\right)_m + \\ + \left[\widetilde{W}_0 \nabla_1^2 (h_{01} + h_{10})\right]_m - \\ - \sigma^{-1}\left[\widetilde{W}_0\left(\frac{\partial}{\partial t} - \nabla^2\right)(L_{01} + L_{10})\right]_m . \tag{15}$$

To this approximation, $R_2$ determines the amplitude $\varepsilon$ .

$$R - R_0 = \varepsilon^2 R_2$$

If $R_2$ were negative, $R < R_0$ , and we could have finite amplitude instability before infinitesimal amplitude instability. The first term on the right of (15) depends only on the eigenfunctions of the linear problem. The second and third terms containing the fluctuating non-linear terms involve the first distortion to the flow. The importance of the third term when the Prandtl number is small is seen in experiments in mercury $(\sigma \simeq \frac{1}{40})$ where a reduction in the initial amplitude is observed.

The above constitutes a formal technique for determining the amplitude and distortion of a disturbance. The linearized problem permits for the same Rayleigh number an infinitely degenerate set of eigenfunctions. The steady state finite solutions also form a multiply infinite set. The stability of these finite amplitude solutions must be explored in order to select the realizable solutions.

We now look at the linear eigenvalue problem

$$\mathcal{L}(W_0) = 0 \tag{11}$$

For this, the boundary conditions can be written as follows: Since $T = 0$ , $z = 0, d$, we also have $\nabla_1^2 T = 0$. From (6) (linearized),

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \nabla^2 W = 0 , \quad z = 0, d.$$

Since all horizontal and time derivatives vanish on the boundaries, it follows that

$$\frac{\partial^4 W}{\partial z^4} = 0, \quad z = 0, d.$$

Also $W = \dfrac{\partial^2 W}{\partial z^2} = 0$ , $z = 0, d$ free boundaries.

Equation (11) can be solved by separation of variables, writing

$$W_0 = F(x, y)\, G(z)\, \phi(t)$$
$$\phi(t) = e^{nt}$$

As the parameters of a system (here the Rayleigh number) are varied, the system goes from stable to unstable conditions through a marginal state. ($\mathcal{R}e\, n$ goes from negative to positive values.) If $\mathcal{I}m\, n > 0$ ,instability sets in as oscillations of increasing amplitude; this is called overstability. If $\mathcal{I}m\, n = 0$ , then the transition from stability to instability takes place through a marginal state exhibiting a stationary pattern of motions. We now investigate the properties of $n$ . Equation (11) expanded is

$$\frac{n^2}{\sigma} \nabla^2 W_0 - n\left(1 + \frac{1}{\sigma}\right) \nabla^4 W_0 + \nabla^6 W_0 - R_0 \nabla_1^2 W_0 = 0$$

Multiplying this by $\widetilde{W}_0$ and integrating over the volume gives

$$\frac{n^2}{\sigma} I_1 + n\left(1+\frac{1}{\sigma}\right) I_2 - (-I_3 + R_0 I_4) = 0 \qquad (16)$$

where

$$\int \widetilde{W}_0 \nabla^2 W_0 \, d\tau = -\int (\nabla W_0)^2 d\tau \quad \text{(for free boundaries } \widetilde{W}_0 = W_0 \text{)}$$

$$\equiv -I_1$$

$$\int W_0 \nabla^4 W_0 \, d\tau = \int (\nabla^2 W_0)^2 d\tau$$

$$\equiv +I_2$$

$$\int W_0 \nabla^6 W_0 \, d\tau = -\int \left(\nabla(\nabla^2 W_0)\right)^2 d\tau$$

$$\equiv -I_3$$

$$\int W_0 \nabla_1^2 W_0 \, d\tau = -\int (\nabla_1 W_0)^2 d\tau$$

$$\equiv -I_4$$

where $I_1$, $I_2$, $I_3$, $I_4$ are positive quantities and the surface integrals associated with each vanishes because of the boundary conditions.

Setting $n = p + iq$, with $p, q$ real in equation (16), the imaginary part gives us

$$2pq\left(\frac{I_1}{\sigma}\right) + q\left(1+\frac{1}{\sigma}\right) I_2 = 0$$

implying that either $q = 0$ or $p < 0$; i.e. instability sets in via a marginal state which is stationary. Thus the marginal state is characterized by setting $n = 0$ , and we have to solve

$$\nabla^6 W_0 - R \nabla_1^2 W_0 = 0$$

$$W_0 = F(x,y)\, G(z)$$

We require that solutions be periodic in the horizontal:

$$\nabla_1^2 F + \alpha^2 \pi^2 F = 0$$

This is satisfied by close-packed cells (squares, rectangles, hexagons, triangles). Then

$$W_0 = F(x,y) \sum_{i=1}^{3} \left[ A_i \cosh 2\mu_i z + B_i \sinh 2\mu_i z \right] \tag{17}$$

where

$$4\mu_i^2 = \alpha^2 \pi^2 \left\{ 1 - \left( R_0 / \alpha^4 \pi^4 \right)^{1/3} \omega_i \right\}$$

$\omega_i$ = three cube roots of unit

$A_i , B_i$ , = arbitrary constants

For two free boundaries, (17) becomes

$$W_0 = A F(x,y) \sin n\pi z , \quad n = 1, 2, 3, \ldots$$

$$R_{0,n} = \pi^4 \frac{(n^2 + \alpha^2)^3}{\alpha^2}$$

$R_{0,1}$ has a minimum value at $\alpha^2 = \frac{1}{2}$ , giving

$$\left(R_{0,1}\right)_{min} = \frac{27}{4}\pi^4 = 657.$$

For two rigid boundaries

$$\left(R_{0,1}\right)_{min} = 1707.8, \quad \alpha^2 \simeq 1.$$

Notes submitted by

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# Variational Procedures

Frederic Bisshopp

I. Preliminary remarks based on the calculus of variations.

Let $SC^n$ denote the set of functions $u \in C^n$, on the interval $[0,1]$ satisfying some specified boundary conditions:

$$\sum_{i=1}^{n} a_i u^{(i)}(0) + b_i u^{(i)}(1) = 0 \qquad (1)$$

Let the function $F(x_0, \cdots x_m)$ possess a sufficient number of derivatives for the following analysis. Functions satisfying (1) are closed under addition. Then, if $u \in SC^n$ and $\bar{u} \in SC^n$, it follows that

$$\delta u \equiv \bar{u} - u \in SC^n \qquad (2)$$

In particular, if $0 \leq \max|\bar{u} - u| = \delta$ then

$$\delta u \equiv \bar{u} - u \in S_\delta C^n$$

If $n > 0$, we can perform differentiation:

$$\frac{d}{dx}\delta u = \bar{u}' - u' = \delta u' \qquad \in S'C^{n-1}$$

Similarly define any higher derivative

$$\left(\frac{d}{dx}\right)^m \delta u = \delta u^{(m)} \qquad m \leq n$$

Now consider $F(x, u, \cdots u^{(n)})$

Define $\Delta F \equiv F(x, \bar{u}, \cdots) - F(x, u, \cdots)$

If $F$ possesses first partial derivatives everywhere, then

$$\Delta F = \delta F + o\left(\|\delta u\|\right)$$

where $\delta F = \sum_j F_{u^{(j)}} \delta u^{(j)}$

and
$$\|\delta u\| = \max_{\substack{j \in [0,n] \\ x \in [0,1]}} |u^{(j)}|$$

Alternatively

$$\lim_{\delta \to 0} \left( \frac{\Delta F - \delta F}{\delta} \right) = 0 \text{ for any } \delta u \ni \|\delta u\| = \delta$$

Consider as a simple example of a functional

$$I(u) = \int_0^1 F(x, u, \cdots)\, dx$$

As above, define

$$\begin{aligned} \Delta I &= I(\bar{u}) - I(u) \\ &= \int_0^1 \Delta F\, dx \\ &= \int_0^1 \delta F\, dx + o\left(\|\delta u\|\right) \\ &= \delta I + o\left(\|\delta u\|\right) \end{aligned}$$

where $\delta I \equiv \int_0^1 \delta F\, dx$

i.e. $\lim\limits_{\delta \to 0} \dfrac{\Delta I - \delta I}{\delta} = 0$ for any $\delta u \ni \|\delta u\| = \delta$

Other kinds of functionals are

(i) $I(u,v) = \int_0^1 F(x, u, \dots u^{(n)}, v, \dots v^{m})\,dx$

(ii) $I(u) = \iint dx\,dy\, F(x, y, u, u_x, u_y, \dots)$

(iii) $G(I_1(u), I_2(u), \dots)$

Many physical problems can be expressed as variational problems; the solution is to be found by requiring that a functional have a stationary value. For example,

$$\delta I = 0 \longrightarrow \int_0^1 \sum F_{u^{(j)}} \left(\frac{d}{dx}\right)^j \delta u\, dx = 0$$

To find the related differential equation, integrate the above by parts, and obtain

$$\delta I = \int_0^1 dx \left(F_u - \frac{d}{dx} F_{u'} + \dots \left(-\frac{d}{dx}\right)^n F_{u^{(n)}}\right) \delta u +$$

$$+ \left[ (F_{u'} \delta u) + \left(F_{u''} \delta u' - \frac{d}{dx} F_{u''} \delta u\right) + \dots \right] \Bigg|_0^1$$

If the class of functions $u$ are picked so that the integrated part vanishes, then $u$ is said to satisfy natural boundary conditions. Then it can be shown that $u$ satisfies the Euler-Lagrange equations.

$$F_u - \frac{d}{dx} F_{u'} + \dots + \left(-\frac{d}{dx}\right)^n F_{u^{(n)}} = 0 \qquad (3)$$

Usually we have the differential equation and want to know the corresponding variational principle. Therefore let us consider -

II. Examples of problems stated in variational form.

(a) Coupled differential equations when we have a functional $I(u,v)$ as in (i)

$$\delta I = \int_0^1 dx \left[ \left( F_u - \frac{d}{dx} F_{u'} + \cdots \right) \delta u + \left( F_v - \frac{d}{dx} F_{v'} + \cdots \right) \delta v \right] + \left. + \left\{ O.I.P. \right\} \right|_0^1 \qquad \text{(Out-Integrated Part)}$$

$\left. \{O.I.P.\} \right|_0^1 = 0$ for natural boundary conditions on $u$ and $v$.

Since $\delta u$ and $\delta v$ are independent,

$$\begin{aligned} F_u - \frac{d}{dx} F_{u'} + \cdots &= 0 \\ F_v - \frac{d}{dx} F_{v'} + \cdots &= 0 \end{aligned} \tag{4}$$

(b) Partial differential equations from functional $I(u)$ as in (ii).

Again $\delta I = 0$ along with natural boundary conditions that make the integrated part (line integral) vanish will give the equation

$$F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} + \cdots = 0 \tag{5}$$

(c) Eigenvalue problems arising from two functionals $I_1$ and $I_2$ as in (iii).

An eigenvalue problem arises when an unknown parameter is introduced. Thus it may arise when we require $I_1$ to be stationary,

$\delta I_1 = 0$ , with a subsidiary condition that $I_2$ be fixed (i.e. Lagrange multiplier is introduced):

$$\delta(I_1 - \lambda I_2) = 0$$

Another way to obtain the same eigenvalue problem is to define

$$\lambda = \frac{I_1}{I_2}$$

Then requiring $\delta\lambda = 0$ leads to

$$\delta\lambda = \frac{1}{I_2}(\delta I_1 - \lambda\, \delta I_2) = 0 \tag{6}$$

as above. The stationary property of the eigenvalue is usually stated in this way. These are not necessarily linear eigenvalue problems.

(d) Linear eigenvalue problems

$$Lu = \lambda\, \ell(z) u, \tag{7}$$

where

$$L = p_0(z)\left(\frac{d}{dz}\right)^n + p_1(z)\left(\frac{d}{dz}\right)^{n-1} + \cdots + p_n(z)$$

with boundary conditions $u_i = 0$, $(u \in SC^n)$ . We wish to find a variational problem whose Euler-Lagrange equation is the above differential equation. In general this can always be done, for, take any function $v$ in some other space

$$v \in S'C^n$$

Consider
$$\lambda = \frac{\int \rho(z) v(z) L u}{\int \rho(z) v \ell u} \tag{8}$$

We take the density function $\rho(z) = 1$ since it can be incorporated into $L$ and $\ell$ while giving the same eigenvalue problem. Now vary $\lambda$, picking $v$ in such a way that we obtain (7) as one of the Euler-Lagrange equations.

$$\lambda = \frac{\int_0^1 v L u}{\int_0^1 v \ell u} \equiv \frac{(v, Lu)}{(v, \ell u)}$$

$$\delta\lambda = \frac{1}{(u, \ell v)} \Big[ (\delta v, (Lu - \lambda \ell u)) + (\delta u, (L^T v - \lambda \ell v)) + \cancel{\{O.I.P.\}_0^1} \Big]$$

where $L^T$ is the adjoint of $L$

$$L^T = \left(-\frac{d}{dz}\right)^n p_0(z) + \left(-\frac{d}{dz}\right)^{n-1} P_1 + \cdots + P_n$$

Here O.I.P. is the bilinear concomitant but it is not needed since it vanishes by choosing $V_i = 0$ .

Thus $\delta\lambda = 0$ implies that given

$$Lu - \lambda \ell u = 0$$

and $L^T v - \lambda \ell v = 0$ , with boundary conditions $U_i = 0$, $V_i = 0$, $\{L^T, V_i\}$ are adjoint to $\{L, u_i\}$. The system is self-adjoint if $L^T = L$ and the $v_i$ will be a rearrangement of the $u_i$ . For the system to be self-adjoint we must have $P_0 = 1$ and $n$ even.

Orthogonality

Suppose the eigenvalues $\lambda$ are the set $\{\lambda_n\}$, we order the eigenfunctions $\{u_n\}$ and $\{v_n\}$ in the same way.

$$Lu_n = \lambda_n \ell u_n \qquad L^T v_n = \lambda_n \ell v_n$$

$$\lambda_n (u_m, \ell v_n) = (u_m, L^T v_n) = (Lu_m, v_n)$$
$$= \lambda_m (\ell u_m, v_n)$$

i.e. $(\lambda_n - \lambda_m)(u_m, \ell v_n) = 0$

For non-degenerate eigenvalues,

$$(u_m, \ell v_n) = N_n \delta_{mn}$$

where $N_n$ is a normalization factor.

Thus, if we want to expand any function that satisfies the $u$ boundary conditions, we expand in terms of the functions $u$, and to get the expansion coefficients, we multiply by $\ell v$ instead of by $u$.

This formalism is used to calculate eigenvalues approximately. In

$$\lambda = \frac{I_1(u,v)}{I_2(u,v)}$$

we choose sets of functions $\varphi_n$ and $\psi_n$ satisfying appropriate boundary conditions and we expand

$$u = \sum A_n \varphi_n$$
$$v = \sum B_n \psi_n$$

giving $\lambda = \lambda(A_n, B_n)$. We differentiate with respect to A, B, and require $\lambda$ be stationary, i.e.

$$\frac{\partial \lambda}{\partial A_i} = \frac{\partial \lambda}{\partial B_i} = 0.$$

This gives

$$\sum_{k=1}^{\infty} A_k \left\{ (\varphi_k, L^T \psi_j) - \lambda (\varphi_k, \ell \psi_j) \right\} = 0$$

$$j = 1, 2, \ldots$$

$$\sum_{k=1}^{\infty} B_k \left\{ (\psi_k, L \varphi_j) - \lambda (\psi_k, \ell \varphi_j) \right\} = 0$$

In most calculations, the problem is truncated at $k = 2$ or 3. The resulting $\lambda$ is an approximate solution of $\delta\lambda = 0$, i.e. where $\lambda$ has a stationary value. Thus the approximation to $\lambda$ will be better than the approximation to $u$.

III. More complicated examples.

(a) Suppose we have

$$Lu = \lambda \ell(z) v \quad , \quad u_i = 0$$

$$Mv = \mu m(z) u \qquad V_j = 0$$

where

$$L = \frac{d^n}{dz^n} + \cdots \quad , \quad M = \frac{d^m}{dz^m} + \cdots$$

In hydrodynamic stability problems one often eliminates $v$ and gets a higher order equation for $u$:

$$M \frac{1}{\ell} L u = (\lambda \mu) m u$$

$$\lambda \mu = \frac{(u^T, M \frac{1}{\ell} L u)}{(u^T, m u)}$$

However, for higher order equations, the boundary conditions are messy; by going to higher order one is not making use of the fact that the boundary conditions factor out.

(b) If we are willing to solve one of the equations, say

$$v = \mu M^{-1} m u \quad \text{(formally)}$$

then

$$\mu\lambda = \frac{(u^T, Lu)}{(u^T, \ell M^{-1} m u)}$$

Define $v^T$ by

$$L^T u^T = \lambda m v^T$$

Then the requirement that $\delta(\mu\lambda) = 0$, for variations $\delta u$ and $\delta u^T$ gives

$$Lu = \lambda \ell v$$

$$M^T v^T = \mu m u^T$$

as required. The method consists in solving half the problem exactly, then the rest by variational method.

(c) Let $\lambda = \dfrac{(u^T, Lu)}{(u^T, \ell v)}$ , $\mu = \dfrac{(v^T, Mv)}{(v^T, mu)}$

Then, setting $\delta(\lambda\mu) = 0$ for independent variations $\delta u$, $\delta u^T$, $\delta v$, $\delta v^T$, again gives

$$Lu = \lambda l v \qquad Mv = \mu m u$$

$$L^T u^T = \lambda m v^T \qquad M^T v^T = \mu l u^T$$

(d) Suppose we have three coupled equations

$$L_i u_i = \sum_j \lambda_{ij} l_{ij} u_j \qquad , \quad i = 1 \cdots 3$$

Which of the $\lambda_{ij}$ are eigenvalues?

The $\lambda_{ii}$ may be set equal to zero since they can be as well incorporated in $L$. We scale the $u_i$

$$u_i \longrightarrow a_i u_i \qquad \text{(not summed)}$$

and look for combinations of the remaining $\lambda'$s which do not change.

$$u_i' = a_i u_i$$

implies

$$L_i \frac{u_i'}{a_i} = \sum_j \lambda_{ij} l_{ij} \frac{u_j'}{a_j}$$

$$L_i u_i' = \sum \left(\frac{a_i}{a_j} \lambda_{ij}\right) l_{ij} u_j$$

$$\equiv \sum \lambda_{ij}' l_{ij} u_j'$$

i.e. $$\lambda_{ij}' = \frac{a_i}{a_j} \lambda_{ij}$$

Consequently certain products of the $\lambda'$s satisfy the requirement of invariance under the above transformations. For example:

1.) $$L_1 u_1 = \lambda_{12} u_2 \qquad \lambda'_{12} = \frac{a_1}{a_2}\lambda_{12}$$

$$L_2 u_2 = \lambda_{23} u_3 \qquad \lambda'_{23} = \frac{a_2}{a_3}\lambda_{23}$$

$$L_3 u_3 = \lambda_{31} u_1 \qquad \lambda'_{31} = \frac{a_3}{a_1}\lambda_{31}$$

Obviously $\lambda'_{12}\lambda'_{23}\lambda'_{31} = \lambda_{12}\lambda_{23}\lambda_{31} = \Lambda$

i.e. the product of the three is invariant to scaling.

2.) $$L_1 u_1 = \lambda_{12} u_2 + \lambda_{13} u_3$$

$$L_2 u_2 = \lambda_{21} u_1$$

$$L_3 u_3 = \lambda_{31} u_1$$

In this case there are two eigenvalues,

$$\Lambda_1 = \lambda_{12}\lambda_{21}$$

$$\Lambda_2 = \lambda_{13}\lambda_{31}$$

Such a case occurs for thermal convection in a rotating system, where one is the Taylor number, the other the Rayleigh number.

3.) $$L_1 u_1 = \lambda_{12} u_2 + \lambda_{13} u_3$$

$$L_2 u_2 = \lambda_{23} u_3$$

$$L_3 u_3 = \lambda_{31} u_1$$

Then the eigenvalues are

$$\Lambda_1 = \lambda_{12}\lambda_{23}\lambda_{31}$$
$$\Lambda_2 = \lambda_{13}\lambda_{31}$$

4.) $$L_1 u_1 = \lambda_{12}u_2 + \lambda_{13}u_3$$

$$L_2 u_2 = \lambda_{21}u_1 + \lambda_{23}u_3$$

$$L_3 u_3 = \lambda_{31}u_1$$

Then the eigenvalues are

$$\Lambda_1 = \lambda_{12}\lambda_{23}\lambda_{31}$$

$$\Lambda_2 = \lambda_{12}\lambda_{21}$$

$$\Lambda_3 = \lambda_{13}\lambda_{31}$$

5.) When all six of the $\lambda$'s are present, the eigenvalues are

$$\Lambda_1 = \lambda_{12}\lambda_{23}\lambda_{31}$$

$$\Lambda_2 = \lambda_{13}\lambda_{32}\lambda_{21}$$

$$\Lambda_3 = \lambda_{12}\lambda_{21}$$

$$\Lambda_4 = \lambda_{13}\lambda_{31}$$

Note $\lambda_{23}\lambda_{32} = \dfrac{\Lambda_1\Lambda_2}{\Lambda_3\Lambda_4}$

Variational formulation of the above cases remains to be carried out.

Notes submitted by

Ruby Krishnamurti

## Thermal Convection: Experiments and Finite Amplitude Effects

Willem V.R. Malkus

In this lecture we will look at some aspects of the effects of finite amplitude on the cellular convection set up in a layer of fluid heated from below.

The effects of finite amplitude come from two kinds of non-linearity in the non-dimensionalized Boussinesq equations. The first comes from the term $\frac{1}{\sigma} \underset{\sim}{V} \cdot \nabla \underset{\sim}{V}$ in the momentum equation and the second from the term $\underset{\sim}{V} \cdot \nabla T$ in the heat equation. We can thus say that at high Prandtl number ( $\sigma$ ) the first effects of non-linearity will come from the heat equation while for low $\sigma$ non-linearities in the momentum equation will dominate. Consistently we expect that for high $\sigma$ ( $\nu$ large) the flow field will be smooth and the temperature strongly disturbed with sharp gradients while for small $\sigma$ we expect a smooth temperature field and a velocity field containing sharp gradients and more small-scale motions.

We thus expect to find at least some dependence of the observables (heat transport, mean fields, etc.) on the Prandtl number for overcritical Rayleigh numbers.

We now turn to a brief description of the experimental results obtained to date.

The first detection of the condition under which convection first starts has been obtained by measurements of the heat flux for different temperature differences over the fluid layer (see Fig.1).

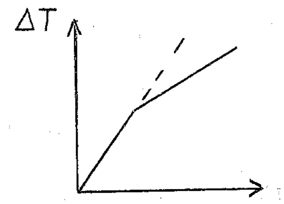



Fig. 1.

It was found that at a certain value of the temperature difference corresponding to a certain universal Rayleigh number there was a relatively well-defined turn in the curve beyond which the heat flux increases over the value corresponding to pure conduction.

More recent experiments on the same phenomenon utilized the setup shown in principle in Fig. 2.

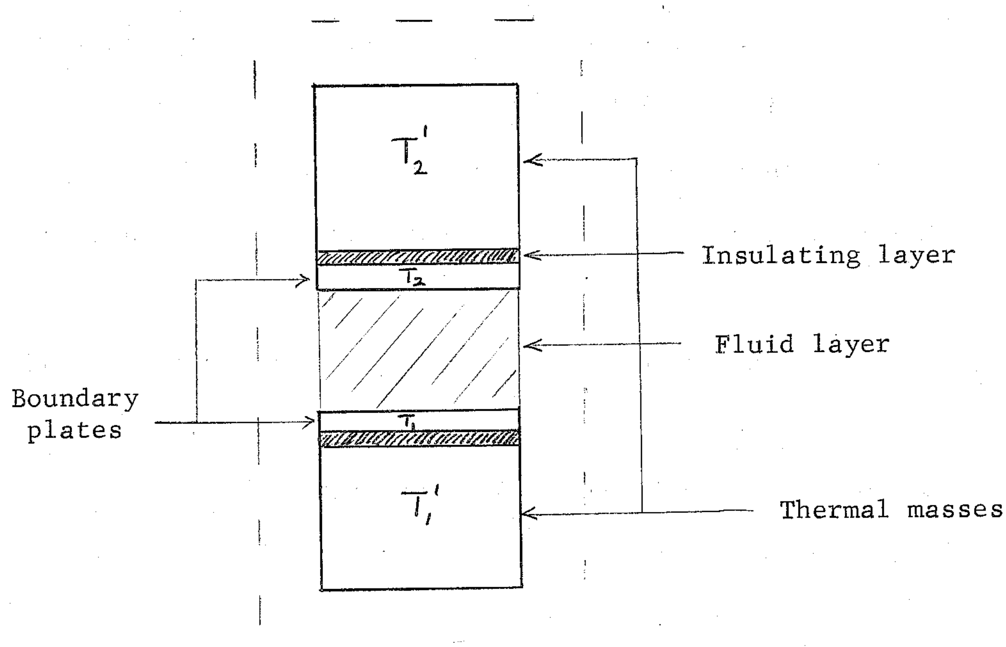



Fig. 2

Initially the thermal masses are brought to a certain temperature difference. The system is then left to decay. During the decay the heat flux and temperature difference through the layer is measured. The heat flux is obtained from the temperature jump over the insulating layer. In this way you can cover a wide range of Rayleigh numbers in a single experiment. The results are indicated in Fig. 3.

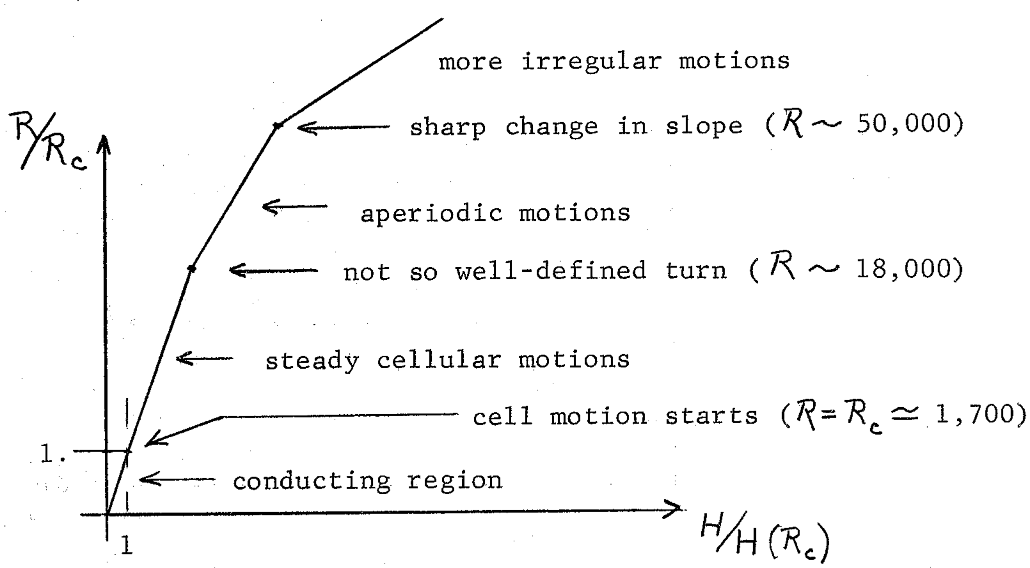



Figure 3.

The curve representing $H/H(\mathcal{R}_c)$ as a function of seems to show a series of more or less well-defined changes in slope (the first of which are shown in the diagram). These discontinuities seem to be related to changes in the character of the flow. Thus the flow at each step turns into a more irregular state finally becoming completely disordered or turbulent. These steps from one state to another more disordered state seem to be of great interest in connection with the general problem of the onset of turbulent motion.

When a wider range of Rayleigh numbers is considered, the heat transport curve is usually drawn with logarithmic scales. The principal features of this curve are shown in Fig. 4.

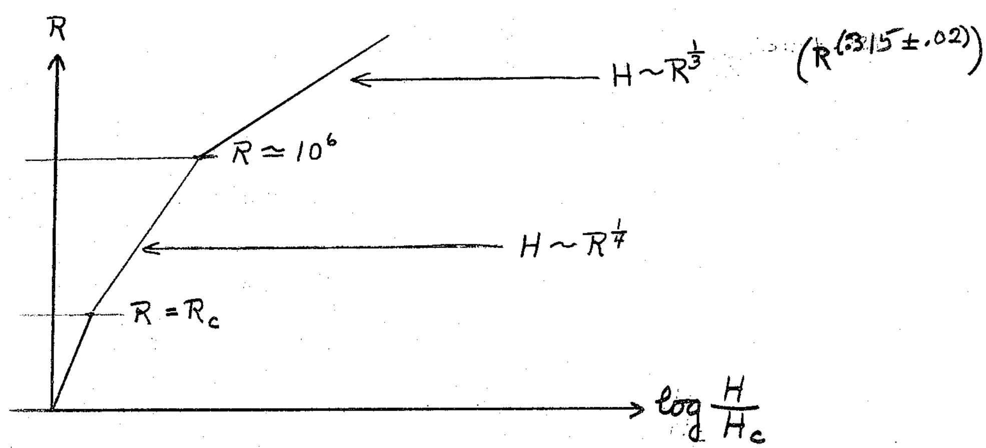



Fig. 4

As can be seen from the diagram the heat transport curve has two principal ranges for $R > R_c$.

The series of steps discussed above are all contained in the region where $H \sim R^{\frac{1}{4}}$ . The region where $H \sim R^{\frac{1}{3}}$ is probably extended to infinity. This power law can be obtained by dimensional arguments by assuming that the heat transport is independent of the layer thickness which might be expected for large enough Rayleigh number. This is, however, a crucial point which is not definitely settled yet.

So far no Prandtl number dependence had been observed on the heat flux. Recently however some experiments using mercury to obtain low $\sigma$ showed a clear dependence. (see Fig. 5).

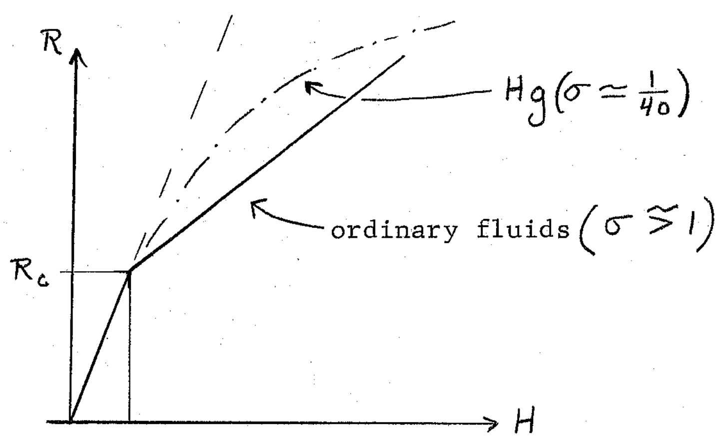



Fig. 5

The lower value on $H$ for small $R - R_c$ seems to depend on the stronger excitation of higher modes in the velocity field which are not correlated to produce the highest heat flux.

In an earlier lecture we presented the formalism by which we can obtain higher approximations to the solutions of the Boussinesq equations than the linear theory yields. By developing after the amplitude $\varepsilon$ of the disturbances we produced an infinite set of non-homogeneous linear equations for the vertical velocity. For this procedure the Rayleigh number $\mathcal{R}$ is also developed after $\varepsilon$.

Thus

$$\mathcal{R} = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 \ldots$$

It will be shown later (coming lecture) that $\mathcal{R}_1 = 0$ quite generally. The first result for the amplitudes of the disturbances are thus given by $\mathcal{R}_2$

$$\varepsilon^2 = \frac{\mathcal{R} - \mathcal{R}_0}{\mathcal{R}_2} \simeq W T \tag{1}$$

As an example we shall give here a sketch of the analysis for two-dimensional rolls with free boundaries. In this case we have

$$\left.\begin{aligned}
W_0 &= 2 \cos \pi\alpha x \sin \pi z \\
T_0 &= (1+\alpha^2)^2 \left(\frac{2\pi^2}{\alpha^2}\right) \cos \pi\alpha x \sin \pi z \\
U_0 &= -\frac{2}{\alpha} \sin \pi\alpha x \cos \pi z, \quad V_0 = 0 \\
R_0 &= \pi^4 \frac{(1+\alpha^2)^3}{\alpha^2}, \quad \left(\alpha^2 = \tfrac{1}{2} \text{ for } R_{0min}\right) \\
h_{00} &= 0, L_{00} = 0 \Rightarrow W_1 = 0, \ R_1 = 0 \\
R_2 &= \frac{\pi^2}{2}(1+\alpha^2)^2 / \alpha^2
\end{aligned}\right\} \tag{2}$$

In this case the first finite amplitude effects appear as a distortion of the mean field of temperature $(\beta)$.

By means of the formula

$$\beta/\beta_m = 1 + \frac{1}{\varkappa\beta_m}\left((\overline{WT})_m - \overline{WT}\right) \tag{3}$$

we obtain

$$\beta/\beta_m = 1 + 2\left(1 - \frac{R_0}{R}\right)\cos 2\pi z . \tag{4}$$

The coefficient (2 in this case) in front of the parenthesis is found to be 1.5 from experiment with rigid boundaries.

From (4) is seen that for $R > 2R_0$, $\beta$ becomes negative in the centre of the layer. This is considered as an unphysical feature which also disappears when higher approximations are taken. The heat transport is easily obtained from (4) with the aid of the general formula

$$H = \varkappa\beta + \overline{WT} \tag{5}$$

because $\overline{WT}$ vanishes at the boundaries ( $z = 0$ or $1$ ).

Figures 6 and 7 show the features of the different approximations. Figure 6 illustrates a curious behavior. It seems as if the curve when we take higher approximations will come back to the curve obtained in the second approximation. Figure 7 shows $\beta/\beta_m$ for $R = 3R_0$.

When three-dimensional disturbancies are considered we

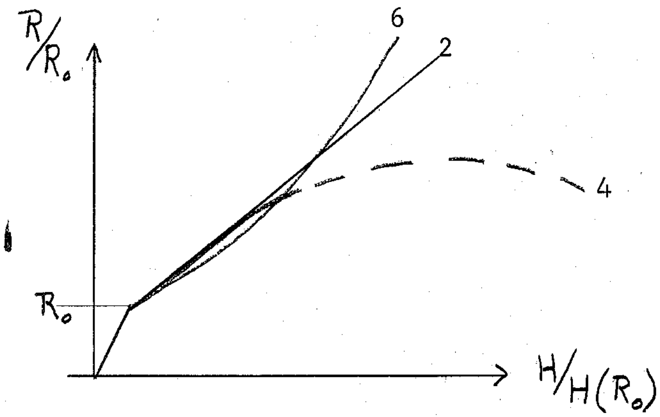



Fig. 6

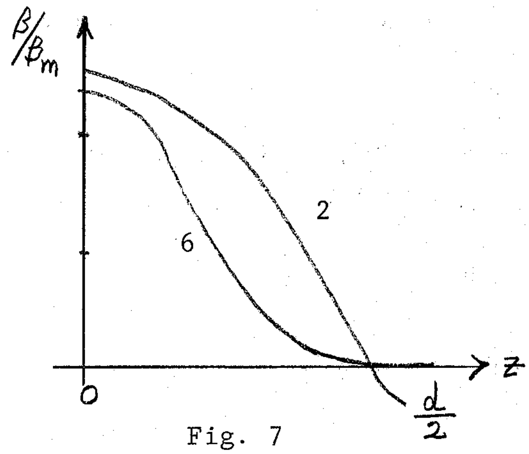



Fig. 7

obtain a different result for $R_2$ given below

$$R_2^{-1} = 2 \Big/ \left[ 1 + \frac{8x^2}{(1+\alpha^2)(32x^3-1)} \left\{ 1 + \frac{1}{4x^2\sigma} + \frac{1}{2x\sigma^2} \right\} \right] \qquad (6)$$

where $x = (1 + \frac{1}{2}\alpha^2) / 1 + \alpha^2$

Fig. 8 shows the result for the heat transport as computed from $R_2$ for different values of $\sigma$ and $\ell/m$ where $\ell$ and $m$ are the two horizontal wave numbers.

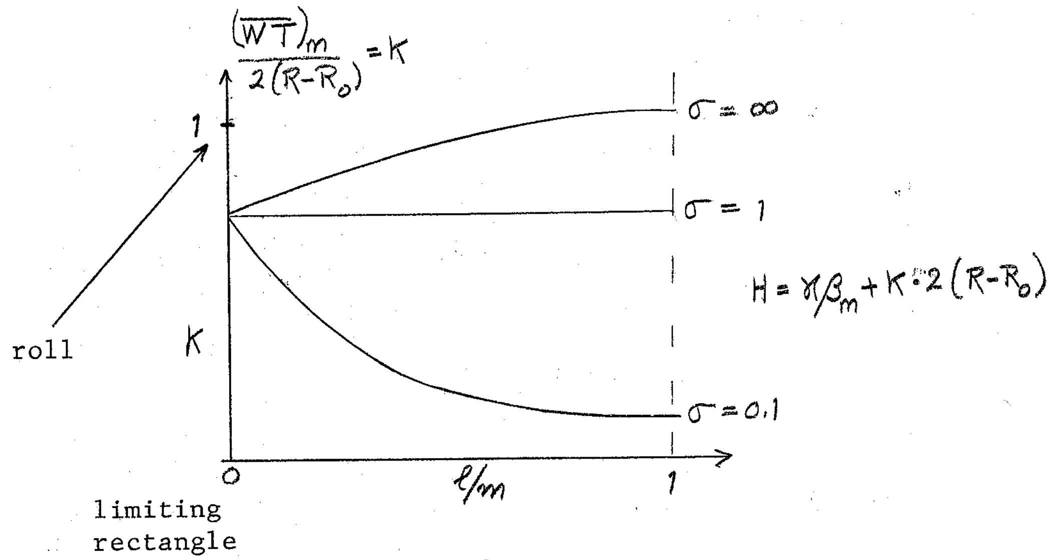



Fig. 8.

The general hope when considering three-dimensional plan forms is that one would be able to find some feature that determines which plan form is preferred. This problem will, however, be treated in the next lecture.

If we accept the limiting rectangle rather than roll as a possible solution, we see from Figure 8 that the plan form which gives maximum heat transport is not the same for different Prandtl numbers. However it is not quite certain that the rolls should be excluded.

When using rigid boundary conditions in the finite amplitude analysis we run into tedious numerical calculations. We have therefore turned our interest to approximate methods to determine amplitudes and heat transport of the convection field.

From the momentum and heat equation we can derive two integral relations:

$$\left.\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\left(v^2\right)_m &= \gamma\left(\overline{WT}\right)_m - \nu\left(\nabla v_i \cdot \nabla v_i\right)_m \\ \frac{1}{2}\frac{\partial}{\partial t}\left(T^2\right)_m &= \left(\beta\,\overline{WT}\right)_m - \kappa\left(\nabla T \cdot \nabla T\right)_m \end{aligned}\right\} \qquad (7)$$

If we assume certain forms $\underline{V}'$ and $T'$ of the fields we can determine their unknown amplitudes A and B from equation (7). Thus we get:

$$\left.\begin{aligned} \frac{B}{A} &= \nu\left(-\overline{V'\nabla^2 V'}\right)_m \Big/ \gamma\left(\overline{W'T'}\right)_m \\ AB &= R - R_s/S \end{aligned}\right. \qquad (8)$$

where $$R_s = \frac{B}{A}\,\frac{\left(-\overline{T'\nabla^2 T'}\right)_m}{\left(\overline{W'T'}\right)_m}$$

and $$S = \left(\overline{W'T'}\right)_m \left\{ \frac{\left(\overline{W'T'^2}\right)_m}{\left(\overline{W'T'}\right)_m^2} - 1 \right\}$$

The convective heat transport $(\overline{WT})$ is given by $\left(AB\left(\overline{W'T'}\right)_m\right)$.

If we choose the form of the initial Rayleigh instability with free boundaries for $\underline{V}'$ and $T'$ we obtain from (8):

$$\left(\overline{W'T'}\right)_m = 1$$

$$\left(\overline{W'T'}\right)_m A\,B = 2\left(R - R_o\right) \qquad (9)$$

which is identical to the result for the roll with free boundaries in the second approximation.

If instead we use forms that approximate the infinitesimal solutions for rigid boundaries for $\underline{V}'$ and $T'$ in (8) we obtain

$$AB = 1.51(\mathcal{R} - 1713.7) \tag{10}$$

This result is very close to the experimental result for the heat transport. This encouraged us to do a more speculative calculation. Suppose the different modes in the convection field do not interact. Then we can derive the total heat transport as the sum of the transport of the different modes. Thus we have:

$$AB = \sum_{1}^{\infty} (\mathcal{R} - \mathcal{R}_{S_i})/S_i \tag{11}$$

In this formula $\mathcal{R}_{S_i}$ denotes the value of $\mathcal{R}$ for which the i'th mode becomes unstable. $S_i$ is determined from (8) with the i'th infinitesimal mode for $V'$ and $T'$.

The heat transport curve obtained in this way shows remarkable resemblance to the experimental results at least qualitatively. The brakes obtained from (11) occur almost exactly for the right value of $\mathcal{R}$.

The reason for the very good results obtained from the integral relations (7) are not quite understood. There is no rational way of choosing the integral statement that will give one the best result.

We will illustrate this by constructing another integral relation from the following form of the heat equation.

$$\mathcal{R} W = h + \left[\overline{WT} - (\overline{WT})_m\right] W - \nabla^2 T \equiv L \tag{12}$$

Squaring and integrating both sides we obtain

$$\mathcal{R}^2 = \frac{(\overline{L^2})_m}{(\overline{W^2})_m} \tag{13}$$

With the infinitesimal roll for $\underline{V}'$ and $T'$ we obtain:

$$AB = (2\mathcal{R}^2 - \mathcal{R}_o)^{\frac{1}{2}} - \mathcal{R}_o \tag{14}$$

which is different from (9). Furthermore the result obtained from (13) is not independent of the plan form which was the case with equations (8).

From the given examples we see that the same trial function gives entirely different results when applied to different integral statements. The problem of how to choose the best integral statement is therefore of great interest.

Notes submitted by

Sten Gösta Walin

# Instability of Finite Amplitude Solutions of the Convection Problem

Dietrich Lortz July 3, 1964

The solution given by the development after the amplitude ( $\varepsilon$ ) to the Boussinesq convection problem is infinitely degenerate.

