Notes on the 1963

Summer Study Program

in

GEOPHYSICAL FLUID DYNAMICS

at

The WOODS HOLE OCEANOGRAPHIC INSTITUTION

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Contents of the Volumes

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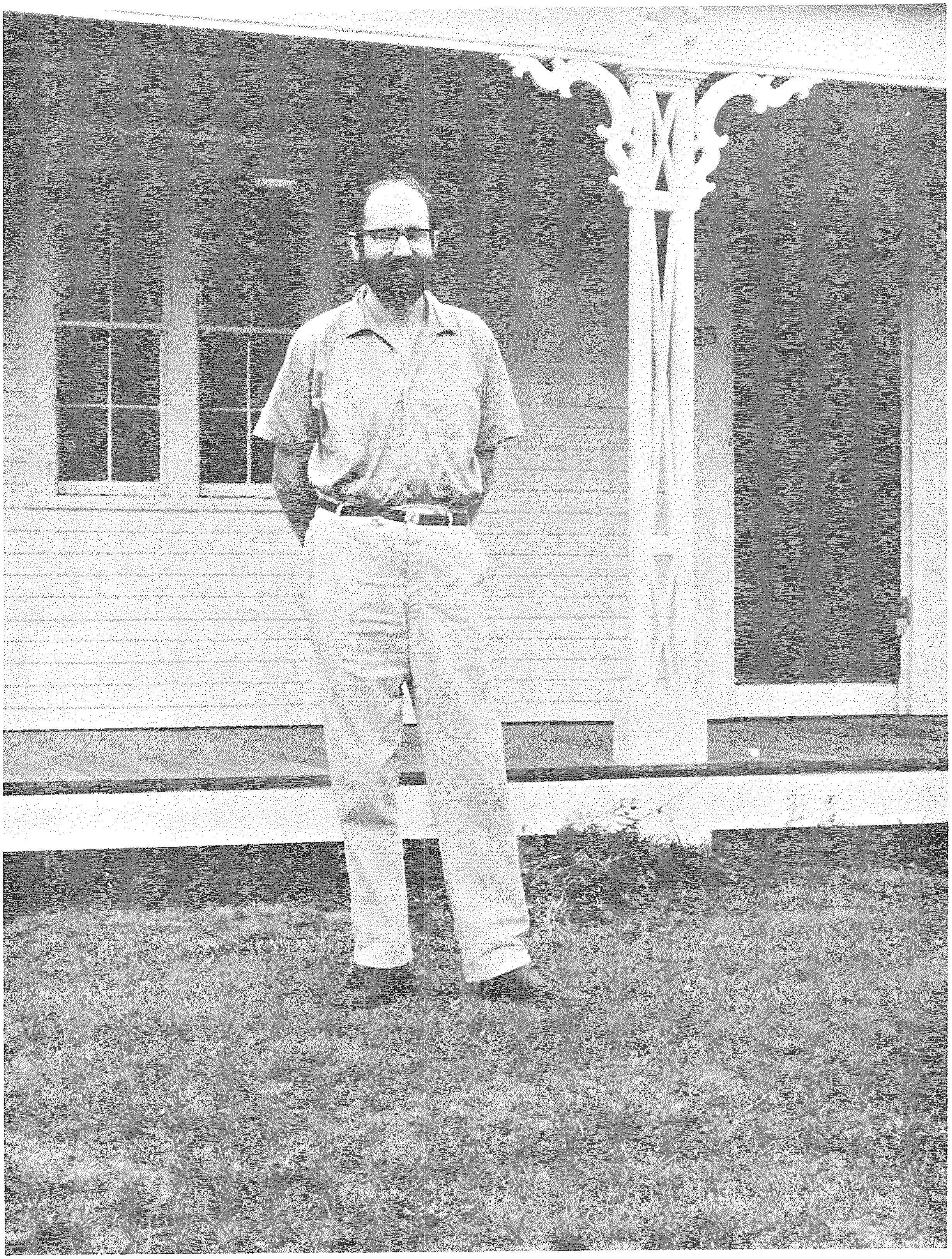
Editor's Preface

This year's lectures by Derek Moore form a detailed report of investigations on the fluid motion caused by the motion of a body in a homogeneous rotating fluid. The emphasis has been on the significance of the Taylor-Proudman theorem and the departure of the fluid from the behavior described by the Taylor-Proudman theorem. The plan was to probe deeply into one problem and thereby acquire information in a wider area of study of rotating fluids.

This volume is a restatement by the students of the lectures. It was not intended that the students' notes be a simple record of the lectures. In some cases the material was reworked by the student and the presentation reflects much of the student's own orientation and interests. In other cases, the notes are much closer to the presentation in the lecture room. Dr. Moore has read the notes and has made alterations so that the material adheres as closely as possible to that of the lectures.

Mrs. Mary Thayer has done all the work in assembling and reproducing the lectures. We are all indebted to her for her remarkable efforts in keeping the summer course running smoothly and to the National Science Foundation for its financial support of the program.

George Veronis



Our invited guest lecturer recently returned from the wilds of J. Banan's.

Geophysical Fluid Dynamics

Derek W. Moore

Lecture I.

Introduction

The interest in problems in rotating fluids arises because of connection with the problems of meteorology, oceanography, and motions in the core of the earth. An excellent account has been given by Hide (1962).

There is a direct mathematical analogy between problems in rotating barotropic fluids, stratified fluids, and magnetohydrodynamic problems in which the induced fields are neglected. Results obtained in one field can be applied to another field to simplify experimental comparisons, since experiments with stratified fluids and magnetohydrodynamic experiments are notoriously difficult to perform.

The main problems to be considered in this course are those of perturbations in the state of uniform rotation of a fluid caused by the motion of obstacles through it.

1. Basic concepts in hydrodynamics

The equations of motion of a fluid will now be obtained in the Eulerian form, in which the fluid velocity is given as a function of position at any given time

$$\text{i.e. } \underset{\sim}{u} = \underset{\sim}{u}(x, t).$$

With the equations of motion in this form, the path of an individual particle in the fluid is not obtained as easily as from the Lagrangian equations, and to bridge the gap we must introduce the idea of the substantive derivative, i.e. differentiation following the motion.

Consider a scalar field specified at every point in space and time, $\theta(\underset{\sim}{x}, t)$. We inquire how θ changes as we move with a given fluid particle.

The change in θ, $\Delta\theta$, observed at a fluid particle originally at $(\underset{\sim}{x}, t)$ is

$$\Delta\theta = \theta(\underset{\sim}{x} + d\underset{\sim}{x}, t + dt) - \theta(\underset{\sim}{x}, t)$$

so that

$$\Delta\theta = dt\left\{\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta + \frac{\partial\theta}{\partial t}\right\} + \cdots\cdot$$

$$\therefore \Delta\theta = \frac{\partial\theta}{\partial t}\cdot dt + d\underset{\sim}{x}\cdot \operatorname{grad}\theta, + \ldots.$$

now

$$d\underset{\sim}{x} = \underset{\sim}{u}(\underset{\sim}{x}, t)\,dt \quad + \text{ higher order terms}$$

Thus the rate of change of θ at a given moving fluid particle is

(1.1) $$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta. \equiv \frac{\partial\theta}{\partial t} + u_i\frac{\partial\theta}{\partial x_i}.$$

Consider the motion of the fluid in the neighbourhood of an observer moving with the fluid: the observer's velocity at $(\underset{\sim}{x}, t)$ is $\underset{\sim}{u}(\underset{\sim}{x}, t)$.

At $(\underset{\sim}{x} + d\underset{\sim}{x}, t)$ the fluid velocity is

$$u_i(\underset{\sim}{x} + d\underset{\sim}{x}, t) = \frac{\partial u_i}{\partial x_j}\,dx_j + u_i(\underset{\sim}{x}, t)$$

so that the fluid velocity relative to the observer is

$$du_i = \frac{\partial u_i}{\partial x_j}\, dx_j .$$

(1.2) $$= \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right] dx_j .$$

The first term is a symmetric tensor, so that there exist three mutually-perpendicular directions along which the relative velocity of the fluid is directly towards or away from the observer; it represents a pure rate of strain in the fluid.

The second term is antisymmetric, and represents a local rigid-body rotation of the fluid.

For $$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_j = \frac{1}{2}\left(\delta_{li}\,\delta_{mj} - \delta_{mi}\,\delta_{lj}\right)\frac{\partial u_l}{\partial x_m}\, dx_j$$

$$= \frac{1}{2}\,\varepsilon_{klm}\,\varepsilon_{kij}\,\frac{\partial u_l}{\partial x_m}\, dx_j$$

$$= -\frac{1}{2}\left(\operatorname{curl} \underline{u}\right)_k \varepsilon_{kij}\; dx_j$$

(1.3) $$= \frac{1}{2}\left\{\left(\operatorname{curl} \underline{u}\right) \times d\underline{x}\right\}_i$$

so the local angular velocity of a fluid element = ½ curl $\underset{\sim}{u}$.

(1.4) The vorticity is defined to be $\underset{\sim}{\omega}$ = curl $\underset{\sim}{u}$.

For an inviscid fluid, the stress acting on the surface of an element is normal to the surface, and so we see that the vorticity of a fluid element is constant since there are no forces acting tangentially to exert torques. (We assume the body force is conservative). In particular, for a motion started from rest $\underset{\sim}{\omega} = 0$.

The Kelvin-Helmholtz Theorems.

The Navier-Stokes equations for an incompressible, homogeneous fluid, where only conservative body forces are acting, are:

$$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho}\underset{\sim}{\nabla} p + \nu \nabla^2 \underset{\sim}{u} \tag{1.5}$$

where the gradient of the potential of the body force is included in the pressure term.

The equation of continuity for such a fluid is

$$\operatorname{div} \underset{\sim}{u} = 0. \tag{1.6}$$

(For derivation see, e.g. Landau & Lifschitz, Fluid Mechanics. Pergamon.)

a) Kelvin's theorem states that in an inviscid fluid, the circulation $K(t)$ around any material curve in the fluid which is simple and closed is a constant.

(1.7) The circulation $K(t) = \int_{c(t)} \underset{\sim}{u} \cdot d\underset{\sim}{x}$, $c(t)$ being the material curve of integration.

Consider an element of the curve of integration. The change in the contribution to the integral

$$\Delta K = \frac{D}{Dt}(\underset{\sim}{u} \cdot d\underset{\sim}{x})$$

$$= d\underset{\sim}{x} \cdot \frac{D\underset{\sim}{u}}{Dt} + \underset{\sim}{u} \cdot \frac{D}{Dt}(d\underset{\sim}{x})$$

where the second term is the rate of extension of the element $d\underset{\sim}{x}$.

Now $\frac{D}{Dt}(d\underset{\sim}{x}) = d\underset{\sim}{u}$, the relative velocity of the ends of the element.

$$\therefore \Delta K = -d\underset{\sim}{x} \cdot \frac{1}{\rho} \underset{\sim}{\nabla} p + \underset{\sim}{u} \cdot d\underset{\sim}{u}.$$

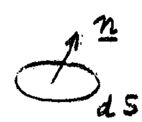
$$\therefore \text{ Change in } K = \int_{C(t)} \left\{ -d\underset{\sim}{x} \frac{1}{\rho} \cdot \underset{\sim}{\nabla} p + d\left(\tfrac{1}{2}\underset{\sim}{u}^2\right) \right\}$$

$$= \int_{C(t)} -\frac{1}{\rho} \frac{\partial p}{\partial s} \, . \, ds + d\left(\tfrac{1}{2}u^2\right)$$

$= 0$ around a closed curve.

Also, for a small material curve, using Stoke's theorem we have

$$\int_{C(t)} \underset{\sim}{u} \, . \, d\underset{\sim}{x} = \int_S \left\{ (\text{curl}\, \underset{\sim}{u}) \cdot \underset{\sim}{n} \right\} dS$$



Thus Kelvin's theorem implies that (curl $\underset{\sim}{u}$. $\underset{\sim}{n}$) dS is conserved.

So that if the motion starts from rest, $\underset{\sim}{\omega} = \text{curl}\, \underset{\sim}{u} \equiv 0.$

The equations of motion in the form

(1.8) $$\frac{D\underset{\sim}{u}}{Dt} = -\frac{1}{\rho} \underset{\sim}{\nabla} p + \nu \nabla^2 \underset{\sim}{u}$$

can be transformed as follows:

using the vector identity, $(\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\underset{\sim}{u} = \underset{\sim}{u} \times \text{curl}\, \underset{\sim}{u} - \frac{1}{2} \underset{\sim}{\nabla} u^2,$

we have $\dfrac{\partial \underset{\sim}{u}}{\partial t} - \underset{\sim}{u} \times \underset{\sim}{\omega} = -\underset{\sim}{\nabla}\left(\frac{p}{\rho} + \frac{1}{2}u^2\right) + \nu\nabla^2\underset{\sim}{u}$, $\underset{\sim}{\omega} = \operatorname{curl} \underset{\sim}{u}$.

Taking the curl, we have

$$\frac{\partial}{\partial t}\underset{\sim}{\omega} + (\underset{\sim}{u}\cdot\underset{\sim}{\nabla})\underset{\sim}{\omega} - (\underset{\sim}{\omega}\cdot\underset{\sim}{\nabla})\underset{\sim}{u} = \nu\nabla^2\underset{\sim}{\omega}$$

where we have used the continuity equation $\operatorname{div} \underset{\sim}{u} = 0$.

(1.9) Thus $\dfrac{D}{Dt}\underset{\sim}{\omega} = \underset{\sim}{\omega}\cdot\underset{\sim}{\nabla}\underset{\sim}{u} + \nu\nabla^2\underset{\sim}{\omega}$, — the vorticity equation.

The first term represents the rate of increase of vorticity of a fluid element. We shall prove that the second represents the intensification of vorticity by the stretching of vortex lines in the fluid. The third represents the diffusion of vorticity by viscous forces.

A <u>vortex line</u> is a curve in the fluid whose direction at every point is parallel to $\underset{\sim}{\omega}$ at that point.

b) Helmholtz' theorem. Vortex lines are material lines in an inviscid fluid. (We give a proof based on Truesdell's discussion in "Kinematics of Vorticity".)

<u>Proof</u>: We remark that if a space curve $\underset{\sim}{x} = \underset{\sim}{x}(\theta, t)$ is a vortex line it must satisfy the equation

$$\frac{d}{d\theta}\underset{\sim}{x} \times \underset{\sim}{\omega}(\underset{\sim}{x}, t) = 0.$$

We assume that at time $t = t_0$ a material curve coincides with a vortex line. We prove that the material curve remains a vortex line for $t > t_0$. Thus $\left.\frac{d\underset{\sim}{x}}{d\theta}\right)_{t_0} \times \underset{\sim}{\omega}(\underset{\sim}{x}, t_0) = 0$ whilst $\underset{\sim}{x}(\theta, t)$ is a material curve for $t \geqslant t_0$.

Consider $\left.\frac{D}{Dt}\left\{\frac{d}{d\theta}\underset{\sim}{x} \times \underset{\sim}{\omega}\right\}\right._{t_0} = \left.\frac{d}{d\theta}\underset{\sim}{x}\right)_{t_0} \times \frac{D}{Dt}\underset{\sim}{\omega} + \frac{D}{Dt}\left(\frac{d}{d\theta}\underset{\sim}{x}\right) \times \underset{\sim}{\omega}(\underset{\sim}{x}, t_0).$

Now $\frac{D}{Dt}(d\underset{\sim}{x}) = d\underset{\sim}{u} = (d\underset{\sim}{x} . \underset{\sim}{\nabla})\underset{\sim}{u}$, since, by hypothesis $\underset{\sim}{x}(\theta, t)$ is a material curve.

Thus $\left.\frac{D}{Dt}\left\{\frac{d}{d\theta} . \underset{\sim}{x} \times \underset{\sim}{\omega}\right\}\right._{t_0} = \left.\frac{d}{d\theta}\underset{\sim}{x}\right)_{t_0} \times \underset{\sim}{\omega} . \underset{\sim}{\nabla}\underset{\sim}{u} + \left.\frac{d}{d\theta}\underset{\sim}{x}\right)_{t_0} . \underset{\sim}{\nabla}\underset{\sim}{u} \times \underset{\sim}{\omega}$

But $\left.\frac{d}{d\theta}\underset{\sim}{x}\right)_{t_0} = \lambda\,\underset{\sim}{\omega}(\underset{\sim}{x}, t_0)$ for some scalar λ.

Thus $\left.\frac{D}{Dt}\left\{\frac{d}{d\theta}\underset{\sim}{x} \times \underset{\sim}{\omega}\right\}\right._{t_0} = \lambda\,\underset{\sim}{\omega} \times (\underset{\sim}{\omega} . \underset{\sim}{\nabla})\underset{\sim}{u} + \lambda(\underset{\sim}{\omega} . \underset{\sim}{\nabla})\underset{\sim}{u} \times \underset{\sim}{\omega}. = 0.$

so that the material curve remains a vortex line.

A vortex tube is a region bounded by all vortex lines through a simple closed curve. It follows that vortex tubes are convected with the flow if the fluid is inviscid.

We can now show that the second term in the vorticity equation arises from the stretching of vortex lines: consider a vortex tube of original length ℓ_0 which is stretched to length ℓ. The cross-sectional area of the tube must change from dS_0 to dS. Since the vortex tube is convected with the fluid it must always contain the same fluid particles, thus

From Kelvin's theorem: $dS_0\,\omega_0 = dS(\omega_0 + d\omega)$

where $d\omega$ is the change in vorticity induced by stretching.

Now $l = l_0 + \frac{\partial u}{\partial l} \cdot l_0 \, dt.$

so that $ds_0 \, \omega_0 = l_0 \left(1 - \frac{\partial u}{\partial l} \cdot dt\right)(\omega_0 + d\omega)$

(1.10) whence $d\omega = \omega_0 \frac{\partial u}{\partial l} \, dt.$

Problem I.

The stability of a columnar vortex.

Ref. Kelvin: Phil.Mag. 1880 5, vol X. 155.

Consider the vortex flow given in cylindrical polar coordinates, by

$$\begin{cases} \bar{u} = \hat{\theta}\,\omega r, & 0 < r \leq a. \\ \bar{u} = \hat{\theta}\,\dfrac{\omega a^2}{r}, & a \leq r. \end{cases}$$

Investigate the stability of the vortex by applying a perturbation of the form

$$\underset{\sim}{u} = \underset{\sim}{\bar{u}} + \underset{\sim}{f}(r)$$

Viscosity is to be neglected.

Lecture # 2.

Rigid Body Rotation.

We first examine the basic velocity field itself.

The velocity field for a rigid rotation is given by

(1.11) $$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{\Omega} \times \underset{\sim}{x}$$

The divergence of a rigid rotation velocity field vanishes.

$$\operatorname{div} \underset{\sim}{u} = \frac{\partial}{\partial x_i} \epsilon_{ijk} \Omega_j x_k = \epsilon_{ijk} \Omega_j \delta_{ik}$$

$$= \epsilon_{iji} \Omega_j = 0$$

The curl is not zero but equal to twice the angular frequency of rotation.

$$(\operatorname{curl} \underset{\sim}{u})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{kpq} \Omega_p x_q$$

$$= \epsilon_{ijk} \epsilon_{kpj} \Omega_p$$

$$= 2 \delta_{ip} \Omega_p = 2 \Omega_i$$

Thus the vortex lines are parallel to the axis of rotation. The presence of this field of vortex lines endows a rotating fluid with a lateral elasticity, since motions perpendicular to the axis of rotation will in general tend to stretch the vortex lines. Thus it will appear that such motions are resisted and that transverse waves can be propagated.