That is, the horizontal dependence of the field can be of the general type

$$w = \sum_n C_n w_n$$

where $w_n = e^{i \underset{\sim}{k}_n \cdot \underset{\sim}{r}}$

and $\underset{\sim}{k}_n = (k_x, k_y)$ with $k_x^2 + k_y^2 = a^2$.

$a^2$ is defined by the linear stability theory and shall be taken to give minimum value of the critical Rayleigh number ( $R_c$ ). However for $R > R_c$ motions with a different value of $a$, might as well exist which makes the system still more degenerate.

We shall now try to remove this degeneracy. This shall be done by considering the stability of the various finite amplitude solutions to infinitesimal disturbances. This is made possible by simultaneously solving the finite amplitude problem and the stability problem for small values of $R - R_c$ where $R_c$ is the Rayleigh number for the onset of convection. Thus we only hope to remove the degeneracy in the immediate vicinity of critical conditions.

We shall use the following notation:

$u_i$ = velocity

$T$ = temperature

$\partial_i = \dfrac{\partial}{\partial x_i}$

$\lambda_i = (0, 0, 1)$

$T - T_0 = \beta x_i \lambda_i + \theta$

$\rho = \rho_0 \left[ 1 - \alpha (T - T_0) \right]$

The non-dimensional Boussinesq equation becomes:

$$\left.\begin{aligned} \partial_t u_i + u_j \partial_j u_i &= -\partial_j \pi + P \lambda_i \theta + P \Delta u_i \\ \partial_t \theta + u_j \partial_j \theta &= R u_j \lambda_j + \Delta \theta \\ \partial_j u_j &= 0 \end{aligned}\right\} \quad (1)$$

where $P = \dfrac{\nu}{\kappa}$ and $R = \dfrac{\alpha g d^4 \beta}{\nu \kappa}$ and

the summation convection has been used.

Suppose that $(u_i, \theta)$ represents a solution to (1) with $\partial_t = 0$. The equation for the disturbances on the steady solution is obtained by inserting ($u_i + \tilde{u}_i$, $\theta + \tilde{\theta}$) instead of $(u_i, \tilde{\theta})$ in (1), where $(\tilde{u}_i, \tilde{\theta})$ are the infinitesimal disturbances whose time evolution we want to know. We obtain

$$\left.\begin{aligned} \sigma \tilde{u}_t + u_j \partial_j \tilde{u}_i + \tilde{u}_j \partial_j u_i &= -\partial_i \tilde{\pi} + P \lambda_i \tilde{\theta} + P \Delta \tilde{u}_i \\ \sigma \tilde{\theta} + u_j \partial_j \tilde{\theta} + \tilde{u}_j \partial_j \theta &= R \tilde{u}_j \lambda_j + \Delta \theta \\ \partial_j \tilde{u}_j &= 0 \end{aligned}\right\} \quad (2)$$

where we have replaced $\frac{\partial}{\partial t}$ by the growth rate $\sigma$ which is allowed because all coefficients are independent of $t$.

The boundary conditions on $(\tilde{u}_i, \tilde{\theta})$ are the same as those for $(u_i, \theta)$, which could be either of the rigid or free types.

The system (2) constitutes an eigenvalue problem for $\sigma$. More precisely: Given $R$, equations (1) give one an infinite set of solutions $(u_i, \theta)$. For which of these solutions does the system (2) give us $\sigma > 0$ ? If $\sigma > 0$ for a particular form of $(\tilde{u}_i, \tilde{\theta})$, instability is proved. If instead we want to prove stability we must show that $\sigma < 0$ for all kinds of disturbances, which clearly is a much more difficult task.

We shall now write equations (1) and (2) in a compact way using the following tensor notation:

$$D_{\varkappa\lambda} = \begin{pmatrix} \Delta & R & 0 & 0 \\ P & P\Delta & 0 & 0 \\ 0 & 0 & P\Delta & 0 \\ 0 & 0 & 0 & P\Delta \end{pmatrix}; \quad v_\varkappa = \begin{pmatrix} \theta \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}; \quad \partial_\varkappa = \begin{pmatrix} 0 \\ \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix}$$

Equations (1) become

$$\left.\begin{aligned} v_\lambda \partial_\lambda v_\varkappa &= D_{\varkappa\lambda} v_\lambda - \partial_\varkappa \pi \\ \partial_\varkappa v_\varkappa &= 0 \end{aligned}\right\} \qquad (3)$$

and equations (2) become

$$\begin{aligned} \sigma \tilde{v}_\varkappa + v_\lambda \partial_\lambda \tilde{v}_\varkappa + \tilde{v}_\lambda \partial_\lambda v_\varkappa &= D_{\varkappa\lambda} \tilde{v}_\lambda - \partial_\varkappa \tilde{\pi} \\ \partial_\varkappa v_\varkappa &= 0 \end{aligned} \qquad (4)$$

The non-linearity in (3) is handled by using the usual development in the amplitude $\varepsilon$.

Thus $v_\varkappa = \sum_1^\infty \varepsilon^m v_\varkappa^m$, $R = \sum_0^\infty \varepsilon^m R^m$ (5)

When (5) is inserted into (4) the interaction terms $v_\lambda \partial_\lambda \tilde{v}_\varkappa$ and $\tilde{v}_\lambda \partial_\lambda v_\varkappa$ appear as perturbations on the linear operator $\mathring{D}_{\varkappa\lambda}$ with $\varepsilon$ as perturbation parameter. Here $\mathring{D}_{\varkappa\lambda}$ means $D_{\varkappa\lambda}$ with $\mathring{R}$ inserted instead of $R$. As usual in perturbation theory we try the developments:

$$\sigma = \sum_0^\infty \varepsilon^m \sigma^m, \quad \tilde{v}_\varkappa = \sum_1^\infty \varepsilon^{m-1} \tilde{v}_\varkappa^m \tag{6}$$

For $D_{\varkappa\lambda}$ we use the expression

$$D_{\varkappa\lambda} = \mathring{D}_{\varkappa\lambda} + (R + \mathring{R}) A_{\varkappa\lambda} \tag{7}$$

where $A_{\varkappa\lambda} = \begin{pmatrix} 0100 \\ 0000 \\ 0000 \\ 0000 \end{pmatrix}$ or $A_{\varkappa\lambda} = \begin{cases} 1 \text{ if } (\varkappa\lambda) = (12) \\ 0 \quad \text{elsewhere} \end{cases}$

<u>Zeroth order in $\varepsilon$</u>

In the lowest order in $\varepsilon$ equations (3) and (4) become identical except for the term containing $\mathring{\sigma}$ in (4).

Equations (4) become

$$\left.\begin{aligned} \mathring{\sigma} \overset{1}{\tilde{v}}_\varkappa &= D_{\varkappa\lambda} \overset{1}{\tilde{v}}_\lambda - \partial_\varkappa \overset{1}{\tilde{\pi}} \\ \partial_\varkappa \overset{1}{\tilde{v}}_\varkappa &= 0 \end{aligned}\right\} \tag{8}$$

The operator $\overset{\circ}{D}_{\varkappa\lambda}$ has a fundamental property of self-adjointness which we express in the form:

$$\langle v'_{\varkappa} , D_{\varkappa\lambda} v''_{\lambda} \rangle = \langle v''_{\varkappa} , D_{\varkappa\lambda} v'_{\lambda} \rangle \tag{9}$$

where the brackets are defined as follows:

$$\langle v'_{\varkappa} , v''_{\varkappa} \rangle = \frac{P}{R} \left( v'_{o} v''_{o} \right)_{m} + \left( v'_{i} v''_{i} \right)_{m} \tag{10}$$

and $(\quad)_m$ denotes averaging over the layer.

(9) holds for any functions satisfying the boundary conditions.

The solutions to (8) are well-known and can be expressed in a single scalar quantity $\overset{\sim}{\overset{1}{v}}$ (the vertical velocity of a more general potential). If however we put $\overset{\circ}{\sigma} = 0$ in (8) we obtain the lowest order equations of (3). This simply means that if we choose the same horizontal wave number $a$ for $\overset{\sim}{\overset{1}{v}}_{\varkappa}$ and $\overset{1}{v}_{\varkappa}$ we have $\overset{\circ}{\sigma} = 0$ . Restricting the wave number of $\tilde{v}_{\varkappa}$ to be the same as that of $v_{\varkappa}$ is allowed as long as we want to prove instability of $v_{\varkappa}$ . When later considering the stability of rolls we will be forced to allow $\tilde{a} \neq a$.

The lowest order solutions can be expressed as

$$\left. \begin{aligned} \overset{1}{v}_{\varkappa} &= f(z) \sum_{n} C_n \omega_n \\ \overset{\sim}{\overset{1}{v}}_{\varkappa} &= f(z) \sum_{n} \tilde{C}_n \omega_n \end{aligned} \right\} \tag{11}$$

where $\overset{1}{v}_{\varkappa} = L \overset{1}{v}$ and $\overset{\sim}{\overset{1}{v}}_{\varkappa} = L \overset{\sim}{\overset{1}{v}}$ and $L$ is a linear differential vector operator.

In (11) $\omega_n = e^{i \underset{\sim}{k}_n \cdot \underset{\sim}{r}}$ , $\underset{\sim}{r} = (x, y)$ (12)

and $$|\underset{\sim}{k}_n| = a, \ \sum |C_n|^2 = 1, \ C_{-n} = C_n^* \tag{13}$$

$C_n^*$ denotes the complex conjugate of $C_n$ . Within the condition (13) $C_n$ can be chosen freely.

<u>First order in $\varepsilon$</u>

For the steady function $\overset{2}{v}_\chi$ we obtain from (3)

$$-\overset{1}{R} A_{\chi\lambda} \overset{1}{v}_\lambda + \overset{1}{v}_\lambda \partial_\lambda \overset{1}{v}_\chi = \overset{0}{D}_{\chi\lambda} \overset{2}{v}_\lambda - \partial_\chi \overset{2}{\pi} \ ; \ \partial_\lambda \overset{2}{v}_\lambda = 0 \tag{14}$$

$\overset{1}{R}$ is determined by forming the scalar product defined in (10) with $\overset{1}{v}_\chi$ .

Now it can be proved rather straightforwardly that

$$\langle \overset{1}{v}{}'_\chi , v_\lambda \partial_\lambda \overset{1}{v}{}''_\chi \rangle = 0 \tag{15}$$

$\overset{1}{v}{}'_\chi$ and $\overset{1}{v}{}''_\chi$ must be solutions to the zeroth order problem but $v_\lambda$ can be a somewhat more general function.

Using (15) and (9) it is easily shown that $\overset{1}{R} = 0$ . This result holds for all kinds of boundary conditions, even of the mixed type.

In essentially the same way we can prove that $\overset{1}{\sigma} = 0$ under the same conditions. We are thus forced to go to the next order in $\varepsilon$ to get any information about $\sigma$ .

The solutions to (14) with $R_1 = 0$ are of the form

$$\overset{2}{v}_{\varkappa} = \sum_{nm} F_{nm}\left((\underset{\sim}{k}_n \cdot \underset{\sim}{k}_m), z\right) \cdot C_n C_m w_n w_m \tag{16}$$

The corresponding solutions for the disturbances (with $\overset{1}{\sigma} = 0$ ) are of the form

$$\tilde{\overset{2}{v}}_{\varkappa} = \sum_{n,m} F_{nm}\left((\underset{\sim}{k}_n \cdot \underset{\sim}{k}_m), z\right) \cdot \tilde{C}_n C_m w_n w_m \tag{17}$$

<u>Second order in $\varepsilon$</u>

Equations (3) and (4) become

$$-\overset{2}{R} A_{\varkappa\lambda} \overset{1}{v}_{\lambda} + \overset{1}{v}_{\lambda} \partial_{\lambda} \overset{2}{v}_{\varkappa} + \overset{2}{v}_{\lambda} \partial_{\lambda} \overset{1}{v}_{\varkappa} = \overset{0}{D}_{\varkappa\lambda} \overset{3}{v}_{\lambda} - \partial_{\lambda} \overset{3}{\pi} \tag{18}$$

and

$$\overset{2}{\sigma} \tilde{\overset{1}{v}}_{\varkappa} - \overset{2}{R} A_{\varkappa\lambda} \tilde{\overset{1}{v}}_{\lambda} + \overset{1}{v}_{\lambda} \partial_{\lambda} \tilde{\overset{2}{v}}_{\varkappa} + \overset{2}{v}_{\lambda} \partial_{\lambda} \tilde{\overset{1}{v}}_{\varkappa} + \tilde{\overset{1}{v}}_{\lambda} \partial_{\lambda} \overset{2}{v}_{\varkappa} + \tilde{\overset{2}{v}}_{\lambda} \partial_{\lambda} \overset{1}{v}_{\varkappa} = \overset{0}{D}_{\varkappa\lambda} \tilde{\overset{3}{v}}_{\lambda} - \partial_{\lambda} \tilde{\overset{3}{\pi}} \tag{19}$$

Forming the scalar product defined in (10) of equations (18) and (19) with one of the solutions $\overset{1}{v}_{\varkappa}$ we obtain two relations of the following types:

$$\overset{2}{R} N_n C_n = T_{nm} C_m^{*} C_m \tag{20}$$

$$\sum_m \left(\overset{2}{\sigma} \delta_{nm} + T_{nm} C_n C_m\right) \tilde{C}_m = 0 \tag{21}$$

(21) is a system of homogeneous equations to determine $\tilde{C}_n$ from the coefficients $C_n$ and the wave number vectors $\underset{\sim}{k}_n$ appearing in the zeroth order solution.

For the system (21) to be solvable we must require

$$\det \left| \sigma^2 \delta_{nm} + T_{nm} C_n C_m \right| = 0 \tag{22}$$

This equation contains the information about $\sigma^2$ we are looking for.

It is possible to give conditions for the existence of at least one positive root $\sigma^2$ to Equation (22) without much information about the detailed structure of the matrix $C_n C_m T_{nm}$. In fact we do not even know the order of the matrix since the number of components $C_n W_n$ in the lowest order solutions is completely free. Thus it is possible to show that for all $T_{nm} C_n C_m$ that represent a three-dimensional flow pattern Equation (22) has at least one positive root $\sigma^2$ and hence all three-dimensional solutions to (3) are unstable. The only solutions left that can possibly be stable are thus the rolls. To show their stability we have to drop the assumption $\tilde{a} = a$. This has been done and in fact the rolls seem to be stable within certain limits of their horizontal wave number a.

Experimentally one generally observes a three-dimensional flow-pattern even when the lateral boundaries are reasonably far away. To explain this one has to look carefully on the approximations made. The only assumption that can not be systematically justified is that the coefficients $\nu$, $\alpha$ and $\gamma$ are independent of $T$. When introducing a slight dependence in these coefficients it is actually possible to show that one gets a region for $\mathcal{R}$ immediately above $\mathring{\mathcal{R}}$ where hexagons are stable. For $\mathcal{R}$ bigger than a certain value $\mathring{\mathcal{R}} + \Delta\mathcal{R}$ the rolls however are the only stable solution.

Summary

We have found the result that neglecting the temperature dependence of $\nu$ , $\alpha$ and $\varkappa$ the only stable finite amplitude solutions to the Boussinesq convection problem are the two-dimensional rolls. When a slight temperature dependence of the coefficients $\nu$ , $\alpha$ and $\varkappa$ is included we get a small region $(\mathring{\mathcal{R}} < \mathcal{R} < \mathring{\mathcal{R}} + \Delta\mathcal{R})$ where hexagons are stable.

Notes submitted by

Sten Gösta Walin

# Finite Amplitude Stability in Water Stratified by Both Salt and Heat

George Veronis

In this study we shall meet two kinds of instability of a fluid system. For a fluid system to be physically stable it must be stable against any kind of small disturbances. Generally we restrict our study to requiring that the system be stable against infinitesimal disturbances. However if the system is unstable to small but finite amplitude disturbances we may not be able to realize the unperturbed state in an experiment. This kind of instability we will call finite amplitude instability.

We will now look at the stability of a layer of water stratified by both salt and heat.

The Boussinesq equations for this system are:

$$\frac{\partial}{\partial t}\underline{V} + \underline{V}\nabla\underline{V} = -\frac{1}{\rho_m}\nabla p + g(\alpha T - \beta S)\underline{k} + \nu\nabla^2\underline{V}$$

$$\rho = \rho_m[1 - \alpha T + \beta S], \quad \alpha = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_S, \quad \beta = \left(\frac{1}{\rho}\frac{\partial\rho}{\partial S}\right)_T$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}w + \frac{\partial}{\partial y}v = 0 \qquad (1)$$

$$\frac{\partial}{\partial t}T + \underline{V}\cdot\nabla T - w\frac{\Delta T}{d} = K_T\nabla^2 T, \quad T_{tot} = T_m - \Delta T\frac{z}{d} + T$$

$$\frac{\partial}{\partial t}S + \underline{V}\cdot\nabla S - w\frac{\Delta S}{d} = K_S\nabla^2 S, \quad S_{tot} = S_m - \Delta S\frac{z}{d} + S$$

Introduce non-dimensional quantities according to:

$$\underline{V} = \frac{K_T}{d}\underline{V}' . \quad t = \frac{d^2}{K_T}t' \quad . \underline{r} = d\, r' \quad T = \Delta T\, T' \quad S = \Delta S\, S'$$

The boundary conditions are similar to the Rayleigh problem with the addition of: $S' = 0$ at $z' = 0$ and $1$. We obtain after dropping the primes:

$$\begin{cases} \frac{1}{\sigma}\left(\frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\nabla\underline{V}\right) = -\nabla p + (RT - R_s S)\underline{K} + \nabla^2\underline{V} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial}{\partial t}T + \underline{V}\cdot\nabla T - W = \nabla^2 T \\ \frac{\partial}{\partial t}S + \underline{V}\cdot\nabla S - W = \tau\nabla^2 S \end{cases} \tag{2}$$

(2) contains the following parameters:

$$R = \frac{g\alpha\Delta T d^3}{K_T\nu}, \quad R_S = \frac{g\beta\Delta S d^3}{K_T\nu}, \quad \sigma = \frac{\nu}{K_T}, \quad \tau = \frac{K_S}{K_T} \tag{3}$$

Let us now consider the disturbances $(V, T, S)$ as infinitesimal. For simplicity we consider only the case with free boundaries. The horizontal dependence is given by the total wave number $\alpha$.

If we consider the marginal case characterized by $\frac{\partial}{\partial t} = 0$ (no overstability), we obtain:

$$R = R_S \cdot \tau^{-1} + \frac{\pi^4(\alpha^2+1)^3}{\alpha^2} \tag{4}$$

Since $\tau \simeq \frac{1}{100}$ this result indicates that the density gradient introduced by the temperature gradient must be a hundred times the gradient introduced by the salt for the system to be unstable.

This result seems wrong so let us allow $\frac{\partial}{\partial t} \sim i\lambda$ ( $\lambda$ real number) for marginal conditions. Then we obtain

$$\mathcal{R} = \frac{\sigma}{\sigma+1} \mathcal{R}_S + \frac{\pi^4(\alpha^2+1)^3}{\alpha^2} \tag{5}$$

Here we can see that the case with overstability gives a drastic change in the critical value of $\mathcal{R}$ . Equation (5) states that instability occurs for $\mathcal{R}$ slightly less than $\mathcal{R}_S$ .

To investigate the situation still closer we now look at the effects of finite amplitude by the same technique that has been described earlier in connection with the Rayleigh problem.

Thus we want to determine $\mathcal{R}_2$ in the expansion for $\mathcal{R} = \mathcal{R}_0 + \varepsilon^2 \mathcal{R}_2 + \ldots$ where $\mathcal{R}_0$ is determined by equation (5) ( $\mathcal{R}_1$ is equal to zero as usual). However we find $\mathcal{R}_2 < 0$ in this case in contrast to the Rayleigh case. This means that the heat flux must behave as indicated in Fig. 1.

This means that the fluid for $\mathcal{R} < \mathcal{R}_0$ ( $\mathcal{R}'$ for example) can have three different states, one resting state, I, and two convecting states II, III. The interpretation of this result is that the system in state I is unstable to finite amplitude disturbances. That is if we kick the system in state I hard enough it will spontaneously jump to state II or III. If we want

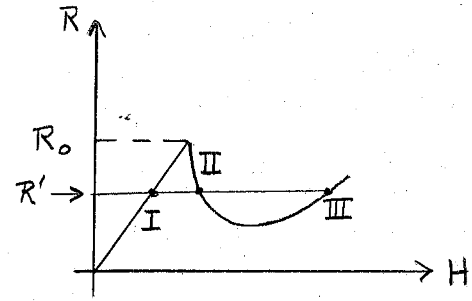



Fig. 1

to obtain features of the system beyond the critical point the computations become very tedious and we shall therefore try another approach.

Let us develop the involved fields in Fourier series and only keep the terms in the representation that is necessary to represent any physics at all. For simplicity we take the flow as two-dimensional (rolls) and define a stream function $\psi$ from:

$$u = \frac{\partial \psi}{\partial z} \quad W = -\frac{\partial \psi}{\partial x}$$

The minimum Fourier representation is found to be (still we use free boundaries):

$$\left.\begin{aligned} \psi &= a_1 \sin \pi \alpha x \sin \pi z \\ T &= a_2 \sin 2\pi z + a_3 \cos \pi \alpha x \sin \pi z \\ S &= a_4 \sin 2\pi z + a_5 \cos \pi \alpha x \sin \pi z \end{aligned}\right\} \qquad (6)$$

From the non-linear equations (2) we then obtain:

$$\left.\begin{aligned}
\dot{a}_1 &= -\sigma\pi^2(\alpha^2+1)a_1 - \frac{\sigma\alpha}{\pi(\alpha^2+1)}\left(Ra_3 - R_s a_5\right) \\
\dot{a}_2 &= -4\pi a_2 + \frac{\pi^2\alpha}{2} a_1 a_3 \\
\dot{a}_3 &= -\pi^2(\alpha^2+1)a_3 - \pi\alpha a_1 - \pi^2\alpha a_1 a_2 \\
\dot{a}_4 &= -4\tau\pi^2 a_4 + \frac{\pi^2\alpha}{2} a_1 a_5 \\
\dot{a}_5 &= -\tau\pi^2(\alpha^2+1)a_5 - \pi\alpha a_1 - \pi^2\alpha a_1 a_4
\end{aligned}\right\} \qquad (7)$$

These equations have been time-integrated for different values of $R$ and different initial conditions.

The time integration shows many interesting features and in particular verifies the interpretation of the finite amplitude calculations.

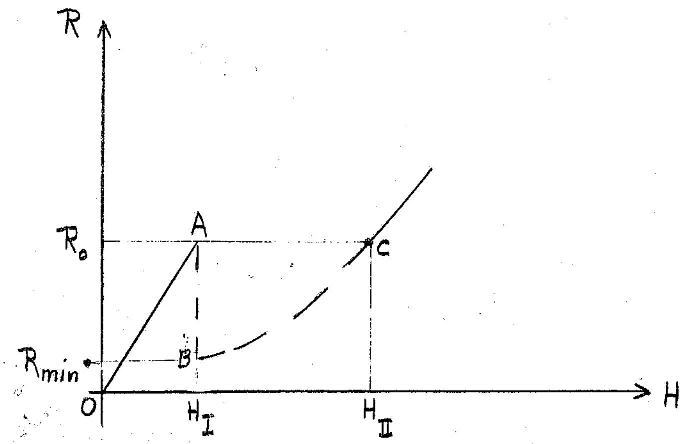



Fig. 2

If one starts the time integration with weak fields and $R < R_o$ the fields decay and one arrives at a point on the line between $O$ and $A$ in Fig. 2. If $\mathcal{R}$ is increased somewhat over $\mathcal{R}_0$ the system will not decay but will settle to the point $C$ in Fig. 2. Then if $\mathcal{R}$ is chosen with a value slightly less than $\mathcal{R}_0$ but with the steady state obtained previously as initial conditions the system will not decay but settle to a point on the curve $B-C$. In this way one can reproduce the curve $B-C$ in Fig. 2.

When we were looking at the stability for infinitesimal disturbances we got the result that oscillatory motions were to be expected for $\mathcal{R} > \mathcal{R}_0$. However the solutions we have obtained in terms of Fourier coefficients are steady and of essentially the same type as in the Rayleigh case.

Knowing that we can expect a steady state let us drop the time derivatives in equations 7. The system can be solved for $a_1$ and we obtain:

$$A_1^4 + 8\left\{\pi^4(\alpha^2+1)(1+\tau^2) + \frac{\alpha^2}{(\alpha^2+1)^2}\left(\tau R_s - \mathcal{R}\right)\right\}A_1^2 +$$

$$+ 64\left[\tau^2\pi^8(\alpha^2+1)^2 + \frac{\pi^4\alpha^2}{\alpha^2+1}\left(\tau R_s - \tau^2 R\right)\right] = 0 \tag{8}$$

where $A_1 = \pi^2\alpha\, a_1$.

This equation can be solved for $A_1^2$ and we require $A_1^2 > 0$. This condition can be expressed in terms of the coefficients of equation (8) and we end up with

$$\mathcal{R} \geq \mathcal{R}_{min} = \tau \mathcal{R}_s + \frac{\pi^4(1+\alpha^2)}{\alpha^2} + 2\left(\tau \mathcal{R}_s \frac{\pi^4(\alpha^2+1)^3}{\alpha^2}\right)^{\frac{1}{2}}$$

This condition we expect to approximately give the point $\mathcal{B}$ in Fig. 2. We can see that $\mathcal{R}_{min}$ is really much smaller than $\mathcal{R}_0$ (approximately $\mathcal{R}_{min} \simeq \frac{1}{100}\mathcal{R}_0$).

The drastic change in the critical $\mathcal{R}$ we obtain when we allow the initial disturbance to be finite can be explained as follows:

Suppose you disturb the system so strongly that the layer becomes isothermal and isohaline.

Then the destabilizing temperature gradient will establish itself a hundred times faster than the salt gradient and convection will start. When the convection has started it will keep the layer isohaline and the stabilizing salt gradient will never develop.

Notes submitted by

S. Gösta Walin

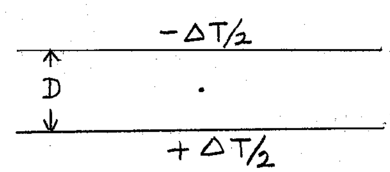
## Turbulence

Robert Kraichnan

As we have studied the problem up to this point we have a layer heated from below. We will use the Boussinesq approximation.

Velocity field $V$

Temperature field $T$




$\alpha$ = coefficient of thermal expansion

Boussinesq Equations:

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) \underset{\sim}{u} = -u \cdot \nabla u - \nabla p + \underset{\sim}{\gamma} T$$

$$\underset{\sim}{\gamma} = \alpha g$$

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = -u \cdot \nabla T$$

When the fluid is heated, the first thing that happens is that there is a steady cellular convection. When $\Delta T$ increases, it is known empirically that the flow pattern becomes irregular and we have turbulence. If the Boussinesq equations are assumed to remain valid only two parameters can affect a steady or statistically steady flow:

(1) Prandtl number $\sigma = \nu/\kappa$

(2) Rayleigh number $R_a = \dfrac{\Delta T \gamma D^3}{\kappa \nu}$.

The static state (<u>no</u> flow) becomes unstable to cellular convection at small $\Delta T$. At larger $\Delta T$ cellular convection becomes unstable to a large variety of different kinds of perturbations. The reason it is not possible to predict the flow exactly at a later time, even if the Boussinesq equations remain strictly valid, is that there are always small perturbations to the boundary conditions which give rise to large fluctuations in the flow.

Def: The turbulent state is one in which the exact flow state is not predictable with certainty.

The question arises to the maximum amount of information that we can obtain about turbulent flow. We do this by constructing a statistical ensemble: we set up the flow many times and measure everything each time and get certain average properties of the flow.

Let $\underset{\sim}{u} = \bar{\underset{\sim}{u}} + \underset{\sim}{V}$

where $\underset{\sim}{u}$ = velocity field, $\bar{\underset{\sim}{u}}$ = mean velocity field and $\underset{\sim}{V}$ = fluctuation velocity. Then $\langle \underset{\sim}{u} \rangle = \bar{\underset{\sim}{u}}$ where $\langle \ \rangle$ = average.

Similarly

$$T = \bar{T} + \psi$$

$$\rho = \bar{\rho} + \rho'$$

We substitute these quantities into the Boussinesq set of equations. First we average the 1st equation over the ensemble:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\bar{\underset{\sim}{u}} = -\langle u\cdot\nabla u\rangle - \nabla\bar{p} + \gamma\bar{T}$$

Subtracting from the unaveraged equation we get:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\underset{\sim}{v} = +\langle u\cdot\nabla u\rangle - u\cdot\nabla u - \nabla p' + \gamma\psi$$

Now $\langle u\cdot\nabla u\rangle = \bar{u}\cdot\nabla\bar{u} + \langle v\cdot\nabla v\rangle - \underset{\substack{\| \\ 0}}{\langle\bar{u}\cdot\nabla v\rangle} + \underset{\substack{\| \\ 0}}{\langle v\cdot\nabla\bar{u}\rangle}$

since $\langle v\rangle = 0$

Then

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\underset{\sim}{v} = -\left[v\cdot\nabla v - \langle v\cdot\nabla v\rangle\right] - \bar{u}\cdot\nabla v - v\cdot\nabla\bar{u} - \nabla p' + \gamma\psi$$

Similarly, for the temperature equation

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)\bar{T} = -\bar{u}\cdot\nabla T - \langle v\cdot\nabla\psi\rangle$$

and for the fluctuating temperature

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)\psi = -\bar{u}\cdot\nabla\psi - v\cdot\nabla\bar{T} - \left[v\cdot\nabla\psi - \langle v\cdot\nabla\psi\rangle\right]$$

To simplify the further discussion we assume that the mean velocity is identically zero. This is consistent with the equations of the problem, since $\bar{p}$ enters only into the equation and can be adjusted to make $\bar{u} \equiv 0$ throughout.

Then

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\underset{\sim}{v} = -\left[v\cdot\nabla v - \langle v\cdot\nabla v\rangle\right] - \nabla p' + \gamma\psi$$

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)\bar{T} = -\langle v\cdot\nabla\psi\rangle$$

$$\left(\frac{\partial}{\partial t} - K\nabla^2\right)\psi = -v\cdot\nabla\bar{T} - \left[v\cdot\nabla\psi - \langle v\cdot\nabla\psi\rangle\right]$$

These equations present the problem that the equation for the mean temperature field includes an interaction term with the fluctuations, and the equations for the fluctuation temperature field include an interaction with the mean temperature.

Remarks: Suppose we have a box containing a flow. As we increase the Rayleigh number the flow increases and we get greater gradients of velocity and temperature at the boundaries. Eventually the situation becomes unstable. The surfaces of constant shear begin to pucker up, convolute and the disturbances begin to grow very large. The disturbed portion of fluid is not standing still and it is being carried about by the large-scale motion. This physical characteristic is particularly difficult to characterize with only the first few Eulerian moments. At this point in the analysis it would be attractive to have simple Lagrangian moments.

Given the statistical ensembles of the flow, how do we describe (or characterize) them?

First we have the mean fields $\overline{T}(x,t)$. These are easy to describe. Then we have the fluctuating fields: (1) We can form the moments

$$\langle \psi \rangle = 0$$

$$\langle \psi(x,t)\,\psi(x',t') \rangle$$

$$\langle \psi(x,t)\,\psi(x',t')\,\psi(x'',t'') \rangle \text{ etc.}$$

Or: (2) We can form the probability distributions defined by

$$P(\psi, x, t)\, d\psi = \text{probability that } \psi - \frac{d\psi}{2} < \psi(x,t) < \psi + \frac{d\psi}{2}$$

$$P(\psi, x, t; \psi_1, x_1, t_1)\, d\psi\, d\psi_1 = \text{probability that } \psi - \frac{d\psi}{2} < \psi(x,t) < \psi + \frac{d\psi}{2}$$

$$\text{and } \psi_1 - \frac{d\psi_1}{2} < \psi(x_1, t_1) < \psi_1 + \frac{d\psi_1}{2}$$

etc.

The trouble with these characterizations is that with only a few of these functions, there are many flows that satisfy the conditions. The problem is that the first few functions do <u>not</u> characterize (or determine) the flow. In particular the second order moments by themselves do not characterize the flow.

For example: We may define a function $f(t)$ which has the properties of a "square root" of the $\delta$-function:

$$f(t) = 0 \quad \forall\, t \neq 0$$

$$\int f(t)^2 dt = 1 \text{ or } \int f(t_1 - t) f(t_2 - t)\, dt = \delta_1(t_1 - t_2)$$

$$\text{where } \delta_1(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Then consider the random function

$$g(t) = f(t - t_1) + f(t - t_2) \text{ where } t_1, t_2$$

are randomly distributed on $[0, 1]$

Then $\langle g(t) g(t') \rangle = \int_0^1 \int_0^1 g(t) g(t')\, dt_1\, dt_2 = 2\,\delta_1(t - t')$

But the different random function $g_0(t) = \frac{1}{2} f(t-t_1) + \frac{\sqrt{7}}{2} f(t-t_2)$

also has the same second order moment $2\,\delta_1(t-t')$.

We deal only with the moments. We wish to construct the equation of motion of the moments.

Multiply the fluctuating temperature equation by $\psi(x',t')$ and take the average:

$$\psi(x',t')\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\psi(x,t) = \psi(x',t')\left(-v(x,t)\cdot\nabla\overline{T}(x,t) - [v\cdot\nabla\psi - \langle v\cdot\nabla\psi\rangle]\right)$$

Then

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\langle\psi(x,t)\,\psi(x',t')\rangle = \langle\psi(x't')\,v(x,t)\rangle\cdot\nabla\overline{T}(x,t) - \langle\psi(x't')\,v(x,t)\cdot\nabla\psi(x,t)\rangle$$

The problem we now see is that the statistical moments of all orders are coupled together. To solve the equations we must find a way of short-circuiting this process of involving the higher order moments.

Argument (Batchellor): of increasing gradients: We consider a small element of fluid at a given time and ask how the fluid element distorts. According to the Boussinesq equations we can assume incompressibility and the volume remains constant. Because of the incoherence of the flow the cube will be drawn out into some shape. If the cube originally had two faces of constant temperature then the tendency would be for the two faces to draw close together and produce sharper gradients of the temperature. Because of the turbulent character the regions of increasing gradients are more probable than regions of decreasing gradients. Similarly for the velocity (but more complicated because of pressure effects).

Statistical mechanical arguments for increasing gradient: Ref. Burgers; T.D.Lee, QAM 1950; Kraichnan, Phys.Rev. 1958.

Make a Fourier analysis of fields in a box:

$$T(x,t) = \sum_{\underset{\sim}{k}} T(\underset{\sim}{k},t)\, e^{i\underset{\sim}{k}\cdot x}$$

If we go to the equation of motion and assume $\chi = 0$ and construct a phase space, if we assume mixing, we find the amplitudes $T(\underset{\sim}{k},t)$ satisfy a Liouville theorem so the motion of the points $T(\underset{\sim}{k},t)$ is incompressible. This suggests by analogy to statistical mechanics that the density in phase space is constant along surfaces of the constants of motion.

One such constant of motion is

$$\int [T(x,t)]^2 d^3x = \sum_{\underset{\sim}{k}} [T(\underset{\sim}{k},t)]^2$$

which implies that in equilibrium all the Fourier terms would be equally excited.

Remark: If we were only considering the Navier-Stokes equation in a box the only constant of motion would be the kinetic energy. But with the thermodynamic equation included the mean square fluctuation temperature is also a constant of the motion. Similarly we can argue for equipartition for the Fourier modes of of the velocity field. But there are an infinite number of such terms. The reason the field can't excite all these terms is that we have neglected the molecular terms $-\chi\nabla^2$.

Then $\left(\frac{\partial}{\partial t} + \chi K^2\right)\widetilde{\psi}(k,t)$ takes the place of $\frac{\partial}{\partial t}\widetilde{\psi}(k,t)$ in the non-dissipative equation.

The molecular term is a damping term which becomes stronger the higher $k$. The non-linear terms tend to produce equipartition and put energy into higher modes but the molecular term tends to draw energy away from the higher Fourier modes. This is consistent with the picture of higher gradients. For, higher gradients come from $k \longrightarrow \infty$. Eventually the molecular processes win out and prevent the formation of infinite gradients.

From this one can get an idea of the way the turbulent process depends on the molecular coefficients. If we decrease $\chi$ by a factor of 100, gradients will increase but the effect back on the large scale motion can be very small, provided $R_a$ is large enough and $\chi$ is small enough.

Reynold's number: $$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u = -u\cdot\nabla u + \dots$$

$$\left(\frac{\partial}{\partial t} - \chi\nabla^2\right)T = -u\cdot\nabla T$$

If we try to estimate the importance of the various terms:

$$\chi = V_{\text{thermal}}\,\lambda_{mol};$$

where $\lambda_{mol}$ = molecular mean free path.

The $u\cdot\nabla T$ term has an effective transport coefficient:

$$\chi_{eff} = u\lambda_{eff} = uD$$

The ratio of these two terms $\frac{uD}{\chi}$ = Peclét number is a measure of the effectiveness of convective transfer of heat to the conductive transfer of heat.

Similarly $\frac{uD}{\nu}$ = Reynolds number = ratio of effectiveness of convective transfer of momentum/molecular transfer of momentum.

Prandtl number = Peclét number/Reynolds number.

Near the boundary layer there are smaller scale (order $\ell$) local motions and we can define a local Reynolds number $u\ell/\nu$.

The Reynolds and Peclét numbers give estimates of the relative importance of the convective and conductive terms. Rayleigh number gives the total tendency of the fluid to be driven over the tendency to be dissipated.

Notes submitted by

Steven Orszag

## Turbulence: Modeling in Statistical Hydrodynamics

Willem V.R. Malkus

We have made a detailed study of a particular irreversible process - the finite amplitude processes. We found in this study that advection of momentum seems to increase the role played by the higher wave numbers. Also we have seen how the temperature field may be decomposed into a mean field and a fluctuation field. We will want to stress the role played by the mean field. But first, let us get some feeling for the type of ideas that will be necessary for the further study of the turbulent process.

In the study of equilibrium statistical mechanics, progress was made by abandoning the hope for a complete solution to the N-body problem and concentrating on general laws which were relatively independent of the specific circumstances. Such was the nature of the statistical mechanics as formulated by Maxwell, Boltzmann, and Gibbs.

The possibility for such a solution to the problems of turbulence can be argued against by noting that turbulence must deal with rates of certain processes and general statements about rates are difficult to come by. But non-equilibrium statistical mechanics, in general, has such problems with rates. In particular, kinetic theory has shown some success in explaining non-equilibrium phenomena such as heat conduction, etc. Further, there is Onsager's result on linear systems (force is proportional to flux) which states that entropy production should be minimized in the steady state. Recent results on non-linear systems seem to show that entropy production should be nearly independent of the system. Even more elementary than this is the reason for the introduction of the entropy in the first place - as a measure of the stability of a system. So there has been some success at formulating principles relating to non-equilibrium phenomena.

Perhaps, then, there is a criterion for statistical stability in turbulence theory. It has been suggested by Malkus that maximum heat transfer (or entropy production) is the criterion by which to distinguish the statistically steady state. To properly formulate this principle, we must define the set of possible solutions amongst which we are to choose the extremum. The point is that we wish to admit only physically reasonable solutions to the problem. One way of delimiting our set of admissible solutions would be as follows:

We expect that the smallest scale of motion is marginally stable on the mean field. Large scales can grow and cascade down. But to delimit the physical problem we assume first that motions that carry heat have a wave number upper bound $n_0$ . A second assertion to delimit the set is $\beta \equiv -\frac{d\bar{\bar{T}}}{dz} \geqslant 0$ . This assumption is reasonable since both conduction and convection tend to remove negative gradients (convective elements will deposit heat at a point with $(\beta < 0)$.

Mathematically, we may formulate the second condition as

$$\frac{\varkappa\beta}{H} = I^2 > 0.$$

Further, we may expand $I$ in a series of orthogonal functions $I = \sum_{n=0}^{\infty} I_n \phi_n$ . We will discuss the case of free boundary conditions. In this case we may take

$$\begin{cases} \phi_n = \sqrt{2} \cos 2n\varphi & n > 0 \\ \phi_0 = 1 \end{cases}$$

The first condition above delimiting our set of possible solutions may be expressed as the requirement that $I_n = 0$ for $n > n_0/2$ . Then $I = \sum_{n=0}^{n_0/2} I_n \phi_n$ Our boundary conditions become $I^2(0) = I^2(\pi) = 1$ or in terms of our series expansion

$\left( \sum_{n=0}^{n_0/2} I_n \delta \right)^2$ where $\delta = \sqrt{2} - (\sqrt{2} - 1)\delta_{n,0}$ , and

$\delta_{i,j}$ is the ordinary delta function.

With this condition as a restraint we want to extremize

$$\frac{\varkappa\beta_m}{H} = \frac{1}{\pi}\int_0^{\pi} \frac{\varkappa\beta}{H}\, d\varphi = \sum_{n=0}^{n_0/2} I_n^2$$

Using a Lagrange multiplier we find easily that the solution to the extremum problem is

$$\begin{cases} I_n = \dfrac{\sqrt{2}}{n_0+1} & n > 0 \\ I_0 = \dfrac{1}{n_0+1} \end{cases}$$

Then using these values for $I_n$ we obtain

$$H = \varkappa\beta_m (n_0+1)$$

and

$$\beta/\beta_m = \frac{1}{n_0+1}\left(\sum_{n=0}^{n_0/2}\delta\phi_n\right)^2 = \frac{1}{n_0+1}\quad\frac{\sin^2(n_0+1)\varphi}{\sin^2\varphi}$$

Integrating this relation, we obtain for $\varphi >> \frac{\pi}{n_0}$

$$\overline{T}(\varphi) - T_m = \frac{\Delta T}{2\pi}\,\frac{1}{n_0+1}\left\{\cot\varphi + o\left(\frac{1}{n_0}\right)\right\}$$

and for $\varphi \sim \frac{\pi}{n_0}$ with $\xi \equiv (n_0+1)\varphi$

$$\overline{T}(\varphi) - T_m = \frac{\Delta T}{2}\left[1 - \frac{2}{\pi}\left\{Si\,(2\xi) - \frac{\sin^2\xi}{\xi}\right\}\right]$$

where $Si\,\xi = \int_0^{\xi}\frac{\sin x}{x}\,dx$

If we set $\overline{WT} = \sum_{n=0}^{n_0} y_n \sin^2 n\,\varphi$ then $y_n = \varphi\chi\beta_m\left(1 - \frac{n}{n_0+1}\right)$.

This has in particular the attractive property that $y_n = 0$ for $n = n_0 + 1$ which indicates the validity of the approximation that information on the large scale processes is not propagated past $n = n_0$.

We have still the problem of determining the value of $O$ relevant to a particular problem. We may do this using a variational statement of the problem for marginal stability,

$$\mathcal{R} = \frac{\int W\nabla^6 W\,d\varphi}{\int\beta W^2\,d\varphi}$$

The requirement is that the field is marginally stable to $n_0 + 1$. All smaller $n$'s (larger scales) would be unstable. Using $W = \sin\xi,\ \xi = (n_0+1)\varphi$ as a trial function, we obtain

$$\mathcal{R} = (n_0+1)^4\mathcal{R}_c \quad \text{where} \quad \mathcal{R}_c = \frac{27}{4}\pi^4 = 656\,.$$

If we, however, look for scales of instability along the boundary, then we obtain, similarly,

$$R = \frac{(n_0+1)^3}{\alpha^2}\pi^4 \frac{\int_0^{\pi/2} W\nabla^6 W d\xi}{\int_0^{\pi/2} \beta W^2 d\xi} \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial \xi^2} - \alpha^2 \text{ and } \beta = \frac{\sin^2 \xi}{\xi}$$

With $W = \sin\xi$ we obtain

$$R = (n_0+1)^3 R_{\text{boundary}} \quad \text{where } R_{\text{boundary}} \cong 1017$$

We may define a Nusselt number, $Nu$, as

$$Nu \equiv \frac{H}{\gamma \beta_m}$$

Then in the first case we obtain $Nu = \left(\frac{R}{RC}\right)^{\frac{1}{4}}$ while in the second case $Nu = \left(\frac{R}{R_{\text{boundary}}}\right)^{\frac{1}{3}}$. In the case of large Rayleigh numbers the cutoff in the spectrum will be due to the first type of process while at smaller Rayleigh numbers the boundary type process may provide the cutoff for the available spectrum.

In the case of a system with rigid boundaries the orthogonal functions to be used were constructed by Chandrasekhar and Reid. These functions modify the free boundary case functions only near the boundary. The procedure then is to extend the boundary by a slight amount and fit into the laminar solution.

This says

$$\frac{R_{\text{boundary, free}}}{R_{\text{boundary, rigid}}} \cong \frac{656}{1708}$$

Then $R_{\text{boundary, rigid}} = 2600$.

Experimentally $R_{\text{boundary, rigid}} = 2000 \pm 100$. Furthermore, it is easily seen that for non-boundary type instability with rigid boundaries the value of $R_C$ should be of the order of $R_C$ for the free case, $R_C = 656$. Experimentally, $R_C = 700 \pm 100$.

Notes submitted by

Steven A. Orszag.

The Role of the Mean Field in Turbulence Theory

Jackson Herring

We will discuss the usual problem of a layer heated from below. In the Boussinesq approximation the equations describing the flow are:

$$(1)\quad \nabla \cdot \vec{V} = 0$$

$$(2)\quad \left(\frac{\partial}{\partial t} - \nu \nabla^2\right)\vec{V} = -V \cdot \nabla \vec{V} - \nabla P + g \hat{k} \theta$$

$$(3)\quad \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)\theta = \beta W - \nabla \cdot \left(\vec{V}\theta - \hat{k}\, \overline{W\theta}\right)$$

$$(4)\quad \left(\frac{\partial}{\partial t} - \kappa \nabla^2\right)\overline{T} = -\frac{\partial}{\partial z}\, \overline{W\theta}$$

where $\theta = T(\vec{r}, t) - \overline{T}(z,t)$, $\overline{T}$ = horizontal average of $T$,

$$\beta = -\frac{d\overline{T}}{dz}$$

In the steady state $H = \kappa\beta + \overline{W\theta}$ = constant. We want to discuss the effect of leaving out terms which have zero average. Throwing out such terms may be pictured as a statistical method for throwing out third order cumulants. That is, we have cut off our system of an infinite set of coupled equations at order 2. In other words, we have effectively set

$$\langle v_1 T_2 T_3 \rangle = \overline{T}_2 \langle v_1 T_3 \rangle .$$

The resulting equations seem to be good because they are amplitude equations, rather than moment equations. The thrown-out quantities are conservative quantities and average out to zero. Further, the system has the desirable physical characteristics that it leads to a positive kinetic energy wave number spectrum (which follows directly from the fact that the system is a real, amplitude system) and satisfies the same conservation laws associated with the complete set of equations.

If we non-dimensionalize the equations, setting

$$\vec{v} = \frac{d}{\kappa}\vec{v}' \qquad (d = \text{thickness of the layer})$$

$$T = T'/\Delta T$$

$$t = \frac{\kappa}{d^2}t'$$

we obtain

$$\left(\frac{1}{\sigma}\frac{\partial}{\partial t} - \nabla^2\right)\vec{V} = -\frac{1}{\sigma}\vec{V}\cdot\nabla\vec{V} - \nabla P + R\hat{k}\theta$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = \beta W - \nabla\cdot\left(\vec{V}\theta - \hat{k}\,\overline{W\theta}\right)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\overline{T} = -\frac{\partial}{\partial z}\overline{W\theta}$$

Leaving out third order terms and going to the limit $\sigma \to \infty$ we have

$$-\nabla^2\vec{V} = -\nabla P + R\hat{k}\theta$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\theta = \beta W$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\overline{T} = -\frac{\partial}{\partial z}\overline{W\theta}$$

We set

$$w(\vec{r},t) = \sqrt{2} \sum_{n,\alpha} w_n^\alpha e^{i\vec{\alpha}\pi.\vec{x}} \sin n\pi z$$

$$\theta = \sqrt{2} \sum_{n,\alpha} \theta_n^\alpha e^{i\vec{\alpha}\pi\cdot\vec{x}} \sin n\pi z$$

$$\beta = 1 + \sum_n \beta_n \cos 2\pi n z$$

Then

$$\omega_n^\alpha = J_n^\alpha \theta_n^\alpha \quad \text{where } J_n^\alpha = \frac{R}{\pi^4} \frac{\alpha^2}{(n^2+\alpha^2)^2}$$

$$\left(\frac{d}{dt} + (n^2+\alpha^2)\right)\theta_n^\alpha = J_n^\alpha \theta_n^\alpha + \frac{1}{2}\sum_p \left(\beta_{|n-p|} - \beta_{n+p}\right) J_p^\alpha \theta_p^\alpha$$

$$\left(\frac{d}{dt} + n^2\right)\beta_n = \sum_{\alpha,p} J_p^\alpha \theta_p^\alpha \left(\theta_{|n-p|}^\alpha \sigma(n-p) - \theta_{n+p}^\alpha\right)$$

$$\text{where} \quad \sigma(\psi_p) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 0 \end{cases}$$

Further the Nusselt number

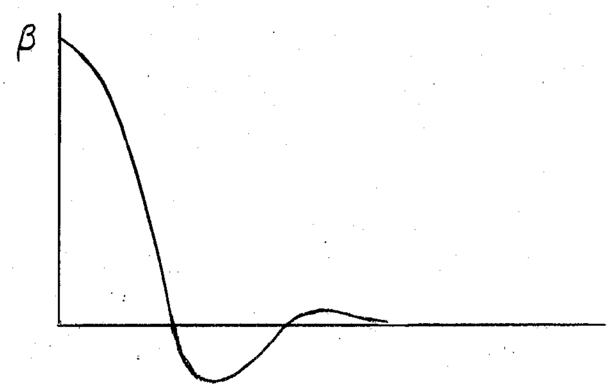
$$N \equiv \frac{H}{\kappa\langle\beta\rangle} = 1 + \sum \beta_n = \sum J_n \theta_n^2 \ .$$

These equations were solved numerically and it was found that

$$N = 0.31 R^{1/3} \qquad R \gtrsim 4000 \qquad \text{for free boundaries}$$

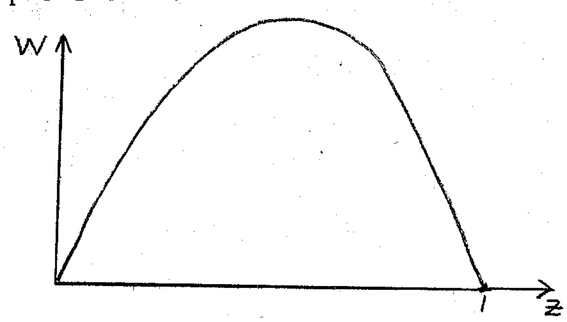
and $N = 0.116 R^{\frac{1}{3}}$ for rigid boundaries. This last result is about 20% off from experiment.

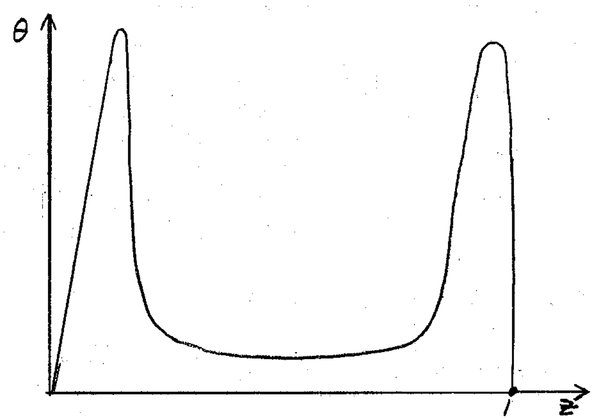
The only physically unreasonable characteristic of the numerical solution is the fact that $\beta$ becomes negative.




Actually it is a priori evident that $\beta$ must become negative at some $z$ because by leaving out the terms we have left out, we have also omitted all other mechanisms for stopping the heat flux $H$ from becoming excessive.

Further results of the numerical calculation are shown below in the plots of $W$ vs $z$ and $\theta$ vs $z$.



It was found that only the first mode $w_1$ is important. Thus we can set $J_n = 0$ for $n \neq 1$ and solve the resulting set analytically.

Then $N = J_1 \theta_1^2$ and in the steady state

$$\beta_n = J_1 \theta_1 (\theta_{n-1} - \theta_{n+1})$$

$$(n^2 + \alpha^2)\theta_n = J_1 \theta_1 \delta_{n,1} + \tfrac{1}{2}(\beta_{n-1} - \beta_{n+1}) J_1 \theta_1$$

Further since the solutions are symmetric about $z = \frac{1}{2}$ we retain only the odd terms in $\theta$, $W$ and only the even terms in $\beta$. Thus

$$\beta_{2n} = J_1 \theta_1 (\theta_{2n-1} - \theta_{2n+1})$$

$$((2_{n-1})^2 + \alpha^2)\theta_{2n-1} = J_1 \theta_1 \delta_{n,1} + \tfrac{1}{2}(\beta_{n-1} - \beta_n) J_1 \theta_1$$

$$1 + \sum_n \beta_{2n} = J_1 \theta_1^2$$

Setting $n = 1$ we obtain marginal stability for the first mode (since $\theta_1$ is a factor of the equation).

Further $\beta_1 = 2\left(1 - \frac{1+\alpha^2}{J_1}\right)$

so $$\frac{\beta_1 - 2}{2} = \frac{-1-\alpha^2}{2J_1^2} \sim 20,\ 30\ \%.$$

For higher $n$

$$\Gamma_{n+1}\beta_{n+1} + \Gamma_{n-1}\beta_{n-1} - \left(\Gamma_{n+1} + \Gamma_{n-1} + \frac{2}{J_1^2 \theta_1^2}\right)\beta_n = 0$$

where $\Gamma_n = \dfrac{1}{(2n-1)^2 + \alpha^2}$

We assume that $\beta_n$ may be pictured as a function of a continuous variable $n$ , and we rewrite the difference equation as a differential equation which we solve.

Approximately

$$\frac{d}{dn}\left(\Gamma_n \frac{d}{dn}\beta_n\right) - \frac{2}{J_1^2\theta_1^2}\beta_n = 0$$

or

$$\frac{d}{dn}\left(\Gamma_n \frac{d}{dn}\beta_n\right) - \frac{2}{J_1 N}\beta_n = 0$$

When $n$ is large $\Gamma_n \sim \frac{1}{4n^2}$ so if we let

$$\beta_n = \int_n^\infty f(n')\,dn' \frac{1}{\Gamma_{n'}}$$

$$\frac{d^2 f}{dn^2} - \frac{2}{J_1 N}\left(4n^2 + \alpha^2\right) f = 0 .$$

First by a dimensional analysis, $n^4 \sim J_1 N$ and since $N^4 \sim n^4$ we have $N \sim J_1^{1/3}$.

More precisely, it is known that the solution to the differential equation for $f$ is a parabolic cylinder function, $f = D_{-\frac{1}{2}}(\xi)$ where $\xi = q\left(\frac{1}{NJ_1}\right)^{\frac{1}{4}} n$ and $q$ is some numerical factor. Then

$$N \cong \frac{1}{2}\left(\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\right)^{\frac{4}{3}} J_1^{1/3}$$

$$\cong 0.27\, R^{\frac{1}{3}},\ \alpha = 1$$

compared with 0.31 $R^{\frac{1}{3}}$ gotten from the machine calculation.

Notes submitted by

Steven A. Orszag.

# Turbulence: Foundations for a Deductive Theory

Robert Kraichnan

A few approaches to the problem of turbulence will be described. The main ingredients and some of the results will be emphasized while the mathematical steps leading from the hypotheses to the results will be left out.

## Mixing Length Approach

Consider a horizontally infinite layer of fluid of thickness $D$ confined between rigid, perfectly conducting top and bottom surfaces at which all components of velocity vanish. A constant temperature difference $\Delta T$ is maintained. We assume the fluid follows the Boussinesq equations of motion.

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2\right) T = - u \cdot \nabla T \tag{1}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) u = -(u \cdot \nabla) u - \rho_m^{-1} \nabla p + \hat{n} \gamma T \tag{2}$$

$$\nabla \cdot u = 0 \tag{3}$$

where

$\kappa \equiv$ thermometric conductivity

$T \equiv$ temperature

$u \equiv$ velocity vector

$\nu \equiv$ kinematic viscosity

$\rho_m \equiv$ mean density

$p \equiv$ pressure

$\hat{n} \equiv$ unit vector pointing vertically upward

$\gamma \equiv$ product of gravitational acceleration and volume coefficient of thermal expansion.

A dimensional analysis reveals that the structure of the thermal convection depends on two dimensionless parameters, the

Prandtl number $\sigma \equiv \nu/\kappa$

and the

Rayleigh number $R_a \equiv \gamma \Delta T D^3/\kappa\nu$

If we write

$$T \equiv \overline{T} + \psi$$
$$\psi \equiv T - \overline{T}$$

where

$$\overline{T} \equiv \text{ensemble mean of } T$$
$$\psi \equiv T - \overline{T}$$

then (1), (2), (3) yield

$$\kappa\left(\frac{d^2\overline{T}}{dz^2}\right) = \frac{d\langle\omega\psi\rangle}{dz} \tag{4}$$

$$\left(\frac{\partial}{\partial t} - \kappa\nabla^2\right)\psi = -\omega\left(\frac{d\overline{T}}{dz}\right) - \left[u\cdot\nabla\psi - \langle u\cdot\nabla\psi\rangle\right] \tag{5}$$

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)u = -(u\cdot\nabla)u - \rho_m^{-1}\nabla p + \hat{n}\,\gamma\,\psi \tag{6}$$

where $\langle\ \rangle$ denotes ensemble mean (which is assumed equivalent to either a horizontal mean or a time mean) and $\omega$ is the vertical velocity component.

Next characteristic roles are attributed to various terms in

(5) and (6). $-w \frac{d\bar{T}}{dz}$ in (5) represents creation or destruction of temperature fluctuations by action of the velocity field on the mean temperature gradient. The terms

$$\chi \nabla^2 \psi \text{ and } -[u \cdot \nabla \psi - \langle u \cdot \nabla \psi \rangle]$$

represent smoothing of temperature fluctuations by molecular and eddy conduction, respectively. $\hat{n} \gamma \psi$ represents buoyancy. $-\nu \nabla^2 u$ and $-(u \cdot \nabla) u$ represent molecular and eddy damping of the velocity field respectively.

Now for the main assumptions of mixing length theory.

In analogy to the case of thermal conduction where

$$\chi = \text{(thermal velocity)(mean free path)}$$

we define

$$\nu_{eddy} \equiv \chi_{eddy} \equiv \text{(characteristic velocity of eddy)} \times \text{(characteristic length of eddy)}$$

Now local Reynolds and Peclét numbers can be defined, which measure the relative importance of convection to molecular processes in the transport of heat and momentum.

$$\mathcal{R}e(z) \equiv \frac{\nu_{eddy}(z)}{\nu}$$

$$\mathcal{P}e(z) \equiv \frac{\chi_{eddy}(z)}{\chi}$$

It will be assumed that the characteristic length of an eddy is equal to its distance to the nearest boundary.

Thus we have

$$\mathcal{R}e(z) = \frac{\tilde{w}(z)\,z}{\nu}$$

$$\mathcal{P}e(z) = \frac{\tilde{w}(z)\,z}{\kappa}$$

where $\tilde{w}$ is $rms$ value of $w$ at height $z$.

Next, transition Reynolds and Peclét numbers can be defined by

$$\mathcal{P}e_T = \frac{\tilde{w}(z_\kappa)\,z_\kappa}{\kappa}$$

$$\mathcal{R}e_T = \frac{\tilde{w}(z_\nu)\,z_\nu}{\nu}$$

where $z_\kappa$ is the height where convection accounts for exactly half the heat transport and $z_\nu$ is the height where eddy viscosity effects are equal to molecular viscosity effects. The numerical values of $\mathcal{P}e_T$ and $\mathcal{R}e_T$ are guessed at after consulting relevant measurements and theoretical studies.

$$\mathcal{R}e_T \sim 30$$

$$\mathcal{P}e_T \sim 3$$

$\mathcal{R}e_T$ and $\mathcal{P}e_T$ are assumed not to depend on $\sigma$ and $\mathcal{R}a$.

It is also assumed that above $z_\kappa$ convection dominates heat transport and below $z_\kappa$ molecular conductivity dominates. Similarly, above $z_\nu$ eddy viscosity effects are assumed to dominate while below $z_\nu$ molecular viscosity is assumed to be the dominant viscosity effect. Since in general $z_\kappa \neq z_\nu$ we will have three regions of flow. In each region we will be able to simplify equations (5) and (6) since certain terms are assumed to dominate. Finally by imposing a rough order-of-magnitude balance of input and dissipation for velocity and temperature fluctuations and energy it is possible to deduce some approximate equations. Two such results are:

$$\tilde{\psi} \sim (2\pi^2 Pe_T^2)^{-\frac{1}{3}} \sigma^{-\frac{1}{3}} \Delta T \left(\partial/\partial x\right)^{-1/3}$$

$$\tilde{w} \sim (\gamma x \Delta T)^{\frac{1}{3}} \left(\partial/\partial x\right)^{\frac{1}{3}}$$

where $z_r < z < \frac{1}{2}D$ , $\sigma$ is large and $Ra$ is sufficiently high. These results are not good to more than a factor of two.

A Formal Approach

Another more formal method of dealing with non-linear terms now will be discussed. Consider the same situation as before except with the temperature removed. So we assume

$$\nabla \cdot V = 0 \tag{7}$$

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) V = -V \cdot \nabla V - \nabla p \tag{8}$$

If we take the divergence of (8) we find

$$\nabla^2 p = -\nabla \cdot [V \cdot \nabla V]$$

Thus if we assume no motion at great distances we can drop $\nabla p$ from (8). Non-dimensionalizing (8) we get

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) V = -R_0 (V \cdot \nabla V)$$

where $R_0 = \frac{L_0 u_0}{\nu}$ the Reynolds number.

We can treat $V$ as an expansion from zero expressing the right-hand side with Green's functions. The expansion has the following form:

$$V_i(x,t) = V^0 + R_0 \int G^0 V^0 V^0 + R_0^2 \iint G^0 G^0 V^0 V^0 V^0 + \ldots \tag{9}$$

where $\left(\frac{\partial}{\partial t} - \nabla^2\right) V^0 = 0$

It is guessed that this series has zero radius of convergence for all values of $\mathcal{R}_0$. Taking a few terms of (9) gives unreasonable results for the decay of the energy spectrum. So we try something slightly different. We regroup the terms of (9) and an associated series for $U_{ij}(x,t,x',t') = \langle v_i(x,t), v_j(x',t')\rangle$ we get

$$U = U^0 + R_0^2 \int G^0 G\, UU + R_0^4 \int G^0\, GGG\, UUU + \cdots$$

If we take two terms of this expansion we get reasonable results for the decay of the energy spectrum at low Reynold's number. But if we take more than two terms we get very unreasonable results. This is because the system with just two terms represents a possible statistical system and thus can give no unphysical results. The equations have been worked out for only the isotropic case. But because of the correctness of the description of $E(k)$ for two octaves suggests that the procedure will work for large-scale and nearly large-scale motion. One drawback of the procedure is that it gives too high results for large $k$ in energy spectrum in comparison to Komogorof's hypothesis. One suggestion on how to correct this is to consider a Lagragian viewpoint where

$V = V(x, t/s) \equiv$ the velocity at time $s$ of the particle that was at position $x$ at time $t$.

Form

$V(x, t/s)\ V(x', t'/s')$ and go through the above expansion procedure.

Notes submitted by

Benjamin R. Halpern

# Turbulence: Lagrangian Basis for Expansion

Klaus Hasselmann

We will first develop a fairly general probabilistic approach to fluid motion. This will essentially consist of writing down the equations of motion in a generalized form, defining the relevant probability distributions, deriving the differential equations for the time development of these probability distributions, and finally exploring the problem of separating out the probability distribution of just a few of the coordinates.

After this general theory is set down we will describe a model and then in some detail describe the procedure by which the probability distributions and covariences are calculated.

General Theory.

Only a finite number of generalized coordinates are necessary to describe any flow of a finite amount of mass since there are at most 6N degrees of freedom, where N is the number of particles.

Let the generalized coordinates be

$$Q = (q_1, q_2, \cdots, q_N)$$

$N$ very large.

The $q$'s may be Fourier amplitudes or velocities at certain discrete points, etc.

We assume that the equations of motion can be written in the form

$$\dot{q}_\nu = F_\nu(Q) \tag{1}$$

It is of little use to follow the motion for a long time since it is unstable and in an experiment uncontrollable perturbations in the initial conditions will grow in time and will eventually dominate the motion. Therefore we consider statistical quantities.

We would like to determine

$$P_p\left(Q(t_1), Q(t_2) \ldots Q(t_p)\right) \tag{2}$$

which is the probability of finding $Q(t_1)$ at $t_1$ and $Q(t_2)$ at $t_2$ and . . . $Q(t_p)$ at $t_p$ for one realization of the experiment.

The problem is how (1) gives (2).

If we hold $Q(t_2), Q(t_3) \ldots Q(t_n)$ and $t_2 \ldots t_n$ constant and vary $t_1$ we obtain a continuity equation since the number of points is constant.

$$\frac{\partial}{\partial t} P_p\left(Q(t_1) \ldots Q(t_n)\right) + \sum_\nu \frac{\partial P_p\left(Q(t_1) \ldots Q(t_n)\right)}{\partial q_\nu(t_1)} = 0 \tag{3}$$

If we tried to solve (3) completely we would end up trying to solve (1). But all we really want are partial answers.

Take (3) with $p=1$

$$\frac{\partial P(Q(t))}{\partial t} + \sum_\nu \frac{\partial}{\partial q_\nu}\left(\dot{q}_\nu P\right) = 0 \tag{4}$$

If $F_\nu$ is such that

$$\sum_\nu \frac{\partial \dot{q}_\nu}{\partial q_\nu} = 0 \tag{5}$$

then

$$0 = \frac{\partial P}{\partial t} + \sum_{\nu} \frac{\partial}{\partial q_\nu}(\dot{q}_\nu P)$$

$$= \frac{\partial P}{\partial t} + \sum_{\nu}\left(\frac{\partial \dot{q}_\nu}{\partial q_\nu} P + \dot{q}_\nu \frac{\partial}{\partial q_\nu} P\right)$$

$$= \underbrace{\frac{\partial P}{\partial t} + \sum_{\nu} \dot{q}_\nu \frac{\partial}{\partial q_\nu} P}_{= \frac{DP}{Dt}} + \underbrace{P \sum_{\nu} \frac{\partial \dot{q}_\nu}{\partial q_\nu}}_{=0}$$

$$\therefore \qquad \frac{DP}{Dt} = 0 \qquad \text{under condition (5).}$$

This is the often-mentioned Liouville equation for special generalized coordinates.

<u>Reduction in the number of coordinates.</u>

We really want to know

$$P(q_1, q_2 \cdots q_n) \qquad n << N \qquad n \approx 2,3$$

Set $\hat{Q} = (q_1 \cdots q_n)$

$\check{Q} = (q_{n+1} \cdots q_N)$

and define

$$P\hat{Q} \equiv \iiint P(Q)\, d\check{Q}$$

$$\equiv \iiint P(q_1 \cdots q_n\, q_{n+1} \cdots q_N)\, dq_{n+1} \cdots dq_N$$

If we integrate (4) with respect to $d\check{Q}$ we obtain:

$$\frac{\partial P(\hat{Q})}{\partial t} + \int d\check{Q}\left[\sum_{\nu=1}^{n} \frac{\partial}{\partial q_\nu}(\dot{q}_\nu P(Q)) + \sum_{\nu=n+1}^{N} \frac{\partial}{\partial q_\nu}(\dot{q}_\nu P(Q))\right] = 0$$

$$\frac{\partial P(\hat{Q})}{\partial t} + \sum_{\nu=1}^{n} \left( \frac{\partial}{\partial q_\nu} \int \dot{q}_\nu P(Q)\, d\check{Q} \right) + \sum_{\nu=n+1}^{N} \int \frac{\partial}{\partial q_\nu} \left( \dot{q}_\nu P(Q) \right) d\check{Q} = 0$$

We define

$$\bar{\dot{q}}_\nu \equiv \frac{\int \dot{q}_\nu P(Q)\, d\check{Q}}{P(\hat{Q})}$$

If $P$ falls off at large values sufficiently fast so that

$$\dot{q}_\nu P(q_1 \cdots q_\nu \cdots q_N) \longrightarrow 0$$

as $|q_\nu| \longrightarrow \infty$

then

$$\sum_{\nu=n+1}^{N} \int \frac{\partial}{\partial q_\nu} \dot{q}_\nu P(Q)\, d\check{Q} = \sum_{\nu=n+1}^{N} \int_{q_{n+1}=-\infty}^{q_{n+1}=+\infty} \int_{q_{n+2}=-\infty}^{q_{n+2}=+\infty} \cdots \int_{q_\nu=-\infty}^{q_\nu=+\infty} \cdots \int_{q_N=-\infty}^{q_N=+\infty}$$

$$\frac{\partial}{\partial q_\nu} \left( \dot{q}_\nu P(q_1 \cdots q_\nu \cdots q_N) \right) dq_{n+1} \cdots dq_\nu \cdots dq_N$$

and thus if we perform the $q_\nu$ integration first we see that this term vanishes.

With the above definition and assumption about $P$ we have the following equation:

$$\frac{\partial P(\hat{Q})}{\partial t} + \sum_{\nu=1}^{n} \frac{\partial}{\partial q_\nu} \left( \bar{\dot{q}}_\nu P(\hat{Q}) \right) = 0 \tag{6}$$

if

$$\dot{\hat{Q}} = F(\hat{Q}) \tag{7}$$

and if

$$P(Q) = P(\hat{Q})\, P(\check{Q}) \tag{8}$$

then by the definition of $\bar{\dot{q}}_\nu$ we have

$$\overline{\dot{q}_\nu} = \frac{\int F_\nu(\hat{Q})P(\hat{Q})P(\check{Q})\,d\check{Q}}{P(\hat{Q})}$$
$$= F_\nu(\hat{Q})$$

and we see that in this case the equations separate and we may solve for $q_1 \cdots q_n$ without solving for $q_{n+1}, \cdots, q_N$. But in general (7) and (8) do not hold and for turbulence (7) and (8) do not hold.

The Model

The object is to find a model which does not depend critically upon the method of closing the equations.

We will use a Lagragian system because such a system has the "memory" of the particles built in and this will represent the convection terms rigorously. Assume that we have $n$ particles. Let $X^i(t)$ be the position of the $i^{th}$ particle at time $t$.

We want to determine

$$P(X^{(1)} \ldots X^{(n)} \dot{X}^{(1)}, \ldots, \dot{X}^{(n)} / a_1 \ldots a_n)$$

where

$$a = X(t=0)$$

The equations of motion can be written as

$$\ddot{X}_i^{(\nu)} = A_{ij}^{\nu\mu} X_j^{(\mu)} + B_{ijk}^{\nu\mu\lambda} \dot{X}_j^{\mu} \dot{X}_k^{\lambda} + \eta_i^{(\nu)}(t) \qquad (9)$$

(summation convection being used) where $X_i^{(\nu)}$ the $i^{th}$ component of the position of the $\nu^{th}$ particle.

The term $\eta_i^{(\nu)}(t)$ is the remainder or error term and makes (9) hold exactly. We would like to make $\eta_i^{(\nu)}(t)$ as small as possible in some sense. More precisely we want

$$\langle \eta_i^{(\nu)}(t)\, \eta_i^{(\nu)}(t) \,|\, X^{(1)}, \ldots, X^{(n)} \rangle = \text{minimum} \qquad (10)$$

$(i, \nu)$ not summed

under variation of $A_{ij}^{\nu\mu}$ and $B_{ijk}^{\nu\mu\lambda}$.

The left-hand side of (10) is calculated (in principle) by starting the system off many times with all possible initial positions and velocities (initial positions are tried uniformly and initial velocities are tried in proportion given by an assumed initial velocity probability distribution). Out of all these realizations of the system at time $t$ we consider as our conditioned sample space only those realizations in which the $i^{th}$ particle is at $X^{(i)}$ for $i = 1, \ldots, n$. We then average $\eta_j(t)\, \eta_j(t)$ on this sample space. In the following we will abbreviate $\langle \text{———} \,|\, X^{(1)} \ldots X^{(n)} \rangle$ by $\langle \text{———} \rangle$. (Also in the calculation of $\langle \eta_i\, \eta_i \rangle$ we take $\ddot{X}_i^{(\nu)}$ to be the correct value as calculated by Navier-Stokes.)

Dropping all indices we have the following necessary condition on $A$ and $B$ for $\eta\,\eta$ to be a minimum.

$$\delta \langle (\ddot{X} - A\dot{X} - B\,\dot{X}\dot{X})^2 \rangle = 0 \qquad (11)$$

$$\therefore \langle (\ddot{X} - A\dot{X} - B\dot{X}\dot{X})\dot{X} \rangle \delta A + \langle (\ddot{X} - A\dot{X} - B\dot{X}\dot{X})\dot{X}\dot{X} \rangle \delta B = 0$$

Finally we get

$$\langle \ddot{X}_i^{(\nu)} \dot{X}_\ell^{(\alpha)} \rangle = A_{ij}^{\nu\mu} \langle \dot{X}_j^{(\mu)} \dot{X}_\ell^{(\alpha)} \rangle + B_{ijk}^{\nu\mu\lambda} \langle \dot{X}_j^{(\mu)} \dot{X}_k^{(\lambda)} \dot{X}_\ell^{(\alpha)} \rangle \tag{12}$$

$$\langle \ddot{X}_i^{(\nu)} \dot{X}_\ell^{(\alpha)} \dot{X}_m^{(\beta)} \rangle = A_{ij}^{\nu\mu} \langle \dot{X}_j^{(\mu)} \dot{X}_\ell^{(\alpha)} \dot{X}_m^{(\beta)} \rangle + B_{ijk}^{\nu\mu\lambda} \langle \dot{X}_j^{(\mu)} \dot{X}_k^{(\lambda)} \dot{X}_\ell^{(\alpha)} \dot{X}_m^{(\beta)} \rangle \tag{13}$$

Since the above expectations are conditional expectations with condition being that the particle must be at fixed places at time $t$ , and also since the flow is assumed incompressible these expectations (or coveriences) are actually Eulerian coveriences.

<u>Navier-Stokes Equation in a Lagragian System</u>.

The Navier-Stokes equation in the Lagragian system is

$$\ddot{X} = -\nabla p + \nu \nabla^2 \dot{X} \tag{14}$$

Since
$$\ddot{X} = \frac{\partial \dot{X}}{\partial t} + (\dot{X} \cdot \nabla) \dot{X}$$

and by incompressibility
$$\nabla \cdot \dot{X} = 0.$$

We obtain by taking the divergence of (14) the following equation

$$\nabla \cdot \left[ (\ddot{X} \cdot \nabla) \dot{X} \right] = -\nabla^2 p \tag{15}$$

This enables us to calculate $p$ from the $\dot{X}$ distribution. Also, we will be able to calculate coveriences of the form $\langle \ddot{X} \dot{X} \rangle$ from coveriences of form $\langle \dot{X} \dot{X} \rangle$, $\langle \dot{X} \dot{X} \dot{X} \rangle$. In fact it will turn out that only triple moments involving just two positions will be needed. An example of such a moment is $\langle \dot{X}_1(\vec{a}) \dot{X}_3(\vec{a}) \dot{X}_2(\vec{b}) \rangle$. In order to derive the desired formulae we first note that

$$\nabla \cdot \left[(\dot{X}\cdot\nabla)\dot{X}\right] = \frac{\partial}{\partial x^l}\dot{X}_j \frac{\partial}{\partial x^j}\dot{X}_l$$
$$= \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^j}\left(\dot{X}_j \dot{X}_l\right) - \frac{\partial}{\partial x^l}\left(\frac{\partial}{\partial x^j}\dot{X}_j\right)\dot{X}_l$$

But $\nabla \cdot \dot{X} = \frac{\partial}{\partial x^j}\dot{X}_j = 0$

So
$$\nabla \cdot \left[(\dot{X}\cdot\nabla)\dot{X}\right] = \frac{\partial}{\partial x^l}\frac{\partial}{\partial x^j}\left(\dot{X}_j \dot{X}_l\right) \tag{16}$$

$$\therefore \; p(\vec{x}) = \int G(\vec{x},\vec{a})\frac{\partial}{\partial a^l}\frac{\partial}{\partial a^j}\dot{X}_j(\vec{a})\,\dot{X}_l(\vec{a})\,d\vec{a} \tag{17}$$

where
$$\vec{a} = a^1\hat{\imath} + a^2\hat{\jmath} + a^3 k$$
$$\vec{x} = x^1\hat{\imath} + x^2\hat{\jmath} + x^3 k$$

and $G(\vec{x},\vec{a})$ is the appropriate Green's function.

Rewriting (14) in component form we have

$$\ddot{X}_k(\vec{x}) = -\frac{\partial}{\partial x^k}p(\vec{x}) + \nu\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\dot{X}_k(\vec{x}) \tag{18}$$

Dot (18) by $\dot{X}_m(\vec{y})$ and average. Making use of (17) we find the desired formula

$$\langle\ddot{X}_k(\vec{x})\dot{X}_m(\vec{y})\rangle = -\frac{\partial}{\partial x^k}\int G(\vec{x},\vec{a})\frac{\partial}{\partial a^l}\frac{\partial}{\partial a^j}\langle\dot{X}_j(\vec{a})\dot{X}_l(\vec{a})\dot{X}_m(\vec{y})\rangle d\vec{a} + \nu\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^i}\langle\dot{X}_k(\vec{x})\dot{X}_m(\vec{y})\rangle \tag{19}$$

<u>Procedure</u>

The actual procedure followed by Dr. Hasselmann made use of simplified versions of equations (9) - (13). These are

$$\ddot{X}_i^{(\nu)} = A_{ij}^{\nu\mu}\, X_j^{(\mu)} + \eta_i^{(\mu)}(t) \tag{9*}$$

(10*) is the same but under variation of $A_{ij}^{\nu\mu}$ only. Taking the variations as in (11) we get a set of equations for $A_{ij}^{\nu\mu}$

$$\langle \ddot{x}_i^{(\nu)} \dot{x}_\ell^{(\alpha)} \rangle = A_{ij}^{\nu\mu} \langle \dot{x}_j^{(\mu)} \dot{x}_\ell^{(\alpha)} \rangle \tag{12*}$$

In connection with the comment following equation (13) we further note

$$\langle \ddot{x}_i^{(\nu)} \dot{x}_\ell^{(\alpha)} \rangle \equiv \langle \ddot{x}_i^{(\nu)} \dot{x}_\ell^{(\alpha)} | \vec{x}\, \vec{y} \rangle = \langle \ddot{x}_i(\vec{x}) \dot{x}_\ell(\vec{y}) \rangle$$

also $\langle \dot{x}_j(\vec{a}) \dot{x}_\ell(\vec{a}) \dot{x}_m(\vec{y}) \rangle = \langle \dot{x}_j^{(\nu)} \dot{x}_\ell^{(\nu)} \dot{x}_m^{(\alpha)} | x^{(\nu)} = \vec{a}, x^{(\alpha)} = \vec{y} \rangle$

$$= \langle \dot{x}_j^{(\nu)} \dot{x}_\ell^{(\nu)} \dot{x}_m^{(\alpha)} \rangle$$

Similarly for all other covariences used in (19) and (12*). This observation and equation (19) enable us to calculate all the covariences used in (12*) from velocity covariences involving only <u>two</u> particles.

An iterative process can now be described which gives this model's prediction for covariences. Subscripts and superscripts will be dropped in the following.

First the values of $\langle \dot{x}\dot{x} \rangle$ and $\langle \dot{x}\,\dot{x}\,\dot{x} \rangle$ as functions of time and pairs of positions are assumed. By (19) $\langle \ddot{x}\ \dot{x} \rangle$ is calculated and then $A$ is calculated using (12*).

The minimization of $\langle \eta\,\eta \rangle$ implies that

$$\langle \eta \cdot \dot{x} \rangle = 0 \tag{20}$$

So $\eta$ is assumed up to some parameters which are adjusted so that

(20) is fulfilled. Now that we have $A$ and $\eta$ for all pairs of positions and time we numerically integrate (9*) from $0$ to $t$ for many different initial conditions. The initial conditions constitute a pair of initial positions and a pair of initial velocities. The initial conditions are chosen by a "Monte Carlo" method using a uniform distribution for positions and a Gaussian distribution for velocities. For each value of $t$ $\langle \dot{X}\dot{X} \rangle$ and $\langle \dot{X}\dot{X}\dot{X} \rangle$ are calculated as functions of pairs of positions. For example to calculate $\langle \dot{X}_3^{(1)} \dot{X}_1^{(2)} \rangle$ at the pair of positions $Y^1, Y^2$ one would select from the total set of pairs of particles those pairs which have the property that

$$\| X^1(t) - Y^1 \| \leq \Delta Y$$

$$\| X^2(t) - Y^2 \| \leq \Delta Y$$

$\Delta Y$ is some small number set at the start of the procedure.

$\| X \|$ represents the maximum of the components of $X$.

Using just the "selected" group of pairs of particles we form the average of $\dot{X}_3^{(1)} \dot{X}_1^{(2)}$.

Now we have completed one cycle of the iterative process and the rest is clear. We just keep cycling until $\langle \dot{X}\dot{X} \rangle$ converges.

Notes submitted by

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# Flow in Rotating Systems

Allan Robinson

## I. The Equations of Motion

This series of lectures will be concerned with large-scale fluid motions having a rotational constraint. Hoping that turbulence as such will not have a dominant effect, we will employ pseudo-laminar models involving eddy coefficients on free parameters. Although we do not pretend to study the real ocean, our models will be constructed with the intention of illuminating actual oceanic processes.

Consider a reference frame $R$, rotating at constant angular velocity $\Omega$ with respect to a fixed frame, $S$, and having its origin coincident with that of $S$. For a general vector $\vec{A}$

$$\left.\frac{d\vec{A}}{dt}\right|_S = \left.\frac{d\vec{A}}{dt}\right|_R + \vec{\Omega} \times \vec{A} \tag{1}$$

Therefore for a radius vector $\vec{r}$

$$\dot{r}_S = \dot{r}_R + \Omega \times r \tag{2}$$

Applying (1) to $\dot{r}_S$ then gives, with (2)

$$\ddot{r}_S = \ddot{r}_R + 2\vec{\Omega} \times \dot{\vec{r}} + \Omega \times (\Omega \times \vec{r}) \tag{3}$$

The acceleration term in the hydrodynamic equations of motion then takes the form

$$\frac{D\vec{V}}{Dt} + 2\Omega \times \vec{V} + \Omega \times (\Delta \times \vec{r}) \tag{4}$$

for motion relative to $R$. The second term in (4) is the Coriolis acceleration and will be of prime concern to us throughout these lectures. For motion on the rotating earth the third term, the centripetal acceleration, has a potential and can be absorbed in the gravity term.

As spherical coordinates are difficult to work with, and tend to complicate the effects under study, we will derive a simpler geometry. We will take a Boussinesq fluid with sub-tropical boundaries, and develop a latitude-dependent approximate geometry.

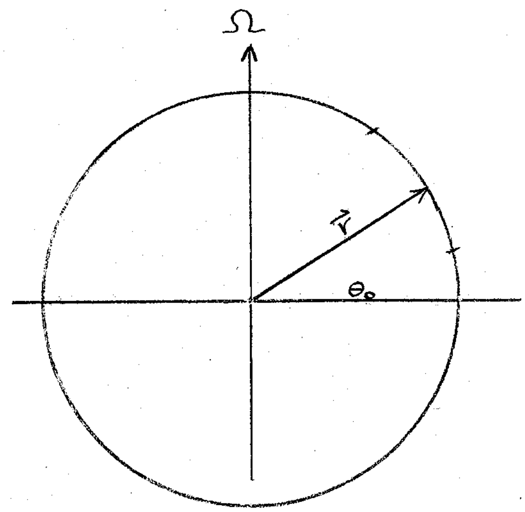
Let $\theta$ = latitude, with $\theta_o$ the reference latitude as shown, $\phi$ = longitude and $\vec{r}$ = radius vector. Define

$x = R\cos\theta_o\phi$ — positive East

$y = R(\theta-\theta_o)$ — positive North

$z = \vec{r}-\vec{R}$ — positive up

where $\vec{R}$ is the radius of the spherical earth.