Since there are no viscous forces in a rigid rotation, the equation of motion of the fluid becomes

$$-\frac{1}{\rho}\frac{\partial P}{\partial x_j} = u_i \frac{\partial u_j}{\partial x_i}$$
$$= \epsilon_{ipq}\Omega_p x_q \frac{\partial}{\partial x_i}\epsilon_{jlm}\Omega_l x_g$$
$$= \epsilon_{ipq}\epsilon_{ijl}\Omega_p\Omega_l x_q$$
$$= (\delta_{pq}\delta_{ql} - \delta_{pl}\delta_{qj})\Omega_p\Omega_l x_q$$

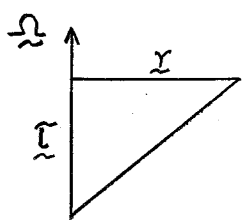
or

(1.12) $$-\frac{1}{\rho}\frac{\partial P}{\partial x_j} = \Omega_j\Omega_q x_q - \Omega_l^2 x_j$$

which has the solution

$$\frac{P}{\rho} = -\frac{1}{2}(\boldsymbol{\Omega}\cdot\boldsymbol{x})^2 + \frac{1}{2}\boldsymbol{\Omega}^2\boldsymbol{x}^2 + \text{const.}$$

Let us introduce the perpendicular distance from the axis of rotation, $\boldsymbol{r}$.



$$\boldsymbol{x} = \boldsymbol{\ell} + \boldsymbol{r}$$

Then we find

(1.13) $$\boxed{\frac{P}{\rho} = \frac{1}{2}\boldsymbol{\Omega}^2\boldsymbol{r}^2 + \text{const.}}$$

2. The Governing Equations in a Steadily Rotating Frame of Reference.

Since Newton's laws of motion only apply in an inertial frame of reference these laws must be modified for a rotating frame.

Let OS' be an inertial frame and OS be a frame rotating with a steady angular velocity $\underset{\sim}{\Omega}$ with respect to OS'.

Consider any vector $\underset{\sim}{P} = \overrightarrow{OP}$ then

$$\boxed{\left(\frac{\partial \underset{\sim}{P}}{\partial t}\right)_{S'} = \left(\frac{\partial \underset{\sim}{P}}{\partial t}\right)_{S} + \underset{\sim}{\Omega} \times \underset{\sim}{P}} \tag{2.1}$$

Proof: Suppose $\underset{\sim}{P}$ is fixed in the S-frame, i.e. P is a fixed point in the S-frame, then due to the motion of S it has a velocity $\underset{\sim}{\Omega} \times \underset{\sim}{P}$ relative to S'. Thus the result easily follows.

Consider a small element of fluid P. In the fixed frame of reference Newton's Law of Motion can be written

$$\rho \underset{\sim}{f} = \underset{\sim}{F}$$

where

$$\underset{\sim}{f} = \text{absolute acceleration of } P$$

and

$$\underset{\sim}{F} = \text{net surface force on the fluid element, } P.$$

Now

$$\underset{\sim}{f} = \left(\frac{\partial^2 \underset{\sim}{x}_P}{\partial t^2}\right)_{S'}, \text{ where } \underset{\sim}{x}_P \text{ is the position vector of } P.$$

Transforming to the rotating frame, using (2.1)

$$\underset{\sim}{f} = \left[\left(\frac{\partial}{\partial t}\right)_S + \underset{\sim}{\Omega} \times\right]\left[\left(\frac{\partial}{\partial t}\right)_S + \underset{\sim}{\Omega} \times\right]\underset{\sim}{x}_P$$

or $$\underset{\sim}{f} = \left(\frac{\partial^2 \underset{\sim}{X}_p}{\partial t^2}\right)_s + 2\underset{\sim}{\Omega} \times \left(\frac{\partial \underset{\sim}{X}_p}{\partial t}\right)_s + \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{X}_p)$$

If the small element $\underset{\sim}{X}_p$ is instantaneously coincident with a reference point $\underset{\sim}{X}$ fixed in the S frame we can write

$$\left(\frac{\partial \underset{\sim}{X}_p}{\partial t}\right)_s = \underset{\sim}{u}(\underset{\sim}{X}, t)$$

where $\underset{\sim}{u}$ is the fluid velocity relative to S.

Then

$$\left(\frac{\partial^2 X_p}{\partial t^2}\right)_s = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u}$$

by the usual argument.

Thus

(2.2) $$\underset{\sim}{f} = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} + \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \wedge \underset{\sim}{X}_p).$$

Now in a homogeneous incompressible fluid

$$\underset{\sim}{F} = -\nabla p + \mu \nabla^2 \underset{\sim}{U}$$

where $\underset{\sim}{U}$ is the absolute fluid velocity.

Thus

$$\underset{\sim}{F} = -\nabla p + \mu \nabla^2 (\underset{\sim}{u} + \underset{\sim}{\Omega} \times \underset{\sim}{X})$$

or

(2.3) $$\underset{\sim}{F} = -\nabla p + \mu \nabla^2 \underset{\sim}{u} \qquad \text{since } \nabla^2 \underset{\sim}{\Omega} \times \underset{\sim}{X} = 0$$

We can easily verify that not only the net viscous force, but the stress tensor itself, are given by the same expressions as in a non-rotating frame.

For $\tau_{ij} = \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$

In the rotating frame this becomes

$$\tau_{ij} = \mu\left[\frac{\partial}{\partial x_j}\left(u_i + \epsilon_{ipq}\Omega_p X_q\right) + \frac{\partial}{\partial x_i}\left(u_j + \epsilon_{jrs}\Omega_r X_s\right)\right.$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu\left\{\epsilon_{ipq}\Omega_p \delta_{qj} + \epsilon_{jrs}\Omega_r \delta_{is}\right\}$$

$$= \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \mu\left\{\epsilon_{ipj}\Omega_p + \epsilon_{jri}\Omega_r\right\}$$

or $\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Thus the stress tensor has the same form in the rotating frame of reference.

Finally we can absorb $\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{X})$ into the pressure term so that

$$P = p - \tfrac{1}{2}\rho\, \underset{\sim}{\Omega}^2 \underset{\sim}{r}^2$$

where $\underset{\sim}{r}$ is the distance from the axis of rotation. Now combining (2.2), (2.3) we obtain the equation of motion governing the fluid velocity $\underset{\sim}{u}(\underset{\sim}{x}, t)$ relative to a rotating frame of reference:

$$\boxed{\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\underset{\sim}{u}} \tag{2.4}$$

The continuity equation for an incompressible fluid in a rotating frame of reference is unchanged

(2.5) $$\boxed{\operatorname{div} \underset{\sim}{u} = 0}$$

Thus the basic equations of hydrodynamics in a rotating frame differ from those in a fixed frame only by the inclusion of the Coriolis term, $2\underset{\sim}{\Omega} \times \underset{\sim}{u}$, in equation (2.4).

We now prove a theorem which shows when rotation is <u>not</u> dynamically significant.

<u>Theorem</u> (G.I.Taylor, 1917).

If the velocity is perpendicular to the axis of rotation and independent of the coordinate parallel to the axis of rotation and if none of the boundary conditions explicitly involve the pressure, then the velocity field is unaffected by the rotation.

<u>Proof</u>: Choosing the z-axis as the axis of rotation we have

$$\frac{Du}{Dt} - 2\Omega v = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \nabla^2 u$$

$$\frac{Dv}{Dt} + 2\Omega u = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \nabla^2 v$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Introducing a stream function ψ

$$u = \frac{\partial \psi}{\partial y} , \quad v = -\frac{\partial \psi}{\partial x}$$

Thus

$$\frac{Du}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial x}\left(P + 2\Omega\psi\right) + \nu\nabla^2 u$$

and $$\frac{Dv}{dt} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(P + 2\Omega\Psi\right) + \nu\nabla^2 v$$

Therefore we can conclude that the flow is not affected by the rotation although the pressure field is altered.

We can now also prove that the basic rigid rotation velocity field is <u>stable</u> to small perturbations (proof due to Dr. G. Veronis). We have for a general finite perturbation

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla\underset{\sim}{u} + 2\underset{\sim}{\Omega}\times\underset{\sim}{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underset{\sim}{u}$$

Thus

$$\frac{1}{2}\frac{\partial(\underset{\sim}{u})^2}{\partial t} + \underset{\sim}{u}\cdot\left[(\underset{\sim}{u}\cdot\nabla)\underset{\sim}{u}\right] = -\frac{\underset{\sim}{u}}{\rho}\cdot\nabla P + \nu\underset{\sim}{u}\cdot\nabla^2\underset{\sim}{u}$$

or

$$\frac{1}{2}\frac{\partial u_i^2}{\partial t} + u_i u_j\frac{\partial u_i}{\partial x_j} = -\frac{u_i}{\rho}\frac{\partial P}{\partial x_i} + \nu u_i\frac{\partial^2 u_i}{\partial x_j{}^2}$$

Using the continuity equation this can be written

$$\frac{1}{2}\frac{\partial u_i^2}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x_j}\left(u_i^2 u_j\right) = -\frac{\partial}{\partial x_i}\frac{(\rho u_i)}{\rho} + \nu\frac{\partial}{\partial x_j}\left(u_i\frac{\partial u_i}{\partial x_j}\right) - \nu\left(\frac{\partial u_i}{\partial x_j}\right)^2$$

If we integrate this equation over all space and assume that the disturbance is small at infinity, the divergence theorem shows that

$$\frac{\partial}{\partial t}\int\frac{1}{2}u_i^2\,dV = -\nu\int\left(\frac{\partial u_i}{\partial x_j}\right)^2 dV$$

so that the kinetic energy of the perturbation decreases in time.

Lecture III

We next examine the orders of magnitude of the various terms in the equation of motion.

Assume that the relative motion has a velocity scale U, a length scale L, and a time scale τ.

The orders of magnitude of the terms in the equation are:

$$\frac{U}{\tau}, \quad \frac{U^2}{L}, \quad \Omega U, \quad \frac{P}{\rho}, \quad \nu\frac{U}{L^2}$$

where P is not scaled since it adjusts itself to balance the other terms. Dividing by ΩU we find the orders of magnitude of the terms are

$$\frac{1}{\Omega\tau}, \quad \frac{U}{L\Omega}, \quad 1, \quad \frac{P}{\rho U\Omega}, \quad \frac{\nu}{\Omega L^2}.$$

$\frac{U}{L\Omega}$ is the Rossby no., $R_o \equiv$ non-linear inertial forces/Coriolis forces. $\frac{\nu^2}{\Omega L^2}$ is $\frac{1}{R_e}$; where R_e = Reynolds number of the rotation $\equiv$ Coriolis forces/viscous forces.

If we now assume a) $R_o \ll 1$

b) $R_e^{-1} \ll 1$ (or $\nu \simeq 0$ i.e. inviscid flow)

c) and $\frac{\partial}{\partial t} = 0$; i.e. motion is steady,

the equation of motion reduces to

$$2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\underline{\nabla} P. \tag{2.6}$$

A steady flow in which the Coriolis and pressure forces balance is called geostrophic. Many large-scale meteorological and oceanographic flows approximately satisfy this condition.

The Taylor-Proudman theorem.

Taking the curl of the equation of geostrophy:

$$\text{curl}\left(2\underline{\Omega} \times \underline{u}\right) = 0$$

$$\text{(2.7)} \qquad \text{i.e. } (\underline{\Omega} \cdot \underline{\nabla})\underline{u} = 0$$

so that $\underline{u}$ is independent of the coordinate parallel to $\underline{\Omega}$, i.e. a slow, steady flow in an inviscid rotating fluid should always be two-dimensional.

Furthermore, if the fluid is bounded by a plane perpendicular to the axis of rotation, then since the axial component of fluid velocity $W = 0$ on this plane, it must be zero everywhere. Thus the fluid velocity is perpendicular to the axis of rotation and is the same in all planes perpendicular to the axis.

Consequently, if a sphere is moved sufficiently slowly through the fluid perpendicular to the axis of rotation the velocity field in all planes perpendicular to the axis will be identical with the velocity field in that plane which passes through the centre of the sphere. Thus the fluid must flow round the circular cylinder whose axis is parallel to the rotation axis and which circumscribes the sphere. We shall refer to such a cylinder of fluid as a Taylor column.

If there is no bounding plane perpendicular to the axis of rotation a sphere moved slowly along the axis of rotation will, according to the Taylor-Proudman theorem, push an infinite circumscribing cylinder of fluid ahead of it, and pull a similar infinite cylinder of fluid behind it.

The Taylor-Proudman theorem never predicts a unique velocity field in such situations e.g. in the case of a sphere moving along the axis of rotation of the fluid, the axial velocity of the fluid outside the Taylor column is arbitrary.

These predictions were tested experimentally by Taylor in two important papers.

In Proc. Roy. Soc. A.102, 180, Taylor described an experiment in which a sphere was moved along the axis of a cylinder of water which was rotating steadily. One of the results obtained was that if the sphere was moved sufficiently slowly, there was a tendency for a column of liquid of approximately the same diameter as the sphere to be pushed ahead of it. The flow behind the sphere in this case was not investigated.

The critical value of the Rossby no. $\left(= \frac{U}{a\Omega}\right)$ for the occurrence of this type of flow was given as 0.32, although Taylor does not give any other experimental details. (a = radius of sphere.)

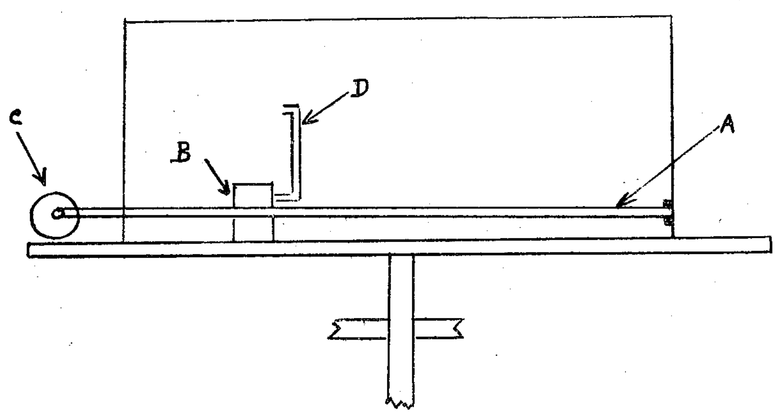
In a further paper (Proc. Roy. Soc. A 104, 213) Taylor discussed the possible classes of motion of a rotating fluid in which a three-dimensional disturbance was created.

Either 1 the motion may be always unsteady (this is unlikely since the disturbance would have to increase indefinitely in all directions away from the obstacle),

or 2 the motion may be steady, but not small in the neighborhood of the obstacle,

or 3 the motion may be steady and two-dimensional: in this case the obstacle would carry with it a Taylor column of fluid.

In order to check which of these possible flows takes place in practice, Taylor performed the following experiment:



A rectangular tank, filled with water, 9" x 12" x 4" deep was rotated on a turntable at a "considerable speed". A cylindrical obstacle B $1\frac{1}{4}$" diameter and 1" high was moved slowly across the bottom of the tank by turning a screwed rod A which passed through its centre. The rod was turned by a small electric motor C mounted on the turntable.

The streamlines of the flow were traced by introducing coloured ink into the fluid through a hypodermic tube D connected to the obstacle which served as a reservoir of ink. The ink was released at the appropriate instant by means of a pressurizing system attached to the tank which could be operated while the tank was in motion.

After allowing the water in the tank to attain solid-body rotation, Taylor performed the experiment and photographed the streamlines from above through the glass lid.

He found that ink released well above and in front of the obstacle divided sharply when it struck the imaginary circumscribing cylinder, flowed around it as though a solid obstacle had extended throughout the fluid i.e. a Taylor column was formed. Also, ink released inside the Taylor column remained there throughout the motion of the obstacle across the tank.

Unsteady motions.

We can ask the question: If a small three-dimensional velocity field is initially imposed on a rotating fluid can we trace the development of the flow into the 2D form implied by the Taylor-Proudman theorem? S.F.Grace (1926) considered the time-dependent motion of a sphere along the axis of a rotating fluid using the unsteady equations.

$$\frac{\partial u}{\partial t} + 2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\underline{\nabla} P$$

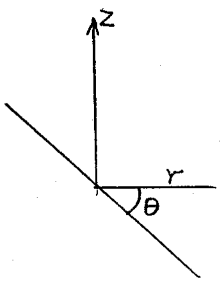
$$\mathrm{div}\, \underline{u} = 0$$

In using these equations we assume that the velocity can change rapidly, so that $1/\Omega r$ is not negligible, but that the velocity remains small, so that the inertia terms are negligible.

$$2\underline{\Omega}\times\underline{\tilde{u}} = -\frac{1}{\rho}\underline{\nabla}\tilde{P} - p\underline{\tilde{u}} + \text{const.}$$

Grace expanded variables in a power series in Ωt, and could thus not determine the asymptotic velocity field as $t \to \infty$. He was able, however, to find the force on an impulsively-started and proved that it attained its asymptotic value essentially in single rotation of the fluid. Göertler (1944) approaches the problem by studying axi-symmetric periodic solutions of the equations of motion, with a view to a Fourier synthesis for non-periodic motions.

In cylindrical polar coordinates, (r, θ, z) with Oz along axis of rotation



$$\frac{\partial u_r}{\partial t} - 2\Omega u_\theta = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

$$\frac{\partial u_\theta}{\partial t} + 2\Omega u_r = 0$$ assuming azimuthal symmetry.

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z}$$

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0.$$

We can introduce ψ such that $u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$, $u_z = +\frac{1}{r}\frac{\partial \psi}{\partial r}$.

and define $u_\theta = \phi/r$.

The equations become: $\dfrac{\partial}{\partial t}\left(D^2\psi\right) + 2\Omega\dfrac{\partial\phi}{\partial z} = 0$, $\left\{D^2 = \dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r} + \dfrac{\partial^2}{\partial z^2} = -r\,\omega_\theta\right\}$

$$\frac{\partial\phi}{\partial t} - 2\Omega\frac{\partial\psi}{\partial z} = 0.$$

(2.8) so that $\dfrac{\partial^2}{\partial t^2}\left(D^2\psi\right) + 4\Omega^2\dfrac{\partial^2\psi}{\partial z^2} = 0.$

If we assume a time dependence $e^{-i\beta t}$,

(2.9) $$\frac{\partial^2\psi}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \left(1 - \frac{4\Omega^2}{\beta^2}\right)\frac{\partial^2\psi}{\partial z^2} = 0.$$

If a) $\beta \gg 2\Omega$, $D^2\psi = 0$. i.e. the motion is irrotational and is unaltered by the presence of rotation.

b) $\beta > 2\Omega$, Equation is elliptic; modified irrotational flow.

c) $\beta = 2\Omega$, Parabolic equation, singular case.

d) $\beta < 2\Omega$, yperbolic equation, whose characteristics are

$$z = \pm\left(\frac{4\Omega^2}{\beta^2} - 1\right)^{1/2} r.$$

i.e. characteristics inclined to axis at angle $\tan^{-1}\left(\dfrac{4\Omega^2}{\beta^2} - 1\right)^{-1/2}$.

Thus for $\beta \ll 2\Omega$, the flow field is divided into two regions by a narrow cone spreading from the centre of disturbance.

Göertler saw this as a possible explanation of the Taylor column, since when $\beta << 2\Omega$ the velocity of the obstacle is practically unchanged over a rotation period and we have quasi-steady motion.

$\beta/2\Omega$ plays the role of a Rossby number for small unsteady motions as we have

$$\frac{\text{inertia force}}{\text{Coriolis force}} = O\left(\frac{1}{\Omega \tau}\right) = O\left(\beta/2\Omega\right).$$

Lecture IV

3. The unsteady solutions of Morgan and Stewartson

Forced Oscillations.

Consider the linearized equation for a flow symmetric about the axis of rotation of the fluid.

$$\frac{\partial^2}{\partial t^2}\left(D^2\psi\right)+4\Omega^2\frac{\partial^2\psi}{\partial z^2}=0, \qquad D^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2} \tag{3.1}$$

where ψ is the stream function as defined previously. Now, assume a time factor of $e^{-i\beta t}$ for the stream function, ψ, then (3.1) becomes

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\left(1-\frac{4\Omega^2}{\beta^2}\right)\frac{\partial^2\psi}{\partial z^2}=0 \tag{3.2}$$

and

$$-i\beta\phi+2\Omega\frac{\partial\psi}{\partial z}=0 \tag{3.3}$$

After the solution of (3.2) is obtained, ϕ may be computed from (3.3) and so the velocity component, u_θ, is found. Let $\psi = f(r)g(z)$; substituting into (2), we have

$$\frac{d^2g}{dz^2}-\frac{\beta^2\lambda^2}{\beta^2-4\Omega^2}g=0 \tag{3.4}$$

and

$$\frac{d^2f}{dr^2}-\frac{1}{r}\frac{df}{dr}+\lambda^2f=0 \tag{3.5}$$

where λ is an arbitrary separation constant. We imagine that the half space $z \geqslant 0$ is occupied by the rotating fluid and that u_z is prescribed on the plane $z = 0$. Then, if $\beta > 2\Omega$, the relevant solutions of (3.2) are of the form

$$\psi = A\, r\, J_1(\lambda r) \exp\left[-\frac{\lambda \beta z}{\sqrt{\beta^2 - 4\Omega^2}}\right], \tag{3.6}$$

the positive exponential solution of (3.4) being eliminated by the obvious requirement of boundness at $z = \infty$ and the $r\,Y_1(\lambda r)$ solution of (3.5) being eliminated by requiring that the velocity is finite on $r = 0$.

If $0 < \beta < 2\Omega$ the solutions are wavelike and we must replace the condition of boundness at ∞ by a radiation condition. We can ensure that (3.6) satisfies this condition by choosing the appropriate branch of the function $\sqrt{\beta^2 - 4\Omega^2}$ in the β plane. Let us suppose that, once again assuming a time factor $e^{-i\beta t}$,

$$u_z = U J_0(\lambda r) \quad \text{on} \quad z = 0 \tag{3.7}$$

[A more general r dependence of u_z can be represented as a superposition of Bessel functions with different λ's since an arbitrary function can be represented in the form

$$f(r) = \int_0^\infty B(\lambda)\, J_0(\lambda r)\, d\lambda\ ] \tag{3.8}$$

The boundary condition (3.7) fixes A and we have

$$\psi = \frac{U}{\lambda} r\, J_1(\lambda r) \exp\left\{\frac{-\lambda \beta z}{\sqrt{\beta^2 - 4\Omega^2}}\right\} \tag{3.9}$$

If $\beta > 2\Omega$ the motion is essentially confined to a region of width $O(\lambda^{-1})$ adjoining the plate, whilst if $\beta < 2\Omega$ the disturbance propagates to $z = \infty$ as progressive waves. We note that as $\beta \to 0$ the z dependence disappears and we approach a two-dimensional motion, just as predicted by Göertler's general analysis.

If $\beta = 2\Omega$ the analysis fails. For

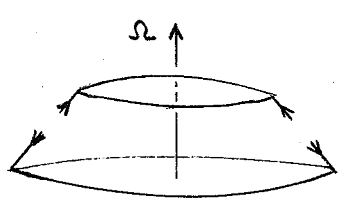
$$u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} = \frac{U\beta}{\sqrt{\beta^2-4\Omega^2}} J_1(\lambda r) \exp\left\{\frac{-\lambda\beta z}{\sqrt{\beta^2-4\Omega^2}}\right\} \tag{3.10}$$

so that $u_r(r,0) \to \infty$ like $\dfrac{1}{\sqrt{\beta-2\Omega}}$ as $\beta \to 2\Omega+$.

This square root singularity is characteristic of resonance. Thus the case $\beta = 2\Omega$ requires consideration of the neglected viscous and inertia terms. If we put $\beta = 2\Omega$ in (3.2) we see that the general solution regular on the axis of rotation is

$$\psi = r^2 f(z)$$

If $f(0) = 0$ we have $u_z = 0$ on the plane $z = 0$, so the solution represents a free oscillation in the region we have been considering. It is this oscillation that is excited as $\beta \to 2\Omega$ leading to the resonance.



Rings of fluid move as shown. If a ring of fluid expands its angular velocity will decrease to preserve its angular momentum and it will be forced back by the pressure gradient, which increases outwards in a rigidly rotating fluid. The oscillation is really an unsteady stagnation flow and we shall find such oscillations near the stagnation points on moving obstacles in a rotating fluid.

Morgan's solution for the unsteady motion of a circular disc

The main objective of the present section is to see whether an initial small three-dimensional disturbance will tend to a two-dimensional form as time proceeds. An interesting general argument is due to Morgan (1951). Consider the basic equation

$$\frac{\partial^2}{\partial t^2}\left(D^2\psi\right) + 4\Omega^2 \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.11}$$

with the initial conditions,

$$\psi(r,z,0) = 0; \phi(r,z,0) = 0 \Rightarrow \frac{\partial \psi}{\partial t}(r,z,0) = 0 \tag{3.12}$$

We suppose that $u_z(r,0,t)$ is prescribed and consider the flow in the half space $z \geqslant 0$. Applying the Laplace transform to (3.11) with the initial conditions (3.12), we have

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \tilde{\psi}}{\partial r} + \left(1 + \frac{4\Omega^2}{p^2}\right)\frac{\partial^2 \tilde{\psi}}{\partial z^2} = 0 \tag{3.13}$$

where $\psi \int_0^\infty e^{-pt}\psi(r,z,t)\,dt.$

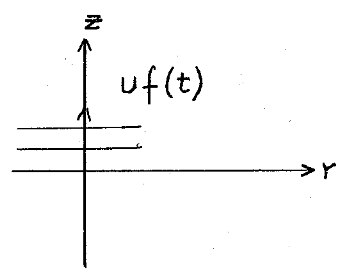
By separation of variables, the solution appropriate to $z > 0$ is

$$\tilde{\psi} = A r J_1(\lambda r) \exp\left\{\frac{-p\lambda z}{\sqrt{p^2+4\Omega^2}}\right\} \tag{3.14}$$

or, in general, a superposition of such solutions. By the Tauberian theorems for the Laplace transform we can infer the behaviour of a function as $t \to \infty$ from the behaviour of its Laplace transform as $p \longrightarrow 0$. We see from (3.14) that as $p \longrightarrow 0$ the motion becomes two-dimensional. However, as Morgan points out, the conditions under which the theorem holds may not be satisfied and one must consider concrete examples.*

He considers the flow set up by a circular disk of a radius a, which is perpendicular to the axis of rotation of the fluid and is given a velocity V(t), along its own normal. We demand that V(0) = 0 and $V(\infty) = V$. We have the same basic equation (3.11), together with the initial conditions

$$\psi(r,z,0) = \phi(r,z,0) = 0$$



and the boundary conditions

$$\begin{aligned} u_z &= V(t) \text{ when } z = 0, 0 \leq r \leq a \\ u_r &= 0 \text{ when } z = 0, \ r \geq a \end{aligned} \tag{3.15}$$

where the second condition is due to the asymmetry of the problem about z = 0.

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* An obvious counter-example is sin t whose behaviour as $t \to \infty$ cannot be inferred from that of its transform $\frac{1}{p^2+1}$.

It is important to note that it is consistent to apply the boundary condition (3.15) at the plane z = 0 rather than at the actual position of the disc. This is because the velocity of disc is arbitrarily small - more precisely, the distance moved by the disc in time T is $O(VT)$ so that the distance it moves as a fraction of its own radius is $O(\frac{VT}{a})$. Now the asymptotic flow must be established in a time of order $\frac{1}{\Omega}$, since $\frac{1}{\Omega}$ is the only time scale in the equations. Thus the fractional distance moved and hence the error in the velocity field, is of order $\frac{V}{a\Omega}$ which is of the same order as the neglected inertia terms. Taking the Laplace transform of (3.11) and (3.15), we have

$$\frac{\partial^2\tilde{\Psi}}{\partial r^2} - \frac{1}{r}\frac{\partial^2\tilde{\Psi}}{\partial r} + \left(1 + \frac{4\Omega^2}{p^2}\right)\frac{\partial^2\tilde{\Psi}}{\partial z^2} = 0 \tag{3.16}$$

and the boundary conditions

$$\begin{aligned} \tilde{u}_z &= \tilde{V}(p) \text{ when } z = 0 \text{ and } 0 \leq r \leq a \\ \tilde{u}_r &= 0 \quad \text{ when } z = 0 \text{ and } r \geq a \end{aligned} \tag{3.17}$$

The solution of (3.16) satisfying the boundary condition at infinity is

$$\tilde{\Psi} = \tilde{V}(p)\int_0^\infty A(\lambda)\, r\, J_1(\lambda r) \exp\left\{\frac{-\lambda p z}{\sqrt{p^2 + 4\Omega^2}}\right\} d\lambda \tag{3.18}$$

To satisfy the boundary conditions (3.17), then we have

$$\left.\begin{aligned}&\int_0^\infty A(\lambda)\lambda J_0(r\lambda)\,d\lambda = 1 \text{ for } 0 \le r \le a\\ \text{and}&\\ &\frac{p}{\sqrt{p^2+4\Omega^2}}\int_0^\infty A(\lambda)\lambda r J_1(r\lambda)\,d\lambda = 0 \text{ for } r \geqslant a\end{aligned}\right\} \tag{3.19}$$

Differentiating the second equation with respect to r gives

$$\begin{aligned}&\int_0^\infty A(\lambda)\lambda J_0(r\lambda)\,d\lambda = 1 \text{ for } 0 \le r \le a\\ \text{and}&\\ &\int_0^\infty A(\lambda)\lambda^2 J_0(r\lambda)\,d\lambda = 0 \text{ for } r \geqslant a\end{aligned} \tag{3.20}$$

Tranter ("Integral Transforms") gives a discussion of these dual integral equations. However, we may note that $A(\lambda)$ is independent of Ω, so that (3.20) must be the same dual integral equations as in the irrotational case. Thus the unknown function $A(\lambda)$ may be determined by comparison with the known solution in this case (Lamb "Hydrodynamics" p.138) and we have

$$A(\lambda) = -\frac{2}{\pi\lambda}\left(a\cos\lambda a - \frac{\sin\lambda a}{\lambda}\right) \tag{3.21}$$

Hence, the solution becomes

$$\begin{aligned}\psi &= -\frac{2}{\pi}\tilde{v}(p)\int_0^\infty\left(\frac{a\cos\lambda a}{\lambda} - \frac{\sin\lambda a}{\lambda^2}\right) r J_1(r\lambda)\exp\left\{\frac{-\lambda p z}{\sqrt{p^2+4\Omega^2}}\right\}d\lambda\\ \text{and } \tilde{u}_z &= -\frac{2\tilde{v}(p)}{\pi}\int_0^\infty\left(a\cos\lambda a - \frac{\sin\lambda a}{\lambda}\right) J_0(r\lambda)\exp\left\{\frac{-\lambda p z}{\sqrt{p^2+4\Omega^2}}\right\}d\lambda\end{aligned} \tag{3.22}$$

We examine the solutions for large values of time.

By Tauber's theorem, $t \longrightarrow \infty$ corresponds to $p \longrightarrow 0$, so we can determine the asymptotic solution by letting $p \longrightarrow 0$ in (3.22). (A more vigorous procedure is to use the inversion formula and evaluate for large but finite t.) Hence

$$U_z = V \quad \text{when} \quad 0 < r < a$$

(3.23)

$$= \frac{2V}{\pi}\left\{ \sin^{-1}\frac{a}{r} - \frac{a^2}{r^2 - a^2} \right\} \quad \text{when} \quad r > a$$

When $r = 0$, however, the integral requires special treatment since the integral obtained by putting $p = 0$ is not convergent in the ordinary sense in this case. (Morgan does not examine this case.) Putting $r = 0$ we find

$$\tilde{u}_z(0, z, p) = -\frac{2V}{\pi}\int_0^{\infty}\left(a\cos a\lambda - \frac{\sin\lambda a}{\lambda}\right)e^{\frac{-pz\lambda}{2\Omega}}\,d\lambda$$

for small p, so that

$$\tilde{u}_z(0, z, p) = -\frac{4Va\Omega}{z}\;\frac{p}{p^2 + \frac{4\Omega^2 a^2}{z^2}} + V \tag{3.24}$$

hence, the inversion gives

$$u_z(0, z, t) = V - \frac{4Va\Omega}{z}\cos\left(\frac{2\Omega a t}{z}\right) \tag{3.25}$$

which oscillates finitely as $t \longrightarrow \infty$. Sarma (1958) has investigated this sort of oscillation in detail.

Similar special treatment is needed for

Morgan's result shows that the small three-dimensional initial motion set up by a disk normal to and moving parallel to the axis of rotation if the fluid tends asymptotically to a small, steady, two-dimensional motion almost everywhere. A Taylor column is eventually formed, just as asserted by the Taylor-Proudman theorem.

We may recall that the problem of determining the flow-field for the case of non-zero rotation was reduced to the same problem for zero rotation. Morgan has developed an interesting similarity theory relating rotating and non-rotating flows. This theory is reminiscent of the Prandtl-Glauert similarity theory for linearized subsonic inviscid flows and indeed the governing equations in the two cases are similar. Details are to be found in Morgan's paper.

Lecture V.

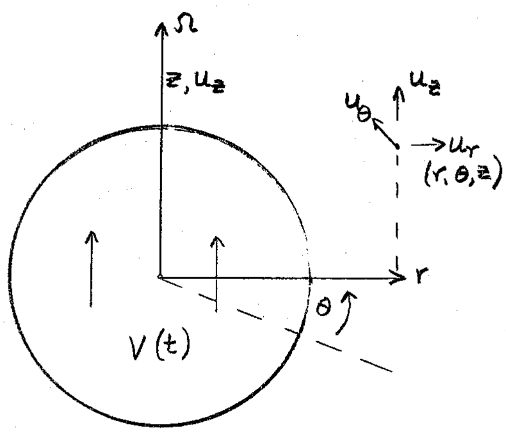
Unsteady motion of a sphere on the axis of a rotating liquid.

(Stewartson 1953).

We adopt cylindrical polar coordinates whose axis coincides with the axis of rotation and with the path of the sphere and whose origin coincides with the initial position of the centre of the sphere.

We assume that at $t = 0$ the sphere starts moving with velocity $V(t)$ where $V(0) = 0$ and $V(\infty) = V$. (Stewartson considers impulsive motion and takes axes moving with the sphere, but the analysis is essentially unaltered.)

We assume the sphere is of radius a.



Then the equations of motion in cylindrical polar coordinates for the axisymmetric motion,

$$\frac{\partial u_r}{\partial t} - 2\Omega u_\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{\partial u_\theta}{\partial t} + 2\Omega u_r = 0 \qquad (3.26)$$

$$\frac{\partial u_z}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

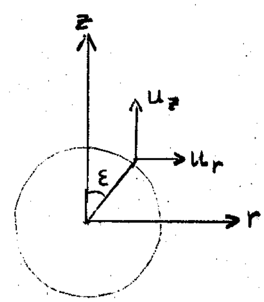
and $$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0$$

with the boundary conditions for all finite t,

$$u_r, u_\theta, u_z \rightarrow 0 \text{ as } z \rightarrow \pm\infty \text{ all } r. \qquad (3.27)$$

and $u_z \cos\varepsilon + u_r \sin\varepsilon = V(t)\cos\varepsilon$ on the body

or $$u_z z + u_r r = V(t) z \quad \text{on } r^2 + z^2 = a^2 \qquad (3.28)$$



Taking the Laplace transform of the equations and boundary conditions we have, since the disturbance is everywhere zero initially*,

$$\left.\begin{aligned} p\tilde{u}_r - 2\Omega\tilde{u}_\theta &= -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r} \\ p\tilde{u}_\theta + 2\Omega\tilde{u}_r &= 0 \\ p\tilde{u}_z &= -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r} \\ \text{and} \quad \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}_r) + \frac{\partial \tilde{u}_z}{\partial z} &= 0 \end{aligned}\right\} \qquad (3.29)$$

with the boundary conditions

*In the case of inpulsive motion, the initial disturbance is the irrotational flow past the sphere (Morgan 1953).

$$\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z \to 0 \quad \text{as } z \to \infty \text{ all } r \tag{3.30}$$

and

$$\tilde{u}_z z + \tilde{u}_r r = \tilde{V}(p) z \quad \text{on } r^2 + z^2 = a^2 \tag{3.31}$$

Solving for the velocity components, we have

$$\tilde{u}_r = \frac{p}{p^2+4\Omega^2}\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r}, \quad \tilde{u}_\theta = \frac{2\Omega}{p^2+4\Omega^2}\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial r}, \quad \tilde{u}_z = -\frac{1}{p}\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial z}$$

then substituting these into the continuity equation, we have

$$\frac{\partial^2 \tilde{p}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{p}}{\partial r} + \left(\frac{p^2+4\Omega^2}{p^2}\right)\frac{\partial^2 \tilde{p}}{\partial z^2} = 0 \tag{3.32}$$

and the boundary conditions become

$$\frac{\partial \tilde{p}}{\partial z} = 0 \quad \text{when } z \to \pm\infty \text{ for fixed } r \tag{3.33}$$

and

$$\frac{r p^2}{p^2+4\Omega^2}\frac{\partial \tilde{p}}{\partial r} + z\frac{\partial \tilde{p}}{\partial z} = -z\rho p \tilde{V}(p) \quad \text{on } r^2 + z^2 = a^2 \tag{3.34}$$

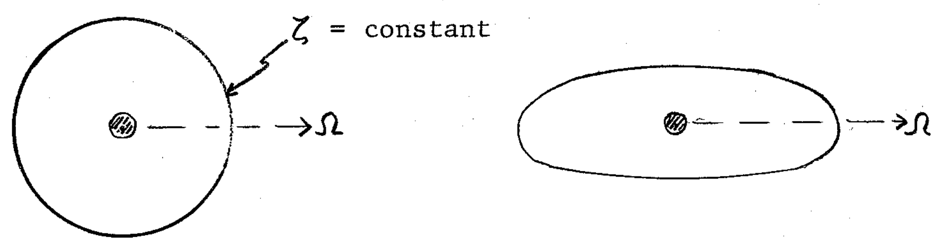
If we absorb the factor $p^2+4\Omega^2/p^2$ in (3.32) by scaling z (3.32) would become Laplace's equation, whilst the sphere would become an ellipsoid of revolution. This suggests the use of ellipsoidal coordinates and we introduce ellipsoidal coordinates (μ, ζ) with

$$\left.\begin{aligned} z &= \frac{2\Omega}{p} a\mu\zeta \\ \text{and} \quad r &= \frac{2\Omega a}{p^2+4\Omega^2}(1-\mu^2)^{\frac{1}{2}}(\zeta^2+1)^{\frac{1}{2}} \end{aligned}\right\} \tag{3.35}$$

It can be shown that ζ = constant is a surface,

$$\frac{r^2}{\left[\dfrac{4\Omega^2 a^2}{p^2+4\Omega^2}\right](\zeta^2+1)} + \frac{z^2}{\left(\dfrac{4a^2\Omega^2}{p^2}\right)\zeta^2} = 1 \tag{3.36}$$

These prolate ellipsoids have the form shown.