Thus the equation of continuity, which in spherical coordinates is

$$\frac{1}{r\cos\theta}\frac{\partial u}{\partial\phi}+\frac{1}{r\cos\theta}\frac{\partial}{\partial\theta}\left(v\cos\theta\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w\right)=0 \qquad (5)$$

becomes

$$R\cos\theta_o U_x - V\sin\left(\theta_o+\frac{y}{R}\right)+\cos\left(\theta_o+\frac{y}{r}\right)\left[RV_y+2w+(R+z)w_z\right] \qquad (6)$$

We now introduce $L$ as a characteristic length of horizontal motion and $H$ as a characteristic length of vertical motion. By Taylor's theorem

$$\cos(\theta_0 + y/r) = \cos\theta_0 \left[1 - \tan\theta_0\, y/r + \cdots\right] \tag{7}$$

and

$$\sin(\theta_0 + y/r) = \cos\theta_0 \left[\tan\theta_0 + y/r + \cdots\right]$$

We will now use $L$ and $H$ to characterize our system in such a way that, when non-dimensional variables are introduced they will be bounded (of O(1)) throughout. This requires that we deal only with "smooth" functions, defined as bounded with all derivatives on ( $0, \infty$ ). Make the estimates

$$\frac{d}{dx} = O\left(\frac{1}{L}\right) \qquad \frac{d}{dz} = O\left(\frac{1}{H}\right)$$

The requirement that we deal with shallow, sub-tropical oceans which do not have too great a lateral extent is now expressed by the following assumptions:

(i) $H/R << 1$

(ii) $L/R \tan\theta_0 << 1$

(iii) $L^2/R^2 << 1$

Using (i) - (iii) and retaining only O(1) terms, (6) becomes

$$U_x + V_y + W_z = 0 \tag{9}$$

(i) - (iii) also serve to reduce $\frac{D\vec{v}}{Dt}$ and $\nabla p$ to cartesian form.

The rotation vector of the earth has the component form $\vec{\Omega} = \Omega(0, \cos\theta, \sin\theta)$ so that the Coriolis term may be written

$$2\vec{\Omega}\times\vec{V} = 2\Omega(-\sin\theta_V + \cos\theta_W, \sin\theta_V, -\cos\theta_V) \tag{10}$$

We now assume that $W$ takes on its maximum values to achieve mass continuity and that the vertical scale of motion is much less than the horizontal. Thus to (i) - (iii) add

(iv) $\frac{W_o}{V_o} \leq \frac{H}{L}$

(v) $\frac{H}{L} < \frac{L}{R}$

where $W_o, V_o$ are vertical and horizontal scales of velocity. The term $\cos\theta_W$ in (10) may now be neglected. Our final assumption (vi) is that the motion is hydrostatic.

The model system under consideration now takes the form

$$U_t + UU_x + VU_x + WU_z - fv + \frac{1}{\rho_o}P_x - F_1 = 0 \tag{11}$$

$$V_t + UV_x + VV_y + WV_z + fU + \frac{1}{\rho_o}P_y - F_2 = 0 \tag{12}$$

$$g(1-\alpha T) + \frac{1}{\rho_o}P_t = 0 \tag{13}$$

$$U_x + V_y + W_z = 0 \tag{14}$$

$$\frac{DT}{Dt} - H = 0 \tag{15}$$

$F_{1,2}$ and $H$ are the divergences of the eddy fluxes of momentum and heat, while $f(y) = 2\Omega(\sin\theta_o + \cos\theta_o\, y/r) = f_o + By$.

The geometry involved in (11) - (15) is that of the

$\beta$-plane, for which there is no simple intuition and which has never been successfully modelled in the laboratory. It may be interpreted as a vertically rotating horizontal plane, with the rotation vector varying linearly with latitude.

## II. Geostrophy

A simple but non-trivial form of I-(11) - (14) may be obtained by considering $\frac{D\vec{v}}{Dt} = 0,\ F_1 = F_2 = 0$ . We see that a class of linearized motion is available to a rotating fluid that is not available to a non-rotating fluid, namely geostrophic hydrostatic motion satisfying

$$-fv + \frac{1}{\rho_o} P_x = 0 \tag{1}$$

$$fu + \frac{1}{\rho_o} P_y = 0 \tag{2}$$

$$g(1-\alpha T) + \frac{1}{\rho_o} P_z = 0 \tag{3}$$

It is found that the characteristic slow motions in the open ocean satisfy (1)-(3) approximately. In fact our more complex analyses will pivot about the geostrophic case. The vertical mode of motion may be exhibited by cross-differentiation of (1) and (2) and use of the continuity equation to get

$$f w_z - \beta v = 0 \tag{4}$$

Cross-differentiation of (16) with (18) and (17) with (18) gives the thermal wind equations

$$f U_z + \alpha g T_y = 0 \tag{5}$$

$$f V_z - \alpha g T_x = 0 \tag{6}$$

For $\alpha = \beta = 0$ we obtain $U_z = V_z = W_z = 0$. The motion obeys the Proudman-Taylor theorem, there being no change in motion in the direction of the rotation vector. This does not imply $W = 0$.

For $\alpha = 0$ $\beta \neq 0$ we conclude that horizontal velocities are independent of depth, but that $W$ is a linear function of depth

$$W = \frac{\beta}{f} V_0 z + W_0 \tag{7}$$

For $\alpha \neq 0$ $\beta \neq 0$ we get

$$W_{zz} = \frac{\alpha \beta g}{f^2} T_x \tag{8}$$

so that longitudinal temperature gradients within the geostrophic velocity fields control the curvature in the vertical velocity profile.

Most of the observational knowledge of the ocean is derived from the observed density field. In our model ocean, $T$ is loosely used for $\rho$, and integration of (5) from any depth $D$ gives

$$U(x,y,z) = U(x,y,D) - \frac{\alpha g}{f} \int_D^z T_y \, dz \tag{9}$$

while from (3)

$$P(z) = \rho_0 g \left[ (H + h - z) - \alpha \int_z^{H+h} T \, dz \right] \tag{10}$$

where $h$ is the departure of the free surface from the mean height $H$. Assuming negligible effects of atmospheric pressure we set $P_{H+h} = 0$. The predicted velocity in the x-direction is then

$$U(x,y,z) = \frac{g}{f}\left[-h_y - \alpha \int_z^{H+h} T_y \, dz\right] \tag{11}$$

and similarly for $V(x,y,z)$. Since $h(x,y)$ is generally not determinable, (11) implies that, from a practical point of view, oceanic velocities are not well known from density measurements. One may, using one of several criteria derived from experience and intuition, take a level $D_{oa}$ as a "level of no motion" and compute velocities up to the vicinity of the surface which agree well with observations. However, a large local transport error may arise due to neglect of baroclinic transports in the deep layers.

## III. Non-dimensional Form of the Equations.

The complete equations of motion for the steady laminar flow of a uniformly rotating homogeneous fluid are

$$-\nu \nabla^2 \vec{V} + \vec{V} \cdot \nabla \vec{V} + 2\Omega \times \vec{V} + \frac{1}{\rho_0} \nabla P = 0 \tag{1}$$

Define the Taylor-Ekman number $\gamma = \frac{\nu}{2\Omega L^2}$ and the Rossby number $\varepsilon = \frac{V_0}{2\Omega L}$.

In order to pivot our investigations about the notion of geostrophy, we divide through by $2\Omega V_0$, the scale of the

Coriolis parameter. (1) may be written

$$-\gamma \nabla^2 \vec{V} + \varepsilon \vec{V} \cdot \nabla \vec{V} + \vec{K} \times \vec{V} + \nabla P = 0 \tag{2}$$

Geostrophy occurs in the limit $\gamma \rightarrow 0$ $\varepsilon \rightarrow 0$ and represents a mathematically singular case.

In the ocean, both temperature $T$ and salinity $S$ have effects on the flow. The Boussinesq approximation in the ocean is then

$$\rho = \rho_0 \left[ 1 - a\,(T - T_0) + b\,(S - S_0) \right] \tag{3}$$

or $\rho = \rho_0 \left[ 1 - aT^* \right]$ where $T^* = (T - T_0) + \frac{b}{a}(S - S_0)$ (4)

is the apparent temperature, and will be what is meant by $T$ in what follows.

For a horizontally isotropic fluid, introduction of eddy-viscosity coefficients $A_h$ and $A_v$ gives

$$F_1 = \frac{\partial}{\partial x}\left(A_h \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(A_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_v \frac{\partial u}{\partial z}\right) \tag{5}$$

and similarly for $F_2$ . We introduce

$$\mu = \frac{A_h}{A_{oh}} \qquad \nu = \frac{A_v}{A_{ov}}$$

where $A_{oh}$ , $A_{ov}$ are for the present undetermined reference eddy viscosity coefficients. Subtraction of the hydrostatic contribution due to the mean density leads to the following non-dimensional form for the pressure

$$p = \frac{P + \rho_0 g z}{\rho_0 f_0 L V_0}$$

With the length scales $L$, $H$ the vertical velocity scale is

$$W_0 = V_0 \frac{H}{L}$$

The non-dimensionlized equations are then

$$\delta U_t + \varepsilon\left[UU_x + VU_y + WU_z\right] - \gamma(\nu U_z)_z - \Gamma\left[(\mu U_y)_y + (\mu U_x)_x\right] - f_V + P_x = 0 \qquad (6)$$

$$\delta V_t + \varepsilon\left[UV_x + VV_y + WV_z\right] - \gamma(\nu V_z)_z - \Gamma\left[(\mu V_y)_y + (\mu V_x)_x\right] + f_U + P_y = 0 \qquad (7)$$

$$\Theta T + P_z = 0 \qquad (8)$$

where $f = 1 + B^* y \qquad \delta = \frac{\omega}{f_0} \qquad \varepsilon = \frac{V_0}{f_0 L}$

$$\Gamma = \frac{A_{oh}}{f_0 L^2} \qquad \gamma = \frac{A_{ov}}{f_0 H^2} \qquad \Theta = \frac{\alpha g \Delta T H}{f_0 L V_0} \qquad B^* = \frac{L}{R} \cot \theta_0 .$$

We introduce as scaling estimates for the large-scale flow

$$A_{oh} = O(10^8) \qquad A_{ov} = O(10^3)$$

$$\varepsilon = O(10^{-4}) \qquad \gamma = O(10^{-4}) \qquad \Gamma = (10^{-4})$$

$$\Theta = O(10) \qquad B^* = O(1)$$

The eddy scalings chosen are roughly those of the largest observed values.

Notes submitted by

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# Frictional Modification of Geostrophy

Allan Robinson

## a) An Exact Solution to the Navier-Stokes Equation

Consider a horizontally infinite plane which is semi-infinite in the vertical, with a uniform stress on the horizontal plane $z = 0$.

The Navier-Stokes equations are:

$$\varepsilon \vec{V}\cdot\nabla\vec{V} - \gamma\nabla^2 V + \hat{k}\times\vec{V} + \nabla P = 0$$

In this model $\vec{V}\cdot\nabla\vec{V} = 0$ since there is complete symmetry in the horizontal plane, and since $-\frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ and $W = 0$ at surface requires $W = 0$ throughout the system, the only available length scale is $L^2 = \frac{\nu}{2\Omega}$.

Since the boundary conditions of the problem are

$$\tau_0 = \nu V_z \quad \text{at} \quad z = 0$$

thus $V_0 = \frac{\tau_0 L}{\nu} = \frac{\tau_0}{\gamma}\left(\frac{\nu}{2\Omega}\right)^{\frac{1}{2}} = \frac{\tau_0}{(\nu 2\Omega)^{1/2}}$

and $\gamma = 1$.

The equations now become

$$-U_{zz} - V = 0$$
$$-V_{zz} + U = 0$$
$$P_z = 0$$

since $W_z = 0$

and boundary conditions at $z = 0$ $\quad V_z = 1$, $\quad U_z = 0$

Then $u^{iv} + u = 0$.

Try $e^{az}$ which gives $a \pm \frac{(1 \pm i)}{\sqrt{2}}$

$$u = e^{-\frac{z}{\sqrt{2}}} \cos\left(\frac{\pi}{4} - \frac{z}{\sqrt{2}}\right)$$

$$v = e^{-\frac{z}{\sqrt{2}}} \sin\left(\frac{\pi}{4} - \frac{z}{\sqrt{2}}\right)$$

This is referred to as Ekman layer flow and constitutes one of the few known exact solutions to the Navier-Stokes equations.

Characteristic of this solution is a velocity at $45^{\circ}$ to the stress to the right in the northern hemisphere and to the left in the southern hemisphere. The integrated transport is at right angles to the stress. It is this last characteristic of the flow that might be expected to be most significant since it is independent of the detached structure of the Austauch coefficient.

b) As a second example of a boundary layer model consider a horizontal laminar boundary layer which is finite in the vertical extent of characteristic depth $L$ and has horizontal boundaries which impose a horizontal scale $L_1$ .

The Navier-Stokes equations are no longer exactly satisfied since the non-linear term $\vec{V} \cdot \nabla \vec{V}$ is no longer necessarily zero and thus the model must be restricted so that $\varepsilon = \frac{\tau_0}{2\Omega\nu} << 1$.

If the vertical (horizontal) length scale

$$L >> \left(\frac{\nu}{2\Omega}\right)^{\frac{1}{2}}.$$

Then $\gamma << 1$ and since the differentialed terms have small coefficients in the interior it is to be expected that a frictional boundary layer will develop to bound an interior geostrophic flow.

Consider $$-\gamma(u_{xx}+u_{zz})-v+p_x=0$$

$$-\gamma(v_{xx}+v_{zz})+u=0$$

$$-\gamma(w_{xx}+w_{zz})+p_z=0$$

$$u_x+w_z=0$$

where because of the infinite extent in the $y$-direction derivatives in that direction are expected to vanish by summetry. With finite boundaries the hydrostatic assumption cannot be assumed _a priori_ from the laminar model with a single length scale.

In the interior it is assumed that $\gamma$ is small enough so that the flow is geostrophic. If the scaling so far is correct the functions must be smooth into the boundary layer. Thus if the geostrophic approximation holds in the interior, then with this scaling it is required to hold at the boundary. This cannot be the case since in the boundary the differentiated velocity terms must contribute and the value of $\gamma$ is no longer sufficient to suppress them.

Let $\xi=\gamma^a z$ then $u_{zz}\to\gamma^{-2a}u_{\xi\xi}$ etc.

The derivatives of the form $u_{xx}$ will always be of an order smaller and thus can be neglected in the boundary layer. This term must balance the coriolis term. Thus $1-2a=0$ and $a=\frac{1}{2}$.

To have a consistent model it is necessary to show that the pressure term is a smaller order in $\gamma$ than the other terms.

Assuming as a first approximation

$$U_{\xi\xi} + V = 0$$

$$-V_{\xi\xi} + U = 0$$

and continuity $U_x(\xi) + \gamma^{-\frac{1}{2}} W_\xi(x,y,\xi) = 0$

If $W_{\frac{1}{2}}(x,y,\xi)$ is of the same order as $U$ then

$$W(x,y,\xi) = \gamma^{\frac{1}{2}} W_{\frac{1}{2}}(x,y,\xi)$$ from the continuity equation.

Now $$-\gamma^{\frac{3}{2}}\left[W_{\frac{1}{2}xx} + \gamma^{-1} W_{\frac{1}{2}\xi\xi}\right] + \gamma^{-\frac{1}{2}} p_\xi = 0$$

and if $p_1(x,y,\xi)$ is defined as the same order as $W_{\frac{1}{2}}$ and thus as $U$ and $V$, then $p(xy\xi) = \gamma\, p_1(x,y,\xi)$ to satisfy the last equation.

Thus $p$ is of order $\gamma$ smaller than the other terms in the boundary layer. We have incidentally shown that the non-hydrostatic terms are of order $\gamma$ smaller than the hydrostatic terms in these equations throughout the whole region.

The velocity is divided into contributions from the interior flow $\vec{V}_I(x,y)$ and from the boundary layer $\vec{V}(x,y,\xi)$ so that the total velocity

$$\vec{V}(x,y,\xi) = \vec{V}_I(x,y) + \vec{V}(x,y,\xi)$$

The surface boundary condition can be set as

1) Velocity field $\vec{V}(x,y,0)$
2) Surface stress $\tau(x,y,0)$

For dimensionful coordinates the interior equations are:

$$-\nu u_{zz} - 2\Omega v = 0$$

$$-\nu v_{zz} + 2\Omega u = 0$$

$$u_x + v_y + w_z = 0$$

Integrating across the boundary layer

$$-\nu u_z \Big|_{-\infty}^{0} = -\tau_{x_0} + \tau_{x_\infty} = 2\Omega V$$

assuming at the bottom of the boundary layer $\tau_{x_\infty} = 0$. Since this is essentially the definition of the boundary layer and where $V$ is the integrated transport in the Ekman layer.

The equations become

$$\tau_x + 2\Omega V = 0$$

$$\tau_y - 2\Omega V = 0$$

$$V_x + V_y + (W_0 - W_\infty) = 0$$

Combining these

$$(W_0 - W_\infty) = \hat{k} \cdot \nabla \times \frac{\tau(x,y)}{2\Omega}$$

This supplies a boundary condition for the interior flow since

at $z = 0$ $\qquad W_{\text{interior}} + W_0 = 0.$

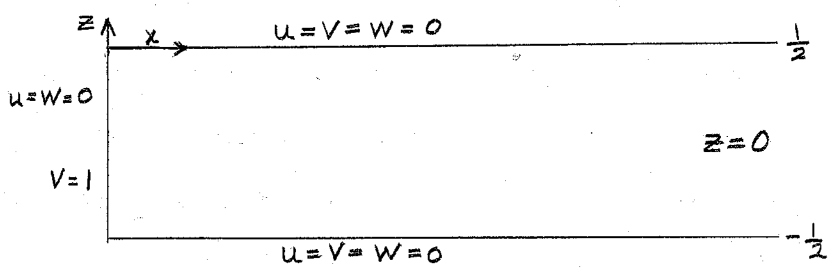
Notes submitted by

Patrick Davis

Frictional Modification of Geostrophy (continued)

c) As a third boundary layer problem consider a system where the boundary velocities are specified rather than the stress.

In particular:




taking $w = \psi_x$ and $u = -\psi_z$ where the equations of motion for the interior are:

$$\gamma \nabla^4 \Psi + V_z = 0$$

$$\gamma \nabla^2 V - \Psi_z = 0$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

where derivatives in the $y$-direction are omitted because of symmetry, and because of the continuity equation a stream function $\psi$ can be introduced into the first two equations. These then give

$$\gamma^2 \nabla^6 \psi + \psi_{zz} = 0$$

and $V(x,z)$, can be determined in terms of $\psi$ from the first equations. Then $\vec{V} = \hat{i}\, \psi_z + \hat{j}\, V - \hat{k}\, \psi_x$ .

The boundary conditions imposed on the interior problem by the Ekman layers at the top and bottom are

(see the next lecture for proof)

at $\chi = 0$ $\quad V = 1$ and $\psi = \psi_\chi = 0$

$$\chi = \infty, \quad \psi_z = V_z = 0$$

and $V = \psi = 0.$

Thus there must be a shear layer at $\chi = 0$ where $V$ decreases from $1$ at the boundary to $0$ in the interior.

First try scaling $\xi = \gamma^{-a}\chi \quad \psi = \gamma^{-b}\phi$ and not changing $V$ or $z$. Then

$$\frac{\partial V}{\partial z} = -\gamma^{1+b-4a}\frac{\partial^4\phi}{\partial\xi^4}, \quad \frac{\partial\phi}{\partial z} = \gamma^{1-b-2a}\frac{\partial^2 V}{\partial\xi^2}$$

$$\therefore a = b = \frac{1}{3}.$$

Now the boundary conditions $\phi = \frac{\pm\gamma^{\frac{1}{6}}}{\sqrt{2}}\left(V + \gamma^{\frac{1}{3}}\phi_z\right) \quad z = \pm\frac{1}{2}$

thus $\phi = 0 + O\left(\gamma^{\frac{1}{6}}\right)$ at $z = \pm\frac{1}{2}$

$\therefore V_z = 0$ at $z = \pm\frac{1}{2}$

at $\chi = 0 \quad \xi = 0 \quad V_{\xi\xi} = V_{\xi\xi\xi\xi} = 0$ (from $\psi$ boundary conditions)

then $$\frac{\partial^6 V}{\partial\xi^6} + \frac{\partial^2 V}{\partial z^2} = 0$$

Since $V$ is even in $z$, we can expand in Fourier series

$$V = V_0(\xi) + \sum_{n=1}^{\infty} V_n(\xi)\cos 2n\pi z$$

$$V_0(\xi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} V_0(\xi, z)\, dz$$

$$V_n(\xi) = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} V(\xi, z)\cos 2n\pi z\, dz$$

at $\xi = 0 \quad V_0 = 1 \quad V_n = 0 \qquad (n \geq 1)$

and $\dfrac{\partial^2 V_n}{\partial \xi^2} = \dfrac{\partial^3 V_n}{\partial \xi^3} = 0$ (all $n$)

Now $(V_n)_{6\xi} - n^2 4\pi^2 V_n = 0$

for $n = 0$.

The solution to the equation for $V_0$ is a polynomial. It is therefore not of boundary layer character (exponential). Thus the scaling must be incorrect at least as far as $V_0$ goes.

That is, the $z$ independent part of $V$ adjusts to the imposed value at $x = 0$ through a boundary layer of some thickness different from $\gamma^{\frac{1}{3}}$.

Therefore rescaling

$$\eta = \gamma^{-c} x \qquad \psi = \gamma^{d} \theta$$

using boundary conditions at $z = \pm \frac{1}{2}$ to set scale

$$V = \pm z^{\frac{1}{2}} \theta \quad \text{if} \quad d = \tfrac{1}{2}.$$

From the differential equation

$$\frac{\partial V}{\partial z} = -\gamma^{1+\frac{1}{2}-4c} \frac{\partial^4 \theta}{\partial \eta^4}$$

then $C = \frac{3}{8}$ and $\dfrac{\gamma^2 V}{\gamma \eta^2} = 0$ which is again not of a boundary layer character.

Alternately scaling from

$$\frac{\partial \theta}{\partial z} = \partial^{1-\frac{1}{2}-2c} \frac{\partial^2 V}{\partial \eta^2}$$

then $C = \frac{1}{4}$

and the other equation becomes $\dfrac{\partial V}{\partial z} = 0$

which is acceptable since the $\gamma^{\frac{1}{3}}$ scaling fails only for the $z$-independent part of $V$.

Now employing the above scaling

$$\text{i.e., } \xi = \gamma^{-\frac{1}{3}}x \quad \eta = \gamma^{-\frac{1}{4}}x$$

$$V = \nu(\eta) + \gamma^{a} v(\xi, z)$$

But

$$\nu(\eta) = \pm z^{\frac{1}{2}}\theta \quad \text{at} \quad z = \pm \tfrac{1}{2}$$

and since $\frac{\partial \nu}{\partial z} = 0$ and $V_{\eta\eta} - \theta_z = 0$

$$\theta = \nu_{\eta\eta} z + G(\eta).$$

Now from the boundary conditions at $z = \pm\frac{1}{2}$

$$\nu(\eta) = +\frac{\sqrt{2}}{2}\nu_{\eta\eta} + G(\eta)$$

$$= +\frac{\sqrt{2}}{2}\nu_{\eta\eta} - G(\eta)$$

then $G(\eta) = 0$ and $\nu(\eta) = \frac{\nu_{\eta\eta}}{\sqrt{2}}$

$$\therefore \nu = e^{-(2)^{\frac{1}{4}}\eta}$$

$$\therefore V = e^{-(2)^{\frac{1}{4}}\eta} + \gamma^{a} v(\xi, z)$$

$$\Psi = \gamma^{\frac{1}{2}}\theta + \gamma^{a}\gamma^{\frac{1}{3}}\phi(\xi, z)$$

$$\Psi = (2\gamma)^{\frac{1}{2}} z\, e^{-(2)^{\frac{1}{4}}\eta} + \gamma^{a+\frac{1}{3}}\phi(\xi, z)$$

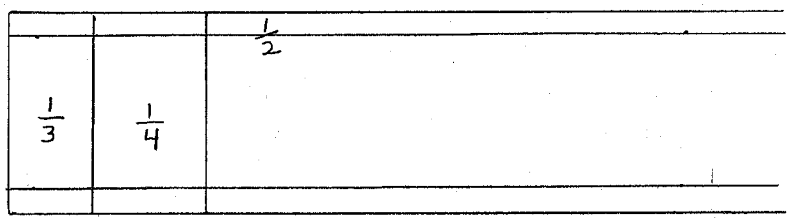
Now to satisfy $\Psi = 0$ at $x = 0$

$$\gamma^{a+\frac{1}{3}}\phi(0, z) = -(2\gamma)^{\frac{1}{2}} z.$$

$$\therefore a = \tfrac{1}{6} \quad \text{and} \quad \phi(0, z) = -2^{\frac{1}{2}} z.$$

also $v(0, z) = 0$.

Now physically in this problem there are three boundary layers.




In the two boundary layers associated with the lateral boundary the scaling of each of the velocities can now be assessed.

| $\gamma^{\frac{1}{3}}$ boundary layer | $\gamma^{\frac{1}{4}}$ boundary layer |
|---|---|
| $V \sim \gamma^{\frac{1}{6}}$ | $\sim 1$ |
| $u \sim \gamma^{\frac{1}{2}}$ | $\sim \gamma^{\frac{1}{2}}$ |
| $w \sim \gamma^{\frac{1}{6}}$ | $\sim \gamma^{\frac{1}{4}}$ |

The effect of the Ekman layers is unimportant for a balance of the normal flow in the $\gamma^{\frac{1}{3}}$ region but is important in the $\frac{1}{4}$ region.

To determine physically what is happening in the $\gamma^{\frac{1}{3}}$ layer it is interesting to follow through the scaling again in each region for the equations

$$\gamma \nabla^2 u - V + P_x = 0$$

$$\gamma \nabla^2 V + u = 0$$

$$-\gamma \nabla^2 w + P_z = 0$$

Discarding higher order terms

| $\gamma^{\frac{1}{3}}$ boundary layer | $\gamma^{\frac{1}{4}}$ boundary layer |
|---|---|
| $-V + P_\xi = 0$ | $-V + P_y = 0$ |
| $-V_{\xi\xi} + \psi_z = 0$ | $-V_{\eta\eta} + \psi_z = 0$ |
| $-\psi_{\xi\xi\xi} + P_z = 0$ | $P_z = 0$ |

In the $\gamma^{\frac{1}{4}}$ boundary layer, geostrophy and hydrostatic balance are maintained. The Ekman layer functions to reduce the value of $V$.

In the $\gamma^{\frac{1}{3}}$ boundary layer geostrophy is maintained, but the hydrostatic balance is upset by the need to satisfy the boundary conditions.

It is interesting to note that the hydrostatic assumption is violated before the geostrophic one. This result is not so surprising if the equations with the Ekman boundary conditions are re-examined.

The existence of an internal geostrophic region free of viscosity depended to a large extent on the ratio $\frac{H}{L} = \lambda$ of the horizontal to the vertical scale.

If the parameter is introduced to the equations it is possible to order the terms with respect to both $\gamma$ and $\lambda$.

$$V(x \pm \tfrac{1}{2}) + \Psi_z = \pm \left(\frac{2}{\gamma\lambda^2}\right)^{\frac{1}{2}} \Psi + V_{\pm\frac{1}{2}}(x)$$

$$-\gamma\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]u - V + P_x = 0$$

$$-\gamma\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]V + u = 0$$

$$-\gamma\lambda^{-2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right]W + P_z = 0$$

Thus $W$ is scaled conversely to $\lambda$ to maintain the continuity condition.

Then for a boundary layer $\gamma << 1$ for hydrostatic balance $\gamma << \lambda^2$.

For the boundary conditions of the last model

$$\text{i.e.,}\quad V_{+\frac{1}{2}}(x) = V_{-\frac{1}{2}}(x)$$

the $V$ is geostrophic and $\psi = 0$ as $x \longrightarrow \infty$.

For boundary conditions

$$V_{\frac{1}{2}}(x) = +1$$

$$V_{-\frac{1}{2}}(x) = -1$$

$$V(0) = 0$$

There is no geostrophic term $V$ and $\psi = 1$. However provided the ordering of the expansion in $\gamma$ and $\lambda$ expansion is correct, a plausible answer is always achieved which is independent of $\lambda$ as a parameter; for every region of the flow that is, if $\gamma << \lambda^2 << 1$. The boundary layer solutions are similar for all $\lambda$ if the system can be shown to boundary layers.

Notes submitted by

Patrick Davis

Problem I.

Verify the boundary condition on the interior fields discussed in the lecture

$$V^I(x, \pm\tfrac{1}{2}) + \psi_z^I = \pm\left(\frac{2}{\gamma}\right)^{\frac{1}{2}} \psi^I \text{ at } \quad z = \pm\tfrac{1}{2} \tag{1}$$

Solution

The exact boundary conditions are

$$u = v = w = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{2}$$

We replace (2) by the more convenient form

$$\psi_z (= u) = v = \psi = 0 \quad \text{at} \quad z = \pm\tfrac{1}{2} \tag{3}$$

Using the additive approcah we write each of the total fields $\psi_z$ , $V$ , $\psi$ as the sum of a boundary layer part $(\psi_z^B, V^B, \psi^B)$ and an interior part $(\psi_z^I, V^I, \psi^I)$.

$$\begin{aligned}
\psi_z &= \psi_z^B(\zeta, x) + \psi_z^I(z, x) \\
V &= V^B(\zeta, x) + V^I(z, x) \\
\psi &= \psi^B(\zeta, x) + \psi^I(z, x)
\end{aligned} \tag{4}$$

where $\zeta$ is the stretched boundary layer coordinate $\zeta = (2\gamma)^{-\frac{1}{2}} z$ and $\gamma << 1$ . At the boundaries $z = \pm\frac{1}{2}$ equations (4) become

$$\begin{aligned}
0 &= \psi_z^B(0, x) + \psi_z^I(\pm\tfrac{1}{2}, x) \\
0 &= V^B(0, x) + V^I(\pm\tfrac{1}{2}, x) \\
0 &= \mp\, \psi^B(0, x) + \psi^I(\pm\tfrac{1}{2}, x)
\end{aligned} \tag{5}$$

In order to write (5) as conditions on the interior fields alone, we must solve the boundary layer problem for $\psi^B_z$, $V^B$, $\psi^B$. To do this we start with the exact equations

$$
\begin{aligned}
\gamma^2 \nabla^6 \psi + \psi_{zz} &= 0 \\
\gamma^2 \nabla^6 V + V_{zz} &= 0 \\
\gamma \nabla^2 V - \psi_z &= 0
\end{aligned}
\qquad (6)
$$

where the third equation is one of the two auxiliary equations used to obtain the first two equations. Replacing $z$ by the stretched coordinate, equations (6) become

$$
\begin{aligned}
\frac{\partial^2}{\partial \zeta^2}\left(\psi_{4\zeta} + 4\psi\right) &= 0 \\
\frac{\partial^2}{\partial \zeta^2}\left(V_{4\zeta} + 4V\right) &= 0 \\
\frac{\partial}{\partial \zeta}\left(\gamma^{\frac{1}{2}} V_\zeta - 2\psi\right) &= 0
\end{aligned}
\qquad (7)
$$

The acceptable solutions of these equations are

$$
\begin{aligned}
\psi^B &= \left[a \cos \zeta + b \sin \zeta\right] e^{-\zeta} \\
V^B &= \frac{2}{\gamma^{\frac{1}{2}}}\left[-(a+b)\cos \zeta + (a-b)\sin \zeta\right] e^{-\zeta}
\end{aligned}
\qquad (8)
$$

where the criterion of acceptability is that $\psi^B$, $V^B$ be smooth functions of $\zeta$ for $0 \leq \zeta \leq \infty$. Now we insert the solutions (8) into equations (5) and equation (1), the desired result, is obtained.

Notes submitted by

Kern Kenyon.

# Wind-driven Flow

Allan Robinson

Measurements show that large regions of the oceans are in approximate geostrophic balance. It is of interest to explore theoretically the different types of boundary layers which can be fitted to an interior geostrophic flow. The type of boundary layer which is discussed in this lecture is one whose dynamics is a balance between Coriolis acceleration and vertical momentum diffusion and whose geometry is a thin flat layer whose normal is parallel to the rotation vector. Such a boundary layer is called an Ekman layer.

As a model we take an ocean with an interior geostrophic flow which is driven by steady winds at the surface. The characteristic non-dimensional parameters are $\delta = \varepsilon = T = \Theta = 0$, $\gamma \ll 1$. The basic equations are

$$\begin{aligned} -\gamma \nu u_{zz} - fv + p_x &= 0 \\ -\gamma \nu v_{zz} + fu + p_y &= 0 \\ p_z &= 0 \\ u_x + v_y + w_z &= 0 \end{aligned} \qquad (1)$$

and the boundary equations at the flat ocean bottom (z = 0) and the flat sea surface (z = 1) are

$$u = v = w = 0 \quad \text{at} \quad z = 0 \qquad (2)$$

$$\left.\begin{aligned} \nu u_z &= \gamma^{-1}\tau_0^* \tau^{(x)}(x,y) \\ \nu v_z &= \gamma^{-1}\tau_0^* \tau^{(y)}(x,y) \\ w &= 0 \end{aligned}\right\} \text{at } z=1 \tag{3}$$

where $\tau^{(x)}$ and $\tau^{(y)}$ are non-dimensional stress components representing wind-stress at the level sea surface, and $\tau_0^*$ is $\gamma \tau_0 H / A_{ov} V_0$.

We describe the interior fluid motion by the geostrophic equations

$$\begin{aligned} u^I &= -\frac{p_y}{f} \\ v^I &= \frac{p_x}{f} \\ w^I &= \left(\frac{\beta p_x}{f^2}\right) z + w(x,y) \end{aligned} \tag{4}$$

where superscript $I$ means interior. The first two equations are obtained by setting $\gamma = 0$ in (1). The third equation is the solution of $w_{zz} = 0$ and $w_z = \beta p_x / f^2$ obtained from eliminating the pressure in (1).

Near the surface and bottom we expect boundary layers in which friction is important. For simplicity we will discuss the laminar case $\nu = 1$. If we assume that the pressure forces are negligible in the boundary layers and try $\zeta_0 = (f/2\gamma)^{1/2} z$ and $\zeta_1 = (f/2\gamma)^{1/2}(1-z)$ as stretched boundary layer coordinates in the lower and upper boundary layers respectively, we find the scaling choice is consistent with the assumption. The boundary layer depth is $O(\gamma^{\frac{1}{2}})$, and we can form a dimensional characteristic scale $\gamma^{\frac{1}{2}}H = (A_{ov}/f)^{\frac{1}{2}}$ which is independent of $H$ and $V_0$. If $\gamma^{\frac{1}{2}}H \sim 100\,m$, then $A_{ov} \sim 10^4\,cm^2/sec$. Dropping the pressure terms in (1) and using $\zeta$ for a boundary layer coordinate we find that the horizontal velocities in both boundary layers satisfy

$$\left(\frac{\partial^4}{\partial \zeta^4} + 4\right)(u, v) = 0. \tag{5}$$

Of the four solutions of this equation only the two which decay exponentially in $\zeta$ are acceptable.

$$\begin{aligned} u &= [a \sin \zeta + b \cos \zeta] e^{-\zeta} \\ v &= [-b \sin \zeta + a \cos \zeta] e^{-\zeta} \end{aligned} \tag{6}$$

We now apply the boundary conditions to determine the coefficients a, b, by formally writing

$$u = u^0 + u^I, \quad v = v^0 + v^I \tag{7}$$

in the lower boundary layer and

$$u = u' + u^I, \quad v = v' + v^I \tag{8}$$

in the upper boundary layer. Using (6) for $u^0$, $u'$, $v^0$, $v'$, and then substituting into (2) and (3) we get

$$\begin{aligned} a &= -p_x/f \\ b &= p_y/f \end{aligned} \tag{9}$$

for the bottom boundary layer, and

$$a = \tau_0^* (2\gamma f)^{-\frac{1}{2}} (\tau^{(x)} - \tau^{(y)})$$
$$b = \tau_0^* (2\gamma f)^{-\frac{1}{2}} (\tau^{(x)} + \tau^{(y)}) \qquad (10)$$

for the surface boundary layer, where $\tau_0^* = \gamma \tau_0 H / A_{ov} V_0$.
From the boundary layer form of the continuity equation we can obtain the vertical velocity by integrating it across the boundary layers. If we require $W^0$ and $W^1$ to vanish exponentially in $\zeta$ and use the boundary condition $W = 0$ at $z = 0, 1$, we get two equations for the interior functions $\mathcal{W}(x,y)$ and $p(x,y)$.

$$-\tau_0^* \hat{k} \cdot \nabla \times (\tau/f) + \frac{\beta p_x}{f^2} + \mathcal{W} = 0$$
$$\left(\frac{\gamma}{2}\right)^{\frac{1}{2}} f^{-\frac{3}{2}} \left[-\nabla^2 p + \frac{\beta}{2f^{1/2}}(p_x + p_y)\right] + \mathcal{W} = 0 \qquad (11)$$

where $\tau = \hat{i}\,\tau^{(x)} + \hat{j}\,\tau^{(y)}$ and $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit eastward, northward, and vertical vectors. We get an equation for the pressure by eliminating $\mathcal{W}$ from (11)

$$\frac{\beta p_x}{f^2} + \left(\frac{\gamma}{2}\right)^{\frac{1}{2}} f^{-\frac{3}{2}} \left[\nabla^2 p - \frac{\beta}{2f^{1/2}}(p_x + p_y)\right] - \tau_0^* \hat{k} \cdot \nabla \times (\tau/f) = 0 \qquad (12)$$

For uniform rotation $\beta = 0$, $f = 1$ and we must take $\tau_0^* = \gamma^{\frac{1}{2}}$ from (12) which implies $V_0 = \tau_0 (A_{ov} f_0)^{-\frac{1}{2}}$. Equation (12) reduces to a Poisson equation for the pressure and (11) shows that there is a uniform vertical velocity $\mathcal{W} = \gamma^{\frac{1}{2}} \hat{k} \cdot \nabla \times \tau$ in the interior region. The vertical velocity in the boundary layers is $O(\gamma^{\frac{1}{2}})$ times the horizontal velocities, and the horizontal velocities have order of magnitude 1 everywhere in the fluid. (The stress in the surface layer has the same order of magnitude as that in the bottom layer). If $\nabla \times \tau = 0$, the horizontal flow may be confined entirely to the upper boundary layer since there is no driving of the interior flow.

More appropriate for the ocean is the case $\beta$ of $O(1)$; then we must choose $\tau_0^* = 1$ in order to balance the two leading terms in (12). Now $V_0 = \tau_0 (f_0 H)^{-1}$ which is independent of $A_{ov}$. Within $O(\gamma^{\frac{1}{2}})$, $w = 0$ and $p_x = f^2/\beta \; \hat{k} \cdot \nabla_x (\tau/f)$ which determines the pressure to within an arbitrary function of $y$ corresponding to an arbitrary meridional distribution of zonal flow. In this case the vertical velocity at the lower edge of the upper boundary is $O(1)$ and the horizontal velocities are $O(\gamma^{-\frac{1}{2}})$ because of $\tau_0^* = 1$ and the continuity equation. However, in the lower boundary layer the vertical velocity is $O(\gamma^{\frac{1}{2}})$ times the horizontal velocities. As a result the stress in the lower layer is $O(\gamma^{\frac{1}{2}})$ times the stress in the upper layer.

As a still closer approach to the real ocean we discuss briefly the effects of incorporating variable density into the model, i.e. $\Theta \neq 0$. We now use the thermal wind equations discussed in the first lecture to describe the interior flow, and we must re-examine the boundary layers and the boundary conditions. If we assume there is no $O(1)$ contribution to the pressure which is a function of $\zeta_1$, the upper boundary layer coordinate, then the dynamics of the upper boundary layer are unchanged. However the surface boundary condition (3) is modified as

$$\nu(u'_z + u^I_z) = \nu(u'_z - \Theta T_y(x,y,1)) = \gamma^{-1}\tau_0^* \tau^{(x)}$$
$$\nu(v'_z + v^I_z) = \nu(v'_z + \Theta T_x(x,y,1)) = \gamma^{-1}\tau_0^* \tau^{(y)} \qquad (13)$$
$$w = 0$$

due to the thermal wind equations.

Now the vertical velocity at the bottom of the surface layer is computed as follows: First, the horizontal momentum equations are put in boundary layer form and are integrated across the boundary layer, and the boundary conditions in (13) are used. This gives the two horizontal transport components. Then the continuity equation, integrated across the boundary layer, is combined with the divergence of the horizontal transport to give

$$w^I(x,y,H) \equiv w_E = \frac{V_0 H}{L}\left\{\tau_0^* \hat{k}\cdot\nabla\times(\tau/f) - \gamma\Theta\nu(1)\nabla^2\left(T(x,y,1)/f^2\right)\right\} \qquad (14)$$

Consider the relative importance of the two terms on the right side of (14) by taking the ratio of the first to the second. This ratio is

$$\frac{\tau_0^*}{\gamma\,\nu(1)\,\Theta} = \frac{\tau_0 f_0 L}{A_{0v}\,\nu(1)\,\alpha g \Delta T} \qquad (15)$$

There is some difficulty in evaluating the ratio because $\nu$ may be very small at the surface but increase with depth. We take the point of view that the second term on the right of (14) is a bottom stress correction in computing the horizontal transport components. Then we evaluate $\nu(1)$ somewhere in the lower part of the constant stress region, i.e. in the lower half of the surface Ekman layer.

Taking the following values in c.g.s. units $\tau_0 = 1$, $f_0 = 10^{-4}$, $A_{ov} = 10^4$, $\nu(1) = 1$, $\alpha g \Delta T = 1$, $L = 10^8$, the ratio in (15) becomes unity. However, in the past the second term on the right side of (14) has always been neglected compared to the first term on the right-hand side.

Notes submitted by

Kern Kenyon.

## Double Boundary Layers

Allan Robinson

The formalism which we have developed up to this point can be applied to situations in which thermal boundary layers are important. In particular, if we consider a typical mean density profile of the ocean (Fig. I) we can recognize two distinct characteristic depths. The smaller of these is $\sim$ 100 m, and can be identified with the Ekman boundary layer; the larger is the main thermocline, and may be treated as a thermal boundary layer.