For large real p (small t) For small p (large t)

Thus the coordinate surfaces tend to become 'two-dimensional' at large t .

Now, in these coordinates, the pressure $\tilde{p}$ satisfies

$$\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \tilde{p}}{\partial \zeta}\right] + \frac{\partial}{\partial \zeta}\left[(\zeta^2+1)\frac{\partial \tilde{p}}{\partial \zeta}\right] = 0 \tag{3.37}$$

and the boundary conditions become

$$\frac{\partial \tilde{p}}{\partial \zeta} = -2\Omega a \mu \tilde{V}(p) \text{ on } \zeta = \frac{p}{2\Omega} \tag{3.38}$$

and

$$\frac{\partial \tilde{p}}{\partial \zeta} \to 0 \text{ as } \zeta \to \infty \tag{3.39}$$

where in obtaining (3.39) from (3.33) we have used the conditions,

$$\frac{\partial \tilde{p}}{\partial \zeta} = \frac{z}{\zeta}\frac{\partial \tilde{p}}{\partial z} + \frac{r\zeta}{\zeta^2+1}\frac{\partial \tilde{p}}{\partial r}$$

and $\zeta = O(z)$ as $z \to \infty$ for fixed r.

The problem imposed by (3.37) and the boundary conditions is similar to that of finding the potential flow past an ellipsoid and the solution (see Lamb, 1932) is given as

$$\tilde{p} = - \frac{2\Omega a \tilde{V}(p)\mu \left\{ \zeta \log \frac{\zeta - i}{\zeta + i} + 2i \right\}}{\log \frac{p - 2\Omega i}{p + 2\Omega i} + \frac{4ip\Omega}{p^2 + 4\Omega^2}}$$

and finally an inversion of (3.39) yields the solutions for p and hence the velocity components (u_r, u_θ, u_z).

The inversion involves complicated integration in complex plane (see Stewartson, 1952) and we shall only give the results.

As $t \longrightarrow \infty$, the solution gives

$$u_z \rightarrow V, \; u_\theta \rightarrow \frac{2Vr}{\pi(a^2-r^2)^{1/2}}, \quad u_r \longrightarrow 0 \qquad \text{when } 0 < r < a$$

and

$$u_z \longrightarrow \frac{2V}{\pi}\left(\sin^{-1}\frac{a}{r} - \frac{a}{\sqrt{a^2-r^2}}\right), \; u_r \rightarrow 0, \; u_\theta \longrightarrow 0 \quad \text{when } r > a$$

while the motion is never steady at $r = 0$ and $r = a$ and on the surface of the sphere.

Thus a Taylor column is formed again confirming the predictions of the Taylor-Proudman theorem.

We note that the singularity on the surface of the Taylor column occurs just as in Morgan's solution. Indeed the asymptotic velocity fields are identical. Thus this singularity is <u>not</u> due to the sharp edge of the disc, but must have a more fundamental significance.

Stewartson (1953) also considered the motion of a sphere perpendicular to the axis of rotation. Once again something resembling a Taylor column is formed, but the streamlines which cross the Taylor column have to turn sharply. Stewartson considered that the flow would be more likely to separate and flow around the Taylor column, rather in the manner Taylor observed experimentally.

In summary, we can conclude that study of the unsteady linear equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} + 2\underset{\sim}{\Omega}\wedge\underset{\sim}{u} = -\nabla P \qquad (3.40)$$
$$\mathrm{div}\,\underset{\sim}{u} = 0$$

tends to confirm the conclusions of the Taylor-Proudman theorem. Furthermore the non-uniqueness inherent in the Taylor-Proudman theorem is removed.

It has been remarked by Crapper (1959) that the examination of unsteady flows in order to remove a non-uniqueness is formally the same as Rayleigh's device of fictitious viscosity. If we take the Laplace transform of (3.40) we have

$$2\underset{\sim}{\Omega}\wedge\tilde{\underset{\sim}{u}} = -\nabla\tilde{P} - p\,\tilde{\underset{\sim}{u}}\,,$$
$$\mathrm{div}\,\tilde{\underset{\sim}{u}} = 0$$

This suggests that similar results will be obtained if we reinsert the real viscous terms i.e. we examine

$$2\underset{\sim}{\Omega} \wedge \underset{\sim}{u} = -\nabla P + \nu \nabla^2 \underset{\sim}{u},$$

$$\operatorname{div} \underset{\sim}{u} = 0.$$

This was done for the case of the circular disc by Morrison and Morgan (1956). Results in agreement with Morgan's (1951) solution were obtained, with the discontinuity on the surface of the Taylor column being replaced by a thin viscous shear layer.

Lecture VI

We have seen how the Taylor-Proudman theorem predicts the formation of a Taylor column and that, as long as inertia forces are negligible, any small three-dimensional initial motion will organize itself into a motion satisfying the Taylor-Proudman theorem. Can we understand this behaviour physically?

If a liquid in solid rotation is thought of as a field of parallel vortex lines, one remarks that the distortion and consequent stretching of these lines will require energy. Thus if a sphere is slowly moving along the axis of rotation, it has small kinetic energy and cannot distort the vortex lines to permit flow around it. The only alternative is that the column ahead of the sphere be pushed along without deformation.

A more direct argument involves the pressure field. A very small ring of fluid about the axis, in front of the sphere, must negotiate a considerable pressure gradient if it is to expand and pass around the moving body. The kinetic energy of the body, however, is again too small to provide the energy for this motion and the Taylor column results.

This last argument brings out the similarity of the phenomenon to the occurrence of 'blocking' when a uniform stream of given depth encounters an obstacle of too great a height

Fluid elements have not enough kinetic energy to overcome the vertical pressure gradient and an increase in depth propagates upstream.

Attempt at higher approximations

Let us consider the ultimate velocity field for a sphere moving at a uniform speed V on the axis of a rotating liquid. As shown above, the first approximation, which satisfies the equation

$$2\,\underset{\sim}{\Omega} \wedge \underset{\sim}{u} = -\frac{1}{\rho}\nabla P$$

is $$u_r^\circ = 0 \;;\; u_\theta^\circ = \frac{2rV}{\pi(a^2-r^2)^{1/2}} \;;\; u_z^\circ = V \text{ for } r < a$$

and

$$u_r^\circ = 0 \;;\; u_\theta^\circ = 0 \;;\; u_z^\circ = \frac{2V}{\pi}\left(\sin^{-1}\frac{a}{r} - \frac{a}{(a^2-r^2)^{1/2}}\right) \text{ for } r > a$$

We might expect that further approximations using the full inviscid equations would lead to an exact solution. We shall see that this is impossible.

Putting

$$\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1 \quad \text{and} \quad p = p_0 + p_1$$

into the steady inviscid equations we get

$$-\frac{(u_\theta^\circ)^2}{r} - 2\Omega u_\theta = -\frac{1}{\rho}\frac{\partial p_1}{\partial r} \tag{3.41}$$

$$2\Omega u_r = 0 \tag{3.42}$$

$$0 = -\frac{1}{\rho}\frac{\partial p_1}{\partial z} \tag{3.43}$$

and the continuity equation is

$$\frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{\partial u_z}{\partial z} = 0.$$

From (3.42) this becomes

$$\frac{\partial u_z}{\partial z} = 0$$

and since

$$u_z = 0 \tag{3.44}$$

on the sphere (3.44) holds everywhere. Now as

$$\frac{(u_\theta^\circ)^2}{r} \sim \frac{1}{r-a} \quad \text{for } r \sim a$$

we see from (3.41) that

$$u_\theta \sim \frac{1}{r-a}$$

Hence u_θ has a worse singularity at r = a than u_θ° . It can easily be seen that $u_\theta^{(n+1)}$ has a worse singularity than $u_\theta^{(n)}$ and so the process fails.

Our failure must make us suspicious of the solutions provided by the Taylor-Proudman theorem. One might expect that the solution of the full inviscid equations $\psi(\underset{\sim}{x}, R_o)$ for flow past an obstable would possess an expansion

$$\psi(\underset{\sim}{x}, R_o) = \psi_o(\underset{\sim}{x}) + R_o \psi_1(\underset{\sim}{x}) + \ldots, \quad R_o \to 0$$

where $\psi_o(\underset{\sim}{x})$ is the Taylor-Proudman solution and R_o the Rossby number. However, this seems not to be the case. It is possible that $\psi(\underset{\sim}{x}, R_0)$ is not analytic in R_0 as $R_0 \to 0$.

Clearly it is desirable to study the non-linear system obtained by re-inserting the non-linear inertia term $\underline{u} \cdot \nabla \underline{u}$. We shall consider what has been achieved in this direction in the next section.

Lecture VII

§ 4. Exact Solutions of the Inviscid Equations

Assuming a steady state, the equations of motion become

$$\underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + 2 \underset{\sim}{\Omega} \wedge \underset{\sim}{u} = -\frac{1}{\rho} \nabla P. \tag{4.1}$$

Going back to a fixed coordinate system, (4.1) becomes

$$\underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\frac{1}{\rho} \nabla p. \tag{4.2}$$

Considering only axi-symmetric solutions, we can define a stream-function and swirl by

$$v_z = \frac{1}{r} \frac{\partial \psi}{\partial r} \; ; \; v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z} \; ; \; v_\theta = \frac{\phi}{r}. \tag{4.3}$$

Then after elimination of p by cross-differentiation we get from (4.2)

$$2 \frac{\phi}{r^2} \cdot \frac{\partial \phi}{\partial z} + \frac{1}{r} \cdot \frac{\partial (\psi, D^2 \psi)}{\partial (r, z)} + \frac{2}{r^2} \frac{\partial \psi}{\partial z} (D^2 \psi) = 0, \tag{4.4}$$

and

$$\frac{\partial (\psi, \phi)}{\partial (r, z)} = 0 \tag{4.5}$$

where $$D^2\psi \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\psi = -r\,\omega_\theta$$

(See Goldstein: Modern Developments in Fluid Dynamics. Vol.1)

If we suppose

$$D^2\psi = \alpha\psi + \beta(r) \tag{4.6}$$

and

$$\phi = \gamma\psi \tag{4.7}$$

where α and γ are constants and $\beta(r)$ is a function of r only then (4.5) is satisfied and we have

$$\frac{\partial(\psi, D^2\psi)}{\partial(r, z)} = -\frac{d\beta}{dr}\cdot\frac{\partial\psi}{\partial z} \tag{4.8}$$

Thus (4.4) becomes

$$2\frac{\gamma^2\psi}{r^2}\frac{\partial\psi}{\partial z} - \frac{1}{r}\frac{d\beta}{dr}\frac{\partial\psi}{\partial z} + \frac{2}{r^2}\frac{\partial\psi}{\partial z}\left(\alpha\psi + \beta(r)\right) = 0$$

or

$$\frac{2\psi}{r^2}\cdot\frac{\partial\psi}{\partial z}\left(\gamma^2+\alpha\right) - \frac{\partial\psi}{\partial z}\left(\frac{1}{r}\frac{d\beta}{dr} - \frac{2\beta}{r^2}\right) = 0 \tag{4.9}$$

If we choose

$$\alpha = -\gamma^2 \quad \text{and} \quad \beta = C r^2 \tag{4.10}$$

where C is a constant then

$$\left.\begin{aligned} D^2\psi &= -\gamma^2\psi + C r^2 \\ \text{and} \quad \phi &= \gamma\psi \end{aligned}\right\} \tag{4.11}$$

lead to exact solutions of the full non-linear equations. This was first found by Long (J.Met. 10 (1953)). A more general treatment is given by Squire : Surveys in Mechanics, C.V.P. 1956.

Long showed that (4.11) could be used to discuss the disturbance to a uniformly rotating uniform stream. Suppose

$$\psi = \tfrac{1}{2} U r^2 \text{ and } \phi = \Omega r^2$$

Then $\gamma = \dfrac{2\Omega}{V}$ and the first of eqn. (4.11) is satisfied if

$$C = \frac{2\Omega^2}{V}.$$

Now let us assume a finite disturbance, setting

$$\psi = \tfrac{1}{2} V r^2 + \psi^* \quad , \quad \phi = \Omega r^2 + \phi^* \tag{4.12}$$

From (4.11) and (4.12) we get

$$\phi^* = \frac{2\Omega}{U} \psi^*,$$

$$D^2 \psi^* + \frac{4\Omega^2}{U^2} \psi^* = 0. \tag{4.13}$$

If we can find solutions of equation (4.13) which tend to zero at upstream infinity we shall have found exact solutions representing the disturbance to the initially uniformly rotating uniform stream.

If we introduce dimensionless coordinates

$r' = \dfrac{r}{L}$ and $z' = \dfrac{z}{L}$, (4.13) becomes

$$D'^2 \psi^* + \frac{4L^2\Omega^2}{U^2} \psi^* = 0, \tag{4.14}$$

where $\dfrac{4L^2\Omega^2}{V^2} = \dfrac{4}{R_0^2} = K^2.$

(4.14) is similar to the Helmholtz wave equation. Thus we infer

(A) We need something like a radiation condition to make the solutions of (4.14) unique

(B) Small R_o is analogous to short wavelengths. Thus we may expect the limit $R_o \rightarrow 0$ to present difficulties. We may note the similarity of a Taylor column to a shadow.

Finally we may note that Long proved that <u>every</u> disturbance to a uniformly rotating uniform stream satisfies (4.13) so long as every stream surface starts at upstream infinity. We can see this most easily from Squire's general analysis. Squire shows that in a general axi-symmetric flow

$$H = p + \tfrac{1}{2}\rho\, \underset{\sim}{u}^2 \quad \text{and} \quad \phi = r\, v_\theta \qquad \text{are functions of } \psi$$

only and that

$$D^2\psi = \frac{r^2}{\rho}\frac{dH}{d\psi} - \phi\frac{d\phi}{d\psi}$$

The functions H(ψ) and K(ψ) are fixed by conditions far upstream and for a uniformly rotating uniform stream we readily recover (4.11). But if there exist closed stream surfaces H(ψ) and K(ψ) will in general be different on these. Thus we cannot treat such disturbances. Long points out that such regions of closed streamlines may be unstable in general, since the circulation, being constant on their surface by (4.11), will not increase outwards.

Lecture VIII

As we will see, a non-uniqueness will occur in the solution of equation (4.13) because of a lack of a radiation boundary condition. A special solution for flow past a sphere was given by G. I. Taylor (P.R.S. A 102, 1922).

He uses the boundary conditions

$$\psi^* = o\left(\tfrac{1}{2} U r^2\right) \text{ as } r \to \infty$$

$$\psi^* = -\tfrac{1}{2} U r^2 \quad \text{on the sphere of radius } a .$$

Adopting spherical polar coordinates, R, Θ, Φ, (4.13) becomes

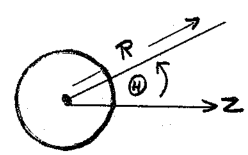
$$\frac{\partial^2\psi^*}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2\psi^*}{\partial\Theta^2} - \frac{\cot\Theta}{R^2}\frac{\partial\psi^*}{\partial\Theta} + K^2\psi^* = 0 \qquad (4.15)$$

where $K = \dfrac{2\Omega}{U}$

and the boundary conditions are now

$$\psi^* = o\left(\tfrac{1}{2} U R^2\right) \text{ as } R \to \infty$$

$$\psi^* = \tfrac{1}{2} U a^2 \sin^2\Theta \text{ on } R = a$$



We seek a solution of the form

$$\psi^* = f(R)\sin^2\Theta$$

and find from (4.15)

$$\frac{d^2 f}{dR^2} - \frac{2f}{R^2} + K^2 f = 0$$

The transformation

$$f = R^{\frac{1}{2}} g$$

will make this Bissel's equation:

$$g'' + \frac{1}{R} g' + \left(K^2 - \frac{(3/2)^2}{R^2}\right) g = 0,$$

with the general solution

$$g = A J_{3/2}(KR) + B J_{-\frac{3}{2}}(KR)$$

which gives

$$f = A'\left(\frac{\sin KR}{KR} - \cos KR\right) + B'\left(\frac{\cos KR}{KR} + \sin KR\right)$$

The resulting ψ^* satisfies the condition at ∞ for all A' and B', and the condition on the body gives only one relation between A' and B'. The problem is thus indeterminate.

Taylor further required that the fluid not slip on the boundary of the sphere, which makes A' and B' unique. But this manner of removing the indeterminacy is not relevant to the inviscid solution.

Taylor performed an experiment consisting of a liquid in solid-body rotation with a light sphere at the axis. The sphere, initially at rest with respect to the liquid, was drawn along the axis of rotation with a cord. He found that for values of the Rossby number $U/a\Omega$ above 0.3 the ball immediately stopped rotating (with respect to a fixed observer), when its axial motion was begun. For R_o less than 0.3, a Taylor column formed.

An explanation of the former case is as follows: for sufficiently large R_o, where a small ring of fluid on the axis of rotation is able to negotiate the pressure gradient and pass around the sphere, conservation of circulation implies that the initial circulation and the circulation at the equator satisfy

$$r_1^2 \Omega = r_1 v_\theta^1 = a v_\theta^2$$

where r_1 is the initial radius of the ring, a the radius of the sphere, v_θ^1 and v_θ^2 the tangential velocities, and Ω the rotation speed of the fluid.

So we have

$$v_\theta^2 << a \Omega$$

since $r_1 << a$, and thus the ring of fluid is rotating much more slowly than the sphere when it is near the equator and viscous shear brings the sphere to rest.

When R_o is less than 0.3, the small ring of fluid cannot negotiate the pressure gradient and pass around the sphere, and upstream blockage - a Taylor column - occurs.

Long (J.Met. 10, 1953) gave a more general solution of equation (4.15):

$$\psi^* = \sin^2\theta \, \frac{d\,P_n(\cos\theta)}{d\,(\cos\theta)} (KR)^{1/2} \left[A J_{n+\frac{1}{2}}(KR) + B J_{-n-\frac{1}{2}}(KR) \right]$$

where P_n (cos θ) is a Legendre polynomial of order n. If

$$A J_{n+\frac{1}{2}}(Ka) + B J_{-n-\frac{1}{2}}(Ka) = 0,$$

we can add this to Taylor's solution without affecting the boundary conditions. Thus the indeterminacy is serious. Long (1953) suggested that the indeterminacy might be removed by making the fluid bounded in the radial direction. He considered the problem of a uniform, rotating stream in a pipe of radius b . He applys equation (4.13) to find the effect of an obstacle on the axis, with the boundary conditions

$$\psi^* = 0 \quad \text{at} \quad r = b$$
$$\psi^* = -\tfrac{1}{2} U r^2 \text{ on the body}$$

A general solution, obtained by separation of variables, is

$$\psi^* = \sum_r \left[A_r \cos \lambda_r z + B_r \sin \lambda_r z\right] J_1\left[(K^2 - \lambda_r^2)^{\frac{1}{2}} r\right]$$

where λ may be complex.

The boundary condition

$$\psi^*(b) = 0$$

leads to an eigenvalue problem. Since this implies

$$J_1\left[(K^2 - \lambda_r^2)^{\frac{1}{2}} b\right] = 0,$$

we get

$$(K^2 - \lambda_r^2)^{\frac{1}{2}} b = \beta_r \qquad (4.16)$$

where

$$\beta_1 < \beta_2 < \beta_3 \ldots\ldots$$

are the non-zero, positive roots of

$$J_1(\beta) = 0$$

From (4.16),

$$\lambda_r^2 = \kappa^2 - \frac{\beta_r^2}{b^2}$$

For $\kappa b < \beta_1$ or $R_0 = \frac{U}{b\Omega} > \frac{1}{\beta_1}$,

the λ_r are pure imaginary. If

$$\beta_2 > \kappa b > \beta_1 \text{ or } \frac{1}{\beta_1} > R_0 > \frac{1}{\beta_2}, \text{ then}$$

λ_1 is real and the λ_r are imaginary for $r > 1$.

In general, for

$$\beta_{n+1} > \kappa b > \beta_n, \text{ or } \frac{1}{\beta_n} > R_0 > \frac{1}{\beta_{n+1}},$$

Then λ_r is real for $r \leq n$

and pure imaginary for $r > n$.

Thus as R_0 becomes smaller, more wave-modes are possible. The first mode appears at $R = \frac{1}{\beta_1} \doteq 0.261$

To obtain a unique solution for the case

$$R_0 < \frac{1}{\beta_1}$$

Long imposes the condition that there be no stationary waves upstream of the body. In experiments Long observed a first wave-mode agreeing with the theory. But for

$$\frac{U}{a\Omega} < .2,$$

where a is the radius of the obstacle, there was upstream influence and the waves disappeared.

There are two arguments for imposing the above condition. First is the analogy to channel flow over a mound or bottom obstacle, where no upstream surface waves occur. Second is an argument due to Fraenkel (P.R.S. A 233, 1956):

One can regard Long's stationary waves in the uniform stream of speed U as progressive waves whose phase velocity is

$$U_r = \frac{2\Omega}{\sqrt{\frac{4\pi^2}{l^2} + \frac{\beta_r^2}{b^2}}} \tag{4.17}$$

where l is the wavelength. The group velocity is

$$G_r = U_r - \frac{dU_r}{dl} = U_r\left[\frac{l^2\beta_r^2}{4\pi^2 b^2 + l^2\beta_r^2}\right] \tag{4.18}$$

and is less than the phase velocity, implying that energy cannot be transmitted upstream to maintain the upstream waves against the small dissipative forces present in a real fluid.

In the paper cited Fraenkel gave some interesting solutions for flow in a pipe. Long's condition of no waves upstream was used and by considering the flow due to sources and sinks flows past finite obstables could constructed. Critical Rossby numbers for solutions free of upstream influence cannot be found by this method.

Lecture IX

Stewartson (Q.J.M.A.M. 11, (1958) tried to justify Long's assumption of the no-waves-upstream condition.

The general solution of (4.13), which has no singularities for $z < 0$, is

$$\psi^* = \int_0^\infty A(\lambda)\, r J_1(\lambda r)\, e^{z(\lambda^2 - K^2)^{\frac{1}{2}}}\, d\lambda$$

where $K = \frac{2\Omega}{v}$. The part of the range of integration for $\lambda < K$ yields waves. Thus the general solution of (4.13), which is free of waves in $z < 0$ is

$$\psi^* = \int_K^\infty A(\lambda)\, r J_1(\lambda r)\, e^{z(\lambda^2 - K^2)^{\frac{1}{2}}}\, d\lambda$$

Stewartson proved, that

$$\psi^* \sim \int_K^\infty r J_1(\lambda r) A(\lambda) e^{z(\lambda^2 - K^2)^{\frac{1}{2}}}\, d\lambda + r\int_0^K J_1(\lambda r) B(\lambda)\, d\lambda, \qquad (4.19)$$

if the motion is started from relative rest. If now $z \to -\infty$, the second term in the above equation does not vanish. Therefore, if we want $\psi^* \to 0$ as $z \to -\infty$, we have to demand

$$B(\lambda) = 0, \qquad 0 < \ell < K$$

and so we recover Long's condition.

Note that we cannot relax the condition $\psi^* \to 0$ as $z \to -\infty$, since (4.13) is based on the assumption of no disturbance far upstream.

Now Stewartson made the following considerations: Let ψ^*

be only a small departure from a uniform stream and uniform rotation. This is clearly valid sufficiently far from the obstacle creating the disturbance. Then we get from the linearized basic equations

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)^2\left(D^2\psi^*\right)+4\Omega^2\frac{\partial^2\psi^*}{\partial z^2}=0$$

as given by Squire (1956). Laplace transformation with respect to the time t leads to

$$\left[\left(p+U\frac{\partial}{\partial z}\right)^2 D^2+4\Omega^2\frac{\partial^2}{\partial z^2}\right]\psi^*=0$$

We assume a solution ψ^* of the form

$$\psi^*=\int_0^\infty r J_1(\lambda r)A(\lambda)\chi(\lambda,p,z)\,d\lambda$$

and find for $\chi(\lambda,p,z)$ the equation

$$\left[\left(p+U\frac{\partial}{\partial z}\right)^2\left(\frac{\partial^2}{\partial z^2}-\lambda^2\right)+4\Omega^2\frac{\partial^2}{\partial z^2}\right]\chi(\lambda,p,z)=0.$$

Thus $\chi(\lambda,p,z)=G(\lambda,p)e^{\alpha z}$

is a solution, if

$$(p+U\alpha)^2(\alpha^2-\lambda^2)+4\Omega^2\alpha^2=0$$

holds. As $p\to\infty$ — which corresponds to small values of t — we get

$$\alpha=\pm\lambda,\quad -\frac{p}{U}\pm iK.$$

As we want bounded solutions as $z\to-\infty$, we must only consider roots α with a positive real part. We see that for large p there is only one root satisfying this condition. Call this root $\alpha_0(p)$, so that $\alpha_0(p) \to \lambda$ as $p \to \infty$. Stewartson shows

(a) $R_e \alpha_0(p) > 0$ for all p with $R_e(p) > 0$.

(b) There is no other root with positive real part for $R_e(p) > 0$.

The details of this analysis will be found in the paper cited.

As $p \to 0$, which corresponds to $t \to \infty$,

$$\alpha_0(p) = \begin{cases} (\lambda^2 - K^2)^{\frac{1}{2}} + o(p) & \text{for } \lambda > K \\ \dfrac{\lambda p}{K-\lambda} + o(p^2) & \text{for } \lambda < K \end{cases}$$

so we get for χ

$$\chi \sim \begin{cases} G e^{(\lambda^2 - K^2)^{\frac{1}{2}} z} & \text{for } \lambda > K \\ G & \text{for } \lambda < K \end{cases}$$

This gives the result (4.14)

If we want $\chi \to 0$ as $z \to -\infty$, we must have no contribution in the Fourier-Hankel-Integral from $\lambda < K$. Stewartson considers in detail the flow past a sphere.

To solve now the problem for the sphere, we make a power series expansion of

$$\psi^* = \int_K^\infty r J_1(\lambda r) A(\lambda) \exp\left\{(\lambda^2 - K^2)^{\frac{1}{2}} z\right\} d\lambda = \frac{1}{2} r^2 \sum_{s=1}^{\infty} A_s z^{-s}$$

where because of the no-wave condition only negative power of z occur. In spherical polar coordinates ψ^* becomes

$$\psi^{*} = \frac{1}{2}(\pi - \Theta)^{2} \sum (-1)^{s} A_{s} R^{2-s}$$

which provides a constraint of a spherical polar solution. Details will be found in Stewartson's paper.

The solution exists when $R_0 > 0.35$.

It fails if $R = R_0 \approx 0.35$.

This is in agreement with Taylor's and Long's experiments, which showed upstream influence, when

$R_0 <$ 0.2 to 0.3.

Lecture X.

Summary

The following topics have thus far been discussed:

$$-\text{For } \nu = 0,\ R_o = 0\left(\frac{\partial u}{\partial t} + 2\,\Omega \wedge u = -\frac{1}{\rho}\nabla p,\ z \parallel \Omega\right),$$

Steady solutions independent of z ; discontinuities along cylinders (axis was $\parallel \Omega$); upstream influence of obstacle, Taylor columns; initially small, three-dimentional perturbation $\longrightarrow$ (steady two-dimensional motion); singularities and unsteadiness on singular cylinders; experimental confirmation. Seems physically reasonable - but can't find higher approximation.

$-$ For $\nu = 0$, R_o finite.

<u>Subset</u> of exact solutions to the non-linear equations for disturbances to a uniform, rotating stream; stationary waves possible.

Non-uniqueness, unless specify no upstream waves. But solutions satisfying this condition may not exist unless $R_o > R_{o\,crit}$. Experiment supports this.

<u>Attempt to Treat Intermediate Case with a Linearized, Complete Equation</u>.

Mrs. K. Trustrum (Cambridge Univ., Ph.D. thesis) has solved a linearized problem for small perturbations in a flow of finite R_o .

Taking this result, and letting R_o tend to zero, we might expect an interesting comparison with Long's and Stewartson's results.

The problem involves an ideal wire mesh or porous plate that gives small axial velocity perturbations to a uniform, rotating stream. If U is the velocity of the uniform stream the assumption of a small departure $\underset{\sim}{u}$ from this state leads to the linear equation

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{U} \cdot \nabla \underset{\sim}{u} + 2\,\underline{\Omega} \wedge \underline{u} = -\frac{1}{\rho} \nabla p. \tag{4.20}$$

Taking the curl of (4.20)

$$\underline{k} \cdot \nabla \left(\nabla_\wedge \underset{\sim}{u} + \frac{2\Omega}{U} \underline{u} \right) = 0 \tag{4.21}$$

where $\underline{k}$ is the unit vector parallel to $\underline{\Omega}$. A sub-class of solutions is thus given by

$$\nabla_\wedge \underline{u} + \frac{2\Omega}{U} \underline{u} = 0$$

Again taking the curl of the equation,

$$\nabla^2 \underset{\sim}{u} + \frac{4\Omega^2}{U^2} \underline{u} = 0$$

which is an encouraging result, since it is the same stationary-wave equation that Long arrived at. Thus the linear model retains the interaction between the inertia and Coriolis forces which leads to stationary waves.

If we restrict the problem to axisymmetric motions, and define a stream function such that

$$u_z = +\frac{1}{r}\frac{\partial\psi}{\partial r}, \quad u_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \; u_\theta = \frac{\phi}{r},$$

we have

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial z}\right)^2 D^2\psi + 4\Omega^2\frac{\partial^2\psi}{\partial z^2} = 0 \tag{4.22}$$

Consider now only solutions of the form

$$\psi(r,z,t) = A(\lambda)\, r\, J_1(r,\lambda)\, \psi(z,t)$$

and, taking the Laplace time transform of (4.22), we have

$$\left[\left(p + U\frac{\partial}{\partial z}\right)^2\left(\frac{\partial^2}{\partial z^2} - \lambda^2\right) + 4\Omega^2\frac{\partial^2}{\partial z^2}\right]\tilde{\psi} = 0$$

so $\tilde{\psi}$ is of the form $e^{\alpha\lambda z}A(p)$ and hence possible values of α are given by

$$(p + U\alpha\lambda)^2(\alpha^2\lambda^2 - \lambda^2) + 4\Omega^2\alpha^2\lambda^2 = 0. \tag{4.23}$$

Now let $s = \dfrac{p}{2\Omega}$ and $R_0 = \dfrac{U\lambda}{2\Omega}$

(The Rossby number is then based on the length scale of the cross-stream flow).

Equation (4.23) becomes

$$(s + R_0\alpha)^2(\alpha^2 - 1) + \alpha^2 = 0$$

which is the same quartic studied by Stewartson (1958).

We may now investigate the behavior in the limit $S \to \infty$

(i.e. $t \to 0$). We find an upstream root,

$$\alpha_0(s) = 1,$$

and three downstream roots,

$$\alpha_1(s) = -1$$
$$\alpha_2(s) = \frac{-s+i}{R_0}$$
$$\alpha_3(s) = \frac{-s-i}{R_0}$$

As s tends to zero, on the other hand, we have the following roots: (details of the analysis are suppressed)

	upstream roots $\alpha_0(s)$	downstream roots $\alpha_1(s)$	$\alpha_2(s)$	$\alpha_3(s)$
$R_0 > 1$	$(1-R_0^{-2})^{1/2}$	$\frac{-s}{R_0+1}$	$\frac{-s}{R_0-1}$	$\frac{-1}{(1-R_0^{-2})}$
$R_0 < 1$	$\frac{-s}{R_0-1}$	$\frac{-s}{R_0+1}$	$\frac{i}{(R_0^{-2}-1)^{1/2}}$	$\frac{-i}{(R_0^{-2}-1)^{1/2}}$

The roots which $\to 0$ as $s \to 0$ $(t \to \infty)$ will represent Taylor columns since the z dependence is lost.

The root $\alpha_2(s)$ for $R_0 > 1$ and both roots $\alpha_1(s)$ represent downstream Taylor columns. $\alpha_2(s)$ for $R_0 < 1$ and $\alpha_3(s)$ for $R_0 < 1$ are downstream wave solutions, while $\alpha_3(s)$ for $(R_0 > 1)$ represents exponential decay. Upstream we have exponential decay if $R_0 > 1$ and a Taylor column if $R_0 < 1$.

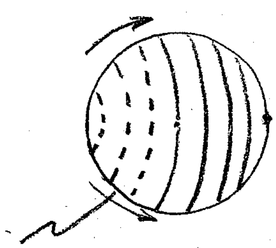
In the most general situation, all values of λ, $0 \leq \lambda \leq \infty$, will enter via the Fourier-Hankel superposition. Therefore since $R_0 = \frac{U\lambda}{2\Omega}$ all values of R_0 will be involved in the general situation. That is, there always exists some upstream disturbance of order one, whatever the Rossby number based on the body size in disagreement with the solutions of Long and of Stewartson.

Mrs. Trustrum points out that the discrepancy between results might be due to a singularity in Long's solution. Consider axially symmetric body initially at rest. The circulation about it is

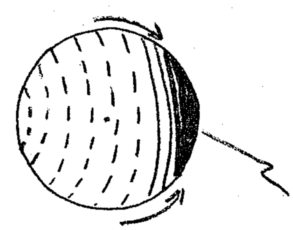
$$\phi = \Omega r^2$$

Now Long's exact solution demands $\psi = \phi$ and also $\psi = 0$ on the body. Therefore we must have $\phi = 0$ on the body.

However material rings are swept back along the body, but must remain on it due to the rear stagnation point. The initial rings are replaced by rings of smaller and smaller circulation, coming from near the front stagnation point, whilst the initial fluid rings are swept to the rear stagnation point. Thus ϕ and hence ψ can never be zero there. In experiments Long never found the downstream Taylor columns, yet he was using a streamlined body to avoid ordinary flow separation.



Rings of lower circulation advance from region of forward stagnation point



Fluid rings initially on body cannot pass rear stagnation point

Mrs. Trustrum derives the conclusion that there will be an upstream disturbance for all Rossby numbers by prescribing the axial velocity on the plane $z = 0$ for all r. It seems possible that, if the axial velocity were prescribed only for $0 < r < a$, say, and the weaker condition that the flow matches the downstream flow applied for $r > a$, the flow might for large enough Rossby numbers be able to adjust to remove the upstream Taylor column. Clearly further work on Mrs. Trustrum's model would be very valuable.

Lecture XI

5. Solutions with Viscosity

The previously found solution for the initial value problem representing a body moving along the axis of rotation had singularities on the cylinder projected by the body. For that solution we assumed viscosity to be absent. It seems reasonable to expect that including viscosity in the treatment will modify the singularities and show viscosity to be important only on the projected cylinder and in the boundary layer. The basic equation becomes:

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} + 2\,\underline{\Omega} \times \underline{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u}\,.$$

One class of exact solutions is due to Ekman* who considered the effects of rotation and viscous stress.

Rather than discussing moving bodies we will examine the effect of the viscous terms on the type of singularity of the previously found solution.

Proudman (1956) considered the problem of a viscous fluid confined between two differentially rotating concentric spheres (fig.1). If the fluid were non-viscous then no motion could be imparted to the fluid so that all effects must be due to viscosity.

Let $R_e = \dfrac{a^2\Omega}{\nu}$ (Taylor number) and assume $R_e >> 1$.

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*See: Prandtl, Fluid Mechanics.

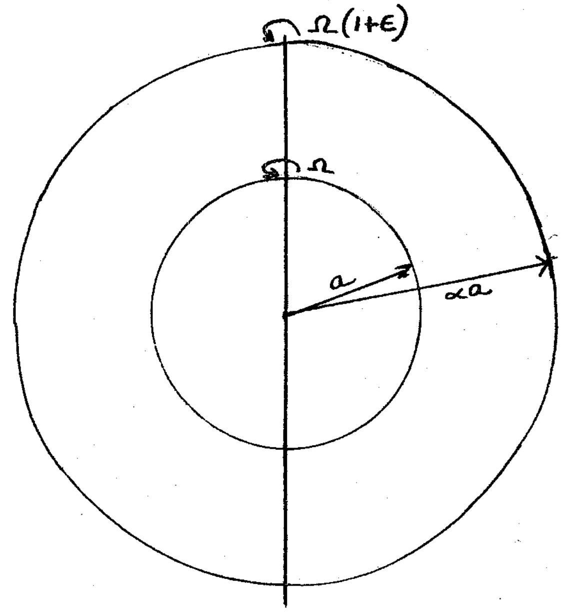
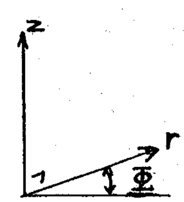


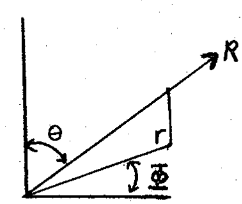
Figure 1. Proudman's spheres.

Two coordinate systems will be used:

cylindrical (r , Φ , z)



spherical (R , θ , Φ).



In both systems, however, the treatment will be independent of the angle Φ due to symmetry. A physical picture may be obtained quite easily. Consider the system depicted in figure 2.

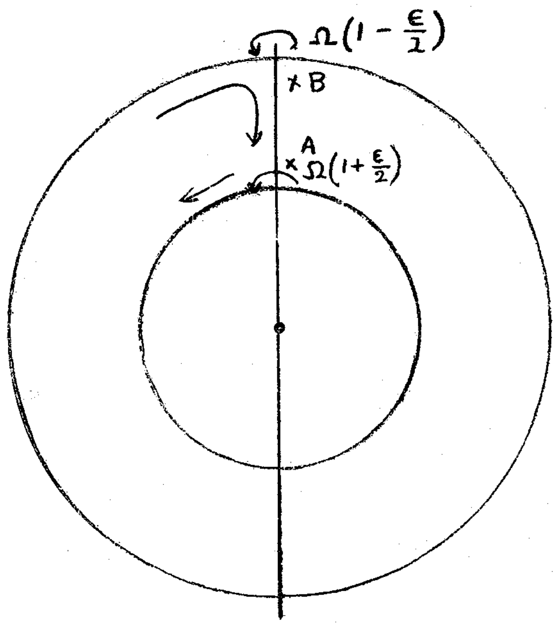


Figure 2.