The existence of the double boundary layer emphasizes that we must consider two independent driving forces in any realistic model, heat transfer and momentum transfer due to wind stresses. For a preliminary treatment, however, we look at the case where no wind stress is present.

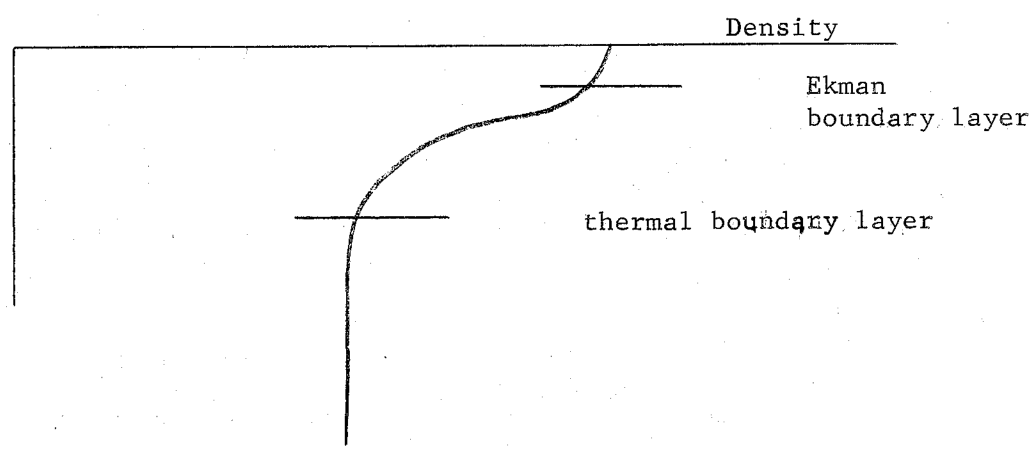



Figure I.

The assumptions of this model are as follows:

a) surface heating produces a thermal boundary layer about 1 km thick;

b) the flow is geostrophic and hydrostatic in the thermal boundary layer;

c) only the vertical component of turbulent transfer is important in balancing the advection of heat.

As a result of assumption b) the momentum and continuity equations may be written

$$-fv + p_x = 0; \tag{1}$$

$$fu + p_y = 0; \tag{2}$$

$$-\theta T + p_z = 0; \tag{3}$$

$$u_x + v_y + w_z = 0; \tag{4}$$

where

$$\theta = \frac{\alpha g \Delta T H}{f_0 L V_0} \tag{5}$$

Also, following assumption c), we may write the heat equation

$$-\left(\frac{KL}{H^2 V_0}\right) T_{zz} + u T_x + v T_y + w T_z = 0 \tag{6}$$

Here $K$ is a thermal eddy conductivity which is assumed to be constant throughout the fluid.

The assumption of geostrophy throughout has an additional consequence. In the case of the Ekman boundary layer, geostrophy was assumed invalid in a small boundary region. The parameter $\gamma$ which appeared in the momentum equations provided a natural way to scale the boundary layer thickness. There the Ekman layer turned out to be independent of the mean depth. Here it is necessary to use a new scale depth $H$ (again not related to the mean depth) which does not arise quite so naturally.

This scale depth, and the scale of the horizontal velocity as well, are determined by setting the free coefficients in Eqs. (3) and (6) equal to 1.

$$\frac{\alpha g \Delta T}{f_0 L} \frac{H}{V_0} = 1 ; \tag{7}$$

$$\frac{KL}{H^2 V_0} = 1 . \tag{8}$$

This gives

$$H = \left(\frac{f_0 L}{\alpha g \Delta T}\right)^{1/3} (KL)^{1/3} ; \tag{9}$$

$$V_0 = \left(\frac{\alpha g \Delta T}{f_0}\right)^{2/3} \left(\frac{K}{L}\right)^{1/3} ; \tag{10}$$

$$W_0 = \frac{V_0 H}{L} = \left(\frac{\alpha g \Delta T}{f_0}\right)^{1/3} \left(\frac{K}{L}\right)^{2/3} . \tag{11}$$

An additional relation which will be found useful is the geostrophic vorticity equation, obtained by appropriate manipulation of Eqs. (1) - (4),

$$W_{zz} = \beta/f^2 \, T_x . \tag{12}$$

The non-linearity of the problem which arises from the advective terms in Eq. (6) makes the present situation considerably more difficult than the Ekman problem. A possible way to proceed in this case is to look for situations which admit similarity solutions. This approach will place some restrictions on the allowed surface temperature distributions, but useful qualitative results can still be obtained.

In particular, we choose transformations of the form

$$\eta = x^r f^s ; \quad (z-1) \tag{13}$$

$$p = x^{2r+1} f^{2s+2} \Pi(\eta) ; \tag{14}$$

$$W = x^r f^s \omega(\eta). \tag{15}$$

(Recall that $f$ is a function of $y$ only.)

These forms are substituted into Eqs. (1) - (4) and Eq. (6), and we obtain, after some manipulation, two coupled, non-linear, ordinary differential equations for $\Pi$ and $\omega$.

$$\omega'' - \beta\left[(3r+1)\Pi' + r\eta\Pi''\right] = 0 ; \tag{16}$$

$$-\Pi''' + \omega\Pi'' + \beta(s-2r)\left[\eta\left(\Pi''\Pi - (\Pi')^2\right) + \Pi'\Pi\right] = 0. \tag{17}$$

Eq. (16) is the geostrophic vorticity equation. Eq. (17) is the heat equation, where the three terms represent, respectively, vertical diffusion, vertical advection, and horizontal advection.

The remarks about surface temperature distributions can be clarified by looking at Eq. (3) in terms of our similarity variables.

$$T_s(x,y) = x^{3r+1} f^{3s+2} \Pi'(0). \tag{18}$$

Since $f = 1 + \beta^* y$, we see that only a restricted class of $x, y$ distributions can be represented by the functional form (18).

The boundary conditions for $\omega$ and $\Pi$ are as follows: At the surface, $\eta = 0$, Eq.(18) shows an order 1 temperature field related to $\Pi'$. Hence $\Pi'(0)$ can be set equal to 1. The velocity boundary condition is $\omega(0)=0$. To see this, we note that: a) an Ekman layer analysis for thermal wind shear, but zero stress gives a small vertical velocity of $\mathcal{O}(\gamma^{\frac{1}{2}})$; b) here we are dealing with an order 1 vertical velocity, so it is sufficient to solve the interior problem with inviscid boundary conditions; c) hence, the $\omega$ field need not contribute at the boundary.

Away from the boundary layer, as $|\eta| \to \infty$, the solution should approach the interior solution. We have a vertical velocity of the same order in the thermal boundary layer and the deep geostrophic region. Hence, from the scaling of the continuity equation, we see that the deep horizontal velocity is reduced by a factor of the order of the ratio of the thermal boundary layer thickness to the depth. In this sense then, $u$, $v$, $T \to 0$ in the interior, as does the pressure gradient which drives those fields. In the similarity variables, this corresponds to $\Pi$ and $\Pi'$ both going to zero. This can be seen by recalling that the differentiations indicated in Eqs. (1) and (2) involve $x$ and $y$ both explicitly, and implicitly, through $\eta$.

Now that the problem is stated, we may ask how $r$ and $s$ are to be chosen. One approach is to make the choice to correspond to a realistic surface temperature distribution. Such a model might depend on latitude only $\left(n = -\frac{1}{3}\right)$. With this choice,

$$\eta = \chi^{-\frac{1}{3}} f^{s}(z-1). \tag{19}$$

so, because of the $\eta$ dependence, the temperature beneath the surface depends on longitude as well as latitude. Reference to Eq. (12) shows that an $\chi$-dependence of $T$ implies $w_{zz} \neq 0$; and it is precisely in this feature that the nature of the thermal boundary layer differs from the interior solutions where $w$ is linear in $z$.

Because of this role of the longitudinal temperature dependence, it is more fruitful to choose $n$, $s$ such that Eqs. (16) and (17) can be solved, rather than to hew too closely to a realistic surface distribution. By choosing $n = 0$, $s = -1$, we can find solutions in the form

$$\omega = a + b e^{a\eta} \tag{20}$$

$$\Pi = ab\beta^{-1} e^{a\eta} \tag{21}$$

Here $a$ represents the limiting value of $\omega$ for below the boundary layer. Application of the boundary conditions at the surface gives

$$a = -b; \quad a^2 b = \beta; \tag{22}$$

or

$$a^3 \equiv \omega_T^3 = -\beta, \tag{23}$$

and

$$\omega_T = -\beta^{\frac{1}{3}} \tag{24}$$

Some arguments about the signs of various terms are now needed. From Eq. (24) we see that $\omega_T$ is negative. For the solution to decay exponentially, $\eta$ must then be positive. The non-dimensional form of $\eta$ in Eq. (13) is negative below the surface. Hence the vertical scaling factor $H$ must be negative to make the dimensionful $\eta$ positive. This, in turn, implies, through Eq. (9), that $\Delta T$ is negative.

The dimensionless temperature field satisfies

$$\frac{\partial T}{\partial x} = \frac{e^{\omega_T \eta}}{f} > 0. \tag{25}$$

The horizontal scaling factor $L$ is positive, so, with $\Delta T < 0$, the dimensionful temperature gradient satisfies

$$\frac{\partial T}{\partial x} < 0, \tag{26}$$

in agreement with the actual conditions in the major subtropical gyres of the real ocean.

As a final remark on this problem, we may point out that the total meridional transport is zero. This is a consequence of assuming geostrophy throughout, and can be seen as follows: Elimination of $p$ and $T$ from Eqs. (1) - (4) gives

$$\beta v = f w_z. \tag{27}$$

If this is integrated over $z$, we find

$$\beta V = \beta \int_0^1 v\, dz = f\left(w(1) - w(0)\right) = 0 \tag{28}$$

We may now turn to the more complicated problem where wind stresses are present in addition to the thermal boundary layer.

Here a model much like that in Fig. I is chosen, with a thin Ekman layer above a thermal boundary layer.

The thickness of the Ekman layer is assumed to be much less than that of the thermal boundary layer. The value of the vertical velocity at the bottom of the Ekman layer is determined in terms of the surface wind stress by the methods discussed in previous lectures. This quantity is then used as the boundary condition on the vertical velocity at the top of the thermal boundary layer. It is further assumed that the thermal wind shear is much less than the stresses in the Ekman layer, and that the term $k \cdot \nabla \times (\tau/f)$ varies as $f^{-1}$. Then the dimensionful velocity at the bottom of the Ekman layer (refer to previous lecture) is

$$W_E = \frac{V_0 H}{L} \tau_0^* \hat{k} \cdot \nabla \times (\tau/f) = \frac{W_E}{f} = \frac{W_0 \omega_E}{f} . \qquad (29)$$

The boundary condition on $\omega$ is now

$$\omega(0) = \omega_E . \qquad (30)$$

Using the same temperature distribution as before ($n = 0$, $s = -1$) we see that the boundary conditions now lead to a more complicated cubic equation

$$\omega_T^2 (\omega_E - \omega_T) = \beta \qquad (31)$$

Rather than attempt a detailed solution of Eq. (31), we look at some limiting cases. If the flow is primarily driven by thermal effects, $\omega_E$ will be small, and Eq. (31) has the approximate solution

$$\omega_T = -\beta^{1/3} + \omega_{E/3} \tag{32}$$

Here the qualitative nature of the solution discussed in the first part of this lecture is preserved. In addition we note that the effect of the wind on the deep motion is reduced by a factor of 1/3 in the presence of the thermal boundary layer.

The opposite limiting case is that in which wind stress is the main effect. This is probably a more realistic case. The approximate solution for large $\omega_E$ is

$$\omega_T \approx \pm\sqrt{\frac{\beta}{\omega_E}} + \Theta\left(\frac{\beta}{\omega_E{}^2}\right) \tag{33}$$

More arguments about signs are now needed. For the solution to show a boundary layer behavior rather than an oscillating behavior, $\omega_T$ must have a real leading term. Since $\beta > 0$ , this means we must have $\omega_E > 0$ also.

From Eq. (29) we find

$$\omega_E = \frac{W_E}{W_0} \sim \text{pos. const. } \frac{W_E}{H} . \tag{34}$$

Hence for $\omega_E > 0$ we have either

a) $W_E > 0$ ; $H > 0$ , or

b) $W_E < 0$ ; $H < 0$ .

A given choice of $H$ fixes the sign in the dimensionful form of $\eta$ and allows the correct sign for the root in Eq. (33) to be chosen. Furthermore, by Eq. (9), $H$ and $\Delta T$ have the same sign, so the two cases may also be characterized as

a) $W_E > 0;\ \Delta T > 0$ or

b) $W_E < 0;\ \Delta T < 0$ .

Both cases lead to the same result for the dimensionful forms of the upwelling velocity and the thermocline $e$ folding depth

$$W_{up} = \beta^{\frac{1}{2}} \frac{K}{L} \left( \frac{\alpha g \Delta T}{f_0 W_E} \right)^{\frac{1}{2}} \tag{35}$$

$$z_T = L \left( \frac{f_0 W_E}{\alpha g \Delta T} \right)^{\frac{1}{2}} . \tag{36}$$

It is important to note here that an increase in surface wind stress $\sim W_E$ leads to a decrease in the deep upwelling.

There is a third root to Eq. (31) which has physical significance if $\omega_E > 0$ . This is

$$\omega_T \cong \omega_E - \beta/\omega_E^2 \tag{37}$$

Here $\omega_T > 0$ so $\eta < 0$, which implies that $H > 0$ and $\Delta T > 0$ for boundary layer behavior. In this case the upwelling is influenced very little by the thermal boundary layer. The $e$ folding depth is

$$z_T = \frac{K}{W_E} . \tag{38}$$

Finally, for $W_E < 0$ , $\Delta T > 0$ , no solutions of the boundary layer type exist.

Notes submitted by

John G. Pierce.

# Ocean Circulation

Allan Robinson

Now that we have discussed solutions for the open ocean circulation, we shall consider regions where the flow is more complicated. We consider the case of vanishing thermal diffusion and thermal forces smaller than wind forces. We know from our discussion of the thermal boundary layer that for the case $K \to 0$ we still have a length scale determined thermally and that the motion is confined to the thermal boundary layer. Thus we consider a two-layer system with the lower layer at rest. The upper layer has an Ekman "sublayer" at the surface; in the interior the geostrophic velocities are independent of depth (provided thermal forces are small) and $w$ decreases linearly to $W(x, y, D)$, which is $O(\gamma^{\frac{1}{2}})$ of $w$ at the bottom of the surface Ekman layer. By considering mass conservation for a column of fluid of height $D(x, y)$ ($D$ is taken to be positive), we obtain the interface condition, $w(x, y, D) = uD_x + vD_y$ (neglecting small differences in density and gradients of sea level with respect to gradients of $D$; we shall presently show that the latter assumption is a very good one).

If we consider the two layers to be homogeneous but of slightly different density $\Delta\rho = \alpha\,\Delta T$, then the condition that the lower layer be motionless implies that $p_x = 0$ and hence $h_x = \alpha \Delta T D_x$ where $h(x, y)$ is the elevation of the mean surface above or below the surface of the undisturbed fluid. Thus horizontal gradients of $h$ are of order $10^{-3}$ of gradients of $D$ (or alternatively we can take the point of view that changes in $h$ are magnified by about 1000 at the interface between the two layers). The pressure gradient in the upper layer is related to $D$ in this case by $p_x = \Theta D_x$ (where this equation has been non-dimensionalized in the same way as in the previous lectures).

There will be an internal shear layer (Ekman layer) at the bottom of the moving layer but, since $w$ is $O(\gamma^{\frac{1}{2}})$ there, to a first approximation, we assume that the normal velocity is zero and that there is no stress at the bottom of the moving layer.

We now introduce volume transport functions by integrating our equations over $z$. This integration has two advantages: it reduces the problem to a two-dimensional one and it makes a detailed knowledge of the turbulent stresses in the body of the fluid unnecessary since the only stress terms which appear are the values of the stress at the surface. It has the disadvantage that we lose information about the details of the three-dimensional flow. We define volume transport functions as $U = \int_S^{1+h} u\,dz$ and $V = \int_S^{1+h} v\,dz$; $P$ is defined by $P = \int_S^{1+h} p\,dz$, where $S(x,y)$ is some arbitrary surface in the fluid. The equations of motion then become (in non-dimensional form)

Continuity: $U_x + V_y = 0$ provided that the normal velocities at the surface and at $S(x,y)$ vanish.

Momentum: $-\tau_0^* \tau^{(x)} - fV + P_x + p_S S_x = 0$

$-\tau_0^* \tau^{(y)} + fU + P_y + p_S S_y = 0$ where $\tau^{(x)}, \tau^{(y)}$ are the surface stresses in the $x$ and $y$ directions, respectively.

Vorticity: obtained by cross differentiation of the momentum equations

$V = (1/\beta)(\tau_0^* \hat{k} \cdot \nabla \times \vec{\tau} + J(p_S, S))$ where $J(p_S, S)$ is the Jacobian $(p_S)_x S_y - (p_S)_y S_x$.

If $S = D$ or $S > D$ everywhere, then $J \equiv 0$ since $p_x = H D_x$, $p_y = H D_y$ in the first case and $p_x = p_y \equiv 0$ in the second. The vorticity equation becomes:

$$\beta V = \tau_0^* \hat{k} \cdot \nabla \times \vec{\tau} \qquad \text{(the Sverdrup equation).}$$

We can consider various two-layer systems, but provided that a lower stationary layer exists our equations are valid. If we consider the case for which $\tau_0^* = 1$ and $H = 1$ then the characteristic scale and velocity are given by $H = [\tau_0^* L(\alpha g \Delta T)^{-1}]^{\frac{1}{2}}$, $V_0 = \tau_0^* (f_0 H)^{-1}$. Putting in values for $L$ and the surface stress of $5 \times 10^8$ and 1 dyne/cm$^2$ gives 300 m and 1/3 cm/s as characteristic values. These are typical of observation.

From the Sverdrup equation we note that $V$ vanishes if $\mathrm{curl}_K \vec{\tau}$ vanishes. Thus it might be possible to consider some small region of the ocean as a closed system rather than having to deal with the whole world ocean at once. Such regions exist in the major gyres of the North Atlantic and North Pacific and

$\text{curl}_K \vec{\tau}$ vanishes along lines which are nearly of constant latitude. We introduce the further simplicity of neglecting coastal geometry. Thus we consider a rectangular ocean and hope that our simple system will illustrate the major features of the circulation to be expected in the real ocean. The boundary conditions are that $V$ vanishes on the latitudinal walls while $U$ vanishes on the north-south walls. The major winds over such a region are westerlies in the northern half and easterlies in the southern half. We take the wind stress to be of the form $\vec{\tau} = \hat{i}\,\tau(y)\,(\tau'(y) \geq 0)$ (often a half-sine wave form is used for $\tau(y)$). Since the equations have been reduced to a two-dimensional form we can introduce a stream function which is defined by $\psi_x = V,\ -\psi_y = U$. The Sverdrup equation then has the form:

$$\psi_x = -\tau'(y)/\beta\ ,$$ which has the solution

$\psi = (\tau'(y)/\beta)(-x + x_0(y))$, where $x_0(y)$ is an arbitrary function of $y$ which is found from the boundary conditions. This solution cannot describe a closed circulation since the flow is always south; alternatively we see that the solution cannot satisfy all the boundary conditions.

We shall regard this solution as an interior solution to a boundary layer problem. The boundary layer dynamics which we assume will have a strong effect on the interior solution since they will determine $x_0(y)$. We must return to our original equations and relax our assumptions. Two very simplified approaches will be considered to illustrate ways in which a boundary layer may be formed. In the first approach, we relax the assumption that the stress at the bottom vanishes. Thus we consider a boundary layer which is frictionally controlled. With the depth $D$ fixed, we know from our discussion of Ekman layers (see earlier notes) that we have the condition:

$$\frac{\beta p_x}{f^2} + \left(\frac{\gamma}{2}\right)^{\frac{1}{2}} \frac{1}{f^{3/2}} \left[\nabla^2 p - \frac{\beta}{2f^{1/2}}(p_x + p_y)\right] = \hat{k} \cdot \nabla \times \frac{\vec{\tau}}{f} \, .$$

We assumed that $p$ was smooth and that $\gamma$ was small, giving the Sverdrup equation. Our description is not complete so we must scale this equation. We replace $\frac{\partial}{\partial x}$ by $\left(\frac{\gamma}{2}\right)^a \frac{\partial}{\partial \xi}$

$$\frac{\gamma^a}{2^a} \frac{\beta p_\xi}{f^2} + \frac{1}{2^{\frac{1}{2}+2a}} \frac{1}{f^{3/2}} \gamma^{\frac{1}{2}+2a} p_{\xi\xi} + \text{other terms} = 0 \, .$$

To make the two pressure terms of the same order we require $a = \frac{1}{2} + 2a$ , that is $a = -\frac{1}{2}$ . All the other terms are $O(1)$ or smaller while the dominant terms are $O(\gamma^{-\frac{1}{2}})$. Thus in the boundary layer the local wind driving is negligible. The scaled equation is, after integration over $z$ :

$$p_{\xi\xi} + \frac{\beta}{f^{1/2}} p_\xi = 0 .$$

The equation $fV = P_x$ holds to order $\gamma^{\frac{1}{2}}$ . Hence $P_x = f\Psi_x$ and the equation becomes:

$$\Psi_{B\xi\xi} + \frac{\beta}{f^{1/2}} \Psi_{B\xi} = 0, \quad \text{with solution}$$

$$\Psi_B = C_1 + C_2 \exp\left(-\frac{\beta}{f^{1/2}} \xi\right)$$

( $\Psi_B$ denotes the boundary layer part of the solution.)

This boundary layer must be on the western side of the ocean if the boundary layer stream function is to have the proper behaviour ($\psi_\beta \to 0$ as $\xi \to \infty$). If we put $x_o(y) = 1$, then the eastern boundary condition is satisfied. The application of the western boundary condition to the total stream function then gives the complete solution:

$$\psi_{total} = \frac{\tau'(y)}{\beta}\left[1 - x - \exp\left(-\frac{\beta}{f^{1/2}}\,\xi\right)\right]$$

The boundary current is driven by mass influx from the wind-driven interior.

Next we consider another possible approach to obtain a boundary layer for our interior solution. We retain the assumption that the stress at the bottom of the moving layer vanishes but now let $\varepsilon$ be non-zero, that is, we consider a boundary layer which is inertially controlled. We neglect local driving forces and assume that the flow is produced by influx (or efflux) of mass. We assume that the flow is two-dimensional, that is, that the velocities in the upper layer are independent of depth. Then in non-dimensional form $u = U$, $v = V$ since the depth of the layer is $1$ in the non-dimensional system. The equations of motion (in non-dimensional form) are:

$$\varepsilon\left[Uu_x + Vu_y\right] - fv + p_x = 0$$
$$\varepsilon\left[Uv_x + Vv_y\right] + fu + p_y = 0$$
$$u_x + v_y = 0$$
$$p_z = 0$$

The vorticity equation is:

$$\varepsilon\left[\psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x\right] + \beta \psi_x = 0.$$

We expect a boundary layer in the $x$-direction. Therefore we put $\xi = \varepsilon^a x$. From continuity we expect $v$ to be $O(\varepsilon)^a$ if $u$ is $O(1)$. From the $y$-momentum equation we require $a = -\frac{1}{2}$ if we are to obtain additional $O(1)$ terms in the equations. Putting $v = \varepsilon^{-\frac{1}{2}} v'(\xi, y)$ ($v'$ is $O(1)$) we have:

$$-f v' + p_\xi = 0$$

$$U v_\xi + v' v'_y + f u + p_y = 0$$

$$u_\xi + v'_y = 0$$

$$\psi_\xi \psi_{\xi\xi\xi y} - \psi_y \psi_{\xi\xi\xi\xi} + \beta \psi_\xi = 0$$

Consider a solution of the form

$$\psi = -y\, u_0 \varphi(\xi), \quad \varphi \to 1 \text{ as } \xi \to \infty$$

$$u = -\psi_y = u_0 \varphi(\xi)$$

$U_0$ is the uniform flow into (or out of) the boundary layer. Try a solution for $\varphi(\xi)$ of the form

$$\varphi = 1 - e^{a\xi}$$

$$u_0\left[-a^3 e^{a\xi}\right] - \beta a e^{a\xi} = 0$$

$$a^2 = -\beta/u_0, \quad a = \pm\sqrt{-\frac{\beta}{u_0}}$$

For a boundary layer type of solution we require that $u_0 < 0$.

For $u_0 > 0$ we obtain oscillatory solutions (standing Rossby waves). The boundary layer can be on either coast for this solution, with an influx of mass on a western boundary or an efflux on an eastern boundary.

The inertial boundary layer theories, with a more detailed formulation than the simple illustrative model discussed, give good agreement with observations of the boundary layer current in the lower half of the ocean regions considered. After the current leaves the coast, it appears to be mainly controlled by topographic effects (for the Gulf Stream). This topographic control is similar to a $\beta$-effect. Approximately, the current follows contours of constant $f/D$. (This result follows from the vorticity equation for a homogeneous layer if local frictional forces are neglected and the relative vorticity, $\mathrm{curl}_K \vec{V}$ is much smaller than $f$; more generally, the condition is that $(f + \mathrm{curl}_K \vec{V})/D$ remains constant).

Now we shall consider another interesting feature of the real ocean which is not predicted by the mean circulations which we have so far considered. Recently, measurements made in the deep water by Swallow using neutrally buoyant floats have shown that the instantaneous velocities are much larger than those obtained from geostrophic calculations (integration of the thermal wind equations). The velocities are of the order of 10 cm/s rather than the 0.1 to 1 cm/s which we might expect. The length scales appear to be of the order of 100 km so that the Rossby number is of the order $10^{-2}$. Thus the motion is still geostrophic but the Rossby number is about 100 times greater than that of the mean circulation. The motions are very deep; apparently, they extend right to the bottom and hence are probably barotropic (in the sense that the velocity extends to the bottom and is well correlated at all depths). There appears to be more north-south than east-west motion. At present there are not enough measurements to do any statistics; individual floats have been followed for several weeks to several months but these are not long times when compared to the time scales of the motions. The method has the difficulty that the information rate is not very high but the measurements are very significant since previously we had no information at all (of a direct sort) about the velocities in the deep water of the ocean.

Consider an ocean containing small scale almost geostrophic transients and suppose that the motion is almost two-dimensional (this should be a fairly good approximation since the motions appear to be approximately independent of depth). We introduce a total stream function $\Psi(x,y,t)$ and separate it into mean and fluctuating parts

$$\Psi = \psi(x,y) + \varphi(x,y,t), \quad \overline{\varphi} \equiv 0$$

The stream function then satisfies the vorticity equation (written in dimensional form)

$$\nabla^2 \Psi_t + \Psi, \Psi + \beta \Psi_x = \hat{k} \cdot \nabla \times \vec{\tau}\,(x,y,t)$$

where $\underline{\Psi}, \underline{\Psi}$ is shorthand for the Jacobian of $\underline{\Psi}$ and $\nabla^2\underline{\Psi}$. The equations for the mean and fluctuating parts are then

$$\overline{\varphi,\varphi} + \psi,\psi + \boxed{\beta\psi_x = \hat{k}\cdot\nabla\times\overline{\overline{\vec{\tau}}}}$$

$$\nabla^2\varphi_t + \varphi,\psi + \varphi,\varphi - \overline{\varphi,\varphi} + \beta\varphi_x = \hat{k}\cdot\nabla\times(\vec{\tau} - \overline{\overline{\vec{\tau}}})$$

where the terms enclosed in the box give the Sverdrup equation.

We consider, as an example, the very simple case $\overline{\overline{\vec{\tau}}} = 0$. The mean motion is then driven by the fluctuating motion only. We consider solutions for the free modes of the fluctuations (no driving forces in the $\varphi$ equation). This case is easy to solve since the zeroth order equation for $\varphi$ is very simple ( $\mathcal{E}$ , the Rossby number, is used as an expansion parameter).

$$\nabla^2\varphi_t + \beta\varphi_x = 0 .$$

We shall assume that $\varphi$ satisfies the same boundary conditions as $\psi$ does in the case of the mean circulation of the rectangular ocean discussed previously, although this is not a good assumption since it depends on the Sverdrup relation which does not hold for the fluctuations. The rectangular basin and the boundary conditions produce quantized solutions of the form:

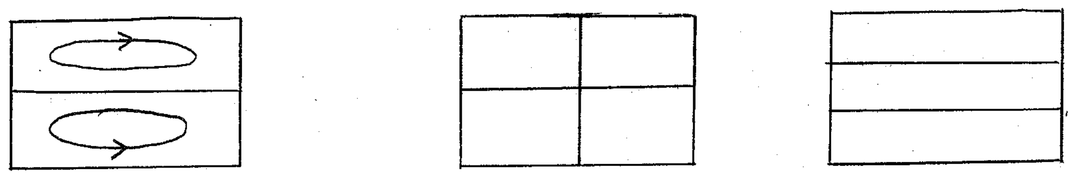
$$\varphi_{n,m} = \sin\left(\frac{\beta t}{2\pi\delta q} + \pi q x\right)\sin n\pi x \sin m\pi y$$

where $q^2 = n^2 + m^2$ and $\delta = \omega/f_0$ . This zeroth order solution is then put into the equation for the mean stream function to give the zeroth order solution for the mean motion which the fluctuations produce.

$$\psi = -\left(E\pi^3 m q^2 \beta^{-1}\right)\left(\sin n\pi x\right)^2 \sin 2m\pi y$$

where $E$ is an arbitrary amplitude since we have a free flow solution.

The sketch below shows the first mode and two second mode solutions for $\psi$ .



We note that the lowest mode contains two gyres while the mean wind driven circulation, which we discussed earlier, had only one. The ratio of the mean to transient amplitudes of the meridional velocities is of the order:

$$\frac{V_0 R \pi^3 m q^2}{2\Omega \cos\theta_0 L^2}$$

( $R$ is the radius of the earth and $V_0 m\pi$ is the transient amplitude).

For $m$ and $n$ equal to 10 this ratio is of order 1; for $n$ equal to 10 and $m$ equal to 1, a 10 cm/s transient amplitude gives a 1 cm/s mean field. The scale of the transients is 1/10 of the total ocean. The free periods are given by $\dfrac{qR}{2\Omega L\cos\theta_0}$ and the miminum period is about 1 month.

The type of motion which will be produced depends strongly on the boundary conditions imposed on $\varphi$ . We don't want to deal with the whole world ocean but putting on the boundary conditions which are used for the mean motion governed by the Sverdrup equation is not very satisfactory either. One approach that is being tried is to put in a wave train of limited extent and then see what motion results; one hopes that this will make it possible to infer boundary conditions for the problem. Another difficulty is that in the presence of strong mean currents such as the Gulf Stream, decoupling of the mean and fluctuating equations, such as the decoupling which occurred in our simple example, does not occur.

Notes submitted by

Stephen Pond.

# Onset of Convection in a Rapidly Rotating Self-gravitating Fluid

Frederic M. Bisshopp

Let us consider first the momentum balance in a self-gravitating system,

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \rho f + \frac{\partial \cdot \tau}{\partial x}$$

with $f(x) = -G \int \frac{x-x'}{|x-x'|^3} \rho(x') d^3x'$

(We shall consistently suppress subscripts and other indicators of transformation properties of familiar quantities. Contractions may be indicated by a dot, and later vector products of two vectors will appear as indicated by the symbol $\times$ .)

When $\rho = \rho_0(r), r \equiv |x|$ , we have

$$f = -\frac{M_0(r) G}{r^3} x$$

where

$$M_0(r) = 4\pi \int_0^r r'^2 \rho_0(r') dr'$$

$$= \frac{4}{3}\pi r^3 \rho_0 \quad \text{if } \rho_0 \text{ is } \underline{\text{constant}}.$$

We take great advantage of the Boussinesq approximation, in which we have, for a liquid

$$\rho(x) = \rho_0(r)\left(1 - \alpha\left(T(x) - T_0(r)\right)\right), \quad |\alpha(T-T_0)| << 1$$

$$\frac{\partial \cdot u}{\partial x} = 0 \qquad \left(= \alpha \frac{d}{dt}(T - T_0)\right)$$

$$\frac{du}{dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\rho}{\rho_0} f + \frac{1}{\rho_0}\frac{\partial}{\partial x} \cdot \left(\mu_0\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}^T\right)\right)$$

$$\frac{\partial T}{\partial t} = \frac{1}{\rho_0 C_0}\frac{\partial}{\partial x}\cdot k_0 \frac{\partial T}{\partial x} + \frac{\mu_0}{\rho_0 C_0}\frac{\partial u}{\partial x} : \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}^T\right) + \frac{Q_0}{C_0}$$

$$\frac{\rho}{\rho_0} f = -\frac{M_0 G}{r^3}x + \alpha G\left(\frac{M_0(T-T_0)}{r^3}x + \rho_0 \int \frac{(T-T_0)(x-x')}{|x-x'|^3} d^3x'\right)$$

where $\rho_0, C_0, \mu_0, k_0, Q_0$ are given functions of $r$.

To obtain the corresponding equations governing the same phenomena, we must postulate what we think happens in a rotating system. Before we become involved in this to the exclusion of the problem at hand, let us just suppose there is an inertial coordinate system in which the equations we have written above work. Then let

$$x' = R^T x \quad \text{with} \quad R^T R = R R^T = I$$

be the equations connecting a rotating coordinate system $(x')$ with a stationary one $(x)$. The motion of a material point then is

$$x(X,t) = R x'(X',t)$$

where $X$ and $X'$ may be taken to be initial position if you like. Now

$$u = \frac{dx}{dt} = R(u' + \Omega x') \quad \text{where } \Omega = R^T \dot{R}$$

$$\frac{du}{dt} = \frac{d^2x}{dt^2} = R\left(\frac{du'}{dt} + 2\Omega u' + (\dot{\Omega} + \Omega^2)x\right)$$

$$f \equiv R f', \quad p \equiv p', \quad \tau \equiv R \tau' R^T$$

The fundamental postulate of continuum mechanics (of late) then may be formulated as:

$$\left.\begin{aligned} f(x'(X',t)) &= f(x(X,t)) \\ p'(x'(X',t)) &= p(x(X,t)) \\ \tau'(x'(X',t)) &= \tau(x(X,t)) \end{aligned}\right\} \text{ when } x'(X',t) = x(X,t)$$

Before anyone here takes this too seriously, let me hasten to point out that these assumptions are already built into the N-S equations. The postulate becomes important in the study of non-newtonian fluids. Note that

$$RR^T = I \rightarrow \Omega^T + \Omega = 0$$

and it is evident that the similarity of our relations with the ones obtained by the formal device

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \Omega \times$$

is no accident. In a familiar notation, we have in the rotating system equations of motion and heat conduction which are modified only insofar as

$$f \rightarrow f - 2\Omega \times u - \dot{\Omega} \times x - \Omega \times (\Omega \times x)$$

We shall take as an equilibrium state of no motion in our rotating fluid, the state where

$$\rho_0,\ k_0,\ \mu_0,\ C_0,\ Q_0 \quad \text{are given \underline{constants}.}$$

Then $\quad k_0 \frac{\partial}{\partial r} r^2 \frac{\partial T_0}{\partial r} = -\rho_0 Q_0$

gives $\quad T_0 = \beta_0 - \beta r^2, \quad \beta = \frac{\rho_0 Q_0}{k_0}$

Let $\Theta = T - T_0$, $\Omega$ = const., $\delta p = p - p_0$;

then $$\frac{\partial u}{\partial t} + 2\Omega u = -\frac{\partial}{\partial x}\left(\frac{\delta p}{\rho}\right) + \frac{4}{3}\pi\alpha G\rho_0 \Theta x + \nu_0 \nabla^2 u + \alpha\Theta\Omega^2 x + \alpha G\rho_0 \int \frac{x - x'}{|x - x'|^3}\,\Theta\, d^3x'$$

$$\frac{d\vartheta}{dt} = -u \cdot \frac{\partial T}{\partial x} + K_0 \nabla^2 \Theta + \frac{\mu_0}{c_0}\Phi$$

$$\frac{\partial \cdot u}{\partial x} = 0$$

where $$K_0 \equiv \frac{k_0}{\rho_0 c_0}, \qquad \nu_0 = \frac{\mu_0}{\rho_0}$$

The term $\alpha\Theta\Omega^2 x$ will be dropped, i.e. we treat the case where
$$\frac{\Omega^2}{G\rho_0} << 1.$$

Also, we shall take $r = R_0$ as the spherical boundary of the fluid even when that boundary is not rigid. The term $\alpha G\rho_0 \int \frac{x-x'}{|x-x'|^3}\Theta d^3x'$, which at first glance looks troublesome, can be written as $-\alpha G\rho_0 \frac{\partial\varphi}{\partial x}$ where $\nabla^2\varphi = 4\bar{u}\Theta$.

Thus it can be put in with $\frac{\delta p}{\rho_0}$, nevermore to be seen.

In the rest of this lecture we shall follow fairly closely a treatment of the problem by P.H.Roberts where the equations are made dimensionless according to,

$$x \to R_0 x,\ t \to R_0^2 t/\nu_0,\ u \to R_0\Omega_0 u,\ \Theta \to 2\beta R_0^4 \Omega_0 \Theta/K_0$$
$$\frac{\delta p}{\rho_0} + \alpha G\rho_0\varphi \to \nu_0 \Omega \bar{u},$$

and linearized to give

$$\frac{\partial u}{\partial t} + \Gamma \hat{z} \times u = -\frac{\partial \bar{u}}{\partial x} + R_a \theta x + \nabla^2 u$$

$$P \frac{\partial \vartheta}{\partial t} = u \cdot x + \nabla^2 \theta$$

$$\frac{\partial \cdot u}{\partial x} = 0,$$

with $\Gamma = T^{1/2} = \frac{2\Omega_0 R_0^2}{\nu_0}$, $R_a = \frac{8}{3} \bar{u} G \alpha \rho_0 \beta R_0^6 / K_0 \nu_0$.

By the usual artifice of twice constructing curl we obtain the set of equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \equiv \xi$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\xi - \Gamma \frac{\partial w}{\partial z} = R_a \left(y \frac{\partial \theta}{\partial x} - x \frac{\partial \theta}{\partial y}\right)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w + \Gamma \frac{\partial \xi}{\partial z} = R_a \left(z \nabla_1^2 \theta - \frac{\partial}{\partial z}\left(2\theta + x \frac{\partial \theta}{\partial x} + y \frac{\partial \theta}{\partial y}\right)\right)$$

$$\left(P \frac{\partial}{\partial t} - \nabla^2\right)\theta = xu + yv + zw$$

Note that the first two equations provide

$$\nabla_1^2 u = -\frac{\partial^2 w}{\partial z \partial x} - \frac{\partial \xi}{\partial y}, \quad \nabla_1^2 v = -\frac{\partial^2 w}{\partial z \partial y} + \frac{\partial \xi}{\partial x}.$$

Here we should point out that the equations are not separable in either rectangular, cylindrical, or spherical coordinates. Things look grim, yet we may hope that in the limit $\Gamma \to \infty$ the equations will become separable in a c.s. appropriate for the limiting form of the boundary conditions.

Several traps must be avoided - not just <u>any</u> solution will do since this is a stability problem and we are therefore interested in the minimum critical value of $R_a$ of which there is a solution. In particular, there are several choices of $M$ for which $R_a = R\Gamma^{M}$ allows solution, and we want the minimum $M$ too.

As far as I know there is no rigorous argument that the next step is the correct one, so we must consider what follows a hueristic theory. If the <u>exact</u> solution of the analogous problem for a rotating plane layer heated from below be analysed for its limiting behaviour as $\Gamma \to \infty$, it is found that

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial y} \sim \Gamma^{\frac{1}{3}} \frac{\partial}{\partial z} \sim \Gamma^{\frac{1}{3}}$$

What we shall do is to introduce this exact result of the plane problem into our analysis of the spherical case? I shall save you all the gore and give the only reasonable transformation I can think of, viz:

$$x \to \Gamma^{-\frac{1}{3}} x, \quad y \to \Gamma^{-\frac{1}{3}} y, \quad t \to \Gamma^{-\frac{2}{3}} t, \quad R_a = R\Gamma^{4/3}, \quad \zeta \to \Gamma \zeta$$

$$u \to \Gamma^{2/3} u, \quad v \to \Gamma^{2/3} v, \quad w \to \Gamma^{2/3} w, \quad \bar{u} \to \Gamma^{4/3} \bar{u}$$

Then $\left(P\frac{\partial}{\partial t} - \nabla^2\right)\theta = z w + \Gamma^{-\frac{1}{3}}(xu + yv)$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\zeta - \frac{\partial w}{\partial z} = \Gamma^{-\frac{1}{3}} R\left(y\frac{\partial \theta}{\partial x} - x\frac{\partial \theta}{\partial y}\right)$$

$$\left(\frac{\partial}{\partial t} - \nabla^2\right)\nabla^2 w + \frac{\partial \zeta}{\partial z} = R\left(z\nabla_1^2\theta - \Gamma^{-\frac{2}{3}}\frac{\partial}{\partial z}\left(2\theta + x\frac{\partial \theta}{\partial x} + y\frac{\partial \theta}{\partial y}\right)\right)$$

$$\nabla_1^2 u = -\frac{\partial \zeta}{\partial y}, \quad \nabla_1^2 v = \frac{\partial \zeta}{\partial x}, \quad \nabla^2 = \nabla_1^2 + \Gamma^{-2/3}\frac{\partial^2}{\partial z^2}$$

The b.c. one to be satisfied at $z = \pm\sqrt{1 - \Gamma^{-\frac{2}{3}}(x^2 + y^2)}$ and we may use cylindrical coords in the 'interior'. Let $(x,y,z)\to(r,\varphi,z)$. $(u_r,u_\varphi,u_z)=(u,v,w)$; then

$$\left(P\frac{\partial}{\partial t}-\nabla^2\right)\theta = zw+\Gamma^{-\frac{1}{3}}ru$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\xi-\frac{\partial w}{\partial z}=\Gamma^{-\frac{1}{3}}R\frac{\partial\theta}{\partial\varphi}$$

$$\left(\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2 w+\frac{\partial\xi}{\partial z}=R\left(z\nabla_1^2\theta-\Gamma^{-2/3}\frac{\partial}{\partial z}\left(2\theta+u\frac{\partial\theta}{\partial r}\right)\right)$$

$$\left(\nabla_1^2-\frac{1}{r^2}\right)u-\frac{2}{r^2}\frac{\partial v}{\partial\varphi}=-\frac{1}{r}\frac{\partial\xi}{\partial\varphi}$$

$$\left(\nabla_1^2-\frac{1}{r^2}\right)v+\frac{2}{r^2}\frac{\partial u}{\partial\varphi}=\frac{\partial\xi}{\partial r}$$

$$\left(\nabla_1^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right.$$

b.c. on $z=\pm\sqrt{1-\Gamma^{-2/3}r^2}$.