If $\epsilon = 0$ we have a steady state system in which the pressure gradient force balances the centrifugal force. Now make $\epsilon < 0$ and small. Then the pressure gradient changes by an amount of order ϵ and as $|\epsilon|$ is small we may assume that the pressure gradient force is unchanged. Due to viscous drag a particle at A will speed up and hence experience a greater centrifugal force. Since the pressure gradient force now provides an insufficient balance the particle will move away from the polar axis. Similarly a particle at B will move toward the axis. Since particles coming from B cannot cross the axis they must move down and a circulation will be set up. In fact, the flow near the axis resembles that set up by a pair of parallel plates rotating at slightly different speeds and is a linearized version of the

Karman-Batchelor similarity solution.

This simplified picture does not explain how the fluid returns to complete the circulation.

Let us adopt dimensionless coordinates so that $R=1$ and $R=\alpha$, $\alpha > 1$, are the spherical surfaces. In cylindrical coordinates we find that:

$$u_z = \epsilon \Omega a \frac{1}{r} \frac{\partial \psi}{\partial r} ,$$

$$u_r = -\epsilon \Omega a \frac{1}{r} \frac{\partial \psi}{\partial z} ,$$

$$u_\varphi = \epsilon \Omega a \frac{\phi}{r} .$$

The boundary conditions are:

$$\phi = \psi = \frac{\partial \psi}{\partial R} = 0 \quad on \quad R = 1,$$

$$\phi = \sin^2 \varphi, \ \psi = \frac{\partial \psi}{\partial R} = 0 \quad on \ R = \alpha,$$

where we now assume the outer surface has $a.v.\ \Omega(1+\epsilon)$ and the inner surface has $a.v.\ \Omega$. Then in cyl. coordinates the equation for the streamfunction becomes:

$$2 \frac{\partial \phi}{\partial z} = \frac{1}{Re} D^4 \psi ,$$

$$-2 \frac{\partial \psi}{\partial z} = \frac{1}{Re} D^2 \phi,$$

where $D^2 = \frac{\partial^2}{\partial v^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

We use cyl. coordinates because the equations do not separate in spherical coordinates. Since $\mathcal{R}_\ell >> 1$, $\frac{\partial\phi}{\partial z} = \frac{\partial \psi}{\partial z} = 0$ except in regions of large shear, that is, we have two-dimensional motion except in regions of large shear. This is a consequence of the Taylor-Proudman theorem. Let us proceed as if $\mathcal{R}_\ell = \infty$ and only invoke viscosity when the inviscid solution has discontinuities. Away from singular surfaces the inviscid solutions are

$$\phi = \phi_0(r),$$

$$\psi = \psi_0(r).$$

It is clear that these cannot satisfy the boundary condition requiring that the tangential velocity vanish on the spheres since these represent an axial flow which collides with the spheres. Thus we have boundary layers on $\mathcal{R} = 1$ and $\mathcal{R} = \alpha$ where gradients with respect to $\mathcal{R}$ become large. The boundary layer equations in spherical coordinates are:

$$\frac{1}{\mathcal{R}_e}\frac{\partial^4\psi}{\partial\mathcal{R}^4} = 2\frac{\partial\phi}{\partial\mathcal{R}}\cos\theta,$$

$$\frac{1}{\mathcal{R}_e}\frac{\partial^2\phi}{\partial\mathcal{R}^2} = -2\frac{\partial\psi}{\partial\mathcal{R}}\cos\theta.$$

We can integrate these equations once:

$$\frac{\partial^3 \psi}{\partial R^3} = 2 R_e \cos\theta \,(\phi - \phi_o),$$

$$\frac{\partial \phi}{\partial R} = -2 R_e \cos\theta \,(\psi - \psi_o).$$

Here we have assumed that $\psi \longrightarrow \psi_o$, $\phi \longrightarrow \phi_o$, and gradients of ϕ, $\psi \longrightarrow 0$ as $R >> R_e^{-\frac{1}{2}}$. This is to insure that outside the boundary layers the different flows match smoothly.

Eliminating ϕ we obtain:

$$\frac{\partial^4}{\partial R^4}\left(\psi - \psi_o\right) = -4 R_e^2 \cos^2\theta \left(\psi - \psi_o\right).$$

A solution is:

$$\psi = \psi_o\left(1 - e^{-\eta}(\cos\eta + \sin\eta)\right)$$

where $\eta = (R-1)\left(2\cos\theta\, R_e^{-\frac{1}{2}}\right)$,

$\psi = \dfrac{\partial\psi}{\partial R} = 0$ at $R = 1$, and goes asymptotically to ψ_o. Note that the solution resembles the Ekman spiral solution.

Solving the other equation we find:

$$\phi = \phi_o - 2\left(R_e \cos\theta\right)^{\frac{1}{2}} \psi_o(\sin\theta).$$

Since $\phi = 0$ on $R = 1$ we get a relation between ϕ and ψ as $r = \sin\theta$ on $R = 1$:

$$\phi_o(\sin\theta) = 2\left(R_e \cos\theta\right)^{\frac{1}{2}} \psi_o(\sin\theta).$$

When $\theta \rightarrow \frac{\pi}{2}$, $\cos\theta \longrightarrow 0$, which means that the boundary layer becomes infinitely thick. Thus the equator of the inner sphere is a singular region. Physically this is because we have two streams flowing toward each other and not influenced by each other's presence until they are one boundary layer thickness apart. Then large gradients with respect to θ are required. We cannot analyze what actually happens in the equatorial region by simple boundary-layer theory. The analysis for the outer sphere is quite similar and one finds another relation between ψ_0 and ϕ_0 :

$$\alpha^2 \sin^2\theta - \phi_0(\alpha\sin\theta) = 2(R_e\cos\theta)^{\frac{1}{2}}\,\psi_0(\alpha\sin\theta)$$

where $r = \alpha\sin\theta$ on $R = \alpha$.

Using the above relations we can solve for ψ_0, ϕ_0, and find:

$$\psi_0(r) = \frac{r^2}{2R_e^{\frac{1}{2}}}\left\{\left(1-\frac{r^2}{\alpha^2}\right)^{\frac{1}{4}} + (1-r^2)^{\frac{1}{4}}\right\}^{-1},$$

$$\phi_0(r) = r^2(1-r^2)^{\frac{1}{4}}\left\{\left(1-\frac{r^2}{\alpha^2}\right)^{\frac{1}{4}} + (1-r^2)^{\frac{1}{4}}\right\}^{-1},$$

valid for $0 < r < 1$, that is, the solution holds for the column C formed by the projection of the inner sphere along the axis of rotation. Note that as $r \to 1-$ the gradient of u_θ becomes ∞, while the axial speed is of order $(\epsilon R^{-\frac{1}{2}})$, and the tangential speed is of order (ϵ). It is interesting to note how the non-uniqueness inherent in the Taylor-Proudman theorem is removed by matching to the boundary-layers on the solid surfaces.

It is difficult to ascertain where the fluid goes. Proudman showed that an equatorial jet is not possible since the governing equation would be of form

$$\frac{\partial^4 \psi}{\partial z^4} = C\psi, \qquad C = const.$$

which does not have solutions bounded for all z . Thus we must conclude that $\psi_0(r)$ is analytic in the whole region outside C except at the surface $R = 1$. Since clearly $\psi_0 = 0$ outside C it follows that

$$\phi_0 = r^2 \text{ for } r > 1.$$

Thus the region outside C rotates solidly with the outer sphere, being decoupled from the inner core C where we find axial motions. The circulation must be completed by a thin jet of fluid returning along the singular surface. Geophysical consequences of the division of the flow field into two distinct regions have been explored by R. Hide (1962).

If $\varepsilon < 0$, the flow pattern is pictured in Fig. 4. Fluid is drawn uniformly down the column, flows out along the inner boundary, flows up along the discontinuity surface C , and then flows in along the outer boundary. Physically speaking, the fluid cannot form an equatorial jet because not enough energy is available to cross the vortex lines. This is in contrast to the case of a sphere rotating rapidly in a fluid at rest, where an equatorial jet is formed.

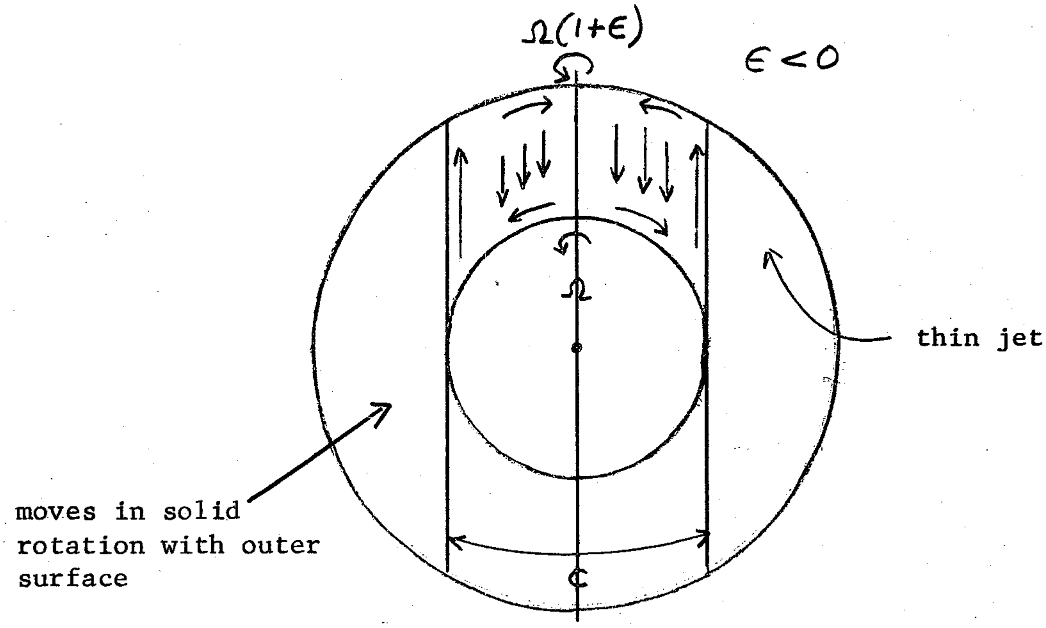


Figure 4.

The flow has been examined experimentally by Fultz and Moore (unpublished). It was confirmed that the flow pattern had the general features predicted by Proudman and that the flow field varied linearly with ε for $0 < \varepsilon < O(10^{-1})$. When $\varepsilon < 0$, so that the inner sphere was rotating more rapidly than the outer, the shear layer sometimes became unstable, but the flow still possessed the same general features.

Lecture XII.

We have seen that the surface C representing a discontinuity in u_θ also carries fluid from the inner to the outer sphere. This discontinuity at $r=1$ must, at large but finite Reynolds numbers, be replaced by some kind of viscous shear layer. Now the exact linear equations are

$$2\frac{\partial \phi}{\partial z} = \frac{1}{\mathcal{R}e} \mathcal{D}^4 \psi,$$

$$-2\frac{\partial \psi}{\partial z} = \frac{1}{\mathcal{R}e} \mathcal{D}^2 \phi,$$

where $\mathcal{D}^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Several cases may be considered.

1) The shear must be such that a balance exists between the viscous terms and the inertia terms. Then

$$2\frac{\partial \phi}{\partial z} = \frac{1}{\mathcal{R}e} \frac{\partial^4 \psi}{\partial r^4},$$

$$-2\frac{\partial \psi}{\partial z} = \frac{1}{\mathcal{R}e} \frac{\partial^2 \phi}{\partial r^2},$$

where $\mathcal{D}^2$ has been replaced by $\frac{\partial^2}{\partial r^2}$ i.e. we assume that large r gradients exist. Assume orders of magnitude:

$$\psi = O(\mathcal{R}e^{\alpha}),$$
$$\phi = O(\mathcal{R}e^{\beta}),$$
$$\frac{\partial}{\partial r} = O(\mathcal{R}e^{\delta}),$$
$$\frac{\partial}{\partial z} = O(1).$$

Then the condition for balance is:

$$\beta + 1 = \alpha + 4\delta,$$

$$\alpha + 1 = \beta + 2\delta.$$

Then $6\delta = 2$ or $\delta = \frac{1}{3}$.

Thus the shear layer has thickness of $\mathcal{O}(Re^{-\frac{1}{3}})$. Furthermore, $\alpha - \beta = -\delta = -\frac{1}{3}$, and by continuity $\alpha = \mathcal{O}(Re^{-\frac{1}{2}})$ since the shear layer must return the fluid, the axial flow being of $\mathcal{O}(Re^{-\frac{1}{2}})$. ϕ changes by $\mathcal{O}(1)$ which means that $\beta = 0$. This last requirement violates the last equation. Thus the $Re^{-\frac{1}{3}}$ shear layer by itself is insufficient for our purposes. Consider now the decoupling of the shear and swirl so as to satisfy the balance between the inertial terms and the viscous terms in one equation only.

2) Let $\alpha = -\frac{1}{2}$, $\beta = 0$, and consider the order of the terms.

1st eq: $1 = 4\delta - \frac{1}{2}$

2nd eq: $\frac{1}{2} = 2\delta$

If the balance is in the second equation, then $\delta = \frac{1}{4}$ so that the shear in the first equation is of $\mathcal{O}(Re^{-\frac{1}{2}})$ which is negligible compared to $O(Re)$ since Re is very large. Then

$$2\frac{\partial \phi}{\partial z} = 0,$$

$$-2Re\frac{\partial \psi}{\partial z} = \frac{\partial^2 \phi}{\partial r^2},$$

and the shear layer thickness is $O(Re^{-\frac{1}{4}})$.

Proudman did not consider the structure of this shear layer in detail since he concluded that it must be determined by matching at the equator of the inner sphere and at the outer sphere.

Stewartson (1957) considered the simpler case represented by two differentially rotating parallel disks as shown in figure 5. In this case an exact solution could be found

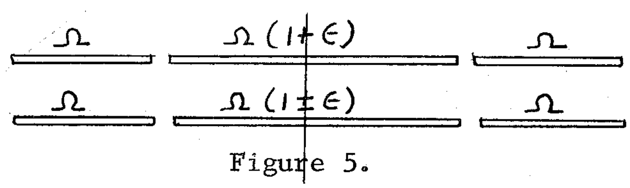


Figure 5.

Results were:

1) If the upper central disk rotating with velocity $\Omega(1+\epsilon)$, and the bottom central disk rotating with velocity $\Omega(1-\epsilon)$, so discontinuities in ϕ occur since the cylinder rotates with the same velocity as the rest of the fluid. Stewartson found a single shear layer of $O(Re^{-\frac{1}{3}})$.

2) If both central disks rotate with velocity $\Omega(1+\epsilon)$ there occurs a discontinuity of $O(1)$ in ϕ. Stewartson found <u>two</u> shear layers:

a) $O(Re^{-\frac{1}{3}})$, which carried mass,

b) $O(Re^{-\frac{1}{4}})$, which allows the zonal velocity to change.

No motion of $\mathcal{O}(Re^{-\frac{1}{2}})$ was found in the core.

If no discontinuity in ϕ occurs in the inviscid solution we get only one shear layer which is of $\mathcal{O}(Re^{-\frac{1}{3}})$.

The equations are:

$$-2\frac{\partial \phi}{\partial z} = \frac{1}{Re}\frac{\partial^4 \psi}{\partial r^4},$$

$$2\frac{\partial \psi}{\partial z} = \frac{1}{Re}\frac{\partial^2 \phi}{\partial r^2}.$$

This has a similarity solution since the bahaviour depends only on the axial distance (Moore, 1958). Then

$$\psi = z^m f\left[\frac{(r-1)R^{\frac{1}{3}}}{2^{\frac{1}{3}}}\left(\frac{2}{3}\right)^{\frac{1}{3}}\right] = z^m f(\eta) \qquad \text{say}$$

the case $m = 0$ is of relevance in a layer of constant mass flux; then f satisfies

$$f^{(6)} + \eta^2 f^{(2)} + 4\eta f^{(1)} = 0.$$

This has only one solution which is bounded for $-\infty < \eta < \infty$:

$$f^{(1)} = A\int_0^\infty e^{-\frac{1}{3}t^3}\cos\eta t\, dt.$$

Thus the shear layer structure is determined once the amount of mass it has to carry is specified.

In general, however, no method of determining the structure of the shear layers without recourse to an exact solution is known, though work in progress by L.N.Howard seems likely to provide such a method.

Lecture XIII

Wave Propagation in a Rotating System

In a rotating fluid there is a vortex field which acts like strings under tension. Therefore the fluid will respond elastically to disturbances and waves will propagate along vortex lines. (Chandrasekhar - Hydrodynamic and Hydromagnetic Stability.) First extensively studied by Bjerknes. Our treatment follows that of O. M. Phillips (1963).

Consider an incompressible, inviscid fluid, then the only scale imposed by the equations is the rotational frequency Ω . Then the period of oscillations will be proportional to $1/\Omega$ since the density isn't involved. Since there is no velocity scale imposed the phase velocity will be a function of the wave-length and the waves will be dispersive.* Viscosity turns out not to affect the propagation, but only causes damping.

If $\underset{\sim}{u}(\underset{\sim}{x},t)$ is the velocity relative to the rotating frame the basic equations are

$$\nabla \cdot \underset{\sim}{u} = 0$$

$$\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} + 2\underset{\sim}{\Omega} \times \underset{\sim}{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \underset{\sim}{u}$$

There exist exact solutions for finite amplitude plane waves, but since the equation is non-linear such solutions cannot be combined. Thus to study interactions we must consider infinitesimal waves. We seek a solution of the form

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* This shows that the analogy between a vortex line and a tense string is very crude.

$$\underset{\sim}{u} = \underset{\sim}{\alpha} e^{-\lambda t} \cos(\underset{\sim}{K} \cdot \underline{x} - \omega t) + \underset{\sim}{\beta} e^{-\lambda t} \sin(\underset{\sim}{K} \cdot x - \omega t),$$

$$p = A e^{-\lambda t} \cos(\underset{\sim}{K} \cdot \underline{x} - \omega t) + B e^{-\lambda t} \sin(\underset{\sim}{K} \cdot x - \omega t).$$

We first note that

$$\frac{\partial u_i}{\partial x_j} = -e^{-\lambda t} \alpha_i K_j \sin(\underset{\sim}{K} \cdot \underline{x} - \omega t) + \beta_i K_j e^{-\lambda t} \cos(\underset{\sim}{K} \cdot \underline{x} - \omega t)$$

so that $\nabla \cdot \underset{\sim}{u} = 0$ implies $\underset{\sim}{\alpha} \cdot \underset{\sim}{K} = \underset{\sim}{\beta} \cdot K = 0$

thus the waves are <u>transverse</u>.

Now consider the non-linear inertia term in the equation of motion. We find

$$(\underset{\sim}{u} \cdot \nabla \underset{\sim}{u})_i = e^{-2\lambda t} [\alpha_j \cos + \beta_j \sin] [-\alpha_i K_j \sin + \beta_i K_j \cos]$$

$$= e^{-2\lambda t} [-\alpha_i \alpha_j K_j + \beta_i \beta_j K_j] \cos \sin$$

$$= 0$$

It is only because this non-linear term drops out that we can get an exact solution for finite amplitude waves. Substituting the assumed forms in the equations of motion gives

$$-\underset{\sim}{\alpha} \lambda \cos + \underset{\sim}{\alpha} \omega \sin - \underset{\sim}{\beta} \lambda \sin - \underset{\sim}{\beta} \omega$$

$$+ 2(\underset{\sim}{\Omega} \times \underset{\sim}{\alpha}) \cos + 2(\underset{\sim}{\Omega} \times \underset{\sim}{\beta}) \sin$$

$$= \frac{1}{\rho} A \underset{\sim}{K} \sin - \frac{1}{\rho} B \underset{\sim}{K} \cos - \nu K^2 \alpha \cos - \nu K^2 \beta \sin$$

If the coefficient of sin and cos are to vanish

$$-\boldsymbol{\alpha}\lambda - \boldsymbol{\beta}\omega + 2\boldsymbol{\Omega}\times\boldsymbol{\alpha} = -\frac{1}{\rho}B\boldsymbol{k} - \nu K^2\boldsymbol{\alpha}$$

$$-\boldsymbol{\beta}\lambda + \boldsymbol{\alpha}\omega + 2\boldsymbol{\Omega}\times\boldsymbol{\beta} = -\frac{1}{\rho}A\boldsymbol{k} - \nu K^2\boldsymbol{\beta}$$

We take the scalar product of the first equation with $\boldsymbol{\alpha}$ and the second with $\boldsymbol{\beta}$ giving

$$-\alpha^2\lambda - \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\,\omega = -\nu K^2\alpha^2$$

$$-\beta^2\lambda + \boldsymbol{\beta}\cdot\boldsymbol{\alpha}\,\omega = -\nu K^2\beta^2$$

Thus $\lambda = \nu K^2$

and $\boldsymbol{\alpha}\cdot\boldsymbol{\beta} = 0$

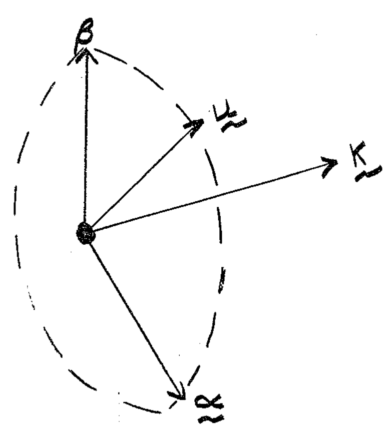
Now we take the scalar product of the first with $\boldsymbol{\beta}$ and the second with $\boldsymbol{\alpha}$ giving

$$-\beta^2\omega + 2(\boldsymbol{\Omega}\times\boldsymbol{\alpha})\cdot\boldsymbol{\beta} = 0,$$

$$+\alpha^2\omega + 2(\boldsymbol{\Omega}\times\boldsymbol{\beta})\cdot\boldsymbol{\alpha} = 0;$$

adding gives $\alpha^2 = \beta^2$,

thus $\boldsymbol{\beta} = \pm\,\boldsymbol{\ell}\wedge\boldsymbol{\alpha} = \phi(\boldsymbol{\ell}\wedge\boldsymbol{\alpha})$, where $\phi = \pm 1$, $\boldsymbol{\ell} = \boldsymbol{k}/K$



Thus the wave is <u>circularly</u> polarized, and the particle paths are circles.

(If $\omega = 0$ we still get $\boldsymbol{\alpha}\cdot\mathbf{K} = \boldsymbol{\beta}\cdot\mathbf{K} = 0,\ \lambda = \nu k^2$

and we are left with the equations

$$2\,\boldsymbol{\Omega}\wedge\boldsymbol{\alpha} = -\frac{1}{\rho}\,B\,\mathbf{K}$$

$$2\,\boldsymbol{\Omega}\wedge\boldsymbol{\beta} = +\frac{1}{\rho}\,A\,\mathbf{K}$$

Thus the flow is purely geostrophic.)

We can now find the frequency from the equation

$$\alpha^2\omega + 2\,(\boldsymbol{\Omega}\wedge\boldsymbol{\beta})\cdot\boldsymbol{\alpha} = 0$$

or

$$\alpha^2\omega + 2\phi\,(\boldsymbol{\Omega}\wedge(\boldsymbol{l}\wedge\boldsymbol{\alpha}))\cdot\boldsymbol{\alpha}$$

$$= \alpha^2\omega + 2\phi\,[\boldsymbol{l}\,(\boldsymbol{\alpha}\cdot\boldsymbol{\Omega}) - \alpha\,(\boldsymbol{\Omega}\cdot\boldsymbol{l})]\cdot\boldsymbol{\alpha}$$

$$= \alpha^2\omega - 2\phi\,\alpha^2(\boldsymbol{\Omega}\cdot\boldsymbol{l}) = 0$$

Thus $\omega = \phi\,(2\,\boldsymbol{\Omega}\cdot\boldsymbol{l})$

Therefore collecting results our solution is

$$\mathbf{u}(\mathbf{x},t) = e^{-\nu k^2 t}\,[\boldsymbol{\alpha}\,(\mathbf{k}\cdot\mathbf{x} \pm 2(\boldsymbol{\Omega}\wedge\boldsymbol{l})\,t)$$

$$\mp (\hat{\mathbf{k}}\wedge\boldsymbol{\alpha})\sin(\mathbf{k}\cdot\mathbf{x} \pm 2(\boldsymbol{\Omega}\cdot\boldsymbol{l})\,t)]$$

If the wave speed is C_p, then we see that

$$C_p = \frac{2\,\boldsymbol{\Omega}\cdot\boldsymbol{l}}{k}$$

The group velocity is defined by the condition that a small change in κ introduces no first order changes in the phase, i.e.

$$\delta\left(\underline{k}\cdot\underline{x} - \omega(k)t\right) = 0, \quad \omega = \phi(2\underset{\sim}{\Omega}\cdot\underset{\sim}{\ell})$$

then

$$(\underline{k} + d\underline{k})\cdot\underline{x} - (\omega + d\omega)t$$

$$= \underline{k}\cdot\underline{x} - \omega t + d\underline{k}\cdot\left(x - \nabla_k \omega(k)t\right)$$

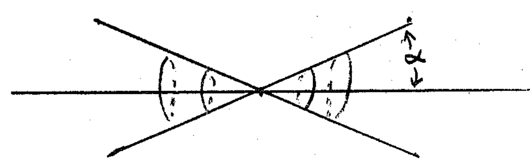
Thus $\underset{\sim}{c}_g = \nabla_k \omega(k) = \nabla_k\left(\frac{2\Omega\cdot\ell}{k}\right) = \frac{2\Omega}{k} - \frac{2(\Omega\cdot\ell)\underline{k}}{k^2}$

Note $\underset{\sim}{c}_g \cdot \underset{\sim}{k} = 0$

This means that energy has to flow perpendicular to the direction of propagation of the disturbance! Clearly, the disturbance created by a point source will in no way resemble the spherical disturbance one finds in more familiar media.

This problem is discussed by Nigam and Nigam (1962).

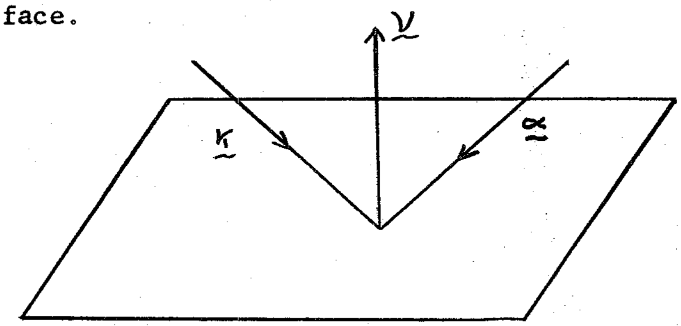
For a rotating fluid at rest, the propagation of disturbances is confined to the <u>surface</u> of a double cone



The wave is proportional $1/r^2$ outside the cone and $1/r^{\frac{1}{2}}$ in the cone. The angle α is

$$\alpha = \tan^{-1}\left(\frac{4\Omega^2}{\omega^2} - 1\right)^{-\frac{1}{2}}$$

Phillips considers how inertial waves are reflected at a fixed rigid plane surface.



Define the plane Π = plane of $(\underline{\kappa}, \underline{\nu})$ and take $\underset{\sim}{\alpha}$ to be in Π. The boundary conditions on the plane are that the normal velocity $= 0$ (inviscid flow boundary condition). Let the incoming wave be

$$\underset{\sim}{u} = \underset{\sim}{\alpha} \cos(\underset{\sim}{\kappa} \cdot \underset{\sim}{x} - 2(\underset{\sim}{\Omega} \cdot \underset{\sim}{\ell}\, t) + (\underset{\sim}{\ell} \wedge \underset{\sim}{\alpha}) \sin(\underline{\kappa} \cdot \underline{x} - 2(\underline{\Omega} \cdot \underset{\sim}{\ell})t)$$

and the outgoing wave be

$$\underline{u}' = \underset{\sim}{\alpha}' \cos(\kappa' \cdot \underline{x} \pm 2(\underline{\Omega} \cdot \underset{\sim}{\ell}')t)$$

$$\pm (\underline{\kappa}' \wedge \underline{\alpha}') \sin(\underline{\kappa} \cdot \underline{x} \pm 2(\underline{\Omega} \cdot \underset{\sim}{\ell}')t)$$

Note - we must leave sign free at this stage.

The boundary condition is

$$\underset{\sim}{\nu} \cdot (\underline{u} + \underline{u}') = 0 \quad \text{for all } \underset{\sim}{x} \text{ on the reflecting plane.}$$

Equating sin and cos terms this condition gives

$$\underline{\nu} \cdot \underline{\alpha}' = -\underline{\nu} \cdot \underline{\alpha}$$

$$\underset{\sim}{\nu} \cdot (\underset{\sim}{\ell}' \wedge \underset{\sim}{\alpha}) = 0$$

where we choose the argument of the cos so terms cancel, i.e.

$$\underset{\sim}{\Omega} \cdot \underset{\sim}{\ell} = \pm \underset{\sim}{\Omega} \cdot \underset{\sim}{\ell}' \, ; \; \underset{\sim}{\ell} = \frac{\underset{\sim}{k}}{|\underset{\sim}{k}|}$$

and

$\underline{x} \cdot \underline{k} = \underline{x} \cdot \underline{k}'$ for all $\underline{x}$ lying in the reflecting plane.

This last condition means

$\underline{x} \cdot (\underline{k}' - \underline{k}) = 0$ for all $\underline{x}$ lying in the reflecting plane, and since $\underline{\nu}$ is perpendicular to all $\underline{x}$ in the reflecting plane, then $\underset{\sim}{k}' - \underset{\sim}{k}$ is parallel $\underline{\nu}$.

Thus we have the conditions on the reflecting plane

$$\underline{\nu} \cdot (\underline{\alpha} + \underline{\alpha}') = 0 \tag{1}$$

$$\nu \cdot (\underset{\sim}{\ell}' \times \underset{\sim}{\alpha}') = 0 \tag{2}$$

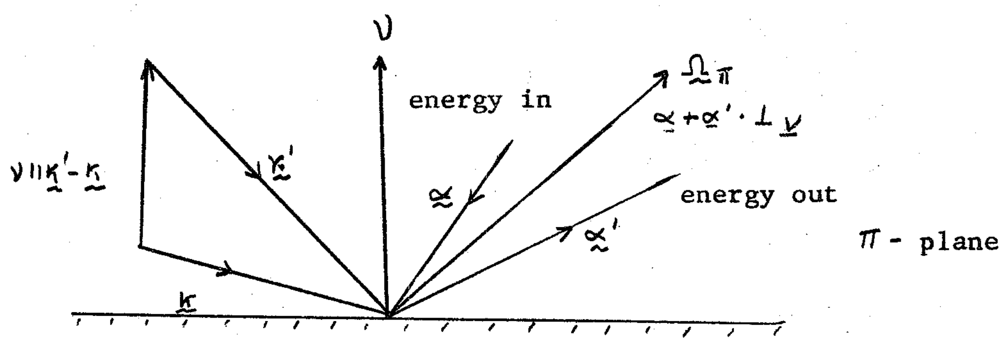
$$\underline{\Omega} \cdot \underset{\sim}{\ell} = \pm \underset{\sim}{\Omega} \cdot \underset{\sim}{\ell}' \tag{3}$$

$$\underline{k}' - \underline{k} \parallel \underline{\nu} \tag{4}$$

By (4), k', $\underline{k}$, $\underline{\nu}$ are coplaner; so $\underline{k}'$ is in π . Then by (2) $\underset{\sim}{\alpha}'$ is in π . Thus all the relevant vectors (except $\underset{\sim}{\Omega}$) lie in the plane π .

Let $\underset{\sim}{\Omega}\,\pi$ be the component of $\underset{\sim}{\Omega}$ in π . Then we see that condition (3) is equivalent to

$$\underset{\sim}{\Omega}_{\pi} \cdot \underset{\sim}{\ell} = \pm \underset{\sim}{\Omega}_{\pi} \cdot \underset{\sim}{\ell}' ,$$



so that the direction of propagation reflects in $\underset{\sim}{\Omega}_\pi$ not ν. Furthermore the direction of propagation of energy $\underset{\sim}{C}_g$ is perpendicular to $\underline{k}$ and coplaner with $\underline{k}$, $\underset{\sim}{\Omega}$. If $\underset{\sim}{\Omega}$ is in the plane π, so $\underset{\sim}{\Omega}_\pi = \underset{\sim}{\Omega}$, then $\underline{C}_g$ is parallel to $\underset{\sim}{\alpha}$, $\underset{\sim}{\alpha}'$ so energy can come in and go out on the same side of the normal $\underline{\nu}$!

Phillips also considers how viscosity affects the reflection. Except at a certain critical angle of incidence, viscous effects are confined to a thin viscous boundary at the plane. At the critical angle, total absorption of the incident wave occurs.

REFERENCES

Crapper, G.D. 1959 J.Fluid Mech. 6, 51

Fraenkel, L.E. 1956 Proc.Roy.Soc. A 233, 506

Goertler, H. 1944 Z. Agnew Math.Mech. 24, 210

Grace, S.F. 1926 Proc.Roy.Soc. A 113, 46.

Hide, R. (1962) unpublished notes

Long, R.R. 1953 Journ. of Met. 10, 197

Moore, D.W. 1958 Camb.Univ.Phd.Thesis

Morgan, G.W. 1951 Proc.Roy.Soc. A 206, 108

Morgan, G.W. 1953 Proc.Camb.Phil.Soc. 49, 362

Morrison, J.A. and G.W.Morgan 1956 Tech.Rep.No.8 Div.Appl.Math. Brown Univ.

Phillips. O.M. 1963 Phys.of Fluids 6, 513

Proudman, I. 1956 J.Fluid Mech. 1, 505

Proudman, J. 1916 Proc.Roy.Soc. A 92, 408

Sarma, L.V.K.V. 1958 J.Fluid Mech. 3, 404

Stewartson, K. 1952 Proc.Camb.Phil.Soc. 48, 168

Stewartson, K. 1953 Quart.J.Mech. 6, 141

Stewartson, K. 1957 J.Fluid Mech. 3, 17

Stewartson, K. 1958 Q.J.M.A.M. 11, 39

Taylor, G.I. 1917 Proc.Roy.Soc. a 93, 99

Taylor, G.I. 1922 Proc.Roy.Soc. A 102, 180

Taylor, G.I. 1923 Proc.Roy.Soc. A 104, 213

Trustram, Mrs. K. 1962 Camb.Univ.Phd.Thesis

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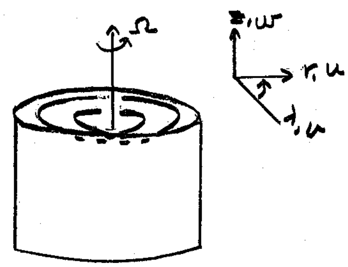
Transformation of the equations from spherical to plane coordinates: the -plane.

1. A simple example of geostrophic flow.



A cylinder of fluid is in solid body rotation with angular velocity $\Omega = \frac{v}{r}$; the free surface will take an equilibrium shape obtained by integrating the equation of motion:

$$-\Omega^2 r = -\frac{1}{\rho}\frac{\partial p}{\partial r} \; ; \quad -\frac{1}{\rho}\frac{\partial p}{\partial z} = g.$$

$$p = -\rho g z + \frac{\Omega^2 r^2}{2}\rho$$

and the curves of constant p are parabolae.

Introduce a disturbance such that now

$$\Omega = \bar{\Omega} + \Omega' \qquad \Omega' << \bar{\Omega}$$

$$p = \bar{p} + p' \qquad p' << \bar{p}$$

the bar denoting averaged quantities which satisfy equation (1), the primes, disturbances. Substitute into (1); to first order,

$$-\bar{\Omega}^2 r - 2\bar{\Omega}\Omega' r = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} - \frac{1}{\rho}\frac{\partial p'}{\partial r}$$

since the first terms on both sides balance, one is left with

$$-2\bar{\Omega}\,\Omega' r = -\frac{1}{\rho}\frac{\partial p'}{\partial r}$$

or $$2\bar{\Omega}\,v' = -\frac{1}{\rho}\frac{\partial p'}{\partial r} \qquad \text{where } v' = r\Omega' \tag{2}$$

Geostrophic flow is then obtained from the balance of Coriolis forces and pressure gradients: it results from a small deviation from the steady rotation provided that local acceleration, inertial forces and friction are of little importance.

2. The Taylor-Proudman Theorem and Geostrophy.

In a barotropic fluid (in which density is a function of pressure only) the geostrophic equations of motion can be written as

$$+2\underset{\sim}{\Omega} \times \underset{\sim}{v} = -\nabla p p'(-\nabla\phi) \tag{3}$$

The density has been incorporated into the pressure, and ϕ is a potential of external forces (e.g. gravity). Take the curl of the above; using the vector identity

$$\nabla\times(2\underset{\sim}{\Omega}\times\underset{\sim}{v}) = 2(\underset{\sim}{v}\cdot\nabla)\underset{\sim}{\Omega} - 2\underset{\sim}{v}(\nabla\cdot\underset{\sim}{\Omega}) + 2(\underset{\sim}{\Omega}\cdot\nabla)\underset{\sim}{v} + 2\underset{\sim}{\Omega}(\nabla\cdot\underset{\sim}{v})$$

and the fact that Ω is constant, (3) becomes

$$2(\underset{\sim}{\Omega}\cdot\nabla)\underset{\sim}{v} - 2\underset{\sim}{\Omega}\,\nabla\cdot\underset{\sim}{v} = 0$$

with $\underset{\sim}{\Omega} = \Omega\,\underset{\sim}{k}$.

$$+2\Omega\frac{\partial \underset{\sim}{v}}{\partial z} - 2\underset{\sim}{\Omega}\,\nabla\cdot\tilde{v} = 0 \tag{4}$$

If the fluid is incompressible, $\nabla \cdot \underset{\sim}{v} = 0$ and $\frac{\partial \tilde{v}}{\partial z} = 0$; in general, whether or not, expansion of (4) in components yield

$$\left[\begin{array}{l} \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \\ \nabla_h \cdot \underset{\sim}{v}_h = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \end{array} \right]$$

However, cross differentiation of the horizontal geostrophic equations

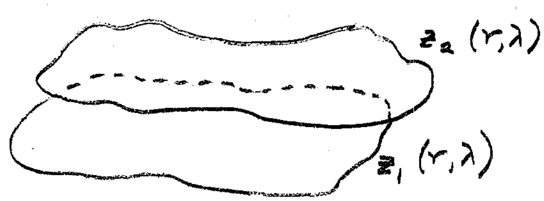
$$2\Omega v = -\frac{\partial p}{\partial r}$$

$$-2\Omega u = -\frac{1}{r}\frac{\partial p}{\partial \lambda}$$

gives only $\nabla_h \cdot \underset{\sim}{v}_h = 0$. It then appears that this is the only result essentially linked with geostrophy; the conditions $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$, although consequences of the Taylor-Proudman theorem are not implied by geostrophic flow.