The interior equations are now fully separable when $\Gamma\to\infty$ and we may let

$$\theta=\tilde{\theta}(z)\,\Phi(r)\,e^{i(m\varphi+\omega t)}+o(1)$$

$$w=\tilde{w}(z)\,\Phi(r)\,e^{i(m\varphi+\omega t)}+o(1)$$

$$\xi=\tilde{\xi}(z)\,\Phi(r)\,e^{i(m\varphi+\omega t)}+o(1)$$

where $m=O(1)$. This last requirement on $m$ is necessary if the solution is to be consistent with the scaling we have adopted where $u$, $v$ and $w$ are all of $O(\Gamma^{\frac{2}{3}})$. (The case $m=0$ presents no difficulty, but $m=O(\Gamma^{\frac{1}{3}})$ is our undoing.)

If $\omega\neq 0$ it follows that

$$\nabla_1^2 \Phi = -\alpha^2 \Phi ,$$

or that $\Phi = Z_m(\alpha r)$,

and that $\tilde{w}'' - \alpha^2(i\omega + \alpha^2)\left(i\omega + \alpha^2 - \frac{R z^2}{i\omega P + \alpha^2}\right)\tilde{w} = 0$

What we do not yet know is the boundary condition on $\tilde{w}$ at $z = \pm 1$. This in fact is all we need from the boundary layer (now to be investigated) near $z = \pm 1$. Given the boundary condition we can solve the eigenvalue problem for $R(\alpha^2, m)$ and $\omega(\alpha^2, m)$ and compute $R_c, \omega_c, \alpha_c^2, m_c$. As it turns out, the boundary condition does not involve $m$ when $m = O(1)$, neither does the equation, and therefore we have only to consider $R(\alpha^2)$ and $\omega(\alpha^2)$ in seeking the critical value $R_c$.

Let us turn now to the problem of putting a lid on our solution. Let

$$\left(\sqrt{1 - \Gamma^{-\frac{2}{3}} r^2} - z\right) \equiv \Gamma^n \tilde{z} \quad \text{define the b.l. coord.}$$

or $\quad z = \sqrt{1 - \Gamma^{-2/3} r^2} - \Gamma^n \tilde{z} \qquad (n < 0)$

$$= 1 - \tfrac{1}{2}\Gamma^{-2/3} r^2 - \Gamma^n \tilde{z} + O(\Gamma^{-\frac{4}{3}})$$

Then $\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial \tilde{z}}\frac{\partial \tilde{z}}{\partial z}$ , $\frac{\partial}{\partial r} \longrightarrow \frac{\partial}{\partial r} + \frac{\partial}{\partial \tilde{z}}\frac{\partial \tilde{z}}{\partial r}$

$$= -\Gamma^{-n}\frac{\partial}{\partial \tilde{z}} \qquad \simeq \frac{\partial}{\partial r} - \Gamma^{-n-\frac{2}{3}} r \frac{\partial}{\partial \tilde{z}}$$

$$\nabla^2 \longrightarrow \Gamma^{-\frac{2}{3}-2n}\frac{\partial^2}{\partial \tilde{z}^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - O(\Gamma^{-n-\frac{2}{3}})$$

$$\left(P\frac{\partial}{\partial t} - \nabla_1^2 - \Gamma^{-\frac{2}{3}-2n}\frac{\partial^2}{\partial \tilde{z}^2}\right)\theta = \tilde{w} + O(\Gamma^{-\frac{1}{3}}, \Gamma^n, \Gamma^{-n-\frac{2}{3}})$$

$$\left(\frac{\partial}{\partial t}-\nabla_1^2-\Gamma^{\frac{2}{3}-2m}\frac{\partial^2}{\partial\tilde{z}^2}\right)\xi+\Gamma^{-n}\frac{\partial w}{\partial\tilde{z}}\simeq\Gamma^{-1/3}R\frac{\partial\theta}{\partial\theta}$$

$$\left(\frac{\partial}{\partial t}-\nabla_1^2-\Gamma^{-\frac{2}{3}-2m}\frac{\partial^2}{\partial\tilde{z}^2}\right)\left(\nabla_1^2+\Gamma^{-\frac{2}{3}-2m}\frac{\partial^2}{\partial\tilde{z}^2}\right)w+\Gamma^{-n}\frac{\partial\xi}{\partial\tilde{z}^2}\sim R\nabla_1^2\theta$$

$n=-\frac{1}{3}$ gives a thermal b.l. where $\frac{\partial w}{\partial\tilde{z}}=\frac{\partial\xi}{\partial\tilde{z}}=0$ and

$\left(iP\omega+\alpha^2-\frac{d^2}{d\tilde{z}^2}\right)\tilde{\theta}=\tilde{w}_0$ where $\tilde{w}_0=\tilde{w}\Big|_{z=0}$

or $\tilde{\theta}-\frac{\tilde{w}_0}{(iP\omega+\alpha^2)}=-\frac{\tilde{w}_0}{(iP\omega+\alpha^2)}e^{-(iP\omega+\alpha^2)^{\frac{1}{2}}\tilde{z}}$

giving $\tilde{\theta}\Big|_{\tilde{z}=0}=0$, $\tilde{\theta}_0=\frac{\tilde{w}_0}{(iP\omega+\alpha^2)}$ as required.

$n=-\frac{1}{2}$, $\tilde{\theta}=O(\Gamma^{-\frac{1}{3}})$ gives an inner, Ekman layer where

$$\frac{\partial^2\xi}{\partial\tilde{z}^2}-\Gamma^{\frac{1}{6}}\frac{\partial w}{\partial\tilde{z}}\simeq 0$$

$$\frac{\partial^4 w}{\partial\tilde{z}^4}+\Gamma^{-\frac{1}{6}}\frac{\partial\xi}{\partial\tilde{z}}\simeq 0$$

thus $w=O(\Gamma^{-\frac{1}{6}})$, $\xi=O(1)$ in this b.l.

The actual b.c. are:

1. Rigid $w=u=v=0\longrightarrow w=\frac{dw}{d\tilde{z}}=\xi=0$ at $\tilde{z}=0$

2. Free $w=\frac{du}{d\tilde{z}}=\frac{dv}{d\tilde{z}}=0\longrightarrow w=\frac{d^2w}{d\tilde{z}^2}=\frac{d\xi}{d\tilde{z}}=0$ at $\tilde{z}=0$.

Again I'll spare you the gore, the transformed $w\to\Gamma^{-\frac{1}{6}}w$ gives a 6th order system with three conditions corresponding to 1. or 2. and three more for boundedness. Corresponding to 1. the Ekman layer is divergent and a correction of $O(\Gamma^{-\frac{1}{6}})$ is propagated to the interior; with 2. the Ekman layer is non-divergent. The bottom boundary layers are of the same type, so we obtain in either case, $$\tilde{w}'' - \alpha^2(i\omega+\alpha^2)\left(i\omega+\alpha^2 - \frac{Rz^2}{i\omega \underline{P}+\alpha^2}\right)\tilde{w} = 0$$

$$\tilde{w}(-1) = \tilde{w}(1) = 0$$

To the present, the case which has been most thoroughly investigated is where $\omega = 0$ .

Then $\tilde{w}'' - \alpha^2(\alpha^4 - Rz^2)\tilde{w} = 0$

and the results are available from two sources.

| Niiler-Bisshopp | Roberts |
|---|---|
| $R_c$ = 24.33 | $R_c$ = 20.7126 |
| $\alpha_c$ = 1.09 | $\alpha_c$ = 1.108 |

The results of N-B are derived from a three-term trial function using trigonometric functions in a variational method, those of Roberts by direct numerical integration on a digital machine. N-B have rechecked their result, Roterts' result is not in doubt, the agreement is very poor for a three-term approximation, and, at the moment, I'm quite mystified by it.

References:

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# The Response of an Unbounded Atmosphere to Point Disturbances

Shoji Kato

Abstract

The response of an unbounded and isothermal atmosphere both to time harmonic disturbances of force (Part I) and to impulsive forces (Part II) are studied mainly for point disturbances. In the former case, the exact solution is obtained in the whole frequency range of disturbances by applying the method of retarded potentials after a simple transformation of coordinates. An interesting phenomenon occurs in the trapped frequency range. For disturbances whose frequencies are just above the lower characteristic frequency of the atmosphere, $\omega_2\left(=(\gamma-1)^{\frac{1}{2}}g/c\right)$, the atmosphere responds very efficiently, especially to vertically oscillating disturbances. Moreover, those oscillations excited by vertically oscillating disturbances consist mainly of a vertical velocity field and the regions of intense response are restricted very sharply to the vertical region from the source. These characteristics of the response still remain even in the frequency range of acoustic waves if the atmosphere is sufficiently stable to convection. In the latter case (impulsive forces), the asymptotic solution is obtained with the use of Lighthill's method. Fronts of disturbances propagate with two different modes corresponding to those

of acoustic waves and of gravity waves. Behind the front resulting from propagation of acoustic waves, intense disturbances are radiated vertically from the source if the atmosphere is sufficiently stable to convection.

## I. Introduction

Problems of wave generation have usually been attacked from two directions. One is to study the coupling between turbulence and wave modes and to obtain the power of waves radiated by turbulence. The aim of this approach is to learn the amount of mechanical energy available for the heating of a stellar corona. Since the original suggestions of Biermann (1946, 1948) and Schwarzschild (1948), several authors have studied this problem (e.g. Lighthill 1952, 1954; Proudman 1952; Kulsrud 1955; Unno and Kato 1962; Kato 1963) although not always in the same context.

The second kind of approach is the attempt to study the generation of waves from given concentrated sources. In this latter case, the attention is focused on the character of the response of the surrounding medium to given simplified disturbances. As is well known, a time-harmonic point force in a homogeneous and isotropic medium gives rise to acoustic dipole radiation (e.g. Lamb 1945). If a medium has a magnetic field,disturbances generate hydromagnetic waves. This was studied by Lighthill (1960) for a non-gravitating compressible medium having a uniform magnetic field, using the method developed by him to obtain the asymptotic solution of an inhomogeneous wave equation in an isotropic medium. If a medium is stratified under the effect of a gravity and the stratification is stable to convection, disturbances generate acoustic waves and gravity waves. This was studied by Moore and Spiegel (1964) for an unbounded and isothermal atmosphere in the case of a time-harmonic disturbance of force, applying Lighthill's method.

In an atmosphere stratified by the effect of gravity, however, disturbances in a certain frequency range cannot propagate (e.g. Bjerknes, Bjerknes, Salbery and Bergeron 1933). For mathematical simplicity, we will restrict our attention to an unbounded and isothermal atmosphere under a constant gravity. Then, the trapped frequency range is specified by two characteristic frequencies of the atmosphere, $\omega_1 (= \gamma g/2c)$ and $\omega_2 (= (\gamma-1)^{1/2} g/c)$, where $\gamma$ , $g$ and $c$ denote the ratio of specific heats, the acceleration of gravity, and the adiabatic sound velocity, respectively. For frequencies above the upper characteristic frequency, $\omega_1$ , disturbances can radiate waves in all directions below the lower characteristic frequency, $\omega_2$ , energy can be radiated within the direction defined by $\cos^2\theta \leq \omega^2/\omega_2^2$ , where $\theta$ is the angle between the vertical direction and the direction of the observer (Moore and Spiegel 1964). On the other hand, disturbances whose frequencies are between $\omega_1$ and $\omega_2$ cannot propagate as waves. They will, however, produce oscillations in the surrounding medium.

In a previous paper, Kato (1965) studied the oscillation in this trapped frequency range under a simplified condition $\omega_2 = 0$. The interest there was to discuss the nature of the source terms due to turbulence and to study the relation with the atmospheric oscillatory motions observed in the solar atmosphere (e.g. Leighton, Noyes and Simon 1962; Evans and Michard 1962, etc.). Hence, that study rather belongs to the former category mentioned at the beginning of this section. So, it is necessary to learn the response of the surrounding atmosphere to disturbances in the trapped frequency range with more general condition $\omega_2 \neq 0$, although the assumed sources are simplified. In this sense, our study corresponds to an extension of Moore and Spiegel's study into the trapped frequency range. Interesting phenomen which were not recognized in Moore and Spiegel's paper occur in this trapped frequency range as we shall see.

We must also recall Lamb's work (1908) as a study about the response of stratified atmospheres to disturbances. Lamb studied the vertical propagation of disturbances both from an impulsive force and from a time-harmonic disturbance of force. In each case horizontal dimensions of the sources were infinite. One of the interesting results obtained by him for the case of an unbounded and isothermal atmosphere is that an impulsive force produces a vertically propagating wave front with a wake whose frequency tends to the upper characteristic frequency of the atmosphere as the front goes to infinity. The other is that there is a resonance phenomenon with response of infinite amplitude for time-harmonic disturbances whose frequencies are just the upper characteristic frequency of the atmosphere. We can say that the study of this paper corresponds also to a generalization of Lamb's study into a three-dimensional case. The results obtained, however, are quite different from Lamb's case because of the increase of dimension. For example, the resonance phenomenon at the upper characteristic frequency disappears in the three-dimensional case and existence of the lower characteristic frequency has important effects upon the response of the atmosphere although this frequency has no effect in Lamb's one-dimensional case.

The purpose of this paper is to study how the efficiency of atmospheric response depends upon the frequency of the disturbing source, what kind of sources (the direction of force) the atmosphere responds to effectively, what the angular dependency of the response is and how the stability of the atmosphere effects upon the response. In Part I, the response of the atmosphere will be studied for time-harmonic disturbances of force. We can obtain the exact solution in the whole frequency range of disturbances as shown later. However, we will mainly focus our attention to the trapped frequency range because Moore and Spiegel (1964) studied the response of the atmosphere in the frequency ranges of waves (although they studied it by means of the asymptotic solution), moreover, because interesting phenomena about the response of the atmosphere are intimately related to the trapped frequency range. In Part II, the response of the atmosphere will be studied for implusive forces, using Lighthill's method to obtain the asymptotic solution at great distances from the source. Finally, in Section VIII a summary of the results will be given and a possibility of application of our results upon the formation of the spicule structure in the solar upper chromosphere will be discussed briefly.

## II. Summary of Fundamental Equations

### a) The Inhomogeneous Wave Equation

We begin with the following linearized equations:

$$\rho_0 \frac{\partial \underset{\sim}{v}}{\partial t} = -\operatorname{grad} p + \rho \underset{\sim}{g} + \underset{\sim}{f} \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho_0 \underset{\sim}{v}) = 0 \tag{2}$$

$$\frac{\partial p}{\partial t} + \rho_0 \underset{\sim}{v} \cdot \underset{\sim}{g} + \rho_0 c^2 \operatorname{div} \underset{\sim}{v} = 0 \tag{3}$$

and

$$c^2 = \gamma p_0 / \rho_0 \tag{4}$$

with $\underset{\sim}{g} = (0, 0, -g)$. Here, $p$ and $\rho$ represent perturbed pressure and density from undisturbed pressure, $p_0$, and density $\rho_0$, and $\underset{\sim}{f}$ denotes the arbitrary forced disturbance of force which will be specified later. Other notations have their usual meanings. For simplicity, we assume that the atmosphere is unbounded and isothermal. Hence, the undisturbed density stratification is given by

$$\rho_0(z) = \rho_{00} \exp\left(-z/H\right), \tag{5}$$

where the scale height, $H$, is represented by

$$H = \mathcal{R} T_0/g = c^2/\gamma g, \; (c^2 = \text{const}). \tag{6}$$

From equations (1) to (6), we obtain the following inhomogeneous wave equation for $p/\rho_0^{1/2}$ (e.g. Moore and Spiegel 1964, Kato 1963)

$$\left[-\frac{\partial^4}{\partial t^4} + \frac{\partial^2}{\partial t^2}\left(c^2\nabla^2 - \frac{c^2}{4H^2}\right) + (\gamma-1)g^2\nabla_1^2\right]\left(p/\rho_0^{1/2}\right) = S(x,y,z,t) \tag{7}$$

where

$$S(x,y,z,t) = -c^2\rho_0^{1/2-\alpha}\frac{\partial^2}{\partial t^2}\left[(\alpha\gamma-1)\frac{g}{c^2} + \frac{\partial}{\partial z}\right]\left(\rho_0^{\alpha-1}f_z\right) -$$

$$-c^2\rho_0^{\frac{1}{2}-\alpha}\left[\frac{\partial^2}{\partial t^2} + (\gamma-1)\frac{g^2}{c^2}\right]\left[\frac{\partial}{\partial x}\left(\rho_0^{\alpha-1}f_x\right) + \frac{\partial}{\partial y}\left(\rho_0^{\alpha-1}f_y\right)\right] \tag{8}$$

and

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{9}$$

In subsequent sections we will discuss effects of $\delta$-functional point sources with respect to $\rho_0^{\alpha-1}\underline{f}$ where $\alpha$ is an unspecified constant. So, as a preparation, $S$ in equation (8) is expressed with the use of a bind form $\rho_0^{\alpha-1}\underline{f}$. Equation (7) is the basic equation in the discussion of wave generation and atmospheric oscillation due to external sources. It should be noted that the dependent variable, namely $p/\rho_0^{1/2}$ is used in the fundamental equation (7) because we can more easily calculate energy density and energy flux with this variable, than with the variable used by Moore and Spiegel (1964).

The dispersion equation obtained from the left-hand side of equation (7) suggests that it is convenient in the later work to introduce the following notation

$$\omega_1 = c/2H = \gamma g/2c \tag{10}$$

and

$$\omega_2 = (\gamma-1)^{1/2} g/c \, . \tag{11}$$

Actually, these are the characteristic frequencies of the atmosphere (e.g. Moore and Spiegel 1964). For frequencies above the upper characteristic frequency $\omega_1$ or below the lower characteristic frequency $\omega_2$ , disturbances can radiate acoustic waves or gravity waves, respectively.

b) The Energy Density and Energy Flux

As the energy conservation equation, from equations (1), (2) and (3), we have (Eliassen and Palm 1954)

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0 \underset{\sim}{v}^2 + \frac{1}{2}p^2/\rho_0 c^2 + \frac{1}{2}(\gamma-1)g^2\zeta^2/c^2\right] + \mathrm{div}(p\underset{\sim}{v}) = 0 \tag{12}$$

where $\zeta$ denotes the displacement in the direction of $z$ -axis.

It is useful to represent the total energy density and the energy flux in equation (12) with the dependent variable, $p/\rho_0^{1/2}$ , adopted in the fundamental equation (7). We decompose the velocity and pressure fluctuation into Fourier components as

$$\rho_0^{1/2} \underset{\sim}{v} = \int_{-\infty}^{\infty} \hat{\underset{\sim}{v}}(\omega) \exp(i\omega t) d\omega \tag{13}$$

and

$$p/\rho_0^{1/2} = \int_{-\infty}^{\infty} \hat{p}(\omega) \exp(i\omega t) d\omega. \tag{14}$$

Then, from equations (1), (2) and (3) we have

$$i\omega \hat{\underset{\sim}{v}}_x(\omega) = -\frac{\partial}{\partial x} \hat{p}(\omega) \tag{15}$$

$$i\omega \hat{\underset{\sim}{v}}_y(\omega) = -\frac{\partial}{\partial y} \hat{p}(\omega) \tag{16}$$

and

$$\frac{1}{i\omega}(\omega^2 - \omega_2^2)\hat{\underset{\sim}{v}}_z(\omega) = \left[\frac{\partial}{\partial z} + (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}/c\right] \hat{p}(\omega). \tag{17}$$

Thus, the total mean energy density (the mean of the terms in the bracket in equation (12)), $E$ , associated with waves or oscillations is represented as

$$E = \int_0^{\infty} E(\omega) d\omega, \tag{18}$$

where

$$E(\omega) = \frac{1}{\omega^2}\left( \left| \frac{\partial \hat{p}(\omega)}{\partial x} \right|^2 + \left| \frac{\partial \hat{p}(\omega)}{\partial y} \right|^2 \right)$$
$$+ \frac{\omega^2 + \omega_2^2}{(\omega^2 - \omega_2^2)^2} \left| \left[ \frac{\partial}{\partial z} + (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}/c \right] \hat{p}(\omega) \right|^2 \tag{19}$$
$$+ \frac{1}{c^2} \left| \hat{p}(\omega) \right|^2.$$

Especially the frequency component of the kinetic energy density, $E_k(\omega)$ , is given by

$$E_k(\omega) = \frac{1}{\omega^2}\left(\left|\frac{\partial \hat{p}(\omega)}{\partial x}\right|^2 + \left|\frac{\partial \hat{p}(\omega)}{\partial y}\right|^2\right) + \\ + \frac{\omega^2}{(\omega^2 - \omega_2^2)^2}\left|\left[\frac{\partial}{\partial z} + (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}/c\right]\hat{p}(\omega)\right|^2 \tag{20}$$

In this equation the first two terms in the left-hand side represent the kinetic energy density of motions in the horizontal plane, and the last one represents that of motions in the vertical direction.

Moreover, the frequency component of the mean energy flux, $\underset{\sim}{q}(\omega)$, defined by

$$\langle p \underset{\sim}{v} \rangle = \int_0^\infty \underset{\sim}{q}(\omega) d\omega \tag{21}$$

can be written as

$$\underset{\sim}{q}(\omega) = \left[\hat{p}(\omega)\hat{\underset{\sim}{v}}^*(\omega) + \hat{p}^*(\omega)\hat{\underset{\sim}{v}}(\omega)\right]. \tag{22}$$

With the help of equation (15), (16) and (17), the quantity, $\underset{\sim}{q}(\omega)$ , can be also expressed by $\hat{p}(\omega)$.

In subsequent sections we will use mainly the kinetic energy density and supplementarily the energy flux as measures of the response of the atmosphere to disturbances. However, the density fluctuation may be a more useful measure if the results should be applied upon the formation of spicules in the solar upper chromosphere (Section VIII). Of course, this can also be expressed by $\hat{p}(\omega)$. A frequency component of the mean square of $\rho_1/\rho_0^{\frac{1}{2}}$, defined by an equation similar with equations (18) or (21), can be written as

$$\frac{2}{c^4(\omega^2-\omega_2^2)^2}\left|\left(\omega^2-\frac{\gamma-1}{2H}g+(\gamma-1)g\frac{\partial}{\partial z}\right)\hat{p}(\omega)\right|^2 \tag{23}$$

Part I. The Response of the Atmosphere to Time-Harmonic Disturbances.

III. The Solution of the Inhomogeneous Wave Equation

The inhomogeneous wave equation for the time-Fourier component can be written from equations (7) and (8) as

$$\left[-\omega^4-\omega^2(c^2\nabla^2-\omega_1^2)+c^2\omega_2^2\nabla_1^2\right]\hat{p}(\omega)=\hat{S}(\omega) \tag{24}$$

where

$$\left.\begin{aligned}\hat{S}(x,y,z,\omega)&=c^2\rho_0^{\frac{1}{2}-\alpha}\omega^2\left[(\alpha\gamma-1)\frac{g}{c^2}+\frac{\partial}{\partial z}\right](\rho_0^{\alpha-1}f_z)+\\&+c^2\rho_0^{\frac{1}{2}-\alpha}(\omega^2-\omega_2^2)\left[\frac{\partial}{\partial x}(\rho_0^{\alpha-1}f_x)+\frac{\partial}{\partial y}(\rho_0^{\alpha-1}f_y)\right].\end{aligned}\right\} \tag{25}$$

We introduce the new coordinate system where the horizontal scale is contracted by the ratio $\omega/(\omega^2-\omega_2^2)^{\frac{1}{2}}$. That is, the new coordinate system $(x', y', z')$ introduced is related to the original coordinate system $(x, y, z)$ by

$$\left.\begin{aligned}\omega x&=(\omega^2-\omega_2^2)^{\frac{1}{2}}x'\\\omega y&=(\omega^2-\omega_2^2)^{\frac{1}{2}}y'\\z&=z'\end{aligned}\right\} \tag{26}$$

Then, equation (24) is reduced to

$$\left[(\omega_1^2-\omega^2)-c^2\nabla'^2\right]\hat{p}'(\omega)=\hat{s}'(\omega)/\omega^2, \qquad (27)$$

where primes added to $\hat{p}(\omega)$ and $\hat{s}(\omega)$ show that these values are represented by the new coordinates. Hereafter, the quantities represented in the new coordinate system will be denoted with the use of the prime.

With the use of the retarded potential method, the solution of equation (27) which satisfies the radiation condition is written as

$$\hat{p}'(\omega)=\frac{1}{4\pi c^2}\int\frac{1}{r'}\exp\left[\mp i\,\frac{(\omega^2-\omega_1^2)^{1/2}}{c}\,r'\right]\frac{\hat{s}'(\omega)}{\omega^2}\,d\underset{\sim}{x}' \qquad (28)$$

for $\omega>\omega_1$, and $\omega<-\omega$, respectively, and

$$\hat{p}'(\omega)=\frac{1}{4\pi c^2}\int\frac{1}{r'}\exp\left[-\frac{(\omega_1^2-\omega^2)^{1/2}}{c}\,r'\right]\frac{\hat{s}'(\omega)}{\omega^2}\,d\underset{\sim}{x}' \qquad (29)$$

for $|\omega|<\omega_1$, where the integral is performed over the whole source region, and $r'$ denotes the distance between the observation point and a point $\underset{\sim}{x}'$ in the source region in the new coordinate system. In equation (29) the frequency range, $|\omega|<\omega_2$, is included. In this frequency range, however, a careful interpretation of this equation is necessary because the transformation (26) implies a transformation into an imaginary coordinate system.

The transformation (26) corresponds to the following one in the polar coordinate systems

$$r'^2 = r^2(\omega^2 - \omega_2^2\cos^2\theta)/(\omega^2 - \omega_2^2) \tag{30}$$

$$\cos^2\theta' = \left[(\omega^2 - \omega_2^2)/(\omega^2 - \omega_2^2\cos^2\theta)\right]\cos^2\theta \tag{31}$$

and

$$\varphi' = \varphi \tag{32}$$

where $\theta$ and $\theta'$ are the angular distances from $z$ and $z'$ axis (the same direction) in each coordinate system. As shown in equation (30), the $r'^2$ becomes negative in the frequency range of $\omega_2^2 > \omega^2 > \omega_2^2\cos^2\theta$, (this frequency range corresponds to that of gravity waves). Therefore, the solution in the frequency range of gravity waves can be expressed by inserting $r' = \pm i\,|r'|$ (the sign $\pm$ is for $\omega \gtrless 0$) into equation (29). That is, we have

$$\hat{p}'(\omega) = \frac{1}{4\pi c^2}\int \frac{\mp i}{|r'|}\exp\left[\mp i\,\frac{(\omega_1^2 - \omega^2)^{1/2}}{c}|r'|\right]\frac{\hat{s}'(\omega)}{\omega^2}\,d\underline{x}' \tag{33}$$

Consequently, the exact solution of the equation in the original coordinate system is

$$\hat{p}(\omega) = \frac{1}{4\pi c^2}\frac{A_1}{|\omega^2 - \omega_2^2|^{1/2}}\int \frac{\hat{s}(\omega)}{r(\omega^2 - \omega_2^2\cos^2\theta)^{1/2}}\exp\left[\mp i\,\frac{|\omega^2 - \omega_1^2|^{1/2}(\omega^2 - \omega_2^2\cos^2\theta)^{1/2}}{|\omega^2 - \omega_2^2|^{1/2}}r\right]d\underline{x} \tag{34}$$

for frequency ranges both of acoustic waves ($\omega^2 > \omega_1^2$) and the sign $\pm$ is for $\omega \gtrless 0$ ) and of gravity waves ($\omega_2^2 > \omega^2 > \omega_2^2\cos^2\theta$ and the sign $\pm$ is for $\omega \gtrless 0$ ). The value of $A_1$ is 1 and $\pm\ i$ (for $\omega \gtrless 0$ ) for acoustic waves and gravity waves respectively. The solution in the trapped frequency ranges $(\omega_1^2 > \omega^2 > \omega_2^2$ and $\omega^2 < \omega_2^2 \cos^2\theta)$ is

$$\hat{p}(\omega) = \frac{1}{4\pi c^2}\frac{A_2}{|\omega^2-\omega_2^2|^{1/2}}\int\frac{\hat{s}(\omega)}{r|\omega^2-\omega_2^2\cos^2\theta|^{1/2}}\exp\left[-\frac{(\omega_1^2-\omega^2)^{1/2}}{c}\frac{|\omega^2-\omega_2^2\omega'\theta|^{1/2}}{|\omega^2-\omega_2^2|^{1/2}}r\right]d\chi \quad (35)$$

where $A_2$ is $\pm$ for $\omega_1^2 > \omega^2 > \omega_2^2$ and for $\omega^2 < \omega_2^2\cos^2\theta$, respectively.

With the use of Lighthill's method, Moore and Spiegel (1964) obtained the asymptotic solution of equation (34) at great distances from the source. We can show that far from the source the exact solution (34) coincides with their asymptotic solution (Appendix A).

## IV. The Response of the Atmosphere to Point Sources of Force.

Some general results about the response of the atmosphere are already shown in equations (34) and (35). The most prominent phenomena are the facts that the response of the atmosphere is very effective to the direction $\cos^2\theta \sim \omega^2/\omega'^2$ in the case of $\omega^2 < \omega_2^2$, and to the vertical direction in the case of $\omega^2 \sim \omega_2^2$. Although the former one was already pointed out by Moore and Spiegel (1964), it is unclear how much of this phenomenon is important in actual media by the reason discussed in the last paragraph of subsection (b) of this section. The main purpose of this section is to stress interesting characteristics of the response of the atmosphere to disturbances whose frequencies are near the lower characteristic frequency $\omega_2$.

To facilitate demonstrating these characteristics, we will study effects of the following two simplified asixymmetric point disturbances. A vertical linear oscillator:

$$\rho_0^{\alpha-1} f_i = A_V \exp(i\omega t)\,\delta(\underset{\sim}{x})\,\delta_{i3},\ (i = 1, 2 \text{ and } 3) \tag{36}$$

and a circular oscillator in the horizontal plane:

$$\rho_0^{\alpha-1} f_i = A_{H,i} \exp(i\omega t)\,\delta(\underset{\sim}{x}),\ (i=1, 2, \text{ and } 3) \tag{37}$$

where $A_{H,3} = 0$ and $A_{H,1}$ and $A_{H,2}$ have a same amplitude.

$$|A_H|^2 \equiv |A_{H,1}|^2 = |A_{H,2}|^2 \tag{38}$$

but their phases differ by $\pi/2$. In equations (36), (37) and (38), suffixes 1, 2 and 3 represent $x$, $y$ and $z$ respectively.

In the following discussion, we shall prefer to use the new coordinate system defined by equation (26) as far as possible because whole calculations can then be done more easily. The point sources given by equations (36) and (37) can be written in the new coordinate system as

$$(\rho_0^{\alpha-1} f_i)' = \frac{\omega^2}{\omega^2 - \omega_2^2} A_V \exp(i\omega t)\,\delta(\underset{\sim}{x}')\,\delta_{i3},\ (i=1, 2 \text{ and } 3) \tag{39}$$

and

$$(\rho_0^{\alpha-1} f_i)' = \frac{\omega^2}{\omega^2 - \omega_2^2} A_{H,i} \exp(i\omega t)\,\delta(\underline{x}'),\ (i=1, 2 \text{ and } 3) \tag{40}$$

where the prime added to $\rho_0^{\alpha-1} f_i$ means that $\rho_0^{\alpha-1} f_i$ is represented by the new coordinates. In addition, from equation

(25) we have

$$\left.\begin{aligned} \hat{s}'(\omega) &= c^2 \rho_0^{\frac{1}{2}-\alpha} \omega^2 \left[ (\alpha\gamma - 1) \frac{g}{c^2} + \frac{\partial}{\partial z'} \right] \left( \rho_0^{\alpha-1} f_z \right)' + \\ &+ c^2 \rho_0^{\frac{1}{2}-\alpha} \omega \left( \omega^2 - \omega_2^2 \right)^{\frac{1}{2}} \left[ \frac{\partial}{\partial x'} \left( \rho_0^{\alpha-1} f_x \right)' + \frac{\partial}{\partial y'} \left( \rho_0^{\alpha-1} f_y \right)' \right]. \end{aligned}\right\} \quad (41)$$

Inserting equation (39) and the first term in equation (41) into equations (28) or (33), and performing partial integrations, we have, for the frequency range of acoustic waves, $\omega^2 > \omega_1^2$, and of gravity waves $\omega_2^2 > \omega^2 > \omega_2^2 \cos^2\theta$,

$$\begin{aligned} \hat{p}_V'(\omega) = -\frac{1}{4\pi r'} \frac{\omega^2}{\omega^2 - \omega_2^2} \rho_{00}^{\frac{1}{2}-\alpha} A_V \exp\left[ \mp i \frac{|\omega^2 - \omega_1^2|^{\frac{1}{2}}}{c} |r'| \right] \\ \times \left[ \frac{(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}}{c} + \left( \frac{1}{|r|} \pm i \frac{|\omega^2 - \omega_1^2|}{c} \right) \cos|\theta'| \right] \end{aligned} \quad (42)$$

where the sign $\pm$ is for $\omega \gtrless 0$, the $\rho_{00}$ denotes the undisturbed density at the position of the source, and, for convenience, $\cos|\theta'|$ denotes $\left[ |\omega^2 - \omega_2^2|^{\frac{1}{2}} / |\omega^2 - \omega_2^2 \cos^2\theta|^{\frac{1}{2}} \right] \cos\theta$. Quite similarly, we have

$$\begin{aligned} \hat{p}_H'(\omega) = \mp \frac{1}{4\pi r'} \frac{\omega |\omega^2 - \omega_2^2|^{\frac{1}{2}}}{\omega^2 - \omega_2^2} \rho_{00}^{\frac{1}{2}-\alpha} \left[ A_{H,1} \cos\varphi' + A_{H,2} \sin\varphi' \right] \\ \times \exp\left[ \mp i \frac{|\omega^2 - \omega_1^2|^{\frac{1}{2}}}{c} |r'| \right] \cdot \left[ + \frac{1}{|r|} \pm i \frac{|\omega^2 - \omega_1^2|^{\frac{1}{2}}}{c} \right] \sin\theta'. \end{aligned} \quad (43)$$

In equations (42) and (43), $\hat{p}_V(\omega)$ and $\hat{p}_H(\omega)$ represent $\hat{p}'(\omega)$ due to point disturbing sources represented by equations (39) and (40) respectively. Using similar procedures, we have, for the frequency range of oscillations, $\omega_2^2 < \omega^2 < \omega_1^2$ and

$\omega^2 < \omega_2^2 \cos^2\theta$,

$$\left.\begin{aligned}\hat{p}_r'(\omega) &= -\frac{1}{4\pi r'}\frac{\omega^2}{\omega^2-\omega_2^2}\rho_{00}^{1/2-\alpha}A_r \exp\left[-\frac{(\omega_1^2-\omega^2)^{1/2}}{c}r'\right] \\ &\times\left[\frac{(\omega_1^2-\omega_2^2)^{1/2}}{c}+\left(\frac{1}{r'}+\frac{(\omega_1^2-\omega^2)^{1/2}}{c}\right)\cos\theta'\right]\end{aligned}\right\} \quad (44)$$

and

$$\left.\begin{aligned}\hat{p}_H'(\omega) &= -\frac{1}{4\pi r'}\frac{\omega|\omega^2-\omega_2^2|^{1/2}}{\omega^2-\omega_2^2}\rho_{00}^{1/2-\alpha}\left[A_{H,1}\cos\varphi'+A_{H,2}\sin\varphi'\right] \\ &\times\exp\left[-\frac{(\omega_1^2-\omega^2)^{1/2}}{c}r'\right]\left[\frac{1}{r'}+\frac{(\omega_1^2-\omega^2)^{1/2}}{c}\right]\sin|\theta'| ,\end{aligned}\right\} \quad (45)$$

where $\sin|\theta'|$ denotes $\left[\omega/|\omega^2-\omega_2^2\cos^2\theta|^{1/2}\right]\sin\theta$.

a) The Frequency Ranges of Oscillations
$(\omega_2^2 < \omega^2 < \omega_1^2,\ \omega^2 < \omega_2^2\cos^2\theta)$.

We will use the kinetic energy density as a measure of response of the atmosphere, as mentioned before. A Fourier component of the kinetic energy density, equation (20), can be written with the use of the new coordinates as

$$\left.\begin{aligned}E_K'(\omega) &= \frac{1}{|\omega^2-\omega_2^2|}\left(\left|\frac{\partial\hat{p}'(\omega)}{\partial x'}\right|^2+\left|\frac{\partial\hat{p}'(\omega)}{\partial y'}\right|^2\right)+ \\ &+\frac{\omega^2}{(\omega^2-\omega_2^2)^2}\left|\left[\frac{\partial}{\partial z'}+\frac{(\omega_1^2-\omega_2^2)^{1/2}}{c}\right]\hat{p}'(\omega)\right|^2\end{aligned}\right\} \quad (46)$$

Therefore, after inserting equations (44) and (45) into equation (46), we have

$$E_{Kr,r}'(r,\theta;\omega) = F'\frac{\omega^6}{|\omega^2-\omega_2^2|^4}|A_r|^2 G_{r,r}'(r',\cos^2\theta';\omega;\omega_1^2,\omega_2^2) \quad (47)$$

$$E_{KH,r}'(r,\theta;\omega) = F'\frac{\omega^4}{|\omega^2-\omega_2^2|^3}|A_r|^2 G_{H,r}'(r',\cos\theta';\omega;\omega_1^2,\omega_2^2)\sin^2|\theta'| \quad (48)$$

$$E'_{KV,H}(r,\theta;\omega) = F' \frac{\omega^4}{|\omega^2-\omega_2^2|^2} |A_H|^2 G'_{V,H}(r',\cos\theta';\omega;\omega_1^2,\omega_2^2)\sin^2|\theta'| \tag{49}$$

and

$$E'_{KH,H}(r,\theta;\omega) = F' \frac{\omega^2}{|\omega^2-\omega_2^2|^2} |A_H|^2 G'_{H,H}(r',\cos\theta';\omega;\omega_1^2,\omega_2^2) \tag{50}$$

where

$$F'(r';\omega;\omega_1^2) = \frac{\rho_{00}^{1-2\alpha}}{16\pi^2 r'^2} \exp\left[-2\frac{(\omega_1^2-\omega^2)^{1/2}}{c} r'\right] \tag{51}$$

and $G'_{V,V}$, $G'_{H,V}$, $G'_{V,H}$ and $G_{H,H}$ are similar terms to each other. For example, $G'_{V,V}$ has the following form:

$$G'_{V,V}(r';\cos^2\theta';\omega;\omega_1^2,\omega_2^2) =$$

$$= \left(\frac{\omega_1^2-\omega_2^2}{c^2} - \left\{\left[\frac{1}{r'} + \frac{(\omega_1^2-\omega_2^2)^{1/2}}{c}\right]^2 + \frac{1}{r'^2}\right\}\cos^2\theta + \frac{1}{r'}\left[\frac{1}{r'} + \frac{(\omega_1^2-\omega^2)^{1/2}}{c}\right]\sin^2|\theta'|\right)^2$$

Moreover, $E'_{KH,V}(r,\theta;\omega)$ for example, denotes a frequency component of the kinetic energy density by the horizontal velocity field excited by the vertical linear oscillator defined by equation (36). Other notations have similar meanings. To obtain the angular distribution and intensity of the kinetic energy density in the original space, the transformations (30) to (32) should be inserted into equations (47) to (50).

If the frequency of the source is near the lower characteristic frequency of the atmosphere, $\omega_2$, there is an interesting phenomenon. That is, $E_{KV,V}(r,\theta;\omega)$ has very large values in the restricted region (the region vertically below or above the source). The values or their angular distributions of other $E'$s have not so interesting characteristics compared with those of $E_{KV,V}$. In other words, near the lower characteristic frequency the atmosphere responds very strongly to a vertically oscillating source, and the respondent region is sharply restricted in the vertical region from the source, and the velocity field in that region consists mainly of a vertical one. These results come partly from the form of equation of the kinetic energy density (equation (20)), and partly from the characteristics of the transformation of coordinates (equation (26)) required to cast the inhomogeneous wave equation into the standard form.

To demonstrate sharpness in the respondent region of the atmosphere to vertically oscillating sources whose frequencies are just above the lower characteristic frequency of the atmosphere, curves of iso-value of $E_{KV,V}(r,\theta;\omega)$ in the original space, are drawn in Figures 1a and 1b. In Figure 1a, the square value of the frequency of the disturbance, $\omega^2$, and that of the lower characteristic frequency, $\omega_2^2$, are 0.6 $\omega_1^2$ and 0.556 $\omega_1^2$ (this corresponds to $\gamma$ = 6/5), respectively. In Figure 1b, $\omega^2 = \omega_1^2$ and $\omega_2^2$ = 0.96 $\omega_1^2$ (this corresponds to $\gamma$ = 5/3). Numerical values in these figures denote those of $E_{KV,V}/[\rho_{00}^{1-2\alpha}|A_r|^2/16\pi^2 H^6\omega_1^2]$. The zero intensity surface cuts the $z$-axis at a great distance from the source (if $\omega^2 < \omega_2^2$, the surface does not cut the $z$-axis), and in the inner region of this surface the vertical velocity of the atmospheric oscillation is in the same phase with that of the source and in the outer region of this surface it is in the anti-phase with that of the source.

To demonstrate that sharpness of the respondent region increases with decreasing of frequency difference between the frequency of the source and the lower characteristic frequency, the angular dependencies of $E_{k\nabla,\nabla}(r, \theta ; \omega)$ at the distance $2H$ from the source are drawn in Figures 2a and 2b for the cases of $\omega^2 = \omega_1^2$ and of $\omega^2 = 0.9\,\omega_1^2$, respectively. Three curves in Figure 2a are for $\gamma = 4/3$ $(\omega_2^2 = 0.75\,\omega_1^2)$, $\gamma = 3/2$ $(\omega_2^2 = 0.89\,\omega_1^2)$ and $\gamma = 5/3$ $(\omega_2^2 = 0.96\,\omega_1^2)$ in order moving inward. Two curves in Figure 2b are for $\gamma = 4/3$ and $\gamma = 3/2$ in order moving inward. In these curves, values at the vertical direction from the sources are normalized to a same value.

b) The Frequency Ranges of Waves $(\omega^2 > \omega_1^2,\ \omega_2^2 > \omega^2 > \omega_2^2 \cos^2\theta)$.