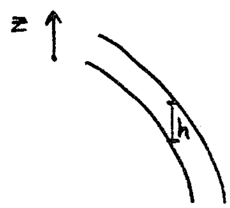
3. Non-geostrophic Motions

Geostrophic flow imposes $\frac{\partial w}{\partial z} = \nabla_h \cdot \underset{\sim}{v}_h = 0$; there can then be no geostrophic motion between two surfaces $z_2(r,\lambda)$ and $z_1(r,\lambda)$ chosen so as to make it impossible for a fluid column to move between them



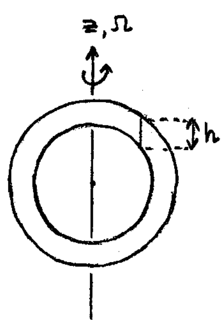
without stretching, since any stretching or compression of a fluid column will make $\nabla_h \cdot \underset{\sim}{v_h} \neq 0$.

Thus, between two spherical shells, the only possible geostrophic flow is zonal flow: a column of liquid cannot move north or south



without changing its length. Since meridional flows are frequent and important in geophysics, they must be studied outside the geostrophic restrictions.

We will consider a set of flows for which the barotropic geostrophy condition will be relaxed so as to make meridional flows between spherical shells attainable. The same conditions are now



$$\left.\begin{aligned} &\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \\ &\nabla_h \cdot \underset{\sim}{v_h} = -\frac{\partial w}{\partial z} = f(r.\lambda) \neq 0 \end{aligned}\right\} \tag{5}$$

Still in cylindrical coordinates, the horizontal equations of motion become

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \lambda} - \frac{v^2}{r} + 2\Omega v = -\frac{\partial P}{\partial r} - \frac{\partial \phi}{\partial r} \\ &\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \lambda} + \frac{uv}{r} - 2\Omega u = -\frac{1}{r}\frac{\partial P}{\partial \lambda} - \frac{1}{r}\frac{\partial \phi}{\partial \lambda} \end{aligned}\right\} \tag{6}$$

The continuity equation will be integrated between the inner and the outer shells; if z_1 and z_2 are their respective coordinates, $z_2 - z_1 = h$ and $\omega_2 - \omega_1 = \dfrac{d(z_2 - z_1)}{dt} = \dfrac{dh}{dt}$

Hence the continuity equation becomes

$$\nabla_h \cdot \underset{\sim}{v}_h = -\frac{1}{h}\frac{dh}{dt} \tag{7}$$

eliminating the pressure from equations (6) by cross-differentiation gives

$$\frac{d\zeta}{dt} + (2\Omega + \zeta)\nabla_h \cdot \underset{\sim}{v}_h = 0 \tag{8}$$

in which $\zeta = \frac{1}{r}\left(\frac{\partial v r}{\partial r} - \frac{\partial u}{\partial \lambda}\right)$: the local z component of vorticity. Since Ω is constant, (8) can be combined with (7) to give

$$\frac{1}{(2\Omega+\zeta)}\frac{d}{dt}(2\Omega+\zeta) = \frac{1}{h}\frac{dh}{dt}$$

or

$$\left[\frac{d}{dt}\left\{\frac{2\Omega+\zeta}{h}\right\} = 0\right] \tag{9}$$

Equation (9) will be called the conservation of potential vorticity equation; note that this is not the same equation as the one by the same name frequently used in oceanography and meteorology: the vertical is along the axis of rotation in (9), and not along the local normal.

The circumstances under which (9) can be approximated by

$$\frac{d}{dt}\left\{\frac{\zeta_h + f}{h_n}\right\} = 0 \tag{10}$$

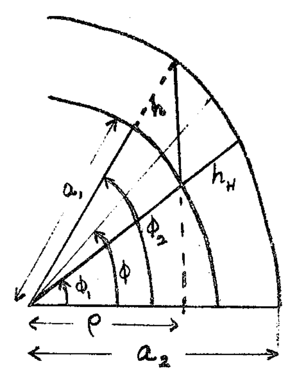
in which
$$h_n = h \sin\phi$$
$$\zeta_n = \zeta \sin\phi$$
$$f = 2\Omega \sin\phi$$

will now be examined.

a) In cylindrical coordinates the vorticity $(\nabla \times \underset{\sim}{v})$ takes the following form: $\left\{ \frac{\partial v}{\partial z} + \frac{\partial u}{\partial z} = 0 \right\}$

$$\frac{1}{r}\frac{\partial w}{\partial \lambda}\,\underset{\sim}{1}_r + \left(-\frac{\partial w}{\partial r}\right)\underset{\sim}{1}_\lambda + \frac{1}{r}\left(\frac{\partial v r}{\partial r} - \frac{\partial u}{\partial \lambda}\right)\underset{\sim}{1}_z$$
$$= \xi\,\underset{\sim}{1}_r + \eta\,\underset{\sim}{1}_\lambda + \zeta\,\underset{\sim}{1}_z$$

The component of vorticity in the normal direction is



$$\zeta_n = \xi \cos\phi + \zeta \sin\phi$$

so that $\zeta_n \simeq \zeta \sin\phi$ only to the extent that

$$\frac{\xi}{\zeta} << \tan\phi$$

ξ involves only w and with small vertical velocities, it is often true that $\frac{\xi}{\zeta} << \tan\phi$; but as $\phi \rightarrow 0$, near the equator this will not hold.

b) When is it permissible to put $h_n = h \sin\phi$?

Referring to the figure, we note

$$h_n = a_2 - a_1$$

and

$$h = a_2 \sin\phi_2 - a_1 \sin\phi_1$$

from the diagram,

$$\cos\phi_1 = \frac{a_2}{a_1}\cos\phi_2$$

$$\sin\phi_1 = \sqrt{a_1^2 - a_2^2\cos^2\phi_2}\,/a_1$$

and

$$h = a_2 \sin\phi_2 - \sqrt{a_1^2 - a_2^2\cos^2\phi_2}$$

If the thickness of the ocean or layer considered is small, we can write

$$a_2 = a_1(1+\delta) \qquad \delta << 1 .$$

then

$$h = a_1\left[(1+\delta)\sin\phi_2 - \sqrt{1-(1+\delta)^2\cos^2\phi_2}\right]$$

$$\simeq a_1\delta\sin\phi_2(1+\cot^2\phi_2)$$ to 1^{st} order in δ .

$$h_n = a_2 - a_1 = a_1\delta \simeq h\sin\phi_2$$ is correct as long as

$$2\delta\cot^2\phi_2 << 1$$

or

$$\frac{a_2-a_1}{a_1} << \frac{\tan^2}{2}\phi_2$$

For the ocean, this is $\tan\phi_2 >> 0.04$. With a margin of an order of magnitude considered sufficient, then

$$\tan \phi_2 > 0.4.$$

or

$$\phi_2 \sim 0.4.$$

This makes the approximation inacceptable south of 25°N. In spite of this, it gives good results when applied much nearer the equator; this is due to density stratification in the oceans: the analysis has been valid only for a barotropic fluid. It will be shown later that this approximation can be used to within ½ degree from the equator when the fluid is baroclinic.

4. The β -plane.

In spherical polar coordinates, the equations of motion and the continuity equation are

$$(11)\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{u}{r\cos\phi}\dfrac{\partial u}{\partial \lambda} + \dfrac{v}{r\cos\phi}\dfrac{\partial}{\partial \phi}(u\cos\phi) + \dfrac{w}{r}\dfrac{\partial}{\partial r}(ru) - 2\Omega v\sin\phi + 2\Omega w\cos\phi = -\dfrac{1}{\rho r\sin\phi}\dfrac{\partial p}{\partial \lambda} \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{u}{r\cos\phi}\dfrac{\partial v}{\partial \lambda} + \dfrac{v}{r}\dfrac{\partial v}{\partial \phi} + \dfrac{w}{r}\dfrac{\partial rv}{\partial r} + \dfrac{u^2}{r}\tan\phi + 2\Omega u\sin\phi = -\dfrac{1}{\rho r}\dfrac{\partial p}{\partial \phi} - \dfrac{1}{r}\dfrac{\partial \Phi}{\partial \phi} \\[2ex] \dfrac{\partial w}{\partial t} + \dfrac{u}{r\cos\phi}\dfrac{\partial w}{\partial \lambda} + \dfrac{v}{r}\dfrac{\partial w}{\partial \varphi} + w\dfrac{\partial w}{\partial r} - \dfrac{u^2+v^2}{r} - 2\Omega u\cos\phi = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} - \dfrac{\partial \Phi}{\partial r} \\[2ex] \dfrac{\partial \rho}{\partial t} + \dfrac{u}{r\cos\phi}\dfrac{\partial \rho}{\partial \lambda} + \dfrac{v}{r}\dfrac{\partial \rho}{\partial \phi} + w\dfrac{\partial \rho}{\partial r} + \rho\left[\dfrac{1}{r\cos\phi}\dfrac{\partial u}{\partial \lambda} + \dfrac{1}{r\cos\varphi}\dfrac{\partial}{\partial \varphi}(r\cos\varphi) + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 w)\right] = 0 \end{cases}$$

The above set is transformed to the one below by putting

$$u^* = r\cos\varphi\, u$$

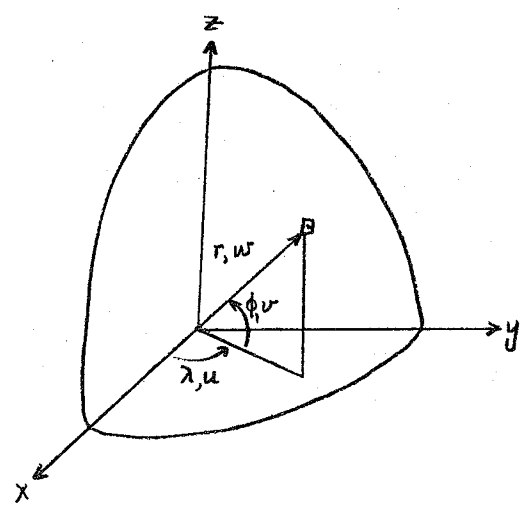
$$v^* = r\cos\varphi\, v$$

$$\frac{d^*}{dt} = \frac{\partial}{\partial t} + \frac{u^*}{r^2\cos^2\varphi}\frac{\partial}{\partial\lambda} + \frac{v^*}{r^2\cos\varphi}\frac{\partial}{\partial\varphi} + w\frac{\partial}{\partial r}$$

$$(12)\begin{cases} \dfrac{d^*u^*}{dt} - 2\Omega v^*\sin\phi + 2\Omega r\cos^2\phi\, w = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial\lambda} \\[2ex] \dfrac{d^*v^*}{dt} + \dfrac{u^{*2}+v^{*2}}{r^2\cos^2\phi}\sin\phi + 2\Omega u^*\sin\phi = -\dfrac{\cos\phi}{\rho}\dfrac{\partial p}{\partial\varphi} \\[2ex] \dfrac{d^*w}{dt} - \dfrac{u^{*2}+v^{*2}}{r^2\cos^2\phi} - \dfrac{2\Omega u^*}{r} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial r} - g \\[2ex] \dfrac{d^*\rho}{dt} + \rho\left[\dfrac{1}{r^2\cos^2\varphi}\dfrac{\partial u^*}{\partial\lambda} + \dfrac{1}{r^2\cos\phi}\dfrac{\partial v^*}{\partial\varphi} + \dfrac{1}{r^2}\dfrac{\partial w r^2}{\partial r}\right] = 0 \end{cases}$$

The Φ terms are the gravity potentials; $\dfrac{\partial\Phi}{\partial\varphi}$ is due to the non-sphericity of the earth and is neglected as small.

The mapping onto a plane will be done by a Mercator projection (Eckart 1960); write



$$d\phi = \cos\phi\, d\mu$$

from which

$$\sec\phi = \cosh\mu$$

$$\tan\phi = \sinh\mu$$

$$\sin\phi = \tanh\mu$$

The system of equations (12) is rewritten as

$$
(13)\quad \begin{cases}
\dfrac{\partial u^*}{\partial t}+\dfrac{1}{r^2\cos^2\phi}\left[u^*\dfrac{\partial u^*}{\partial\lambda}+v^*\dfrac{\partial u^*}{\partial\mu}\right]+w\dfrac{\partial u^*}{\partial r}-2\Omega v^*\sin\phi+2\Omega w r\cos^2\phi=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial\lambda}\\[2ex]
\dfrac{\partial v^*}{\partial t}+\dfrac{1}{r^2\cos^2\phi}\left[u^*\dfrac{\partial v^*}{\partial\lambda}+v^*\dfrac{\partial v^*}{\partial\mu}+(u^{*2}+v^{*2})\sin\phi\right]+w\dfrac{\partial v^*}{\partial r}+2\Omega u^*\sin\phi=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial\mu}\\[2ex]
\dfrac{\partial w}{\partial t}+\dfrac{1}{r^2\cos^2\phi}\left[u^*\dfrac{\partial w}{\partial\lambda}+v^*\dfrac{\partial w}{\partial\mu}-\dfrac{u^{*2}+v^{*2}}{r}\right]+w\dfrac{\partial w}{\partial r}-\dfrac{2\Omega u^*}{r}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial r}-g\\[2ex]
\dfrac{\partial\rho}{\partial t}+\dfrac{1}{r^2\cos^2\phi}\left[u^*\dfrac{\partial\rho}{\partial\lambda}+v^*\dfrac{\partial\rho}{\partial\mu}\right]+w\dfrac{\partial\rho}{\partial r}+\rho\left[\dfrac{1}{r^2\cos^2\phi}\left(\dfrac{\partial u^*}{\partial\lambda}+\dfrac{\partial v^*}{\partial\mu}\right)+\dfrac{1}{r^2}\dfrac{\partial r w}{\partial r}\right]=0
\end{cases}
$$

Equations (13) are still exact. Two approximations will now be made:

(a) The spherical shell studied (ocean or atmosphere) is shallow:

put $\quad r = R + z, \qquad z << R.$

$$x = R\lambda, \quad y = R\mu$$

Replace r by R and ∂r by ∂z in (13). Studying the effect of this approximation on the first equation only for brevity, one has

$$
(14)\quad \frac{\partial u^*}{\partial t}+\frac{1}{R\cos^2\phi}\left[u^*\frac{\partial u^*}{\partial x}+v^*\frac{\partial u^*}{\partial y}\right]+w\frac{\partial u^*}{\partial z}-2\Omega v^*\sin\phi+2\Omega R w\cos^2\phi=-\frac{R}{\rho}\frac{\partial p}{\partial x}
$$

(b) The second approximation is an expansion of $\sin\phi$ and $\cos\phi$ in a MacLaurin Series.

$$\sin\phi = \phi - \frac{\phi^3}{6} + \cdots$$
$$\cos\phi = 1 - \frac{\phi^2}{2} + \cdots$$

Only linear terms in ϕ will be kept; needless to say, this is valid only near the equator, $\phi = 0$.

Dropping the * from u and v after a division by R, the complete system is now

$$(15)\quad \begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} - 2\Omega v\phi + 2\Omega w = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z} + 2\Omega u\phi = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} \\ \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z} - 2\Omega u = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} - g \\ \dfrac{\partial \rho}{\partial t} + \rho(\nabla\cdot \underset{\sim}{v}) = 0 \end{cases}$$

These equations are most frequently used without the $-2\Omega u$ and $2\Omega w$ terms; strictly, these terms cannot be neglected when ϕ is smaller than a certain value. In a baroclinic fluid this value of ϕ is much smaller than for a barotropic fluid; it will now be shown that the extra terms can be rejected to within one-half degree of the equator when the fluid is baroclinic.

In vector form,

$$(16)\quad \begin{cases} \dfrac{\partial \underset{\sim}{v}}{\partial t} + (\nabla\times\underset{\sim}{v})\times\underset{\sim}{v} + \nabla\left(\tfrac{1}{2}v^2\right) + 2\underset{\sim}{\Omega}\times\underset{\sim}{v} = -\dfrac{1}{\rho}\nabla p - \underset{\sim}{g} \\ \dfrac{\partial \rho}{\partial t} + \underset{\sim}{v}\cdot\nabla\rho + \rho\nabla\cdot\underset{\sim}{v} = 0 \end{cases}$$

and there is a conservative property $S(p,\rho)$ such that

$$(17)\quad \frac{\partial S}{\partial t} + \underset{\sim}{v}\cdot\nabla S = 0$$

Taking the curl of equations of motion, the vorticity equation is

$$\frac{d}{dt}(\nabla\times\underset{\sim}{v}+2\underset{\sim}{\Omega})\ (\nabla\times\underset{\sim}{v}+2\underset{\sim}{\Omega})\cdot\nabla\underset{\sim}{v}+(\nabla\times\underset{\sim}{v}+2\underset{\sim}{\Omega})\nabla\cdot\underset{\sim}{v}=-\nabla\frac{1}{\rho}\times\nabla p$$

using continuity, write the third term as $-\frac{\underset{\sim}{\omega}}{\rho}\frac{d\rho}{dt}$ where

$\underset{\sim}{\omega} = \nabla\times\underset{\sim}{v} + 2\underset{\sim}{\Omega}$; therefore,

$$\frac{d\underset{\sim}{\omega}}{dt}-\underset{\sim}{\omega}\cdot\nabla\underset{\sim}{v}-\frac{\underset{\sim}{\omega}}{\rho}\frac{d\rho}{dt}=-\nabla\frac{1}{\rho}\times\nabla p \qquad (18)$$

Take the gradient of (17), multiply by $\frac{\underset{\sim}{\omega}}{\rho}$ and add to (18) multiplied by $\left\{\frac{\nabla S}{\rho}\cdot\right\}$. This yields

$$\frac{d}{dt}\left\{\frac{[2\underset{\sim}{\Omega}+\nabla\times\underset{\sim}{v}]\cdot\nabla S}{\rho}\right\}=0 \qquad (19)$$

(Ertel, Meteorologic Zeitschrifz, 1942)

Back in spherical coordinates, (19) is

$$\left\{\frac{1}{\rho r\cos\varphi}\left[\frac{\partial v}{\partial\lambda}-\frac{\partial}{\partial\varphi}(\cos\varphi u)\right]+2\Omega\sin\varphi\right\}\frac{\partial s}{\partial r}+\left\{\frac{1}{r\cos\varphi}\left[\frac{\partial}{\partial r}(r\cos\varphi u)-\frac{\partial w}{\partial\lambda}\right]-2\Omega\cos\varphi\right\}\frac{1}{r\rho}\frac{\partial s}{\partial\varphi}$$

$$+\frac{1}{\rho r}\left(\frac{\partial w}{\partial\varphi}-\frac{\partial}{\partial r}rv\right)\frac{1}{r\cos\varphi}\frac{\partial s}{\partial z}=0 \qquad (20)$$

The terms in $2\Omega\cos\varphi$ can be neglected if

$$\frac{\frac{1}{r}\frac{\partial s}{\partial\varphi}}{\frac{\partial s}{\partial r}}\ll\tan\phi$$

introducing scale lengths D and L for the vertical and horizontal variations of S,

$$\frac{D}{L}\ll\tan\phi.$$

For the equatorial current for example, this is

$$\frac{100\,Km}{100\,m} = 10^{-3} \ll \tan\phi$$

then, $\tan\phi_c > 10^{-2}$ is sufficient, or $\phi_c > 10^{-2}$, which is $\phi_c \sim 30'$. The excluded region of applicability of the equations without the $\Omega\cos\phi$ terms is then very small.