Generation of waves from a vertically oscillating point source (see equation (36)), was studied by Moore and Spiegel (1964), using the asymptotic solution at great distances from the source. However, they overlooked interesting effects of the degree of the atmosphere upon generation. The purpose of this subsection is to emphasize this point. As a measure of response of the atmosphere, we use different ones (energy density or energy flux) from that used by Moore and Spiegel, because the former will be more fundamental.

If the degree of stability of the atmosphere increases, the lower characteristic frequency approaches to the upper characteristic frequency. So, in that case, we can expect that phenomena obtained near the lower characteristic frequency (see the previous subsection) will remain even in the frequency range of acoustic waves. For example, in the case of $\gamma = 5/3 \; (\omega_2^2 = 0.96\,\omega_1^2)$ we obtained the curves of iso-value of $E_{KT_1T}(\omega)$ at $\omega = \omega_1$ in Figure 1b. It is obvious that the characteristics shown in this figure (strong response and sharpness of the respondent region) are conserved continuously to higher frequency range, although these characteristics are weakened gradually with increasing frequency. So, we will mainly study the effects of the degree of stability of the atmosphere upon another interesting measure of response of the atmosphere in the frequency ranges of waves; namely, energy flux radiated from disturbing sources.

To avoid unnecessary complications, energy flux at great distances from a source will be calculated. Using equations (15), (16), (17), (42) and (43), we can calculate a Fourier component of the velocity field. Consequently, the energy flux defined by equation (22) can be expressed in the original coordinate as

$$\mathfrak{F}_T(r,\theta;\omega) = \frac{1}{8\pi^3 r^2}\,\frac{\omega^2}{c^3}\,\rho_{00}^{1-2\alpha}\,|A_T|^2\,\frac{\omega^3|\omega^2-\omega_1^2|^{1/2}}{|\omega^2-\omega_2^2|^{3/2}(\omega^2-\omega_2^2\cos^2\theta)^{3/2}}$$
$$\times\left[(\omega_1^2-\omega_2^2)+\frac{|\omega^2-\omega_2^2||\omega^2-\omega_1^2|}{(\omega^2-\omega_2^2\cos^2\theta)}\cos^2\theta\right]\frac{\mathbf{r}}{r} \tag{53}$$

and

$$\mathfrak{F}_H(r,\theta;\omega) = \frac{1}{8\pi^3 r^2}\,\frac{\omega^2}{c^3}\,\rho_{00}^{1-2\alpha}\,|A_H|^2\,\frac{\omega^3|\omega^2-\omega_1^2|^{3/2}}{|\omega^2-\omega_2^2|^{1/2}(\omega^2-\omega_2^2\cos^2\theta)^{5/2}}\sin^2\theta\cdot\frac{\mathbf{r}}{r} \tag{54}$$

for two kinds of disturbing sources defined by equations (36) and (37). At a great distance from the source, energy is radiated radially in a dependency of $1/r^2$ as generally expected. Moreover, far from the source we can show after a calculation that the energy density, $E(\omega)(=2E_k(\omega))$, energy flux, $\underset{\sim}{q}(\omega)$, and group velocity, $\underset{\sim}{U}(\omega)$ are related by

$$E(\omega)\,\underset{\sim}{U}(\omega) = \underset{\sim}{q}(\omega) \tag{55}$$

as generally expected, where the magnitude of the group velocity is given by (Moore and Spiegel 1964)

$$U(\omega,\theta) = \frac{|\omega^2-\omega_2^2|^{3/2}\,|\omega^2-\omega_1^2|^{1/2}\left(\omega^2-\omega_2^2\cos^2\theta\right)^{1/2}}{\omega\left[(\omega^2-\omega_2^2)^2+\omega_2^2(\omega_1^2-\omega_2^2)\cos^2\theta\right]}\,c\,. \tag{56}$$

The total energy radiated by unit time, $F(\omega)$, defined by

$$F(\omega) = 2\pi\int_0^{\pi} q(r,\theta;\omega)\,r^2\sin\theta\,d\theta \tag{57}$$

can be easily calculated from equations (53) and (54) for acoustic waves. The results are

$$F_T(\omega) = \frac{1}{6\pi}\frac{\omega^2}{c^3}\rho_{00}^{1-2\alpha}|A_T|^2\left[\frac{\omega(\omega^2-\omega_1^2)^{1/2}}{(\omega^2-\omega_2^2)^2}\left\{3(\omega_1^2-\omega_2^2)+(\omega^2-\omega_1^2)\right\}\right] \tag{58}$$

and

$$F_H(\omega) = \frac{1}{3\pi}\frac{\omega^2}{c^3}\rho_{00}^{1-2\alpha}|A_H|^2\left[\frac{(\omega_2^2-\omega_1^2)^{3/2}}{\omega(\omega^2-\omega_2^2)}\right]. \tag{59}$$

We do not calculate the total energy radiated by gravity waves because their intensity becomes infinite at the critical angle

$\theta_c\left(\cos^2\theta_c = \omega^2/\omega_2^2\right)$. In Figures 3a and 3b, values in the large bracket of equations (58) and (59) times the value of $\omega^2/\omega_1^2$ are drawn as a function of frequency for some cases of the degree of stability of the atmosphere. The vertically oscillating disturbances whose frequencies are very near the upper characteristic frequency are good generators of waves if the atmosphere becomes sufficiently stable to convection (see Figure 3a). This excess energy is mainly radiated to the vertical difection from the source (see equation (53)).

Finally, we should mention about the critical direction $\left(\cos\theta_c^2 = \omega^2/\omega_2^2\right)$ for radiation of gravity waves. At this direction energy flux radiated by gravity waves becomes infinite (equations (53) and (54) , originally shown in equations (34) and (35)). Moore and Spiegel (1964) suggested this fact may be important in delivering energy to the outer region of the sun. However, a careful discussion whether this is actually important is necessary because the wave number of waves radiated to the critical direction is infinite. So, the exponential damping due to dissipations becomes dominant for waves radiated near the critical direction even if the coefficient of dissipation is small. The fact that the wave number of waves radiated to the critical direction is infinite and its direction is perpendicular to the critical direction (the directions of phase velocity and group velocity are perpendicular in each other) may be also seen from the dispersion equation.

# Part II. The Response of the Atmosphere to Impulsive Disturbances.

## V. Lighthill's Method

It will be useful to study quantitatively the response of the atmosphere to impulsive disturbances to demonstrate effects of the degree of stability of the atmosphere, although we can roughly anticipate results because we have studied the response of the atmosphere to time-harmonic disturbances.

In our own case, an impulsive disturbance can be written as

$$\rho_0^{\alpha-1} f_i = A_i \delta(\underset{\sim}{x}) \delta(t) , \quad (i = 1, 2 \text{ and } 3) \tag{60}$$

and the suffixes 1, 2 and 3 denote $x$ , $y$ and $z$ , respectively. In Part I we have studied about effects of two kinds of disturbances whose directions are vertical and horizontal separately and have known the difference between effects of these two types of disturbances. So, in Part II we will adopt a hypothetical pulse whose direction is statistically isotropic. That is, the amplitudes of these $A_i$ are the same

$$|A|^2 \equiv |A_1|^2 = |A_2|^2 = |A_3|^2 , \tag{61}$$

but the phases of them are uncorrelated. Adopting this kind of disturbance, we can concentrate our attention to effects of the degree of stability of the atmosphere upon the response of the atmosphere to disturbances. For, the response of the atmosphere to this kind of disturbance will be spherically symmetric if the atmosphere is convectively neutral $(\gamma = 1)$. As a measure of the response we will use mainly the kinetic energy density, as in Part I.

We can write, in principle, the Green function of the fundamental wave equation (7) in an integral form, because we have obtained the exact solution for the time-harmonic case in Part I. The form, however, is complicated and not practical. We will be satisfied with the asymptotic solution obtained with the use of Lighthill's method (1960). Lighthill studied the asymptotic solution of inhomogeneous wave equations in an isotropic media mainly in the case of time-harmonic disturbing sources. In an appendix of his paper, Lighthill mentioned the asymptotic solution in the following non-time harmonic case:

$$P\left(\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}, \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left(p_1/\rho_0^{1/2}\right) = S(x, y, z, t) \tag{62}$$

under the condition that dependent variable, $p_1/\rho_0^{1/2}$, is everywhere zero until an initial instant, $t = 0$, after which the source, $S$, begins to operate. In equation (62), $P$ is a polynomial and $S$ is a function vanishing outside a restricted region.

We will introduce a four-dimensional space in which a point is represented by $(\underset{\sim}{r}, ct)$ and also the corresponding four-dimensional wave vector space $(\underset{\sim}{k}, \omega/c)$. Thus, the Fourier transform of $S$ is

$$\Phi(\underset{\sim}{k}, \omega/c) = \frac{1}{(2\pi)^4}\iiiint S(x, y, z, t) \exp\left[-i(\underset{\sim}{k}\underset{\sim}{r} + \omega t)\right] d\underset{\sim}{r}\, d(ct) \tag{63}$$

and the dispersion relation can be represented by the hyper-surface $G=0$ in the four-dimensional wave vector space, where

$$G = P(-k_x^2, -k_y^2, -k_z^2, -\omega^2/c^2). \tag{64}$$

Then, using the approximation of stationary phase as $r \to \infty$ and $t \to \infty$ where $r$ is the distance between the observation point and the source, Lighthill obtained the following asymptotic solution of equation (62):

$$p/\rho_0^{1/2} = \frac{(2\pi)^{3/2}}{\hat{r}^{3/2}} \sum \frac{C_0 \Phi \exp[i(\underset{\sim}{k}\,\underset{\sim}{r} + \omega t)]}{|\Box G| \cdot |K|^{1/2}} \tag{65}$$

where $\hat{r} = (r^2 + c^2t^2)^{1/2}$ . The sum $\sum$ is over all points on the hyper-surface $G=0$ whose normal is parallel to the four-dimensional observation point $(\underset{\sim}{r}, ct)$ : that is, over all points where

$$\underset{\sim}{r} : ct = \nabla G : c\,\partial G/\partial\omega \,. \tag{66}$$

Moreover, in equation (65), $C_0$ is a phase factor of modulus 1; $\Box G_T$ is the gradient of $G$ in Cartesian four-space; and $K$ is the Gaussian curvature of the hyper-surface $G$ - each element of the matrix of products of first derivatives of $G$ being multiplied into the corresponding cofactor of the matrix of second derivatives, and divided by $|\Box G|^5$.

## VI. The Fronts of Propagating Disturbances

Without the help of the asymptotic solution in the inhomogeneous wave equation, the concept of the group velocity makes the calculation of the fronts of propagating disturbances possible. In general, energy propagates from sources with the group velocity (e.g. Lighthill 1960; Whilham 1961). The group velocity of the wave whose frequency is $\omega$ is given by (c.f. equation (66))

$$\underset{\sim}{U}(\theta, \omega) = \nabla G \Big/ \frac{\partial G}{\partial \omega} . \tag{67}$$

The explicit expression of the group velocity in our case is given in equation (56).

A pulse has practically all frequency components. So, the fronts of disturbances propagating from a pulse can be considered as the envelopes of the wave fronts resulting from waves of all frequencies. In the atmosphere which we are treating here, two kinds of wave modes are possible: the acoustic wave mode and the gravity wave mode. Thus, we can also expect two kinds of fronts of disturbances propagating from a pulse.

A polar diagram of magnitude of the group velocity for acoustic waves is drawn in Figure 4 for the cases of $\gamma = 1$, 4/3 and 5/3. Similar curves are drawn in Figure 5 for gravity waves for the cases of $\gamma = 4/3$ (the left part of the figure) and $\gamma = 5/3$ (the right part of the figure). The curves in Figures 4 and 5 are drawn so that the magnitude of the adiabatic sound velocity, $(\gamma R T_0)^{1/2}$, becomes a same unit even for different cases of $\gamma$. In other words, figures whould be expanded by the ratio of 1, $(4/5)^{\frac{1}{2}}$ and $(5/3)^{\frac{1}{2}}$, to obtain the actual ratio of the group velocities for the cases of $\gamma = 1$, 4/3 and 5/3.

Some characteristics of disturbances propagating from a pulse are shown in Figures 4 and 5. One front propagates spherically with just the sound velocity $(\gamma R T_0)^{1/2}$, from the source. This front consists of the infinite frequency component in the pulse. After the passage of the front there remains a wake whose frequency tends rapidly to the upper characteristic frequency of the atmosphere. This is qualitatively same with the results in the one-dimensional case treated by Lamb (1908). The other front is that determined by the envelope of disturbances propagating from a pulse as gravity waves, as shown in Figure 5. After the passage of the front, the atmospheric motion consists of wave motions with two different frequencies. This second type of front, of course, does not appear in Lamb's one-dimensional case.

VII. Intensity and Angular Dependency of Propagating Disturbances.

We put in Appendix B the actual expressions of each of the terms in equation (65). Inserting the expression of equation (65) into equation (26) under the assumption of great distances from the source, we have finally the kinetic energy density at the observation point $(\mathbf{r}, ct)$ as

$$E_K(\mathbf{r}, t) = \frac{\rho_{00}^{1-2\alpha}}{4(2\pi)^3 t^3} \frac{|A|^2}{c^2} \sum \frac{\omega^3 |\omega^2 - \omega_1^2|}{(\omega^2 - \omega_2^2 \cos^2\theta)^2} \frac{1}{V^2 \, \partial V^2 / \partial \omega^2}, \qquad (68)$$

where $t$ is the time interval after the operation of the impulsive force $V(\theta, \omega)$ is the group velocity (see equation (56)), and the sum should be performed over all positive values of $\omega$ which are related to the observation point $(\underset{\sim}{r}, ct)$ by

$$r^2/c^2t^2 = V^2(\theta, \omega)/c^2 \tag{69}$$

If $r^2 > c^2t^2$, no point contributes to the sum in equation (68). That is, no disturbance can reach the observation point. If $r^2 < c^2t^2$ on the other hand, one or three frequencies contribute to the sum (see Figures 4 and 5). One is that behind the front resulting from propagation of acoustic waves, and the other two are those behind the front resulting from propagation of gravity waves.

Quite similarly, using the expression (23) we can obtain the mean square value of $\rho_1/\rho_0^{1/2}$, namely $\langle(\rho_1/\rho_0^{1/2})^2\rangle$, in a form similar with equation (68). The ratio of the corresponding terms, $R^2(\omega, \theta)$, in the expression of $\langle(\rho_1/\rho_0^{1/2})^2\rangle$ and of $E_K$ is given by

$$R^2(\omega,\theta) = \frac{1}{c^2}\;\frac{(\omega^2-\omega_2^2\cos^2\theta)(\omega^2-\frac{\gamma}{2}\omega_2^2)^2+(\gamma-1)\omega_2^2(\omega_2-\omega_1^2)(\omega^2-\omega_2^2)\cos^2\theta}{\omega^2\left[(\omega^2-\omega_2^2)^2+\omega_2^2(\omega_1^2-\omega_2^2)\sin^2\theta\right]} \tag{70}$$

Finally, it should be mentioned that the attenuation of intensity like $t^{-3}$ or $r^{-3}$ ( $r^{-n}$ in the $n$-dimensional space) along any radius vector is characteristic of pulses propagated three-dimensionally outward in a dispersive medium. For a pulse contains a range of frequencies, so disturbances propagate as a volume and their volume increases as $r^3$ (e.g., Lighthill 1960).

a) Disturbances behind the Front resulting from Acoustic Waves.

Equation (68) can be written as

$$E_K(\underset{\sim}{r},t) = \frac{\rho_{00}^{1-2\alpha}}{4(2\pi)^3 t^3} |A|^2 \frac{\omega_1^3}{C_{\gamma=1}^6} \sum \left[ \frac{\omega^3 |\omega^2 - \omega_1^2|}{\omega_1^3 (\omega^2 - \omega_2^2 \cos^2\theta)^2} \frac{\gamma^{-1} C_{\gamma=1}^4}{\upsilon^3 \partial \upsilon^2 / \partial \omega^2} \right] , \quad (71)$$

where $C_{\gamma=1}$ denotes the isothermal sound velocity. The large bracket in equation (71) is non-dimensional and its values behind the front resulting from acoustic waves (the sum in equation (71)) is done only for one frequency) are drawn in Figures 6a, 6b and 6c for the cases of $\gamma$ = 1, 4/3 and 5/3 respectively. In these figures the radii of the spheres representing the fronts are drawn in a same radius for the above three cases of $\gamma$ . At a time $t$ after the operation of a pulse, the radius of the sphere representing the front is $ct$ , so that the actual ratio of radii is 1: $(4/3)^{\frac{1}{2}}$:$(5/3)^{\frac{1}{2}}$ for the above three cases of $\gamma$ . We do not calculate intensity of disturbances near the front because the approximation of stationary phase used to obtain equation (65) becomes bad near the front. Another example of this situation is shown in Appendix C. That is, in the one-dimensional case, the intensity on the front is finite (Lamb 1908); however, it becomes infinite if Lighthill's method is applied.

The Figure 6a shows that intensity of disturbances propagating behind the front decreases monotonically and spherically if the atmosphere is convectively neutral ( $\gamma$ = 1). This fact is natural because a hypothetical isotropic pulse is operated in an atmosphere having no particular direction (convective neutrality). Figures 6b and 6c, however, show that behind the front strong disturbances propagate vertically from the impulsive source if the degree of stability of the atmosphere increases, as expected from results in Part I. On the $z$-axis, the bracket in equation (71) has a simpler expression:

$$\gamma^{-3}(1-\alpha^2)^{-5/2}(1-\beta^2+\alpha^2\beta^2)^{-2} \tag{72}$$

where $\alpha = r/r_{front} = r/ct$ and $\beta^2 = \omega_2^2/\omega_1^2$.

At the origin $(\alpha = 0)$, this value becomes 1, 6.75 and 131 for the cases of $\gamma$ = 1, 4/3 and 5/3 respectively, although our approximate solution is not valid near the source.

Intensity of the density fluctuation may be a more useful measure of the response of the atmosphere if we want to apply results upon the spicule structure in the solar upper chromosphere. The r.m.s. of $\rho_1/\rho_0^{1/2}$ is given by

$$\langle(\rho_1/\rho_0^{1/2})^2\rangle^{1/2} = \left(\frac{\rho_{00}^{1-2\alpha}}{4(2\pi)^2 t^3}|A|^2 \frac{\omega_1^3}{c_{\gamma=1}^{\gamma}}\right)^{1/2}$$
$$\times \sum\left[\left(\frac{\omega^3|\omega^2-\omega_1^2|}{\omega_1^3(\omega^2-\omega_2^2\cos^2\theta)^2}\,\frac{\gamma^{-1}C_{\gamma=1}^4}{V^2\partial V^2/\partial\omega^2}\right)^{1/2}\frac{c}{\gamma^{1/2}}R(\omega,\theta)\right]. \tag{73}$$

Behind the front resulting from acoustic waves, the non-dimensional quantity behind the symbol of sum in equation (73) is calculated and drawn in Figure 7 for the case of $\gamma$ = 5/3. The vertical phenomena and efficiency of response represented by the measure $\langle (\rho_1/\rho_0^{1/2})^2 \rangle^{1/2}$ are not so prominent compared with those represented by the measure $E_K$, because the measure corresponding to $E_K$ is not $\langle (\rho_1/\rho_0^{1/2})^2 \rangle^{1/2}$ but $\langle (\rho_1/\rho_0^{1/2})^2 \rangle$.

b) Disturbances behind the front resulting from gravity waves.

These are calculated only for the case of $\gamma = 5/3$. The intensity curves corresponding to those for acoustic waves drawn in Figure 6 are added in Figure 5. On the front $\partial v^2/\partial \omega^2$ becomes zero (because the front is the envelope of disturbances resulting from gravity waves of all frequency components) and intensity becomes infinite. This results from the fact that our approximate solution is not valid until the front is the same as in the case of acoustic waves mentioned in the previous subsection. The intensity shown in Figure 5 comes mainly from waves radiated near the critical angle. So, we may not be able to accept these values literally in actual cases by the reason mentioned at the last subsection of Section IV.

## VIII. Summary and Discussions

We have studied the response of an unbounded and isothermal atmosphere both to time-harmonic disturbances of force and to impulsive forces. We hope some essential characteristics about the response of stratified atmospheres to disturbances are involved even in this simplified model of the atmosphere. Our main purpose in this study is to know how the efficiency of atmospheric response depends upon the frequency of disturbing sources, to what kind of disturbances (the direction of disturbing forces) the atmosphere responds effectively, what the angular distribution of the response is, and what kind of effects the degree of stability of the atmosphere has. As measures of the response of the atmosphere, we have adopted mainly kinetic energy density and supplementary energy flux or density fluctuation.

In Part I, effects of time-harmonic disturbances of force have been studied. The exact solutions of the inhomogeneous wave equation have been obtained in the whole frequency range (frequency ranges of acoustic waves, oscillations and gravity waves). Further study has been made for two kinds of point sources: a vertical linear oscillator and a circular oscillator in the horizontal plane, separately. An interesting phenomenon appears in the frequency range of oscillations. That is, the atmosphere responds very strongly to disturbances whose frequencies are just above the lower characteristic frequency of the atmosphere, especially to vertically oscillating disturbances. The vector velocity field excited by the vertically oscillating point source consists mainly of a vertical velocity field and the region of the response is sharply restricted in the vertical region from the source (Figures 1a, b and 2a, b). These results near the lower characteristic frequency are preserved continuously to the frequency range of acoustic waves across the upper characteristic frequency if stability of the atmosphere is large, because the lower characteristic frequency is near the upper characteristic frequency in this case. Tendencies such as increasing efficiency of response and sharpness of the respondent region also appear upon the energy flux radiated from disturbances whose frequencies are very near the upper characteristic frequency if the atmosphere is sufficiently stable (Figures 3a,b).

In Part II, effects of impulsive forces have been studied. The asymptotic solution at a point of great distance from the source in four-dimensional space-time has been obtained with the use of Lighthill's method. Two kinds of fronts with wakes propagate from the source. One is the front resulting from propagation of acoustic waves, and the frequency of the velocity field in the wake tends rapidly to the upper characteristic frequency (Figure 4). The other is the front resulting from propagation of gravity waves, and the velocity field in the wake has double periods (Figure 5). Further study has been made for a hypothetical isotropic disturbance of force because we already knew effects of the direction of force in Part I. Intensity of kinetic energy density behind the front resulting from propagation of acoustic waves has been figured for some cases of $\gamma$ (Figures 6a, b and c). These figures show that intensity of the kinetic energy density behind the front increases and the region of the response is restricted to the vertical region from the source as the degree of stability of the atmosphere increases. As another measure of the response, the root mean square of $\rho_1/\rho_0^{1/2}$, namely $\langle(\rho_1/\rho_0^{1/2})^2\rangle^{1/2}$, has been calculated only for the case of $\gamma$ = 5/3 (Figure 7). Intensity of the kinetic energy density behind the front resulting from propagation of gravity waves has been calculated for the case of $\gamma$ = 5/3 and added in Figure 5.

An interesting phenomenon in a convectively stable atmosphere is an increasing response of the atmosphere to vertically oscillating disturbances whose frequencies are near the lower characteristic frequency of the atmosphere. In addition, the strong respondent region is sharply restricted in the vertical region from the source. Mathematically speaking, these results are related to the form of energy equation (20) and to the character of the transformation of coordinates (see equation (26)) required to cast the equation into the standard form. For an impulsive disturbance of force, we have a strong disturbance propagating vertically from the source if the atmosphere is sufficiently stable to convection. This reason is obvious, because behind the front resulting from acoustic waves the frequency of oscillation tends rapidly to the upper characteristic frequency (as in Lamb's one-dimensional case) and the lower characteristic frequency of the atmosphere is near the upper characteristic frequency if the atmosphere is sufficiently stable to convection.

The fact that our results were obtained for an isothermal atmosphere is a restriction in applying our results to phenomena in actual atmospheres. However, our results may be of interest in connection with the spicule structure in the transition layer between the solar corona and the chromosphere. There are some mechanisms proposed to the origin of the spicule structure; materials shot upward, wave (shock) path from below, and condensations due to thermal instability (e.g. Field 1963, although Field does not suggest explicitly this possibility in his paper). Here, we will omit the first possibility because the mechanism as to why materials are shot upward is unclear. In the shock hypothesis, the fact that the spicules are denser and cooler than the interspicule media can be explained by increasing radiation cooling with increasing density (Uchida 1961). In the condensation hypothesis, that is a direct result of thermal instability due to radiation cooling which increases with increasing density. Why are spicules vertical? In the former hypothesis waves are assumed to be confined along the line of force; in the latter hypothesis magnetic field is assumed to help condensations along the line of force because thermal conductivity (which acts in the direction to prevent condensations) is reduced in the direction perpendicular to the line of force.

Spicules distribute over the whole surface of the sun, although it is difficult to suppose that the magnetic field is nearly vertical everywhere on the surface of the sun. The strong response of the atmosphere in the vertical region from a source obtained in this paper may become triggers to make condensation due to thermal instability(sometimes with the help of magnetic field) which may not occur without such finite initial disturbances.

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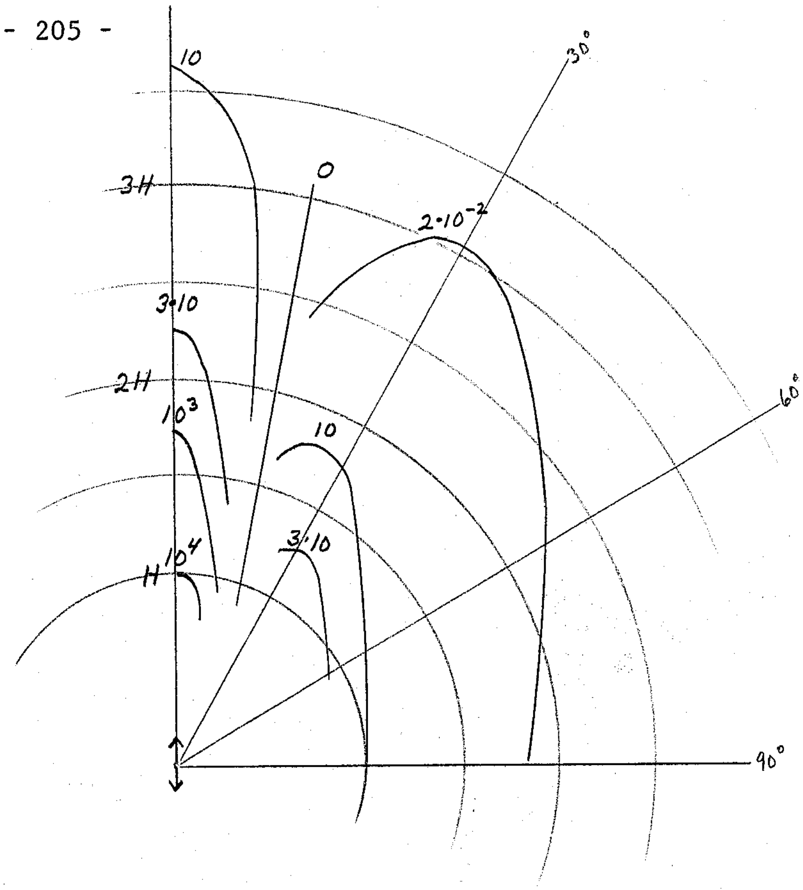



Fig. 1a

$\omega^2 = 0.6\,\omega_1^2$

$\omega_2^2 = 0.566\,\omega_1^2 (\gamma = 6/5)$

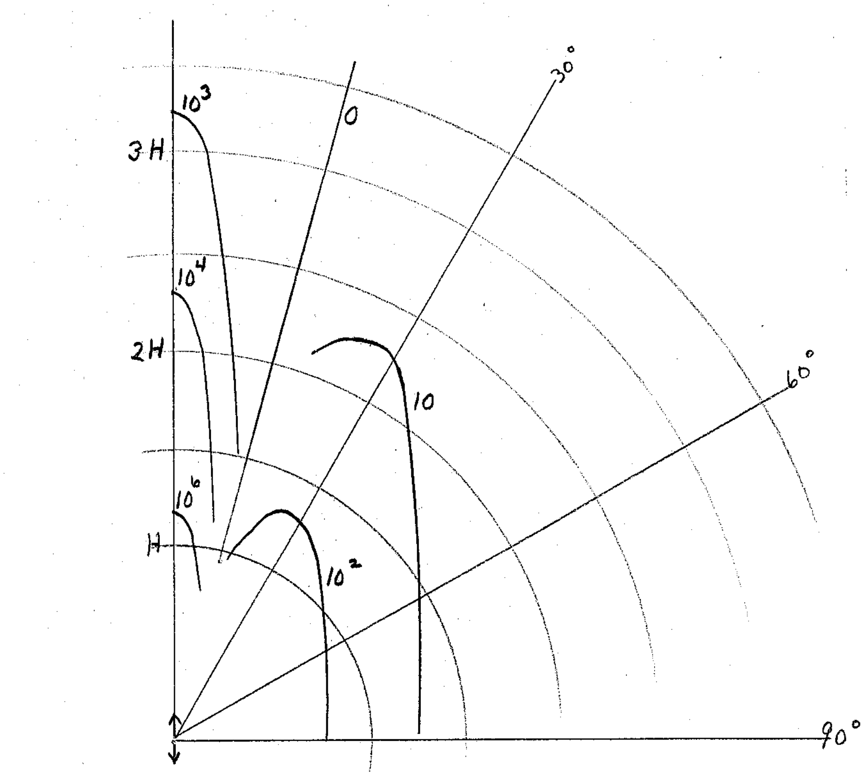



Fig. 1b

$\omega^2 = \omega_1^2$

$\omega_2^2 = 0.96\,\omega_1^2 \ (\gamma = 5/3)$

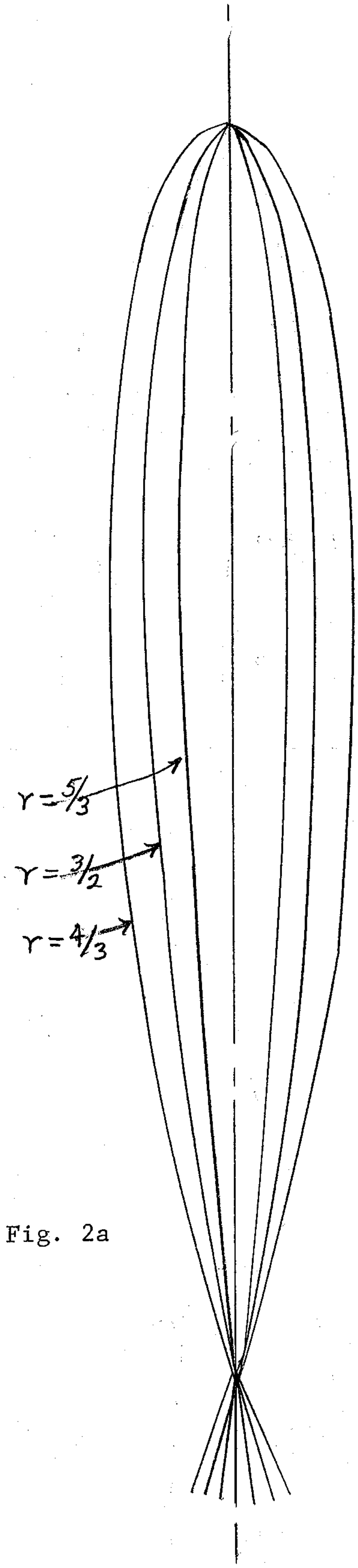



Fig. 2a

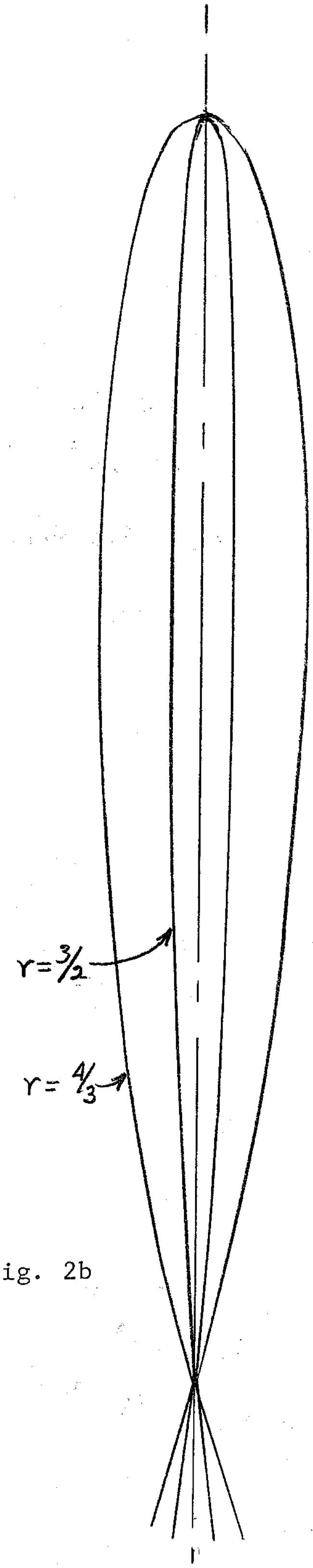



Fig. 2b

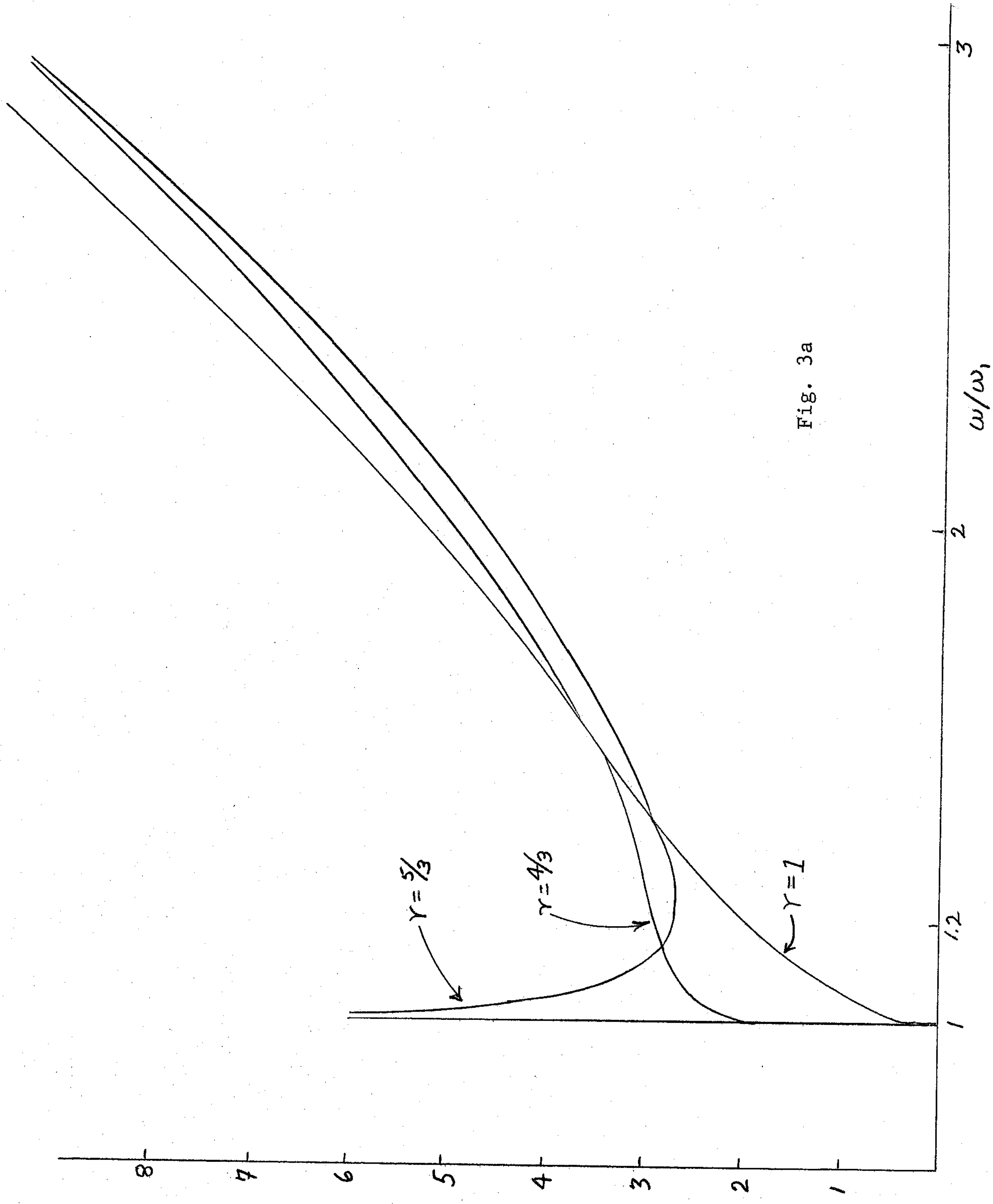



Fig. 3a

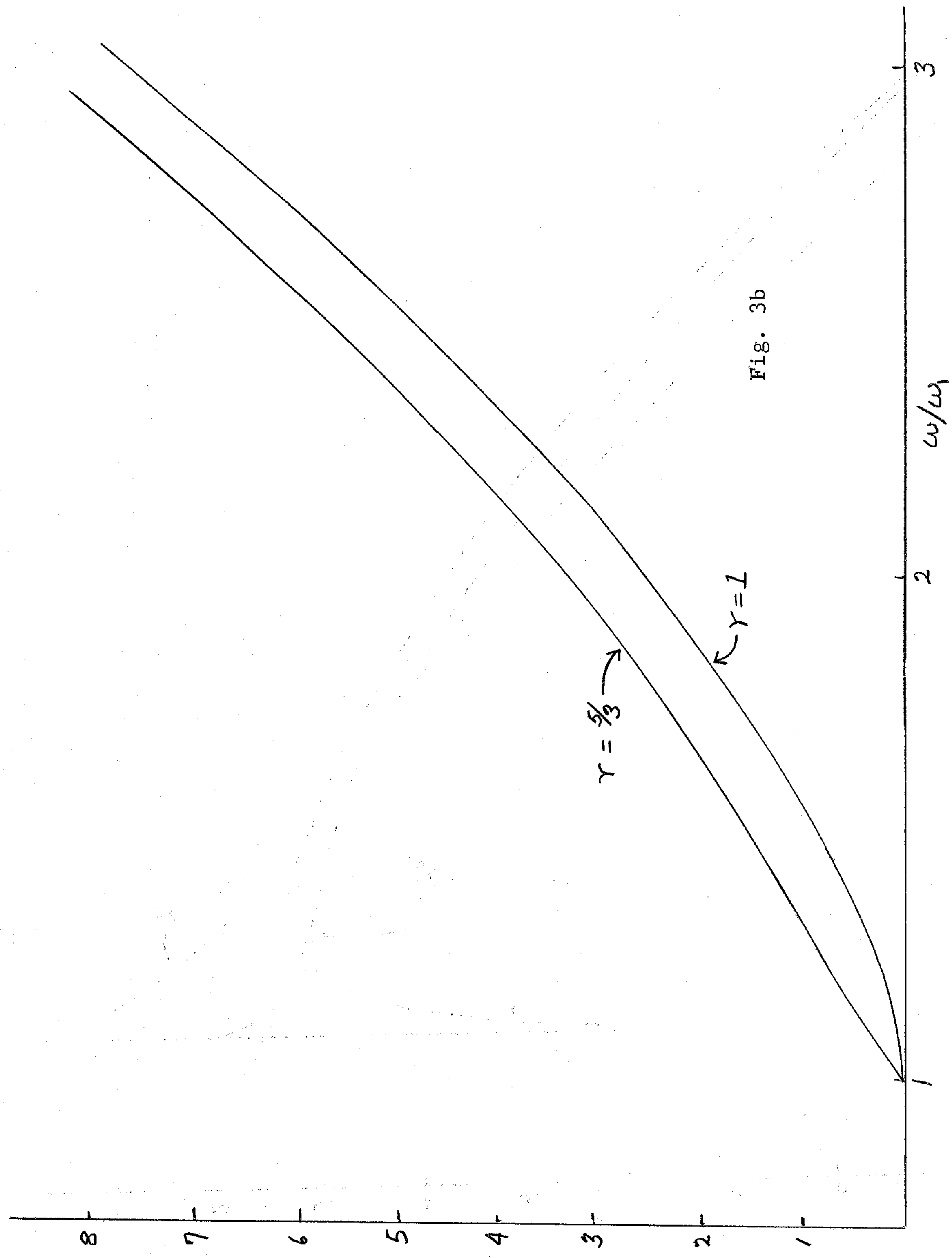



Fig. 3b

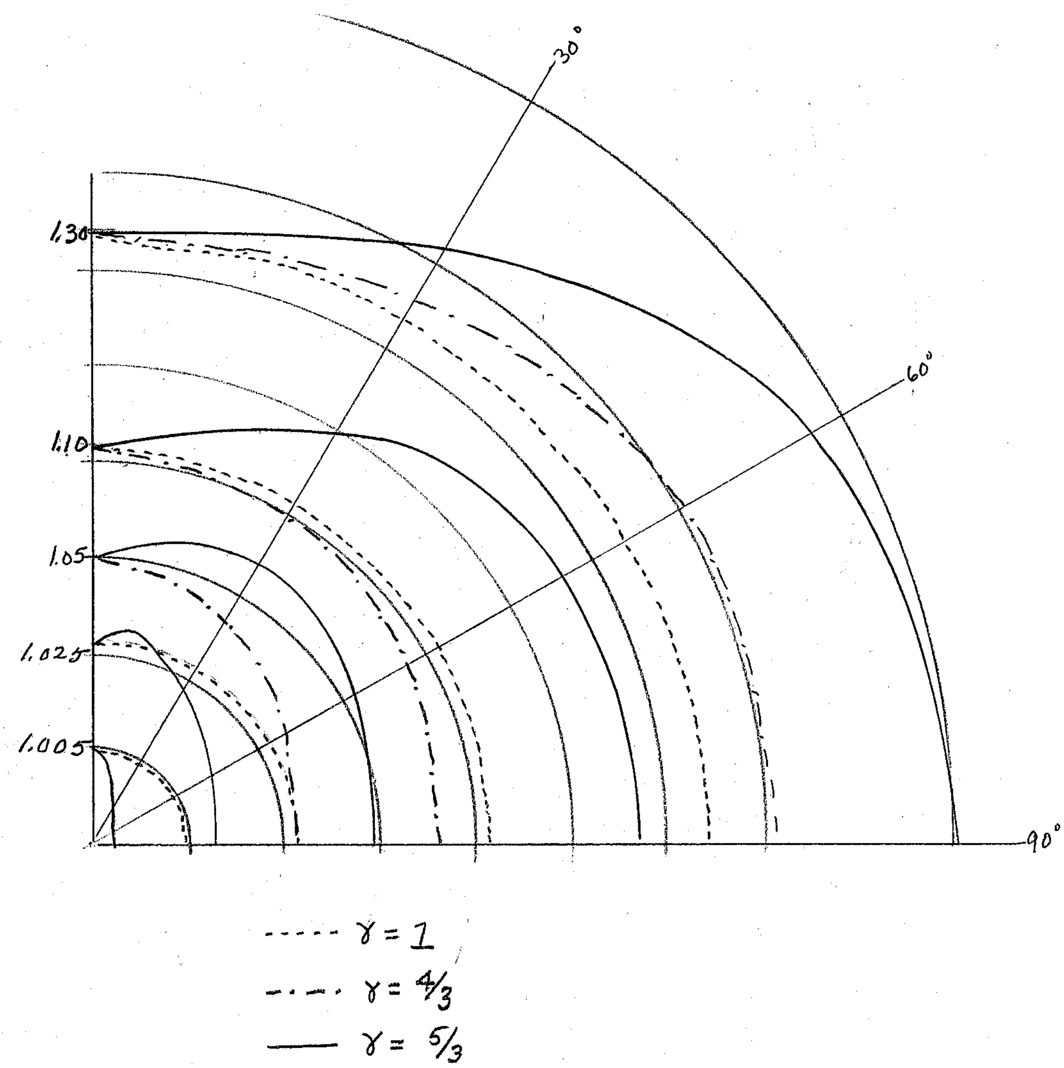



Fig. 4

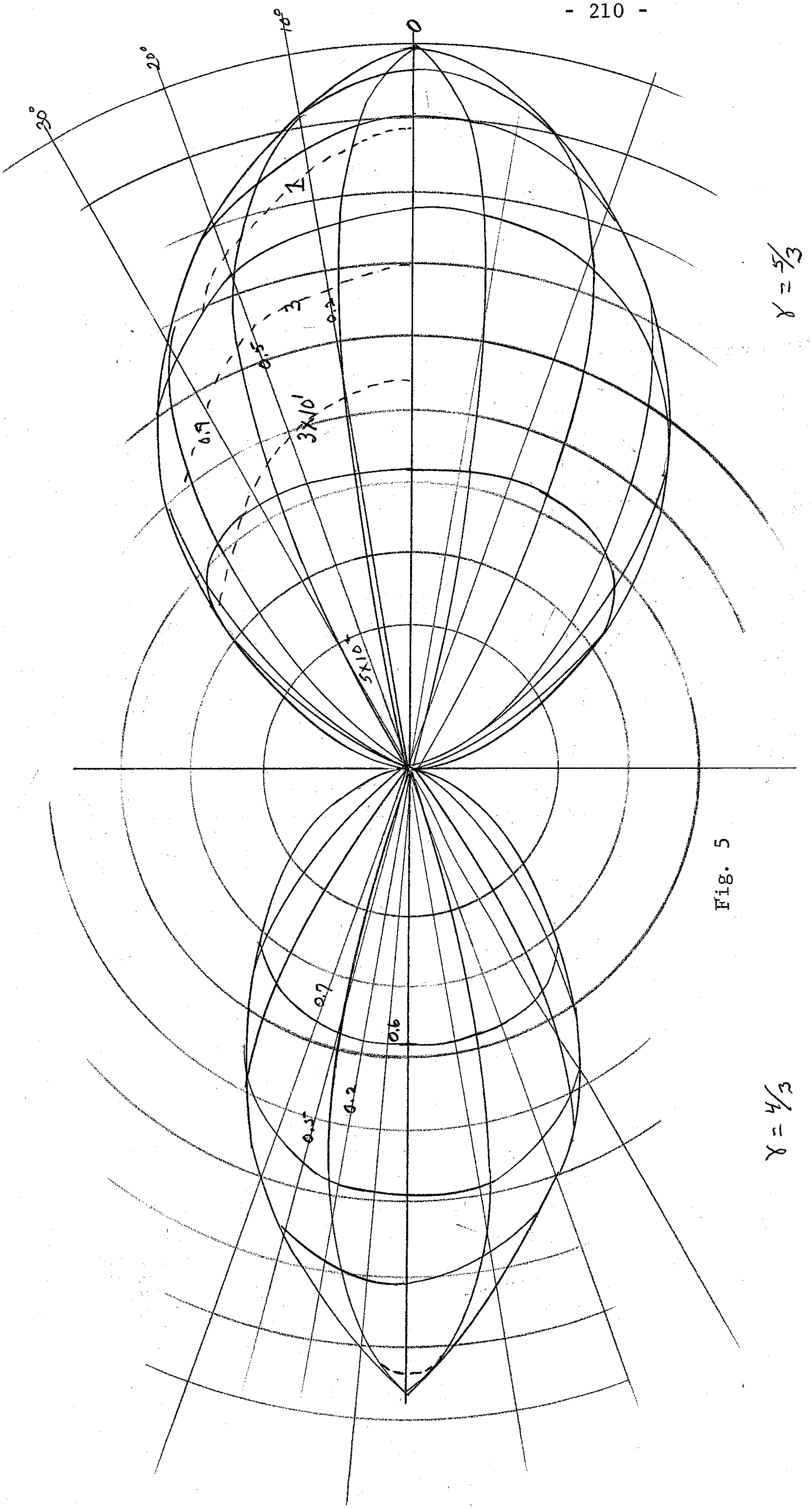



Fig. 5

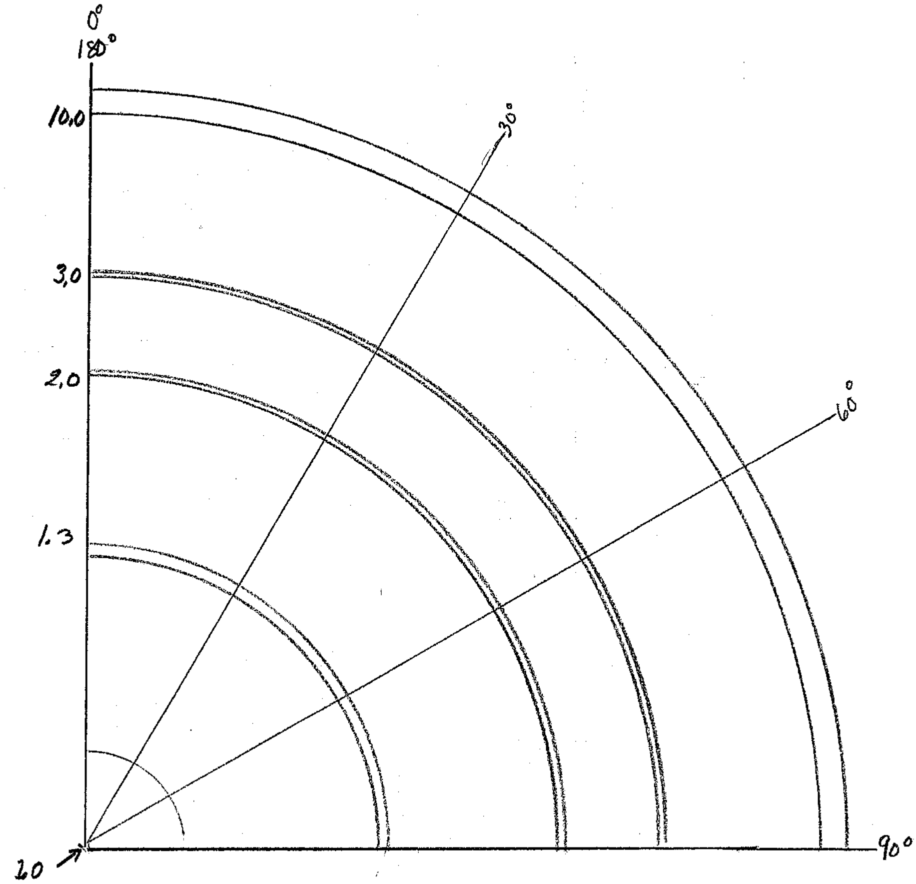



Fig. 6a

$\gamma = 1$

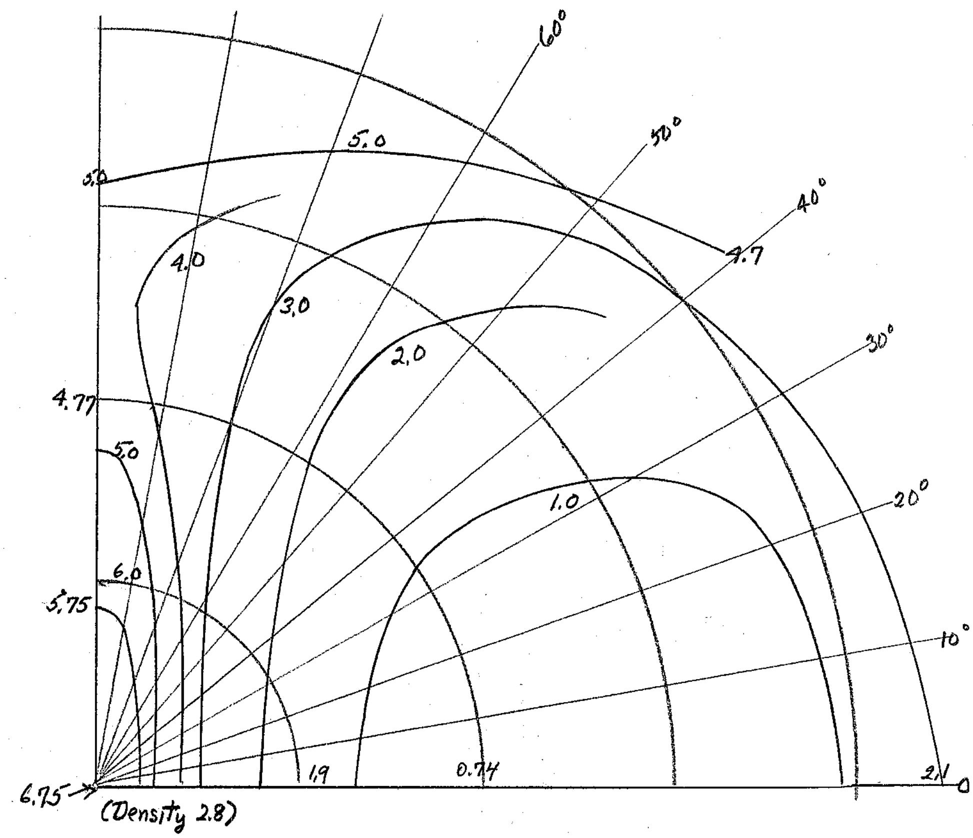



Fig. 6b

$\gamma = 4/3$

Fig. 6c

$\gamma = 5/3$

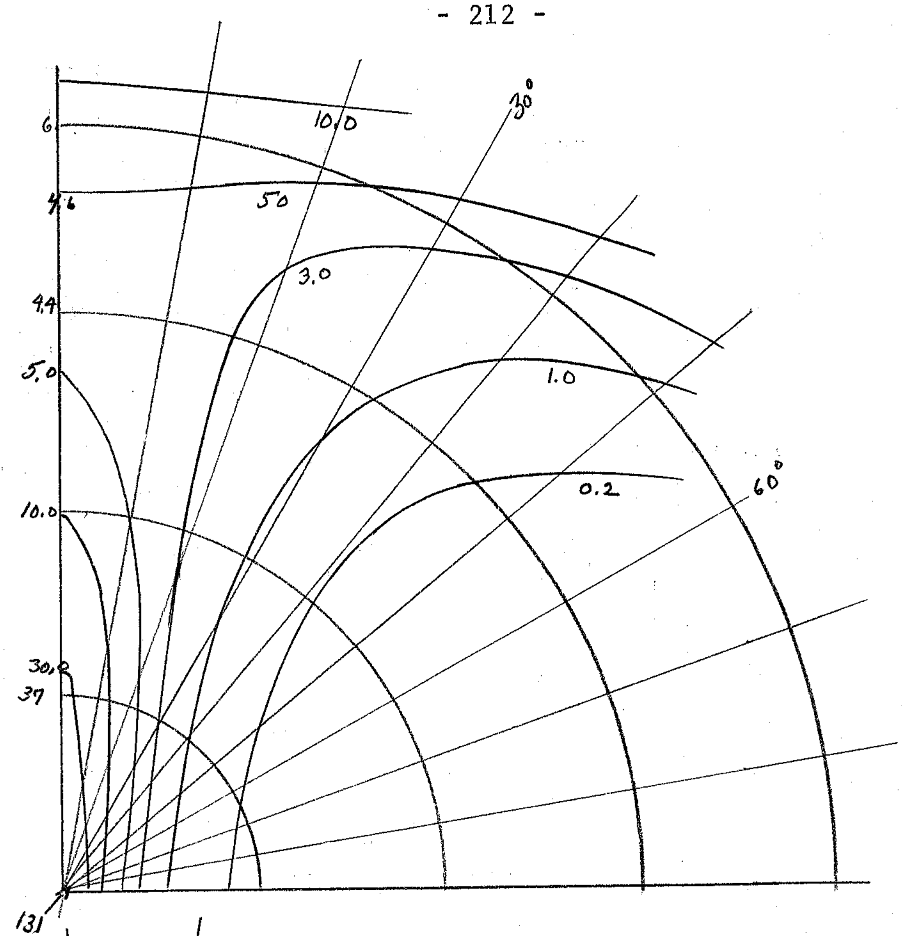
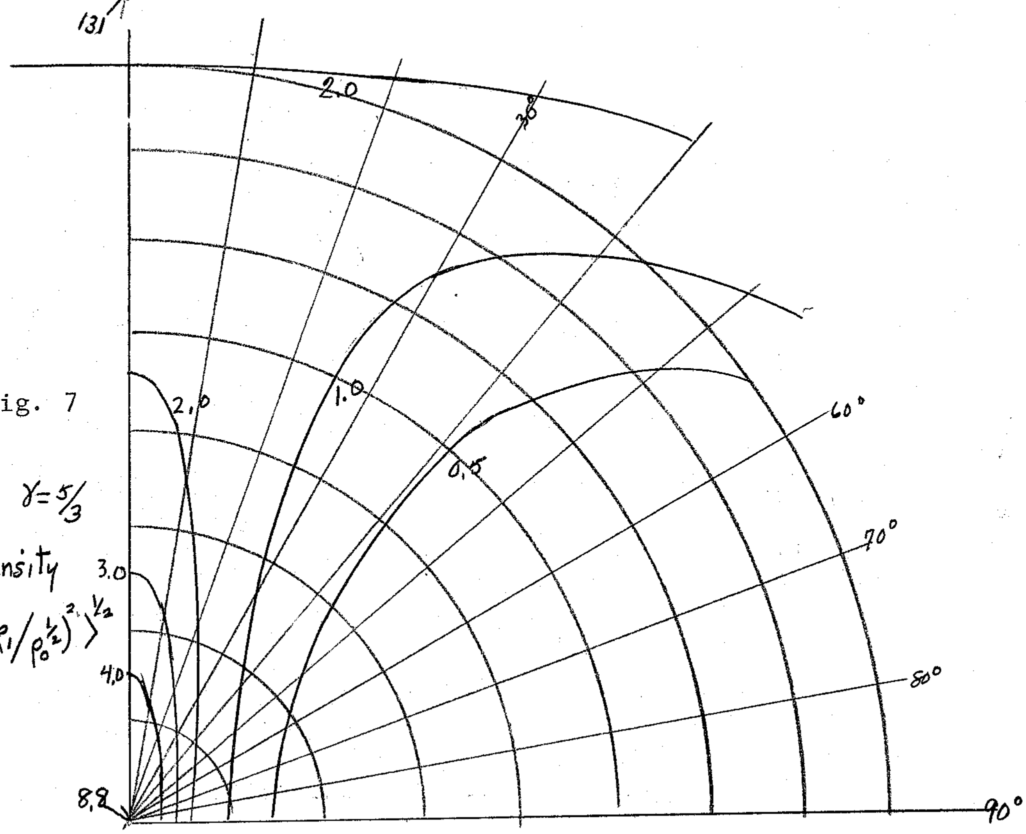



Fig. 7

$\gamma = 5/3$

Density $\langle (\rho_1/\rho_0^{1/2})^2 \rangle^{1/2}$



## Figure Captions

Figs. 1a,1b: Iso-intensity curves of the kinetic energy density of motions in the vertical direction excited by the vertically oscillating point source (36) whose frequency is just above the lower characteristic frequency of the atmosphere, $\omega_2$. (a): $\omega^2 = 0.6\,\omega_1^2$ and $\omega_2^2 = 0.566\,\omega_1^2$ (or $\gamma = 6/5$), and (b): $\omega^2 = \omega_1^2$ and $\omega_2^2 = 0.96\,\omega_1^2$ (or $\gamma = 5/3$) where $\omega_1$ is the upper characteristic frequency of the atmosphere. The numerical values added on curves represent $E_{K\mathrm{V,V}}/[\rho_{00}^{1-2\alpha}|A_{\mathrm{V}}|^2/16\pi^2H^6\omega_1^2]$.

Figs. 2a,2b: Showing angular dependency of $E_{K\mathrm{V,V}}(r,\theta;\omega)$ at the distance 2H (H is the scale height) from the source. Intensities are normalized so that they become unity at the vertical direction, separately. (a): $\omega^2 = \omega_1^2$, and $\omega_2^2 = 0.75\,\omega_1^2 (\gamma = 4/3)$, $0.89\,\omega_1^2 (\gamma = 3/2)$ and $0.96\,\omega_1^2 (\gamma = 5/3)$ in order moving inward. (b): $\omega^2 = 0.9\,\omega_1^2$ and $\omega_2^2 = 0.75\,\omega_1^2$ and $0.89\,\omega_1^2$ in order moving inward.

Fig. 3a: Total energy radiated by unit time from the vertically oscillating point source (equation (36)) in the frequency range of acoustic waves as a function of $\omega/\omega_1$. Value in the vertical axis is $F(\omega)/[\rho_{00}^{1-2\alpha}\omega_1^2|A|^2/6\pi c^3]$. Three curves are for $\gamma = 5/3$, 4/3 and 1.

Fig. 3b: The similar curves with Figure 3a for the circular oscillator in the horizontal plane equation (37). Two curves are for $\gamma = 5/3$ and 1.

Fig. 4: Frequencies of motions in the wake behind the front resulting from propagation of acoustic waves generated by a pulse. In other words, polar diagrams of magnitude of group velocity for acoustic waves. Three curves are $\gamma = 1$, 4/3 and 5/3. Curves are drawn so that the magnitude of the adiabatic sound velocity, $(\gamma R T_0)^{1/2}$, becomes a same unit even for differenc cases of $\gamma$.

Fig. 5: Curves for gravity waves similar to Figure 4. The right part of this figure is for $\gamma = 4/3$ and the left part is for $\gamma = 5/3$. The unit about the magnitude of the adiabatic sound velocity is the same as that in Figure 4. In curves for $\gamma = 5/3$, intensity of propagating disturbances is added with the same unit as in Figure 6 for acoustic disturbances.

Fig. 6: Iso-intensity curves of the kinetic energy density in the wake behind the front resulting from acoustic waves. The numerical values represent

$$E_K/[\frac{\rho_{00}^{-2\alpha}}{4(2\pi)^3t^3}|A|^2\frac{\omega_1^3}{C^6}$$

$\gamma = 1$ (a); $\gamma = 1$ (b); $\gamma = 4/3$ and (c): $\gamma = 5/3$.

Figure Captions (continued)

Fig. 7: Iso-intensity curves of the root mean square of $\rho_1/\rho_0{}^{1/2}$ in the wake behind the front resulting from acoustic waves. The numerical values represent those of the quantity behind the symbol of the sum in equation (73) in the case of $\gamma = 5/3$.

# ABSTRACTS

# Stability of Finite Amplitude Solutions

Dietrich Lortz

## Abstract

## Introduction

If a horizontal layer of fluid is heated from below and the corresponding temperature gradient exceeds a certain critical value, the static state of the fluid becomes unstable because the density decreases in the direction of gravity acceleration. That the instability happens at a finite temperature gradient is due to the fact that the buoyancy forces have to overcome the dissipative forces. The arising convective motions and temperature fluctuations lead to an increase of the heat flux through the layer. If the temperature difference between the top and the bottom of the layer is not too high the motions are steady and, as we shall see, there is an infinite number of solutions to the stationary problem. So we are in general confronted with three stability problems. I. At which temperature gradient does the static state become unstable so that convection is possible? II. Which of the various stationary motions is stable? III. What is the temperature difference beyond which all motions are unstable so that we have unsteady convection? The first problem is completely solved, see for instance (8). We know almost nothing about the third one. In this paper we shall be concerned with the second stability problem.

If the temperature gradient exceeds the critical value only slightly, the amplitude of the motions and temperature fluctuations should be small. We try to find expansions in a small amplitude parameter as Malkus and Veronis (6) did. (Our aim is to handle the whole manifold of steady solutions in contrast to Malkus and Veronis who treated special solutions.) The stability problem is treated in a similar way. We shall prove that all three dimensional motions are unstable, if the volume expansion coefficient, the thermal conductivity, and the kinematic viscosity do not depend on temperature.

Conclusion

We found the result that all three dimensional convective motions are unstable with respect to infinitesimal disturbances. The only stable motion is two-dimensional and has the form of rolls. Since in most convection experiments three-dimensional cellular motions with almost hexagonal cell pattern are observed, let us discuss the assumptions which led to our fundamental equations. The boundary conditions of an ideally conducting surface and the horizontally infinite layer should for rigid boundaries be good approximations to the experiments. Also the Boussinesq approximation, taking into account the density variation only in the buoyancy term is valid if the relative change of density in the convective layer is small. The latter condition is fulfilled in most convection experiments. But the assumption that $\alpha$, $\nu$, and $\chi$ do not depend on temperature is too ideal in most experimental cases. The works (2), (7), and (11) take into account slight dependences of the material properties on temperature. The main results are that the corresponding vertical unsymmetry in the layer leads to the stability of hexagonal cell patterns in a range between the critical Rayleigh number and a certain over-critical value. Beyond this range rolls are again stable, and the stability range of hexagonal cell patterns tends to zero if the material properties become independent of temperature.

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# On a Lagrangian Approach to Turbulence

Klaus Hasselmann

## Abstract

In the problem of turbulence one is concerned with unstable solutions of a system with an infinite number of degrees of freedom $(q_1, q_2, \ldots)$ $\underline{q}$ If a system is unstable it is in practice meaningful to consider only statistical quantities defined as averages over an ensemble of realizations. The system is then said to be completely determined statistically if the joint probability distribution $P_n(\underline{q}(t_1), \underline{q}(t_2) \ldots \underline{q}(t_n))$ is known for any $n$. In principle $P_n$ is determined as the solution of a Liouville equation whose characteristic equations are the equations of motion.

The distributions $P_n$ contain much more information than one is actually interested in. Knowledge of $P(q_1(t_1), q_2(t_2) \ldots q_r(t_r))$, where $r$ is a small number of order 2 or 3, is normally sufficient to determine the gross features of the turbulent fluid. The main difficulty in turbulence is that all components of the flow are non-linearly coupled, so that it is not possible to derive a closed set of equations for a finite number of coordinates only. To do this one is forced to make some form of "closure" assumption about the interaction of the components $q_1, \ldots q_r$ with the remaining infinite set of components.

Turbulent theories hitherto have been characterized by their strong dependence on the closure procedure. The goal of the present approach is to minimize the dependence on closure by a) choosing a Lagrangian, rather than the generally employed Eularian representation; b) extracting the maximum amount of information out of the finite set of coordinates considered before making a closure assumption about the remaining, unattainable information.

In particular, we attempt to determine the joint probability distribution of the positions and velocities of $n$ particles imbedded in the fluid, for arbitrary given initial positions, by setting up approximate equations of motion for a closed system of $n$ particles. The coefficients in the model equations are chosen so that the error between the true and the approximate equations is minimized. The coefficients and the mean square value of the error can be determined from the true Eularian mean velocity products at not more than $n$ points with the aid of the equations of motion. If the approximating equations plus a (closure) assumption about the actual form of the error (consistent with its known mean square value) are good statistical approximations, we may substitute for the true Eularian moments the approximate ones determined from the solution of the closed system. (It can be shown that for incompoessible fluids the Eularian mean velocity products can be derived from the Lagrangian joint position-velocity distributions.) We thus have a closed problem in which we must determine simultaneously a set of coefficients for the model equations and the Eularian moments which are consistent with these equations. An iterative procedure for deriving the solution is described. The transformation from Lagrangian to Eularian moments is performed by a Monte Carlo method. The method is currently being applied to the case of isotropic turbulence.

# Some Interesting Oceanographic Observations

Nicholas P. Fofonoff

Abstract

The theory of wind-driven ocean circulation implies that geostrophic motion is induced by a vertical component of velocity from the surface Ekman layer. Can the presence of the vertical motion be detected by routine oceanographic observations? Is the magnitude consistent with theoretical estimates?

The vertical motion cannot be observed directly. Its magnitude is measured in meters per year. In the steady state, the geostrophic circulation must be such that no vertical displacements of isopycnal surfaces occur. However, the driving field of surface stress is not constant with time; a strong annual period may be present in the variations. Furthermore, precipitation at the surface can provide a tracer to detect the presence of vertical flow.

Oceanographic data from Ocean Station "P" in the Gulf of Alaska provide a useful check on the theory. The observations are taken year round on an alternate six-week schedule. The mean vertical velocity calculated from divergence of the Ekman transport is always positive (upward) and averages 15 meters per year. The annual variation is sufficient to yield slightly negative values in summer.

The surface layer (100 meters) in the region is diluted by 1 part in 20 by net precipitation. From an estimated 1 meter per

year of net precipitation, it follows that the vertical flow into the surface layer must be of the order of 20 meters per year and that the layer contains about 5 years' accumulation of fresh water. These estimates are in good agreement with the calculated 15 meters per year considering uncertainties in estimating wind stress.

Furthermore, the annual variation of the depth of isopycnals, is in phase with the vertical displacement calculated from the departure of the divergence of Ekman transport from its mean. The observed magnitude, however, is much greater possibly because of horizontal movement of the density field associated with variations of the vertical velocity.

The conclusion is reached that the wind-driven circulation model is consistent with available observations in the mean and can provide a useful insight into the time-dependent changes of the circulation in the interior of an ocean.

# Theory of Two-dimensional Waves

Tiruvalam Krishnamurti

## Abstract

A review of the steady, inviscid two-dimensional mountain wave problem is presented. The lee wave problem for a two-layer atmosphere with constant reflection coefficient in each layer is shown for the case where there is one single lee wave.

It is shown that the lee wave problem can be expressed in a frame where entropy is a vertical coordinate. In this frame the lower boundary condition can be exactly represented at the mountain surface. The x-equation of motion and the continuity equation are readily integrable in this frame, hence the non-linear mountain wave problem appears simpler.

A scale analysis of the equations in the entropy frame for a wide mountain is presented with certain non-dimensional expansion parameters as a basis. These are: An inverse Froud number, and a ratio of the height to the width of the mountain. It is shown that the lowest order system has no asymmetry for a symmetric mountain and describes a potential flow.

The lowest order and all the higher order systems are non-linear. Linearising the higher order equations, it can be shown that the lee waves in the wide mountain problem are excited by terms of very high order. Hence we might consider flows in this example to be quasi-static.

# Integral Constraints on the Wind-driven Oceanic Circulation

Pierre Welander

Abstract

The oceans are subjected to wind stresses that drive the main oceanic currents. Because of the stratification of the water the currents decay vertically and the bottom stresses are, accordingly, quite small. It has been argued earlier that the torque on the ocean due to the wind stress must be balanced by lateral stresses at the coasts, to allow for a steady state. Morgan has, however, pointed out that the torque can also be balanced by pressure and Coriolis forces. If one computes the torque with respect to the earth's axis of rotation the Coriolis forces drop out, and only pressure and frictional stresses balance. (This is an inertial system.) Palmén has pointed out that there is a strong difference between the conditions in the Northern Atlantic and the Southern Atlantic. In the Northern Atlantic one has the larger fetch at low latitudes, where the distance from the earth's axis is large, while in the Southern Atlantic the fetch is larger at high latitudes. The torque created by the easterlies at low latitudes and westerlies at high latitudes has a larger net value for the Northern than the Southern Atlantic, and this may explain why the boundary currents, that can balance the torque by either pressure or friction effects, are better developed in the Northern Atlantic.

Further information about the role of boundary friction can be obtained from the integrated vorticity equation. One finds that the integrated wind stress curl is balanced, not by the friction stress, but by the friction force at the boundary. One may have completely slipping boundary and still achieve a balance as long as there is curvature in the velocity profile, created, say, by turbulence. The vorticity created by the wind stress curl can diffuse out to allow a balanced state as soon as there is a vorticity gradient at the wall.

As an example, one may give steady solutions to the oceanic model by Munk: $A_H \nabla^4 \psi - \beta \frac{\partial \psi}{\partial x} = curl_z \vec{\tau}_z^{wind}(y)$ assuming slipping boundaries for small friction $\left( \left( \frac{\beta}{A_H} \right)^{1/3} a >> 1 \right.$ where $a$ is a east-west dimension of the ocean). The solution is

$$\psi \sim \frac{1}{\beta}(a-x)\, curl_z \vec{\tau}_z^{wind}(y) \quad \text{in the interior}$$

$$\psi \sim \frac{a}{\beta} curl_z \vec{\tau}_z^{wind}(y) \left\{ 1 - \frac{2}{\sqrt{3}} e^{-\frac{1}{2}\left(\frac{\beta}{A_H}\right)^{1/3} x} \left[ \sin \frac{\sqrt{3}}{2} \left(\frac{\beta}{A_H}\right)^{\frac{1}{3}} x - \frac{\pi}{3} \right] \right\} \quad \text{at the western boundary}$$

The picture is quite realistic, with a slightly increased Gulf Stream.

It is of interest to note that vorticity can diffuse out also through a free boundary. As an example, consider an infinite fluid cylinder of radius $a$ acted on by a tangential force $F_\varphi = C\left(r - \frac{3}{4}a\right)$. This generates a net vorticity (but no net angular momentum). With a free boundary Navier-Stokes equation gives the steady solution $u = -\frac{Cr^2}{8\mu}(r-2a)$ + solid rotation. Vorticity is diffused to the boundary and is "destroyed" there.

# Time Dependent Motions in the Ocean

Francis P. Bretherton

## Abstract

After introducing the Brunt Väissäla frequency and the frequency relation for inertio-gravitational waves in a uniformly stratified liquid, attention was concentrated on an equational beta-plane, and in particular on inviscid, adiabatic, linear motions which are perturbations from a state of rest in which the potential density is a prescribed function of depth independent of horizontal position. The equations of motion are then separable and the horizontal and vertical structure may be considered separately. If the Coriolis parameter varies linearly with latitude oscillations can occur trapped within a range of latitude about the equator. Physical mechanisms behind this trapping were described for several cases. For inertio-gravitational waves the frequency always exceeds the local value of the Coriolis parameter, and northward-moving waves of given frequency are reflected at a critical latitude. Rossby waves at the equator are ageostrophic but non-divergent - further away the horizontal divergence is more important, and results in a reflection of northward-moving waves. Finally, the consistency of the assumptions made in this theory was examined. It appeared that although the hydrostatic relation fails under certain circumstances, this failure does not significantly affect the horizontal motion.

# Resonant Interactions between Waves

Francis P. Bretherton

## Abstract

A partial differential equation describing one-dimensional weakly non-linear, dispersive waves was taken as a model to examine the process of energy transfer between different wavenumbers by resonant interactions, which was suggested for ocean waves by Phillips. It allows of quadratic resonances between three modes, and the interaction coefficients are obtained after negligible algebra.

The simplified equations for three interacting sinusoidal infinite wave trains nearly satisfying the resonance condition are similar to those proposed by Benney. They are, however, completely soluble and there is slow periodic transfer of energy between the modes, with period depending on the initial relative amplitudes and phases but inversely proportional to the overall amplitudes. If the resonance condition is poorly satisfied, the transfer of energy is small.

For a continuous spectrum of waves these equations are inadequate, and different ones were proposed in which the wave amplitudes are regarded as slowly varying functions of position and time, propagating with the appropriate group velocity and interacting resonantly with other wave groups.

The statistical theory of resonantly interacting waves proposed by Hasselmann assumes that the different wavenumbers

are statistically independent. This may be shown not to remain a consistent assumption after time intervals comparable with those over which a significant transfer of energy takes place However, the manner in which it breaks down initially is such that to a first approximation the energy transfer is not affected, and the precise position is still obscure.

The model of interacting sinusoidal waves may be justified by an asymptotic expansion in which a typical wave amplitude is the small parameter only for discrete oscillations, in which the spatial region under consideration is closed by reflecting boundaries and the spectrum for infinitesimal amplitude is discrete. For a continuous spectrum the wave group representation must be used, but has not yet been satisfactorily justified.

# Ocean Circulation

Kirk Bryan

## Abstract

Solutions are obtained for the ocean circulation in an enclosed basin of planetary scale by means of a numerical method. The model is geostrophic, and includes the effect of horizontal and vertical diffusion, as well as horizontal and vertical advection of heat. The flow is driven by a meridional temperature gradient imposed at the upper surface, and the effects of wind stress and bottom stress are neglected. Vertical mixing is taken to be infinite for unstable stratification, and a constant for all stable cases. The interior solution is quite similar to those obtained in earlier thermocline studies. Although the net mass transport is everywhere zero, a strong northward boundary current forms in the upper layers near the western wall. Sinking motion is concentrated in the northeast corner.

In the asymptotic case of small horizontal diffusion, scale analysis indicates that the total meridional heat transport should be proportional to $\varkappa \Delta\vartheta^* L^2/d$

where $\varkappa$ is the vertical diffusion, $\Delta\vartheta^*/L$ the imposed meridional temperature gradient at the surface, and $d$ the scale depth of the thermocline. The constant of proportionality is estimated from the numerical solutions. Errors arising from the neglect of viscous and inertial terms of the momentum equations in the boundary current are discussed.

# On the Theory of Transient Motions of a Rotating Fluid

Harvey P. Greenspan

## Abstract

These lectures consider the transient motion of a viscous fluid in a container rotating with constant angular velocity. The principal objective is to study the manner in which an arbitrary initial state of motion becomes a rigid rotation. In order to concentrate on the effects of viscosity, only the spherical container is studied in great detail.

Several sources of non-uniform behaviour make the analysis difficult and complex. In particular, there are three important time scales, viscous boundary layers, boundary layer resonances at critical latitudes and intricate side wall effects. The basic approach consists of an expansion procedure by means of which the general inviscid solution is corrected for viscous effects and is made uniformly valid in time through the critical spin-up phase. Uniform validity is effected through the elimination of secular terms, with unacceptable growth rates arising from the asymptotic perturbation series.

The interior (inviscid) motion leads to a non-self-adjoint partial differential equation eigenvalue problem with many intriguing properties. The general expansion theorem, orthogonality relationships, and viscous decay factors are deduced and used to solve the arbitrary initial value problem. It is shown that the mean component of angular momentum parallel to the

rotation axis is extracted from the fluid in the spin-up time scale $T = L(\Omega \nu)^{-\frac{1}{2}}$. This is accomplished by a secondary non-oscillatory convective motion produced by suction into the Ekman layer. The angular momentum not eliminated in this way excites inviscid inertial oscillations which are also caused to decay by the boundary layers in the same time scale. Some very small residual effects decay in the ordinary viscous diffusion time, but all the essential processes are concluded in the much shorter interval. All modal oscillations in a sphere are determined and several specific calculations of frequency and decay rate are made and compared to experimental data. Perhaps the most important of these concerns the mode corresponding to rigid internal motion about another axis which can be produced by impulsively changing the rotation axis of the container. Agreement between theory and experiment is very good in all cases compared thus far.

# Unsteady Ocean Circulations

Joseph Pedlosky

## Abstract

The unsteady motions of a homogeneous bounded ocean on the $\beta$-plane are studied. Both the free normal modes and the forced solutions for the linearized problem are computed. The non-linear response is computed by a perturbation analysis. Of particular interest is the steady (time-dependent) circulations produced by a fluctuating wind stress with zero time-mean due to the non-linearities of the dynamics. It is shown that the structure of the resulting circulations, their strength, and their sense are strong functions of the frequency of the forcing stress. Depending on the magnitude of the frequency the resulting circulations may have: 1) only a western boundary layer (low frequency), 2) no boundary layers (frequencies less than a typical Rossby wave frequency for the basin), 3) boundary layers on both eastern and western walls (very high frequency).

# The Diffusion of a Line Vortex Normal to a Stationary Plane

Albert I. Barcilon

## Abstract

We will investigate the diffusion of a semi-infinite line vortex normal to an infinite stationary plane. Far from the axis and from the plate each fluid particle describes a circular orbit: the centrifugal force balances exactly the horizontal pressure gradient. At the rigid boundary, fluid is decelerated and driven inwards by the unbalanced horizontal pressure gradient. The plate boundary layer erupts near the axis and fluid is forced upward. Because no sink or source are postulated at infinity, the axial boundary layer discharges fluid into the interior. A vertical circulation cell is thus created.

The fluid is assumed of constant properties and the axially symmetric problem is formulated in a "mathematical plane" where the horizontal coordinate is $\xi = r/\sqrt{\Gamma t}$ and the vertical coordinate is $\eta = z/\sqrt{\Gamma t}$. $r, z$ are the cylindrical coordinates in physical space, $t$ is the time, and $\Gamma$ is the circulation of the potential vortex at $r = \infty$, $\epsilon = \nu/\Gamma$ is the only dimensionless constant entering the equations. The problem is thus reduced from a parabolic problem having three independent variables: $r$, $z$, $t$, to an elliptic problem having two independent variables: $\xi$, $\eta$. The solution in the plate boundary layer is obtained as a formal asymptotic expansion in power of $1/\xi$. The $O(\sqrt{\epsilon})$ inviscid flow solution is determined up to a set of unknown constants. These constants are

to be found by using an integral flux condition across a hemispherical surface $\xi^2 + \eta^2$ = constant. Knowledge of the axial boundary layer is necessary for the evaluation of these constants.

# Coupled Convection of Heat and Salt

J. Stewart Turner

## Abstract

When gradients of temperature and salinity are present simultaneously in a liquid, several new phenomena can be observed. The most interesting cases are those fow which one component causes the density to increase, and the other to decrease with depth, and the net gradient is hydrostatically stable. Hot water containing a little salt, when placed above cold fresh water, gives rise to the instability known as "salt fingers". These are alternately downward- and upward-moving filaments of salt and fresh water, shich are driven by the small salinity difference between them while the temperature is kept nearly uniform in the horizontal because of molecular diffusion. Their formation depends on the large difference in the diffusivities of salt and heat.

If, on the other hand, a stably stratified salt solution is heated from below, a series of discrete layers forms from the bottom up. Each of these is in turbulent convective motion, driven by the unstable temperature field, and again the phenomenon depends on the different rates of diffusion of salt and heat. As successive layers grow, heat escapes through the top and begins a new convective process above, while salt is transported more slowly and serves to maintain the stability of the interface.

Layers can form when the differences of density $\beta \Delta S$ due to salinity are far greater than those due to temperature $\alpha \Delta T$.

Experiments have been conducted to investigate in detail a simple system of this second kind, consisting of two layers of salt solution heated from below. Both the heat transport and the ratio of the rates of transport of salt across the single sharp interface between the layers have been found to depend systematically on the density ratio $\beta \Delta S / \alpha \Delta T$. A dimensional argument suggests that the form of functions found should be applicable over a wider range of heat fluxes than those used in these experiments. Another implication of the results is that the potential energy changes in the upper layer due to the transports of salt and heat across the interface are in constant ratio, over a wide range of density differences. This result, and many others suggested by these experiments, are badly in need of theoretical study, but so far theory and experiment in this field have not been very closely related.

# Generation of Ordered Circulation by Fluctuating Winds

George Veronis


## Abstract

The equations for a barotropic model of wind-driven ocean circulation are treated approximately by expanding the stream function in a double Fourier series. A solution with a limited number of the Fourier coefficients is obtained by numerical integration for the case where the wind stress curl is steady. A fluctuating wind (zero time mean) is then superimposed and it is found that the maximum transport with the fluctuating wind is approximately 35% greater than that caused by the steady wind alone. The increase in transport depends on the amplitude and frequency of the fluctuating wind.

# Some Transformations for the Mountain Wave Problem for Incompressible and Compressible Atmospheres

Tiruvalam Krishnamurti

Abstract

A review of some of the recent contributions of Yih and Claus on the mountain wave problem is presented with a view of extension to generalized transformations. The transformations have the property that one solves for new dependent variables (pseudo-stream function) for which the finite amplitude flow is governed by a linear differential equation. A simple example for a particular upstream condition is illustrated in which the precise role of the non-linear terms is found to be that of "undamping" the amplitude of the waves. In this example the solutions of the linearized equations and finite amplitude equations are found to be very similar even for finite-sized mountains. The upstream conditions are defined by a constant value of density stratification and kinetic energy.

Similar pseudo-stream functions are found to exist for compressible fluid motion in a steady inviscid atmosphere under certain simple assumptions of neglect of dynamic compressibility. An infinite class of upstream conditions exist for which the finite amplitude flows are given by a linear differential equation.

The transformations are obtained by dividing the dimensional velocities by a conservative property of the problem like

density or potential temperature in the respective cases of incompressible and compressible flow. It is suggested that if the transformations are based on arbitrary functions of density and potential temperature, one could extend the analogy to a still wider class of mountain-wave problems. These arbitrary unspecified functions of density and potential temperature are determined by the upstream conditions in a given problem such that the differential equation for a pseudo-stream function is linear.